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THE CONDITION OF CLEBSCH FOR MIXED CONSTRAINED OPTIMAL CONTROL PROBLEMS

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Abstract. The second order necessary condition of Legendre in the calculus of variations, which playes a crucial role in the theory, has been generalized in several directions. In particular, for certain classes of optimal control problems, it corresponds to the condition of Clebsch, which can be derived as a consequence of the maximum principle. However, as we show in this paper, the usual normality assumptions imposed for problems involving mixed equality and inequality constraints can be weakened, leaving the condition of Clebsch unchanged. A simple example, which lies beyond the scope found in the literature, illustrates the usefulness of our main result.

Keywords. Clebsch condition; Equality and inequality mixed constraints; Mathematical programming; Optimal control.

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1. INTRODUCTION

In the calculus of variations, one of the best known necessary conditions satisfied by a solution to the problem is that of Legendre (1786) which, in turn, is a consequence of the (stronger) Weierstrass condition (1880). For problems involving constraints, the Legendre condition was generalized by Clebsch (1858) and, later, it was established for certain classes of optimal control problems in terms of the Hamiltonian function.

For singular problems in optimal control, there has been a great interest in generalizing the condition of Legendre-Clebsch, mainly due to the appearance of such problems in the aerospace field and the chemical industry. In particular, we refer the reader to the results of Kelley, Kopp, Moyer, Johnson, Goh, Tait, Robbins (see [8, 9, 10, 13, 14, 16] and references therein). For variational problems with state constraints that do not include $\dot{x}$, an extended Legendre-Clebsch condition has been obtained in [15].

In this paper, we shall deal with optimal control problems involving mixed state-control equality and inequality constraints. We shall derive the condition of Clebsch by applying, on

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one hand, second order necessary conditions for mathematical programming problems and, on the other, the maximum principle. This approach will allow us to weaken the usual normality assumptions imposed on optimal control problems involving equality and inequality constraints.

In Section 2, we provide a brief summary of first and second order necessary conditions for problems in the finite dimensional case, and state and prove a fundamental result applicable to problems involving piecewise continuous functions. In Section 3, we state the optimal control problem we shall deal with, and derive the condition of Clebsch as a consequence of the maximum principle. The assumptions imposed for this derivation are the standard, classical normality conditions expressed in terms of the linear independence of gradients for active constraints. Section 4 provides a new approach for the derivation of the condition of Clebsch which allows us to weaken the previous assumptions on the constraints. The gist of the argument appears in the proof of Theorem 3.7 in [19]. A simple example is given, which lies beyond the scope found in the literature. It illustrates how the usual normality assumption does not hold, while the one proposed can be applied and the second order condition is then easily verified.

2. Basic results

Our starting point will be a brief summary of well-known first and second order necessary conditions, under standard smoothness assumptions, for the basic nonlinear programming problem involving equality and inequality constraints. The main ideas are based on two classical books by Hestenes [11, 12].

Suppose we are given functions $f, g_1, \ldots, g_m$ mapping $\mathbf{R}^n$ to $\mathbf{R}$, $A = \{1, \ldots, p\}$, $B = \{p+1, \ldots, m\}$, and

$$S := \{x \in \mathbf{R}^n \mid g_\alpha(x) \leq 0 \ (\alpha \in A),\ g_\beta(x) = 0 \ (\beta \in B)\}.$$

Our problem, which we label (N), will be that of minimizing f on S.

We assume that the functions delimiting the problem are continuous on a neighborhood of given point $x_0 \in S$ and possess second differentials at x_0.

We begin with well-known first order conditions. In the following definition, the real numbers $\lambda_1, \ldots, \lambda_m$ are the *Kuhn-Tucker* or *Lagrange multipliers*, the function F is the standard *Lagrangian*, g is the function mapping $\mathbf{R}^n$ to $\mathbf{R}^m$ whose components are $g_1, \ldots, g_m$, and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbf{R}^m$ so that $\langle \lambda, g(x) \rangle = \sum_{i=1}^m \lambda_i g_i(x)$.

Definition 2.1. Denote by $\Lambda(f, x_0)$ the set of all $\lambda \in \mathbf{R}^m$ satisfying the *Karush-Kuhn-Tucker conditions* (or *first order Lagrange multiplier rule*)

i. $\lambda_\alpha \geq 0$ and $\lambda_\alpha g_\alpha(x_0) = 0$ $(\alpha \in A)$.
ii. If $F(x) := f(x) + \langle \lambda, g(x) \rangle$ then $F'(x_0) = 0$.

In general, if x_0 affords a local minimum to f on S, these conditions may not hold at x_0, and some additional assumptions should be imposed to guarantee that $\Lambda(f, x_0) \neq \emptyset$. Assumptions of this nature are usually referred to as *constraint qualifications* (see, in particular, [7, 17, 21] and, for a broader definition in terms of critical directions, see [2]).

Definition 2.2. We say x_0 is *normal relative to S* if $\lambda_i = 0$ $(i \in A \cup B)$ is the only solution of

i. $\lambda_\alpha \geq 0$ and $\lambda_\alpha g_\alpha(x_0) = 0$ $(\alpha \in A)$.
ii. $\sum_{i=1}^m \lambda_i g_i'(x_0) = 0$.

Our first result states that normality is a constraint qualification (see, for example, [7, 12]).

Theorem 2.3. *If x_0 solves* (N) *locally and is normal with respect to S, then* $\Lambda(f,x_0) \neq \emptyset$.

Let us briefly explain how this result can be proved. Let $I(x_0) := \{\alpha \in A \mid g_\alpha(x_0) = 0\}$ denote the set of *active indices* at x_0 and define the set of vectors satisfying the *tangential constraints* at x_0 as

$$R_S(x_0) := \{h \in \mathbf{R}^n \mid g'_\alpha(x_0;h) \leq 0 \ (\alpha \in I(x_0)),\ g'_\beta(x_0;h) = 0 \ (\beta \in B)\}.$$

From the theory of convex cones (see, for example, [11, 12]) or using the Farkas-Minkowski theorem of the alternative (see [7]), we know that the first order Lagrange multiplier rule holds at x_0 if and only if $f'(x_0;h) \geq 0$ for all $h \in R_S(x_0)$. In other words, $\Lambda(f,x_0) \neq \emptyset$ if and only if $-f'(x_0) \in R_S^*(x_0)$, where, for $B \subset \mathbf{R}^n$, the closed convex cone $B^* = \{z \in \mathbf{R}^n \mid \langle y,z\rangle \leq 0$ for all $y \in B\}$ is the *dual* or *polar cone* of B.

On the other hand, let $T_S(x_0)$ be the *tangent cone of S at* x_0, defined as the (closed) cone determined by the unit vectors h for which there exists a sequence $\{x_q\}$ in S converging to x_0 in the direction h, in the sense that $x_q \neq x_0$, and

$$\lim_{q\to\infty} |x_q - x_0| = 0, \qquad \lim_{q\to\infty} \frac{x_q - x_0}{|x_q - x_0|} = h.$$

We note that, if $\{x_q\}$ converges to x_0 in the direction h and f has a differential at x_0, then

$$\lim_{q\to\infty} \frac{f(x_q) - f(x_0)}{|x_q - x_0|} = f'(x_0;h).$$

This implies that, if x_0 solves (N) locally, then $f'(x_0;h) \geq 0$ for all $h \in T_S(x_0)$, that is, $-f'(x_0) \in T_S^*(x_0)$. It follows that, if $T_S(x_0)$ and $R_S(x_0)$ coincide, in which case x_0 is said to be *regular with respect to S*, then $\Lambda(f,x_0) \neq \emptyset$. Finally, as shown in [12], if x_0 is normal relative to S then it is regular relative to S, and Theorem 2.3 is proved.

A straightforward, alternative proof, can be given if we invoke the well-known first order conditions due to Fritz John (see, for example, [18] based on the theory of augmentability, or [7] using Motzkin theorem of the alternative).

Theorem 2.4. *If x_0 solves* (N) *locally, then* $\exists \lambda_0 \geq 0$ *and* $\lambda \in \mathbf{R}^m$*, not both zero, such that*

i. $\lambda_\alpha \geq 0$ *and* $\lambda_\alpha g_\alpha(x_0) = 0$ $(\alpha \in A)$.
ii. *If* $F_0(x) := \lambda_0 f(x) + \langle \lambda, g(x)\rangle$ *then* $F'_0(x_0) = 0$.

The Karush-Kuhn-Tucker conditions correspond to (i) and (ii) above with $\lambda_0 = 1$ (in which case, as mentioned above, the function F_0 is the standard Lagrangian), and a constrained qualification is a condition that guarantees the nonvanishing of the cost multiplier.

We note that, if we consider the set of equality constraints for active indices, that is,

$$S_0\ [= S_0(x_0)]\ := \{x \in \mathbf{R}^n \mid g_\gamma(x) = 0 \ (\gamma \in I(x_0) \cup B)\},$$

then $x \in S_0(x_0)$ is normal relative to $S_0(x_0)$ if $\lambda = 0$ is the only solution of

i. $\lambda_\alpha g_\alpha(x_0) = 0$ $(\alpha \in A)$.
ii. $\sum_{i=1}^m \lambda_i g'_i(x) = 0$.

In other words, the linear equations $g'_i(x;h) = 0$ $(i \in I(x_0) \cup B)$ in h are linearly independent, a condition usually known as the linear independence constraint qualification.

Clearly, it implies normality relative to S and, as one readily verifies, it also implies uniqueness of the Lagrange multiplier (we refer to [1, 4, 5, 17, 21] where the question of uniqueness of multipliers is studied).

Theorem 2.5. *If x_0 solves* (N) *locally and is normal with respect to $S_0(x_0)$, then there exists a unique $\lambda \in \Lambda(f,x_0)$.*

For second order necessary conditions, if $\{x_q\}$ converges to x_0 in the direction h and f has a second differential at x_0, then

$$\lim_{q\to\infty} \frac{f(x_q)-f(x_0)-f'(x_0;x_q-x_0)}{|x_q-x_0|^2} = \frac{1}{2}f''(x_0;h).$$

Thus, if x_0 solves (N) locally and $f'(x_0)=0$, then $f''(x_0;h)\geq 0$ for all $h\in T_S(x_0)$. This yields the second order Lagrange multiplier rule (see also [2, 6, 7, 17]).

Theorem 2.6. *Suppose $\lambda \in \Lambda(f,x_0)$ and $S_1 := \{x\in S \mid F(x)=f(x)\}$ where $F(x)=f(x)+\langle\lambda,g(x)\rangle$. If x_0 is normal relative to S_1 and solves* (N) *locally, then $F''(x_0;h)\geq 0$ for all $h\in R_{S_1}(x_0)$.*

Note that S_1 depends on the Lagrange multiplier $\lambda\in\mathbf{R}^m$ and, if $\Gamma=\{\alpha\in A \mid \lambda_\alpha>0\}$, then

$$\begin{aligned} S_1\ [=S_1(\lambda)] &:= \{x\in\mathbf{R}^n \mid g_\alpha(x)\leq 0\ (\alpha\in A,\ \lambda_\alpha=0),\ g_\beta(x)=0\ (\beta\in\Gamma\cup B)\} \\ &= \{x\in S \mid g_\alpha(x)=0\ (\alpha\in\Gamma)\}. \end{aligned}$$

By definition of tangential constraints, we have

$$\begin{aligned} R_{S_1}(x_0) &= \{h\in\mathbf{R}^n \mid g'_\alpha(x_0;h)\leq 0\ (\alpha\in I(x_0),\ \lambda_\alpha=0),\ g'_\beta(x_0;h)=0\ (\beta\in\Gamma\cup B)\} \\ &= \{h\in R_S(x_0) \mid g'_\alpha(x_0;h)=0\ (\alpha\in\Gamma)\} \\ &= \{h\in R_S(x_0) \mid f'(x_0;h)=0\}. \end{aligned}$$

Moreover, given $\lambda\in\Lambda(f,x_0)$, x_0 is a normal relative to $S_1(\lambda)$ if $\mu=0$ is the only solution of

i. $\mu_\alpha\geq 0$ and $\mu_\alpha g_\alpha(x_0)=0$ $(\alpha\in A,\ \lambda_\alpha=0)$.
ii. $\sum_{i=1}^m \mu_i g'_i(x_0)=0$.

To prove Theorem 2.6, we note that, since the point x_0 minimizes F locally on S_1 and $F'(x_0)=0$, we have $F''(x_0;h)\geq 0$ for all $h\in T_{S_1}(x_0)$. Normality relative to S_1 implies regularity relative to S_1, and the result follows.

Some relations between the different sets we have treated are straightforward. If $\lambda\in\mathbf{R}^m$ with $\lambda_\alpha\geq 0$ and $\lambda_\alpha g_\alpha(x_0)=0$ $(\alpha\in A)$, we have $R_{S_0}(x_0)\subset R_{S_1}(x_0)\subset R_S(x_0)$ where

$$R_{S_0}(x_0)=\{h\in\mathbf{R}^n \mid g'_\gamma(x_0;h)=0\ (\gamma\in I(x_0)\cup B)\}.$$

Also, if x_0 is a normal point of $S_0(x_0)$, then it is a normal point of $S_1(\lambda)$, and hence a normal point of S.

We end this section with a basic result, which will be used in an optimal control context, showing how the necessary conditions derived above can be applied to an optimization problem involving a set of piecewise continuous functions. In what follows, '$*$' denotes transpose.

Theorem 2.7. *Suppose $f,\varphi_1,\ldots,\varphi_q\colon \mathbf{R}\times\mathbf{R}^m\to\mathbf{R}$ are continuous functions having continuous derivatives with respect to u, and let*

$$\mathscr{A}=\{(t,u)\in\mathbf{R}\times\mathbf{R}^m \mid \varphi_\alpha(t,u)\leq 0\ (\alpha\in R),\ \varphi_\beta(t,u)=0\ (\beta\in Q)\}$$

where $R = \{1,\dots,r\}$, $Q = \{r+1,\dots,q\}$. Assume that the $q \times (m+r)$-dimensional matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k} \quad \delta_{i\alpha}\varphi_\alpha\right) \qquad (i = 1,\dots,q;\ \alpha = 1,\dots,r;\ k = 1,\dots,m)$$

has rank q on $\mathscr{A}$, where $\delta_{\alpha\alpha} = 1$, $\delta_{\alpha\beta} = 0$ $(\alpha \neq \beta)$. Define

$$\mathscr{C}(\mathscr{A}) := \{u\colon T \to \mathbf{R}^m \mid u \text{ is piecewise continuous and } (t,u(t)) \in \mathscr{A}\ (t \in T)\},$$

and $F(t,u,\mu) := f(t,u) + \langle \mu, \varphi(t,u)\rangle$. Let $u_0 \in \mathscr{C}(\mathscr{A})$ and suppose

$$f(t,u) \geq f(t,u_0(t)) \quad (t \in T) \quad \text{whenever } (t,u) \in \mathscr{A}.$$

Then there exists a unique $\mu\colon T \to \mathbf{R}^q$, piecewise continuous on T and continuous at each point of continuity of u_0, such that

i. *$\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(t,u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$.*
ii. *$F_u(t,u_0(t),\mu(t)) = 0$ $(t \in T)$.*
iii. *If also f and φ are of class C^2, then $\langle h, F_{uu}(t,u_0(t),\mu(t))h\rangle \geq 0$ for all $h \in \mathbf{R}^m$ satisfying*

$$\frac{\partial \varphi_\alpha}{\partial u}(t,u_0(t))h \leq 0 \quad \text{if } \alpha \in R \text{ with } \mu_\alpha(t) = 0 \text{ and } \varphi_\alpha(t,u_0(t)) = 0,$$

$$\frac{\partial \varphi_\beta}{\partial u}(t,u_0(t))h = 0 \quad \text{if } \beta \in R \text{ with } \mu_\beta(t) > 0, \text{ or } \beta \in Q.$$

Proof. Consider the $p \times m$ matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k}\right) \qquad (i = i_1,\dots,i_p;\ k = 1,\dots,m)$$

where $i_1,\dots,i_p$ are the indices $i \in R \cup Q$ such that $\varphi_i(t,u) = 0$. Our rank assumption is equivalent to having this matrix rank p, that is, that the row vectors

$$\frac{\partial \varphi_{i_1}(t,u)}{\partial u},\dots,\frac{\partial \varphi_{i_p}(t,u)}{\partial u}$$

are linearly independent (see [6] for details).

Fix $t \in T$, let $g(u) := f(t,u)$, $\psi_i(u) := \varphi_i(t,u)$ $(i = 1,\dots,q)$, and consider the set

$$S := \{u \in \mathbf{R}^m \mid \psi_\alpha(u) \leq 0\ (\alpha \in R),\ \psi_\beta(u) = 0\ (\beta \in Q)\}.$$

Thus $u_0(t)$ affords a minimum to g on S. Let

$$I_a(u) := \{\alpha \in R \mid \psi_\alpha(u) = 0\}.$$

By our rank assumption, the set

$$\{\psi'_\gamma(u_0(t)) \mid \gamma \in I_a(u_0(t)) \cup Q\}$$

is linearly independent. Thus, $u_0(t)$ is normal with respect to the set of equality constraints for active indices, that is,

$$S_0\ [= S_0(u_0(t))]\ = \{u \in \mathbf{R}^m \mid \psi_\gamma(u) = 0\ (\gamma \in I_a(u_0(t)) \cup Q)\}.$$

In view of Theorem 2.5, there exists a unique $\mu(t) \in \mathbf{R}^q$ such that $F_u(t,u_0(t),\mu(t)) = 0$ and $\mu_\alpha(t) \geq 0$ $(\alpha \in R)$ with $\mu_\alpha(t) = 0$ in case $\varphi_\alpha(t,u_0(t)) < 0$. This proves (i) and (ii). Also, (iii) follows from Theorem 2.6 since normality relative to S_0 implies normality relative to S_1 given by

$$S_1\ [= S_1(\mu(t))]\ = \{u \in S \mid F(t,u,\mu(t)) = f(t,u)\}.$$

To prove the continuity properties of μ, let $s \in T$ and set $v = u_0(s)$. At a point of discontinuity of u_0 choose one of the two values $u_0(s+0)$ and $u_0(s-0)$ for v. Let $P = \{\alpha_1, \dots, \alpha_l\}$ be the set of indices α in $R \cup Q$ such that $\varphi_\alpha(s,v) = 0$. On an interval $[a,b]$ containing s we have $\varphi_\alpha(t,u_0(t)) < 0$ ($\alpha \in R \sim P$) and hence $\mu_\alpha(t) = 0$ ($\alpha \in R \sim P$, $t \in [a,b]$). Let $\hat{\mu} = (\mu_{\alpha_1}, \dots, \mu_{\alpha_l})$ and $\hat{\varphi} := (\varphi_{\alpha_1}, \dots, \varphi_{\alpha_l})$ so that $\hat{\varphi}_u(s,v)$ has rank l and $|\hat{\varphi}_u(s,v)\hat{\varphi}_u^*(s,v)| \neq 0$. Diminish $[a,b]$ if necessary so that this inequality holds on $[a,b]$ and so that u_0 is continuous on this interval. We then have

$$f_u(t,u_0(t)) + \mu^*(t)\varphi_u(t,u_0(t)) = f_u(t,u_0(t)) + \hat{\mu}^*(t)\hat{\varphi}_u(t,u_0(t)) = 0 \quad (t \in [a,b])$$

and so, multiplying by $\hat{\varphi}_u^*(t,u_0(t))$ and summing, we find that

$$f_u(t,u_0(t))\hat{\varphi}_u^*(t,u_0(t)) + \hat{\mu}^*(t)\hat{\varphi}_u(t,u_0(t))\hat{\varphi}_u^*(t,u_0(t)) = 0 \quad (t \in [a,b]).$$

Thus this equation has a continuous solution μ on the interval $[a,b]$. □

3. MIXED CONSTRAINED OPTIMAL CONTROL PROBLEMS

We turn now to optimal control problems with equality and inequality constraints. The problem we shall be concerned with, involving mixed state-control constraints and fixed endpoints, will be posed over piecewise C^1 trajectories and piecewise continuous controls.

Our aim in this section is to provide a simple proof of the condition of Clebsch by an application of Theorem 2.7.

To state the problem, suppose we are given an interval $T := [t_0,t_1]$ in $\mathbf{R}$, two points ξ_0, ξ_1 in $\mathbf{R}^n$, and functions L, f, $\varphi = (\varphi_1, \dots, \varphi_q)$ mapping $T \times \mathbf{R}^n \times \mathbf{R}^m$ to $\mathbf{R}$, $\mathbf{R}^n$ and $\mathbf{R}^q$ respectively.

Denote by X the space of piecewise C^1 functions mapping T to $\mathbf{R}^n$, and by $\mathscr{U}_k$ the space of piecewise continuous functions mapping T to $\mathbf{R}^k$ ($k \in \mathbf{N}$). Let $Z := X \times \mathscr{U}_m$, and consider the following sets:

$$D := \{(x,u) \in Z \mid \dot{x}(t) = f(t,x(t),u(t))\ (t \in T),\ x(t_0) = \xi_0,\ x(t_1) = \xi_1\},$$

$$U := \{(t,x,u) \in T \times \mathbf{R}^n \times \mathbf{R}^m \mid \varphi_\alpha(t,x,u) \leq 0\ (\alpha \in R),\ \varphi_\beta(t,x,u) = 0\ (\beta \in Q)\}$$

where $R = \{1, \dots, r\}$, $Q = \{r+1, \dots, q\}$. The problem we shall deal with, which we label (P), is that of minimizing the functional

$$I(x,u) = \int_{t_0}^{t_1} L(t,x(t),u(t))dt$$

over $S := \{(x,u) \in D \mid (t,x(t),u(t)) \in U\ (t \in T)\}$.

We assume that L, f and φ are C^2 and the $q \times (m+r)$ matrix

$$\begin{pmatrix} \dfrac{\partial \varphi_i}{\partial u^k} & \delta_{i\alpha}\varphi_\alpha \end{pmatrix} \qquad (i = 1, \dots, q;\ \alpha = 1, \dots, r;\ k = 1, \dots, m)$$

has rank q on U. As mentioned before, this condition is equivalent to the condition that, at each point (t,x,u) in U, the $p \times m$ matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k}\right) \qquad (i = i_1, \dots, i_p;\ k = 1, \dots, m)$$

has rank p, where $i_1, \ldots, i_p$ are the indices $i \in \{1, \ldots, q\}$ such that $\varphi_i(t,x,u) = 0$. This rank assumption will be labelled (A).

Given (x_0, u_0) in Z, we use the notation $[t]$ to represent the point $(t, x_0(t), u_0(t))$.

Define the Hamiltonian function H mapping $T \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}$ to $\mathbf{R}$ as

$$H(t,x,u,p,\mu,\lambda) := \langle p, f(t,x,u)\rangle - \lambda L(t,x,u) - \langle \mu, \varphi(t,x,u)\rangle.$$

First order necessary conditions for a solution to the problem, in the form of a maximum principle, are well established. For our purposes, we shall find convenient to quote the following version from [11]. For a more general setting, both in the statement of the problem and the regularity assumptions, we refer to [3, 20] and references therein, where a clear and rigorous nonsmooth analysis approach has been developed.

Theorem 3.1. *Suppose* (x_0, u_0) *solves* (P). *Then there exist* $\lambda_0 \geq 0$, $p \in X$ *and* $\mu \in \mathscr{U}_q$ *continuous on each interval of continuity of* u_0, *not vanishing simultaneously on* T, *such that*

a. $\mu_\alpha(t) \geq 0$ *and* $\mu_\alpha(t)\varphi_\alpha[t] = 0$ $(\alpha \in R,\ t \in T)$*;*

b. $\dot{p}(t) = -H_x^*([t], p(t), \mu(t), \lambda_0)$ *on every interval of continuity of* u_0.

c. $H_u([t], p(t), \mu(t), \lambda_0) = 0$ $(t \in T)$.

d. $H(t, x_0(t), u, p(t), 0, \lambda_0) \leq H([t], p(t), 0, \lambda_0)$ *for all* (t,u) *such that* $(t, x_0(t), u) \in U$.

The condition of Clebsch, under the rank assumptions stated above, corresponds to the following result. The proof we provide is based on the existence of multipliers appearing in Theorem 2.7.

Corollary 3.2. *Suppose* (x_0, u_0) *solves* (P). *Let* λ_0, p *and* μ *be as in Theorem 3.1. Then*

$$\langle h, H_{uu}([t], p(t), \mu(t), \lambda_0)h\rangle \leq 0$$

for all $h \in \mathbf{R}^m$ *satisfying*

$$\frac{\partial \varphi_\alpha}{\partial u}[t]h \leq 0 \quad \text{if } \alpha \in R \text{ with } \mu_\alpha(t) = 0 \text{ and } \varphi_\alpha[t] = 0,$$

$$\frac{\partial \varphi_\beta}{\partial u}[t]h = 0 \quad \text{if } \beta \in R \text{ with } \mu_\beta(t) > 0, \text{ or } \beta \in Q.$$

Proof. Define $\psi(t,u) := \varphi(t, x_0(t), u)$ and

$$g(t,u) := -H(t, x_0(t), u, p(t), 0, \lambda_0) = \lambda_0 L(t, x_0(t), u) - \langle p(t), f(t, x_0(t), u)\rangle$$

for all $(t,u) \in T \times \mathbf{R}^m$ and set

$$\mathscr{A} := \{(t,u) \in T \times \mathbf{R}^m \mid \psi_\alpha(t,u) \leq 0\ (\alpha \in R),\ \psi_\beta(t,u) = 0\ (\beta \in Q)\}.$$

Therefore,

$$g(t,u) \geq g(t, u_0(t)) \quad (t \in T) \quad \text{whenever } (t,u) \in \mathscr{A}.$$

By Theorem 2.7, there exists a unique $\nu \in \mathscr{U}_q$ such that, if

$$G(t,u,\nu) = g(t,u) + \langle \nu, \psi(t,u)\rangle,$$

then $G_u(t, u_0(t), \nu(t)) = 0$ $(t \in T)$. Moreover,

$$\langle h, G_{uu}(t, u_0(t), \nu(t))h\rangle \geq 0$$

for all $h \in \mathbf{R}^m$ such that

$$\frac{\partial \psi_\alpha}{\partial u}(t,u_0(t))h \leq 0 \quad \text{if } \alpha \in R \text{ with } \nu_\alpha(t) = 0 \text{ and } \psi_\alpha(t,u_0(t)) = 0,$$

$$\frac{\partial \psi_\beta}{\partial u}(t,u_0(t))h = 0 \quad \text{if } \beta \in R \text{ with } \nu_\beta(t) > 0, \text{ or } \beta \in Q.$$

Now, by Theorem 3.1(c), $G_u(t,u_0(t),\mu(t)) = 0$ $(t \in T)$ and so, by uniqueness, $\nu \equiv \mu$. This implies that

$$G(t,u,\mu) = -H(t,x_0(t),u,p(t),\mu,\lambda_0)$$

and the result follows. □

This result has, in particular, an immediate consequence for problems linear in the state. Given $(x_0,u_0,p,\mu,\lambda_0) \in Z \times X \times \mathscr{U}_q \times \mathbf{R}$, let

$$J((x_0,u_0,p,\mu,\lambda_0);(y,v)) := \int_{t_0}^{t_1} 2\Omega(t,y(t),v(t))dt \quad ((y,v) \in Z)$$

where, for all $(t,y,v) \in T \times \mathbf{R}^n \times \mathbf{R}^m$,

$$2\Omega(t,y,v) := -\left[\langle y,H_{xx}(t)y\rangle + 2\langle y,H_{xu}(t)v\rangle + \langle v,H_{uu}(t)v\rangle\right]$$

and $H(t)$ denotes $H([t],p(t),\mu(t),\lambda_0)$. Suppose (x_0,u_0) solves (P). Let λ_0, p and μ be as in Theorem 3.1. If the problem is linear in the state, that is,

$$f(t,x,u) = A(t)x + f(t,u), \quad L(t,x,u) = M(t)x + L(t,u), \quad \varphi(t,x,u) = \psi(t)x + \varphi(t,u),$$

then

$$H(t,x,u,p,\mu,\lambda) = \langle p,A(t)x + f(t,u)\rangle - \lambda(M(t)x + L(t,u)) - \langle \mu,\psi(t)x + \varphi(t,u)\rangle$$

and so $2\Omega(t,y,v) = \langle v,H_{uu}(t)v\rangle$. By Corollary 3.2,

$$J((x_0,u_0,p,\mu,\lambda_0);(y,v)) \geq 0$$

for all $(y,v) \in Z$ with v satisfying

$$\frac{\partial \varphi_\alpha}{\partial u}[t]v(t) \leq 0 \quad \text{if } \alpha \in R \text{ with } \mu_\alpha(t) = 0 \text{ and } \varphi_\alpha[t] = 0,$$

$$\frac{\partial \varphi_\beta}{\partial u}[t]v(t) = 0 \quad \text{if } \beta \in R \text{ with } \mu_\beta(t) > 0, \text{ or } \beta \in Q.$$

4. A new approach

In this section we obtain the condition of Clebsch for mixed inequality and equality constrained optimal control problems under assumptions weaker than the previous ones.

The data is as before with

$$U = \{(t,x,u) \in T \times \mathbf{R}^n \times \mathbf{R}^m \mid \varphi_\alpha(t,x,u) \leq 0 \ (\alpha \in R),\ \varphi_\beta(t,x,u) = 0 \ (\beta \in Q)\}.$$

Thus, we are interested in minimizing $I(x,u) = \int_{t_0}^{t_1} L(t,x(t),u(t))dt$ subject to $(x,u) \in Z$ and

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)) \ (t \in T); \\ x(t_0) = \xi_0,\ x(t_1) = \xi_1; \\ \varphi_\alpha(t,x(t),u(t)) \leq 0,\ \varphi_\beta(t,x(t),u(t)) = 0 \ (\alpha \in R,\ \beta \in Q,\ t \in T). \end{cases}$$

In the previous section, we assumed that at each point (t,x,u) in U, the matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k}\right) \qquad (i = i_1,\dots,i_p;\ k = 1,\dots,m)$$

has rank p, where $i_1,\dots,i_p$ are the indices $i \in I_a(t,x,u)\cup Q$ and

$$I_a(t,x,u) := \{\alpha \in R \mid \varphi_\alpha(t,x,u) = 0\}$$

denotes the set of *active indices at* (t,x,u). Let us first provide a new proof of Corollary 3.2 without changing this assumption.

Proof. We note first that (d) in Theorem 3.1 can be put in the alternate form

$$H(t,x_0(t),u,p(t),\mu(t),\lambda_0) + \langle \mu(t),\varphi(t,x_0(t),u)\rangle \le H([t],p(t),\mu(t),\lambda_0)$$

for all (t,u) such that $(t,x_0(t),u) \in U$. Fix $t \in T$ and define $g\colon \mathbf{R}^m \to \mathbf{R}^q$ as $g(u) := \varphi(t,x_0(t),u)$ and $G\colon \mathbf{R}^m \to \mathbf{R}$ as

$$G(u) := H([t],p(t),\mu(t),\lambda_0) - H(t,x_0(t),u,p(t),\mu(t),\lambda_0) - \langle \mu(t),g(u)\rangle \quad (u \in \mathbf{R}^m).$$

We emphasize the fact that both G and g depend on the fixed point t in T.

We have $G(u) \ge 0 = G(u_0(t))$ for all $u \in \tilde{S}$ where

$$\tilde{S} := \{u \in \mathbf{R}^m \mid g_\alpha(u) \le 0\ (\alpha \in R),\ g_\beta(u) = 0\ (\beta \in Q)\}.$$

Denote the set of active indices (with respect to g) as before,

$$I(u) = \{\alpha \in R \mid g_\alpha(u) = 0\}$$

and note that

$$I(u_0(t)) = \{\alpha \in R \mid \varphi[t] = 0\} = I_a[t].$$

Define the sets $\tilde{S}_0$ and $\tilde{S}_1$ as follows:

$$\tilde{S}_0\ [= \tilde{S}_0(u_0(t))]\ := \{u \in \mathbf{R}^n \mid g_\gamma(u) = 0\ (\gamma \in I(u_0(t))\cup Q)\},$$

and, for $\lambda \in \mathbf{R}^q$ with $\lambda_\alpha \ge 0$ and $\lambda_\alpha g_\alpha(u_0(t)) = 0$ $(\alpha \in R)$, let

$$\begin{aligned}\tilde{S}_1\ [= \tilde{S}_1(\lambda)]\ &:= \{u \in \mathbf{R}^m \mid g_\alpha(u) \le 0\ (\alpha \in R,\ \lambda_\alpha = 0),\ g_\beta(u) = 0\ (\beta \in \Gamma\cup Q)\}\\ &= \{u \in \tilde{S} \mid g_\alpha(u) = 0\ (\alpha \in \Gamma)\}\end{aligned}$$

where $\Gamma = \{\alpha \in R \mid \lambda_\alpha > 0\}$. By our rank assumption, the set $\{g'_\gamma(u_0(t)) \mid \gamma \in I(u_0(t))\cup Q\}$ is linearly independent, that is, $u_0(t)$ is normal relative to $\tilde{S}_0(u_0(t))$. In other words, $\lambda = 0$ is the only solution of

i. $\lambda_\alpha g_\alpha(u_0(t)) = 0$ $(\alpha \in R)$.
ii. $\sum_1^q \lambda_i g'_i(u_0(t)) = 0$.

In particular, this implies that $u_0(t)$ is normal with respect to $\tilde{S}$, that is, $\lambda = 0$ is the only solution of

i. $\lambda_\alpha \ge 0$ and $\lambda_\alpha g_\alpha(u_0(t)) = 0$ $(\alpha \in R)$.
ii. $\sum_1^q \lambda_i g'_i(u_0(t)) = 0$.

By Theorem 2.3, $\Lambda(G,u_0(t)) \ne \emptyset$, that is, there exists $\lambda \in \mathbf{R}^q$ such that

i. $\lambda_\alpha \ge 0$ and $\lambda_\alpha g_\alpha(u_0(t)) = 0$ $(\alpha \in R)$.
ii. If $K(u) := G(u) + \langle \lambda, g(u)\rangle$ then $K'(u_0(t)) = 0$.

As we know, the normality of $u_0(t)$ with respect to $\tilde{S}_0(u_0(t))$ also implies that $u_0(t)$ is normal with respect to $\tilde{S}_1(\lambda) = \{u \in \tilde{S} \mid K(u) = G(u)\}$. Therefore, $\nu = 0$ is the only solution of

i. $\nu_\alpha \geq 0$ and $\nu_\alpha g_\alpha(u_0(t)) = 0$ ($\alpha \in R,\ \lambda_\alpha = 0$).

ii. $\sum_1^q \nu_i g_i'(u_0(t)) = 0$.

By Theorem 2.6, $K''(u_0(t);h) \geq 0$ for all $h \in R_{\tilde{S}_1}(u_0(t))$.

As mentioned before, if $\Gamma = \{\alpha \in R \mid \lambda_\alpha > 0\}$, then

$$\begin{aligned}\tilde{S}_1 &= \{u \in \mathbf{R}^m \mid g_\alpha(u) \leq 0\ (\alpha \in R,\ \lambda_\alpha = 0),\ g_\beta(u) = 0\ (\beta \in \Gamma \cup Q)\} \\ &= \{u \in \tilde{S} \mid g_\alpha(u) = 0\ (\alpha \in \Gamma)\}.\end{aligned}$$

Therefore

$$\begin{aligned}R_{\tilde{S}_1}(u) &= \{h \in \mathbf{R}^m \mid g_\alpha'(u;h) \leq 0\ (\alpha \in I(u),\ \lambda_\alpha = 0),\ g_\beta'(u;h) = 0\ (\beta \in \Gamma \cup Q)\} \\ &= \{h \in R_{\tilde{S}}(u) \mid g_\alpha'(u;h) = 0\ (\alpha \in \Gamma)\} \\ &= \{h \in R_{\tilde{S}}(u) \mid G'(u;h) = 0\}.\end{aligned}$$

Now, since

$$G'(u) = -H_u(t,x_0(t),u,p(t),\mu(t),\lambda_0) - \mu^*(t)g'(u),$$

we have $G'(u_0(t)) = -\mu^*(t)g'(u_0(t))$, implying that

$$0 = K'(u_0(t)) = G'(u_0(t)) + \lambda^* g'(u_0(t)) = (\lambda^* - \mu^*(t))g'(u_0(t)).$$

Since $u_0(t)$ is normal with respect to $\tilde{S}_0(u_0(t))$, we have $\lambda = \mu(t)$, and we conclude that

$$K(u) = G(u) + \langle \mu(t), g(u)\rangle = H([t],p(t),\mu(t),\lambda_0) - H(t,x_0(t),u,p(t),\mu(t),\lambda_0).$$

Hence, $K'(u) = -H_u(t,x_0(t),u,p(t),\mu(t),\lambda_0)$ and therefore

$$0 \leq K''(u_0(t);h) = -H_{uu}([t],p(t),\mu(t),\lambda_0;h)$$

for all $h \in \mathbf{R}^m$ satisfying

$$\varphi_{\alpha u}[t]h \leq 0 \quad (\alpha \in I_a[t],\ \mu_\alpha(t) = 0), \quad \varphi_{\beta u}[t]h = 0 \quad (\beta \in \Gamma(t) \cup Q)$$

where $\Gamma(t) = \{\alpha \in R \mid \mu_\alpha(t) > 0\}$. □

Note that, under assumption (A), if $(x_0,u_0) \in S$, then $\nu \equiv 0$ is the only solution of

i. $\nu_\alpha(t)\varphi_\alpha[t] = 0$ ($t \in T,\ \alpha \in R$).

ii. $\sum_1^q \nu_i(t)\varphi_{iu}[t] = 0$ ($t \in T$).

This assumption was used in the derivation of Corollary 3.2. We shall now derive it under a weaker assumption and in a much simpler way.

Given $(x_0,u_0) \in S$ and $\mu \in \mathscr{U}_q$, our new assumption, labelled (B), states that $\nu \equiv 0$ is the only solution of

i. $\nu_\alpha(t) \geq 0$ and $\nu_\alpha(t)\varphi_\alpha[t] = 0$ ($t \in T,\ \alpha \in R,\ \mu_\alpha(t) = 0$).

ii. $\sum_1^q \nu_i(t)\varphi_{iu}[t] = 0$ ($t \in T$).

Theorem 4.1. *Let $(x_0,u_0) \in S$ and suppose (p,μ,λ_0) is as in Theorem 3.1. If (x_0,u_0) solves* (P)*, then the conclusion of Corollary 3.2 holds under assumption* (B).

Proof. Fix $t \in T$, let $\lambda := \mu(t)$, and define G and g as before, that is, $g\colon \mathbf{R}^m \to \mathbf{R}^q$ as $g(u) := \varphi(t, x_0(t), u)$ and $G\colon \mathbf{R}^m \to \mathbf{R}$ as

$$G(u) := H([t], p(t), \mu(t), \lambda_0) - H(t, x_0(t), u, p(t), \mu(t), \lambda_0) - \langle \mu(t), g(u) \rangle \quad (u \in \mathbf{R}^m).$$

We have $G(u) \geq G(u_0(t))$ for all $u \in \tilde{S}$ where

$$\tilde{S} := \{u \in \mathbf{R}^m \mid g_\alpha(u) \leq 0 \ (\alpha \in R),\ g_\beta(u) = 0 \ (\beta \in Q)\}.$$

By Theorem 3.1(a) and (c), $\lambda_\alpha \geq 0$ and $\lambda_\alpha \varphi_\alpha[t] = 0$ $(\alpha \in R)$ and $H_u([t], p(t), \lambda, \lambda_0) = 0$ $(t \in T)$. But this means that

i. $\lambda_\alpha \geq 0$ and $\lambda_\alpha g_\alpha(u_0(t)) = 0$ $(\alpha \in R)$.
ii. If $K(u) := G(u) + \langle \lambda, g(u) \rangle$ then $K'(u_0(t)) = 0$.

In other words, $\lambda \in \Lambda(G, u_0(t))$. By assumption (B), $u_0(t)$ is normal with respect to $\tilde{S}_1(\lambda) = \{u \in \tilde{S} \mid K(u) = G(u)\}$ and so, by Theorem 2.6, $K''(u_0(t); h) \geq 0$ for all $h \in R_{\tilde{S}_1}(u_0(t))$. We now deduce that

$$K'(u) = -H_u(t, x_0(t), u, p(t), \mu(t), \lambda_0)$$

and therefore

$$0 \leq K''(u_0(t); h) = -H_{uu}([t], p(t), \mu(t), \lambda_0; h)$$

for all $h \in \mathbf{R}^m$ satisfying

$$\varphi_{\alpha u}[t]h \leq 0 \quad (\alpha \in I_a[t],\ \mu_\alpha(t) = 0), \quad \varphi_{\beta u}[t]h = 0 \quad (\beta \in \Gamma(t) \cup Q)$$

where $\Gamma(t) = \{\alpha \in R \mid \mu_\alpha(t) > 0\}$. □

5. Example

In this section we provide an example illustrating how our main result can be applied to a problem with inequality constraints, but the classical rank assumption is not satisfied.

Example 5.1. Consider the problem of minimizing $I(x,u) = \int_0^1 \{u_2^2(t) - u_1^2(t)\}dt$ subject to $\dot{x}(t) = u(t)$ $(t \in [0,1])$, $x(0) = x(1) = 0$ and

$$u_2(t) \geq u_1(t), \quad u_1(t) + u_2(t) \geq 0, \quad u_2(t) \geq u_1^2(t) \quad (t \in [0,1]).$$

Setting $T = [0,1]$, $f(t,x,u) = u$, $L(t,x,u) = u_2^2 - u_1^2$, $\xi_0 = \xi_1 = (0,0)$,

$$D = \{x \in Z \mid \dot{x}(t) = f(t, x(t), u(t)) \ (t \in [0,1]),\ x(0) = \xi_0,\ x(1) = \xi_1\},$$

$$\varphi_1(t,x,u) = u_1 - u_2, \quad \varphi_2(t,x,u) = -u_1 - u_2, \quad \varphi_3(t,x,u) = u_1^2 - u_2$$

the problem is that of minimizing $I(x,u) = \int_0^1 L(t, x(t), u(t))dt$ over the set

$$S = \{x \in D \mid \varphi_\alpha(t, x(t), u(t)) \leq 0 \ (t \in T,\ \alpha = 1,2,3)\}.$$

For this problem, the Hamiltonian is given by

$$H(t,x,u,p,\mu,\lambda) = p_1u_1 + p_2u_2 - \lambda(u_2^2 - u_1^2) - \mu_1(u_1 - u_2) - \mu_2(-u_1 - u_2) - \mu_3(u_1^2 - u_2).$$

Clearly (x_0, u_0) with $x_0 = (x_{01}, x_{02}) = (0,0)$ and $u_0 = (u_{01}, u_{02}) = (0,0)$ solves the problem. By Theorem 3.1, there exist $\lambda_0 \geq 0$, $p \in X$ and $\mu \in \mathscr{U}_3$ not vanishing simultaneously on T, such that

a. $\mu_\alpha(t) \geq 0$ $(t \in T,\ \alpha = 1,2,3)$.
b. $\dot{p}_1(t) = \dot{p}_2(t) = 0$ $(t \in T)$.

c. $p_1-\mu_1(t)+\mu_2(t)=p_2+\mu_1(t)+\mu_2(t)+\mu_3(t)=0\ (t\in T)$.

d. $p_1u_1+p_2u_2\le\lambda_0(u_2^2-u_1^2)$ for all $u\in\mathbf{R}^2$ satisfying $u_2\ge u_1$, $u_1+u_2\ge 0$ and $u_2\ge u_1^2$.

Note that the rank assumption (A) applied to (x_0,u_0) does not hold since it corresponds to the condition that the matrix

$$M=\left(\frac{\partial\varphi_i[t]}{\partial u^k}\right)\quad (i=1,2,3;\ k=1,2)$$

has rank 3 and, for this problem,

$$M=\begin{pmatrix}1 & -1\\ -1 & -1\\ 0 & -1\end{pmatrix}.$$

In other words, $\lambda\equiv 0$ is not the only solution of the system

i. $\lambda_\alpha(t)\varphi_\alpha[t]=0\ (t\in T,\ \alpha=1,2,3)$.

ii. $\sum_1^3\lambda_i(t)\varphi_{iu}[t]=0\ (t\in T)$.

Clearly, (i) holds for any λ, but (ii) corresponds to

$$(\lambda_1(t)-\lambda_2(t),-\lambda_1(t)-\lambda_2(t)-\lambda_3(t))=(0,0)\quad (t\in T)$$

which has nontrivial solutions.

On the other hand, the rank assumption (B) is satisfied if, for example, we consider the multipliers $\mu_1=\mu_2=\mu_3\equiv 0$. In this event, $\lambda\equiv 0$ is the only solution of

$$\lambda_\alpha(t)\ge 0\ (t\in T,\ \alpha=1,2,3)\quad\text{and}\quad \lambda_1(t)-\lambda_2(t)=-\lambda_1(t)-\lambda_2(t)-\lambda_3(t)=0\ (t\in T).$$

If we also set $p_1=p_2=0$ and take any $\lambda_0\ge 0$, then (a)–(d) above hold and (p,μ,λ_0) is as in Theorem 3.1. Thus all the assumptions of Theorem 4.1 are satisfied and the second order necessary condition $\langle h,H_{uu}([t],p(t),\mu(t),\lambda_0)h\rangle\le 0$ holds for all $h\in\mathbf{R}^2$ such that $\varphi_{iu}[t]h\le 0$ $(t\in T,\ i=1,2,3)$. This last statement can also be verified directly. We have

$$H_u(t,x,u,p,\mu,\lambda)=(p_1+2\lambda u_1-\mu_1+\mu_2-2\mu_3u_1,p_2-2\lambda u_2+\mu_1+\mu_2+\mu_3)$$

and so

$$H_{uu}(t,x,u,p,\mu,\lambda)=\begin{pmatrix}2\lambda-2\mu_3 & 0\\ 0 & -2\lambda\end{pmatrix}$$

so that $2\lambda_0(h_1^2-h_2^2)\le 0$ for all h_1,h_2 satisfying $h_1-h_2\le 0$, $-h_1-h_2\le 0$, $h_1^2-h_2\le 0$.

Author Note

This paper is dedicated to Richard Bertrand Vinter on the occasion of his 76th birthday.

Such wilt thou be to me, who must,
Like th' other foot, obliquely run;
Thy firmness makes my circle just,
And makes me end where I begun.
John Donne

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