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DYNAMIC PROGRAMMING APPROACH FOR MINIMAX CONTROL PROBLEMS AND OPTIMALITY CONDITIONS

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Dedicated to Professor Richard Vinter on the occasion of his 75th birthday

Abstract. In this paper, we investigate optimality conditions for minimax problems using a dynamic programming approach. We present original results on necessary and sufficient optimality conditions for a broad class of problems, covering both finite and infinite set of parameters. Initially, we derive necessary optimality conditions for the case of finite parameters. Subsequently, we extend these results to the case of an uncountable number of parameters through a rigorous convergence analysis. The proofs of the main results are presented in detail. In addition to necessary conditions, we also explore sufficient optimality conditions and analyze the properties of the value function associated with the minimax problem. Our results contribute significantly to the theory of optimal control and have broad applications in fields like economics, engineering, and computer science.

Keywords. Convergence analysis; minimax optimal control problems, Hamilton-Jacobi equations, value function, nonsmooth analysis, verification functions.

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1. Introduction

Consider the following minimax optimal control problem (MOCP):

$$(P_{t,z}) \quad \begin{cases} \text{Minimize} \quad \max\limits_{\alpha\in\mathscr{A}} g(x(T;\alpha),\alpha) \\ \text{s.t. } u:[t,T]\to\mathbb{R}^m \text{ such that } u(s)\in\Omega(s) \text{ a.e. } s\in[t,T] \\ \text{and arcs} \quad \{x(\cdot,\alpha):[t,T]\to\mathbb{R}^n/\alpha\in\mathscr{A}\} \quad \text{such that, for each} \quad \alpha\in\mathscr{A} \\ \dot{x}(s,\alpha)=f(s,x(s,\alpha),u(s),\alpha), \quad \text{a.e.} \quad s\in[t,T] \\ x(t,\alpha)=z. \end{cases}$$

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Here $(\mathscr{A},\rho)$ is an abstract metric space, $\mathbb{R}^k$ denotes the k-dimensional Euclidean space, $f : [S,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathscr{A}\to\mathbb{R}^n$ and $g:\mathbb{R}^n\times\mathscr{A}\to\mathbb{R}$ are given functions, $t\in[S,T]$ and $z\in\mathbb{R}^n$, $\Omega(s)\subset\mathbb{R}^m$ is a time-dependent set.

Note that this problem formulation takes into account parameter uncertainties both in the cost function and in the control system constraints. The goal is to find control strategies that perform optimally under the worst-case scenario for these parameters.

Richard Vinter, in his pioneering paper [1], developed a framework for deriving necessary conditions for the minimax optimal control problem, known as the minimax maximum principle, which generalizes the classical maximum principle to minimax problems. This framework addresses robust control problems, where the control strategy must be effective despite uncertainties in the system parameters. This is crucial for designing systems that can withstand variations and uncertainties in real-world applications. Vinter's minimax maximum principle captures several special cases, including optimal control problems with minimax costs, problems with semi-infinite endpoint constraints, and state-constrained optimal control problems [1]. Vinter's results were then extended by Karamzin et al. in [2], where the method employed in the proof allows for the inclusion of state constraints in the minimax control problem formulation. Later, Aquino, De Pinho, and Silva [3] further extended the minimax principle to include mixed equality and inequality constraints with dependence on both the state and the control. Their necessary optimality conditions is known as a weak minimax principle.

We point out that currently, the literature offers only necessary conditions in the form of the minimax principle. Typical sufficient conditions require some kind of (generalized) convexity and are challenging to verify in practice. This article proposes a new approach to minimax problems, offering both necessary and sufficient conditions of optimality for this class of problems using Dynamic Programming theory via Hamilton-Jacobi-Bellman (HJB) equations.

Hamilton-Jacobi-Bellman's theory is well established for the following optimal control problem.

$$(P')\quad\begin{cases}\text{Minimize } \; g(x(T)) \; \text{ over control functions}\\ u:[S,T]\to\mathbb{R}^m \text{ and arcs } x:[S,T]\to\mathbb{R}^n \text{ obeying}\\ \dot{x}(t)=f(t,x(t),u(t)),\;\text{ a.e. } t\in[S,T];\\ x(0)=x_0;\text{and}\\ u(t)\in\Omega(t),\;\text{ a.e } t\in[S,T].\end{cases}$$

The value function for this problem is $V(t,z):=\inf(P'_{t,z})$, where $(t,z)\in[S,T]\times\mathbb{R}^n$ and

$$(P'_{t,z})\quad\begin{cases}\text{Minimize }\; g(x(T)) \text{ over,}\\ \dot{x}(s)=f(s,x(s),u(s)),\;\text{ a.e. } s\in[t,T];\\ x(t)=z;\text{and}\\ u(t)\in\Omega(t),\;\text{ a.e. } t\in[S,T].\end{cases}$$

A continuously differentiable function ϕ is a solution of the Hamilton-Jacobi-Bellman Equation (HJB) when

$$\phi_t(t,z)+\inf_{u\in\Omega(t)}\phi_x(t,z)\cdot f(t,z,u)=0\quad\forall(t,z)\in[S,T]\times\mathbb{R}^n,$$
$$\phi(T,z)=g(z),\quad\forall z\in\mathbb{R}^n.$$

The Hamilton-Jacobi-Bellman (HJB) equation is a fundamental partial differential equation in optimal control theory, central to the theory of dynamic programming developed by Richard

Bellman [4], albeit in a somewhat different context. If the value function were continuously differentiable, it would uniquely solve the HJB equation. However, in general, the value function is not continuously differentiable ([5], p.36). Thus, over the past four decades, several authors have contributed with generalized solutions to HJB equations, which allow for solutions that are not differentiable in the classical sense. Crandall and Lions [6] introduced the concept of viscosity solutions in the early 1980s and showed that the value function is the only viscosity solution for the equation Hamilton-Jacobi in the class of uniformly continuous functions. In their work the derivative is replaced by superdifferentials and subdifferentials, which coincide with the derivative when it exists. Alexander Subbotin made notable contributions to the theory of viscosity solutions, especially in the context of differential games and optimal control [7]. In fact, his research demonstrated how viscosity solutions can be applied to Hamilton-Jacobi-Isaacs equations, which are crucial in game theory, and bridged the gap between dynamic programming and viscosity solutions, showing that the value functions in optimal control problems are indeed viscosity solutions of the associated PDEs. Together, these contributions have laid the foundation for modern approaches to solving complex PDEs, particularly in fields like dynamic programming, optimal control, and differential games. The concept of viscosity solutions continues to be a vital tool in mathematical analysis and applications. Other types of continuous solutions, such as Dini solutions or Proximal solutions, which are also very useful, can be found in [5, 8, 9], for example. We note that due to the uniqueness of solutions for the HJB equation, all solutions coincide, allowing us to use the one best suited for the particular application.

A function ϕ is a proximal solution to the HJB equation if for all $(t,z) \in ([S,T] \times \mathbb{R}^n) \cap \text{dom}\phi$ with $\partial^P \phi(t,z) \neq \emptyset$,

$$\xi + \inf_u \eta \cdot f(t,z,u) = 0 \quad \text{for each } (\xi,\eta) \in \partial^P \phi(t,z);$$
$$\phi(T,z) = g(z) \quad \text{for each } z \in \mathbb{R}^n.$$

Here $\partial^P \phi(t,z)$ denotes the proximal subdifferential of ϕ in (t,z). For example, consider the function $f(t,x,u) = |x|u$, where $f : [0,1] \times \mathbb{R} \times [-1,1] \longrightarrow \mathbb{R}$, $g(z) = z$, $\Omega(t) \equiv [-1,1]$, $S = 0$, and $T = 1$. It is straightforward to verify that the function

$$\phi(\tau,a) = \begin{cases} a \cdot e^{\tau-1} & \text{if } a > 0, \\ a \cdot e^{1-\tau} & \text{if } a \leqslant 0 \end{cases}$$

is a continuous, nondifferentiable function that serves as a proximal solution to the Hamilton-Jacobi-Bellman (HJB) equation.

Reference [10] introduces the well-known generalized solutions that involve generalized gradients, which coincide with the derivative when the function is strictly differentiable. The approach used in [10] allows for a verification technique even when the HJB equation does not have a classic solution. Reference [11] provides a comparison between viscosity solutions and generalized solutions, showing that a locally Lipschitz function is a viscosity solution of the HJB equation if and only if the function is a generalized solution. A characterization of the value function of an optimal control problem of Bolza with state constraints as the only lower semicontinuous solution of the associated HJB equation is given in [12]. However, regardless the type of solution used, under fairly general hypotheses, it can be proved that the value function is a solution of the HJB equation.

It is now clear by our previous discussion that there are general techniques for studying optimal control problems via dynamic programming and HJB equations and the theory is almost a closed chapter. Thus, it is also natural to address minimax control problems using Dynamic Programming, typically employing Hamilton-Jacobi-Bellman equations, as is done for standard optimal control problems [5]. This approach is justified by the fact that Dynamic Programming provides not only necessary conditions but also sufficient conditions of optimality without requiring convexity. Di Marco, González [13, 14, 15] work in the sense of Dynamic Programming, but dealing with minimax control problems in which the cost function to be minimized is not given in the integral form. Instead, it is taken as an essential supremum over time $t \in [S,T]$ with dependence on the state and the control. The minimax optimal control problems we consider involve data dependent on unknown parameters within the set $\mathscr{A}$. We aim to provide necessary and sufficient conditions for these problems, drawing on classical theory as outlined by [5] and [9]. When $\mathscr{A}$ is finite, the minimax control problem can be transformed into a standard optimal control problem, for which established results are available. Our proof technique begins by addressing the finite case, formulating the problem as a standard optimal control problem. This allows the application of known dynamic programming techniques to achieve the desired results. Subsequently, we tackle the general minimax control problem through the approximation of finite subsets of $\mathscr{A}$.

The significance of initially addressing the finite case lies in its ability to approximate the more complex general control problem where $\mathscr{A}$ is a compact metric space. By approximating this space with finite sets $\mathscr{A}_i$, we conduct a rigorous convergence analysis to extend the results to the general problem. This method demonstrates that as $\mathscr{A}_i$ approaches the compact metric space $\mathscr{A}$, the results remain robust, thereby bridging the gap between finite and infinite considerations in minimax control problems.

This paper proceeds as follows. Section 2 introduces the necessary notation and reviews some preliminary results that will be used throughout the analysis. In Section 3, we present the main results of this paper, namely the necessary optimality conditions for the minimax problem via dynamic programming approach. We begin by deriving these conditions for the case of finitely many parameters and then extend the results to the more general case of an uncountable number of parameters through a careful convergence analysis. The detailed proof of the latter result is provided in Section 4. Section 5 explores sufficient conditions for optimality and investigates the properties of the value function associated with the minimax problem. Finally, Section 6 concludes the paper with a summary of the key findings and a discussion of potential future research directions.

2. Notation and Preliminarily Results

In this section, we gather some basic notions and preliminary results that will be needed in this work. $|\cdot|$ denotes the Euclidean norm and $\mathbf{B}$ represents the closed unit ball centered at the origin. The vector space $L^1([S,T];\mathbb{R}^p)$ denotes the space of integrable functions, and $L^\infty([S,T];\mathbb{R}^p)$ the space of essentially bounded functions and $W^{1,1}([S,T];\mathbb{R}^p)$ denotes the vector space of absolutely continuous functions from $[S,T]$ to $\mathbb{R}^p$, respectively.

The set

$$\limsup_{i\to\infty} A_i$$

the Kuratowski lim sup) comprises all points $x \in \mathbb{R}^n$ satisfying the condition: there exist a subsequence $\{A_{i_j}\}$ of $\{A_i\}$ and a sequence $x_j \to x$ such that $x_j \in A_{i_j}$, for all j. For any continuous function $x : [a,b] \to \mathbb{R}^n$ and $\delta > 0$, we define the set $T(x,\delta)$, called the δ-tube around x, as $T(x,\delta) := \{(t,y) \in [a,b] \times \mathbb{R}^n \mid t \in [a,b], |y - x(t)| \leqslant \delta\}$. The graph of a multifunction $D : A \rightrightarrows \mathbb{R}^k$ is denoted by GrD, $GrD := \{(x,y) \in A \times \mathbb{R}^k \mid y \in D(x)\}$. Given a closed set $C \subset \mathbb{R}^k$ and a point $x \in C$, the proximal normal cone to C at x, written $N_C^P(x)$, is the set

$$N_C^P(x) := \left\{ p \in \mathbb{R}^k : \exists M > 0 \text{ such that } p \cdot (y - x) \leqslant M |y - x|^2 \ \forall y \in C \right\}.$$

Elements in $N_C^P(x)$ are called proximal normals to C at x. The limiting normal cone to C at x, written $N_C(x)$, is the set

$$N_C(x) := \left\{ p \in \mathbb{R}^k : \text{there exists } x_i \xrightarrow{C} x,\ p_i \to p \text{ such that } p_i \in N_C^P(x_i) \text{ for all } i \right\}.$$

Elements in $N_C(x)$ are called limiting normals to C at x. Take a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in domf$. Here, $domf$ is the set $domf = \{y \in \mathbb{R}^n \mid f(y) < +\infty\}$. The epigraph of f is the set $epif := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqslant y\}$. The proximal subdifferential $\partial^P f(x)$ of $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point $x \in domf$ is the set

$$\partial^P f(x) := \{\eta \mid (\eta, -1) \in N^P_{epif}(x, f(x))\}.$$

The limiting subdifferential $\partial^L f(x)$ de $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point $x \in domf$ is the set

$$\partial^L f(x) := \{\eta \mid (\eta, -1) \in N_{epif}(x, f(x))\}.$$

Let $d_C : \mathbb{R}^n \to [0,\infty)$ be the distance between the point $x \in \mathbb{R}^n$ and a set (i.e. $d_C(x) = \inf_{a \in C} |x - a|$.) This is a Lipschitz continuous function with Lipschitz constant 1. Given $\delta > 0$, the "truncation" $tr_\delta : \mathbb{R}^n \to \mathbb{R}^n$, is defined by

$$tr_\delta(\xi) := \begin{cases} \xi, & \text{if } \ |\xi| \leqslant \delta \\ \frac{\xi}{|\xi|}\delta, & \text{if } \ |\xi| > \delta. \end{cases}$$

A feasible control is a measurable function $u : [t,T] \to \mathbb{R}^m$ satisfying $u(s) \in \Omega(s)$ almost everywhere (a.e.). The set of feasible control functions is denoted by $\mathscr{U}$. A feasible control process $(u, \{x(\cdot;\alpha) \mid \alpha \in \mathscr{A}\})$ comprises a feasible control function u and a family $\{x(\cdot;\alpha) \in W^{1,1}([t,T];\mathbb{R}^n) \mid \alpha \in \mathscr{A}\}$ of arcs satisfying, for each $\alpha \in \mathscr{A}$,

$$\begin{cases} \dot{x}(s;\alpha) = f(s, x(s,\alpha), u(s), \alpha), & \text{a.e.} \\ x(t,\alpha) = z. \end{cases} \tag{2.1}$$

Given $u \in \mathscr{U}$, we occasionally use the notation $\{x(\cdot,u,\alpha)/\alpha \in \mathscr{A}\}$, for the corresponding family of trajectories.

A feasible control process $(\bar{u}, \{\bar{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$ is said to be optimal (or minimal) if

$$\max_{\alpha \in \mathscr{A}} g(x(T;\alpha),\alpha) \geqslant \max_{\alpha \in \mathscr{A}} g(\bar{x}(T;\alpha),\alpha)$$

for all feasible processes $(u, \{x(.;\alpha) \mid \alpha \in \mathscr{A}\})$. A feasible process $(\bar{u}, \{\bar{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$ is said to be a local minimum when there exists $\varepsilon > 0$ such that

$$\max_{\alpha \in \mathscr{A}} g(x(T;\alpha),\alpha) \geqslant \max_{\alpha \in \mathscr{A}} g(\bar{x}(T;\alpha),\alpha)$$

for all feasible processes $(u,\{x(.;\alpha)\mid\alpha\in\mathscr{A}\})$ satisfying $\|x(\cdot,\alpha)-\bar{x}(\cdot;\alpha)\|_{W^{1,1}}\leqslant\varepsilon$ for all $\alpha\in\mathscr{A}$. A feasible process $(\bar{u},\{\bar{x}(.;\alpha)\mid\alpha\in\mathscr{A}\})$ is said to be a strong local minimum when there exists $\varepsilon>0$ such that

$$\max_{\alpha\in\mathscr{A}} g(x(T;\alpha),\alpha)\geqslant\max_{\alpha\in\mathscr{A}} g(\bar{x}(T;\alpha),\alpha)$$

for all feasible processes $(u,\{x(.;\alpha)\mid\alpha\in\mathscr{A}\})$ satisfying $\|x(\cdot,\alpha)-\bar{x}(\cdot;\alpha)\|_{L^\infty}\leqslant\varepsilon$ for all $\alpha\in\mathscr{A}$.

2.1. **Preliminary results.** Next, we state a result about measure convergence which we need in our analysis; for more details see ([5], Proposition 9.2.1) or ([1], Proposition 6.1).

Proposition 2.1. *Take a compact metric space X, a sequence $\{\mu_i\}$ of nonnegative Radon measures in $C^*(X)$, a sequence $\{D_i : X\rightsquigarrow\mathbb{R}^n\}$ of multifunctions and a sequence of Borel measurable functions $\{\gamma_i : X\to\mathbb{R}^n\}$. Take also a measure $\mu\in C^*(X)$ and a multifunction $D : X\rightsquigarrow\mathbb{R}^n$. Assume that GrD is compact, $D(x)$ is convex for each $x\in X$, $\limsup_{i\to\infty} GrD_i\subset GrD$, $\gamma_i(x)\in D_i(x)$ μ_i−a.e. $x\in X$ for i=1,2,... and $\mu_i\to\mu$ weakly*. Define $\eta_i\in C^*(X;\mathbb{R}^n)$ according to $\eta_i(dx)=\gamma_i(x)\mu_i(dx)$ $i=1,2,\ldots$. Then, along a subsequence, $\eta_i\to\eta$ weakly* for some $\eta\in C^*(X;\mathbb{R}^k)$ and some Borel measurable function γ such that $\eta(dx)=\gamma(x)\mu(dx)$, and $\gamma(x)\in D(x)$, μ−a.e.*

2.1.1. *The value function.* We define the value function $V:[S,T]\times\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ of Problem $P:=P(S,x_0)$ to be the infimum of the parametrized problem $(P_{t,z})$, for $(t,z)\in[S,T]\times\mathbb{R}^n$:

$$V(t,z):=\inf\left\{\max_{\alpha\in\mathscr{A}} g(x(T;u,\alpha),\alpha)\mid(u,\{x(.;\alpha)\mid\alpha\in\mathscr{A}\})\text{ is feasible process of }P(t,z)\right\}.$$

Most of the time, we simplify notation and write:

$$V(t,z)=\inf_{u\in\mathscr{U}}\max_{\alpha\in\mathscr{A}} g(x(T,\alpha),\alpha).$$

V satisfies the following terminal condition

$$V(T,z)=\max_{\alpha\in\mathscr{A}} g(z,\alpha).$$

The next proposition states that the value function is Lipschitz continuous if the cost function is Lipschitz continuous. This result extends the concept of the value function for standard optimal control problems. To understand this, note that when the parameter set is a singleton, the minimax problem reduces to a standard control problem.

Consider the following hypotheses, where $(\bar{u},\{\bar{x}(.;\alpha)\mid\alpha\in\mathscr{A}\})$ is a feasible control process and $\delta>0$.

H1): The function $f(\cdot,x,\cdot,\cdot)$ is $\mathscr{L}\times\mathscr{B}^m\times\mathscr{B}^{\mathscr{A}}$ mensurable for each $x\in\mathbb{R}^n$ ($\mathscr{L}$ denotes the Lebesgue subsets of $[S,T]$, $\mathscr{B}^m$ denotes the Borel subsets of $\mathbb{R}^m$ and $\mathscr{B}^{\mathscr{A}}$ denotes the Borel subsets of $\mathscr{A}$).

H2): There exists a function k_f integrable and $C>0$ such that, for each $\alpha\in A$,

$$|f(s,x,u,\alpha)-f(s,x',u,\alpha)|\leqslant k_f(s)|x-x'|\quad\text{ and }\quad|f(s,x,u,\alpha)|\leqslant C$$

for all $x,x'\in\bar{x}(s,\alpha)+\delta\mathbf{B}$, $u\in\Omega(s)$ and for almost every $s\in[S,T]$.

H3): There exists $\theta:[0,+\infty)\to[0,+\infty)$ such that $\lim_{s\to0}\theta(s)=0$ and, for all $\alpha,\alpha'\in\mathscr{A}$,

$$\int_S^T\sup_{u\in\Omega(s)}|f(s,x,u,\alpha)-f(s,x,u,\alpha')|ds\leqslant\theta(\rho_{\mathscr{A}}(\alpha,\alpha')),\quad\forall x\in\bar{x}(s,\alpha)+\delta\mathbf{B}.$$

H4): The multifunction $\Omega : [S,T] \rightrightarrows \mathbb{R}^m$ has a Borel mensurable graph.

H5): The function g is continuous and $g(\cdot,\alpha)$ is Lipschitz with Lipschitz constant k_g, for all $\alpha \in \mathscr{A}$.

H6): For each $\alpha \in \mathscr{A}$, the set $F(s,x,\alpha) := \{f(s,x,u,\alpha) : u \in \Omega(s)\}$ is closed and convex, for each $(s,x) \in [S,T] \times \mathbb{R}^n$.

If $\delta = \infty$, then in the above hypotheses $\delta\mathbf{B}$ is interpreted as $\mathbb{R}^n$. Given a control function, a parameter $\alpha \in \mathscr{A}$ and a initial condition; the conditions H1-H5 above guarantee the existence and uniqueness of trajectories for the control system (2.1). The convexity of $F(s,x,\alpha)$ is to guarantee the existence of optimal processes.

The next two lemmas gather some useful facts regarding the dependence of the arc trajectories on the initial condition, controls and parameters.

We remark that given an arbitrary point $(t,z) \in [S,T] \times \mathbb{R}^n$ and any control function $u \in \mathscr{U}$, it is common to make explicit the dependence of the arc trajectory over them by writing $x(\cdot;\alpha,t,z,u)$. However, to avoid notation overloading we will use simplified notation which will be evident in each situation below, emphasizing only when needed.

Lemma 2.2. *Let $(t,z) \in [S,T] \times \mathbb{R}^n$ be given and let $(u,\{x(.;\alpha) \mid \alpha \in \mathscr{A}\})$ be a feasible process. If the hypotheses $(H1)-(H5)$ are satisfied, then:*

i): *The function $\alpha \to x(s;\alpha)$ for all $s \in [t,T]$ satisfies*

$$|x(s;\alpha) - x(s;\alpha')| \leqslant e^{\int_t^s k_f(\tau)d\tau}\omega(\rho_{\mathscr{A}}(\alpha,\alpha')), \quad \forall \alpha,\alpha' \in \mathscr{A}.$$

ii): $\lim_{\alpha\to\alpha'} |x(s;\alpha) - x(s;\alpha')| = 0$ *and* $\lim_{\alpha\to\alpha'} |g(x(T,\alpha),\alpha) - g(x(T,\alpha'),\alpha')| = 0.$

Proof. Assertion i) follows from the combination of (H3), (H4) and Gronwall Lemma. Assertion ii) an immediate consequence of i), (H5) and of the continuity property of the composition of continuous functions. □

Let $\Delta : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ be the metric on $\mathscr{U}$ defined as

$$\Delta(u,u') = \int_t^T |u(\tau) - u'(\tau)|d\tau.$$

The following lemma consolidates some useful facts regarding the dependence of trajectories on controls and parameters.

Lemma 2.3. *Assume that $H1)-H4)$ are valid. Let $(t,z) \in [S,T] \times \mathbb{R}^n$ be given. Then, for any $\sigma > 0$, we can choose a finite subset $\widetilde{\mathscr{A}} \subset \mathscr{A}$ and $\rho > 0$ such that*

i) $\sup_{u\in\Omega(t)} \sup_{\alpha\in\mathscr{A}} \inf_{\alpha'\in\mathscr{A}'} \|x(\cdot;\alpha,u) - x(\cdot;\alpha',u)\| < \sigma$

ii) $\sup_{\alpha\in\mathscr{A}} \left\{ \|x(\cdot;\alpha,u) - x(\cdot;\alpha,u')\| : u,u' \in \mathscr{U}, \Delta(u,u') < \rho \right\} < \sigma.$

Proof. Let $(t,z) \in t \in [S,T] \times \mathbb{R}^n$, $u \in \mathscr{U}$ and $\alpha \in \mathscr{A}$ be given. Then there exists only one trajectory $x(\cdot;t,z,u,\alpha)$ that satisfies (2.1). Let $\sigma > 0$ and take $\delta' = \dfrac{\sigma}{L}$, where $L := \exp\left(\int_t^s k_f(\tau)d\tau\right)$. As $\lim_{s\to 0}\omega(s) = 0$, there exists $\delta'' > 0$ such that $|\omega(s)| < \delta'$, whenever $|s| < \delta''$. Since $\mathscr{A}$ is compact there exists a finite set $\widetilde{\mathscr{A}} = \{\alpha_1,...,\alpha_N\}$ such that $\mathscr{A} = \bigcup_{j=1} B(\alpha_j,\delta'')$. So, there exists $\alpha' \in \widetilde{\mathscr{A}} \subset \mathscr{A}$ such that $\rho(\alpha,\alpha') < \delta''$.

It folllows from Lemma 2.2(i) that

$$|x(s;\alpha,u)-x(s;\alpha',u)|\leqslant\omega(\rho(\alpha,\alpha'))\exp\left(\int_t^s k_f(\tau)d\tau\right)=\delta'\cdot L=\sigma,$$

which validates assertion (i).

The proof of the assertion (ii) follows standard arguments involving two applications of the Dunford-Pettis Theorem [16].

□

3. The Main Results: Necessary Optimality Conditions for Problem (P)

In Subsection 3.1, we derive necessary optimality conditions using the Hamilton-Jacobi-Bellman inequality for a finite set of parameters $\mathscr{A}$. The general case, where $\mathscr{A}$ is an infinite compact metric space, is addressed in Subsection 3.2, culminating in Theorem 3.3. Due to the length and complexity of the proof for the general case, it is deferred to the next section.

3.1. **Case where the Set $\mathscr{A}$ is Finite.** Let $\mathscr{A}=\{\alpha_1,\alpha_2,\dots,\alpha_N\}$ be a finite set of parameters. Suppose that $((\bar{u},\{\bar{x}(.;\alpha)\mid\alpha\in\mathscr{A}\}))$ is a strong local minimum for the MOCP and, for some $\delta>0$, the following hypotheses hold.

H1'): The function $f(\cdot,x,\cdot,\alpha)$ is $\mathscr{L}\times\mathscr{B}^m$ measurable for each $(x,\alpha)\in\mathbb{R}^n\times\mathscr{A}$.

H5'): The function g is bounded and there exists $k_g>0$ such that

$$|g(x,\alpha)-g(y,\alpha)|\leqslant k_g|x-y|$$

for all $x,y\in\bar{x}(T,\alpha)+\delta\mathbf{B}$ and $\alpha\in\mathscr{A}$.

We define the minimized Hamiltonian by

$$h(s,x,p,\alpha):=\min_{u\in\Omega(t)}p\cdot f(s,x,u,\alpha).$$

Let $\bar{x}=\text{col}\{\bar{x}(\cdot;\alpha_1),\bar{x}(\cdot;\alpha_2),\dots,\bar{x}(\cdot;\alpha_N)\}$ be the collection of trajectories corresponding to $\bar{u}$. Then $(\bar{u},\bar{x})$ is a minimum process of the optimal control problem $(\widetilde{P})$:

$$\text{minimize}\quad \tilde{g}(x(1)) \tag{3.1a}$$

$$\text{subject to}\quad \dot{x}(t)=\tilde{f}(t,x(t),u(t))\quad\text{a.e. } t\in[S,T], \tag{3.1b}$$

$$x(S)=x_0, \tag{3.1c}$$

$$u(t)\in\Omega(t)\quad\text{a.e. } t\in[S,T], \tag{3.1d}$$

where each component is in $\mathbb{R}^n$ and

$$\begin{aligned}
x&=\text{col}\{x_1,x_2,\dots,x_n\},\\
\tilde{f}(t,x,u)&=\text{col}\{f(t,x_i,u,\alpha_i)\}_{i=1}^N,\\
\tilde{x}_0&=\text{col}\{x_0,x_0,\dots,x_0\},\\
\tilde{g}(x)&=\max_i g(x(\cdot;\alpha_i),\alpha_i).
\end{aligned}$$

As $(\bar{u},\bar{x})$ is a strong local minimum for $(\widetilde{P})$, let us choose $\varepsilon'\in(0,\delta)$ such that $\|x(\cdot)-\bar{x}(\cdot)\|\leqslant\varepsilon'$. Then, by (Proposition 12.4.3b, [5]) there is a locally Lipschitz continuous function Φ :

$T(\bar{x}(\cdot),\varepsilon) \to \mathbb{R}$ (with $\varepsilon \in (0,\varepsilon'))$ such that for each $(t,\bar{z}) \in \text{int}T(\bar{x},\varepsilon)$ in which $\partial^P\Phi(t,\bar{z}) \neq \emptyset$, we have

$$\xi + \min_u \eta \cdot \tilde{f}(t,\tilde{x},u) \geqslant 0 \quad \text{for all } (\xi,\eta) \in \partial^P\Phi(t,\tilde{z}), \tag{3.2}$$

$$\Phi(T,y) \leqslant \tilde{g}(y) \quad \text{for all } y \in \bar{x}(T) + \delta\mathbf{B}, \tag{3.3}$$

$$\Phi(S,\tilde{x}_0) = \tilde{g}(\bar{x}(T)). \tag{3.4}$$

Let $z \in \bar{x}(t,\alpha) + \delta B$, for all $t \in [S,T]$ and for all $\alpha \in \mathscr{A}$. Define the following application $i_j : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n$, for each $j = 1,2,\ldots,N$, as being $i_j(a) = (z,z,\ldots a,\ldots,z)$, where a is in the $j-$th position. Denote by $\phi_j = \Phi \circ (id \times i_j)$.

Claim 3.1. The functions ϕ_j are locally Lipschitz on $T(\bar{x}(\cdot,\alpha_j),\varepsilon)$, and, for $(a_j,b_j) \in \partial^P\phi_j(s,a)$,

$$a_j + \min_u b_j \cdot f(s,a,u,\alpha_j) \geqslant 0, \quad \forall j = 1,\ldots,N.$$

Proof. Indeed, take $(s,a),(s',a') \in T(\bar{x}(\cdot,\alpha_j),\varepsilon)$. We may write

$$\begin{aligned}|\phi_j(s,a) - \phi_j(s',a')| &= |\Phi(s,z,..,a,...z) - \Phi(s',z,...,a',...,z)| \\ &\leqslant k_\Phi|s-s',a-a'| = k_\Phi|(s,a)-(s',a')|.\end{aligned}$$

This implies that the functions ϕ_j obey the required locally Lipschitz condition. If $(a_j,b_j) \in \partial^P\phi_j(s,a)$, then there exists $M_j > 0$ such that

$$(a_j,b_j)((s',a')-(s,a)) \leqslant \phi_j(s',a') - \phi_j(s,a) + M_j|(s',a')-(s,a)|^2$$

for all $(s',a') \in T(\bar{x}(\cdot,\alpha_j),\varepsilon)$. By definition of the ϕ_j, we have

$$\begin{aligned}&(a_j,0,..,b_j,..,0)((s',z,..,a',..,z)-(s,z,..,a,..z)) \leqslant \\ &\quad \Phi(s',z,..,a',..,z) - \Phi(s,z,..,a,..z) + M_j|(s',z,..,a',..,z)-(s,z,..,a,..,z)|^2,\end{aligned}$$

which implies that $(a_j,(0,..,b_j,..,0)) \in \partial^P\Phi(s,z,..,a,..z)$. It follows from (3.2) that

$$a_j + \min_u b_j \cdot f(s,a,u,\alpha_j) \geqslant 0, \quad \forall j = 1,...,N,$$

concluding the proof. □

We now pause to make some considerations. Note that each ϕ_j satisfies $\phi_j(t,z) = \Phi(t,z,...,z)$ for all $(t,z) \in T(\bar{x}(\cdot,\alpha_j),\varepsilon)$. In particular, for $(t,z) = (S,x_0)$, we have

$$\phi_j(S,x_0) = \Phi(T,x_0,...,x_0) = \max_{1\leqslant j\leqslant N} g(\bar{x}(T,\alpha_j),\alpha_j).$$

Thus we may define the following application

$$\varphi(t,z) := \max_{1\leqslant j\leqslant N} \phi_j(t,z), \quad \forall\ (t,z) \in \bigcap_j T(\bar{x}(\cdot,\alpha_j),\varepsilon),$$

which is locally Lipschitz and satisfies

$$\begin{aligned}\varphi(S,x_0) &= \max_{1\leqslant j\leqslant N} g(\bar{x}(T,\alpha_j),\alpha_j); \text{and} \\ \varphi(T,z) &\leqslant \max_{1\leqslant j\leqslant N} g(z,\alpha_j).\end{aligned}$$

Let us write $\phi(t,z,\alpha_i) := \phi_i(t,z)$ for all i and define

$$\mathscr{B} := \{\alpha \in \mathscr{A} / \max_{\alpha'\in\mathscr{A}} g(\bar{x}(T,\alpha'),\alpha') = g(\bar{x}(T,\alpha),\alpha)\}$$

and $\mathscr{B}(t,z) := \{\alpha \in \mathscr{A} / \max_{\alpha' \in \mathscr{A}} \phi(t,z,\alpha') = \phi(t,z,\alpha)\}$. Then the max rule ([5], Theorem 5.5.2) implies that

$$\partial^L \varphi(t,z) \subset \Big\{ \sum_{i=1}^{N} \beta_i \partial^L \phi(t,z,\alpha_i) : \beta_i \geqslant 0, \sum_{i=1}^{N} \beta_i = 1, \text{ and } \beta_i = 0 \text{ if } \alpha_i \notin \mathscr{B}(t,z) \Big\}. \tag{3.5}$$

Bearing the above in mind, let us define the probability measure $\Lambda = \sum_{i=1}^{N} \beta_i \delta_{\alpha_i}$ which satisfies $\operatorname{supp} \Lambda \subset \mathscr{B}(t,z)$. For any subset E of $\mathscr{A}$, $P(E)$ denotes the collection of Probability Radon measures supported on E. So it follows from (3.5) that

$$\partial^L \varphi(t,z) \subset \Big\{ \int_{\mathscr{A}} \partial^L \phi(t,z,\alpha) \Lambda(d\alpha) \ : \ \Lambda \in P[\mathscr{B}(t,z)] \Big\}. \tag{3.6}$$

Each $(\xi,\eta) \in \partial^L \varphi(t,z)$ corresponds to an application $\alpha \to \xi_\alpha \in \partial^L \phi(t,z,\alpha)$ and $\Lambda \in P[\mathscr{B}(t,z)]$ such that, for each $(v_1,v_2) \in [S,T] \times \mathbb{R}^n$,

$$\langle (\xi,\eta),(v_1,v_2) \rangle = \int_{\mathscr{A}} \langle (\xi_\alpha,\eta_\alpha),(v_1,v_2) \rangle \Lambda(d\alpha).$$

It is also known that $\xi_i + \min_u \Big\{ \eta_i \cdot f(t,z,u,\alpha_i) \Big\} \geqslant 0$ for all $(\xi_i,\eta_i) \in \partial^L \phi_i(t,z)$ and for all $i = 1,...,N$, i.e.,

$$\xi + \min_u \eta \cdot f(t,z,u,\alpha) = \int_{\mathscr{A}} (\xi_\alpha + h(t,z,\eta_\alpha,\alpha)) d\Lambda(\alpha) \geqslant 0,$$

where

$$\partial_0 \phi(t,z,\alpha) = \begin{cases} \partial^L \phi(t,z,\alpha) & \text{if} \quad \phi(t,z,\alpha) = \max_{\alpha' \in \mathscr{A}} \phi(t,z,\alpha') \\ \varnothing & \text{otherwise.} \end{cases}$$

As it is for all (t,z), in particular for (S,x_0) we have that there is a Radon probability measure such that $\operatorname{supp} \Lambda \subset \mathscr{B}$ and

$$\phi(S,x_0,\alpha) = g(\bar{x}(T,\alpha),\alpha), \quad \Lambda - \text{a.e} \quad \alpha \in \mathscr{A}.$$

Thus we have the following proposition.

Proposition 3.2. *Let $(\bar{u}, \{\bar{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$ be a minimum for the MOCP (P). Assume that $\mathscr{A}$ is a finite set and, for some $\delta > 0$, satisfies the hypotheses $H1'), H2) - H4), H5')$ and $H6)$. Then, for some $\varepsilon > 0$, there exist a family of uniformly bounded Lipschitz continuous functions $\{\phi(\cdot,\cdot,\alpha) : T(\bar{x}(\cdot,\alpha),\varepsilon) \to \mathbb{R} / \alpha \in \mathscr{A}\}$ and a Radon probability measure $\Lambda \in C^*(\mathscr{A})$ satisfying:*

(i) The application $\alpha \mapsto \phi(t,z,\alpha)$ is continuous.

(ii) $\operatorname{supp} \Lambda \subseteq \{\alpha \in \mathscr{A} \mid \max_{\alpha' \in \mathscr{A}} g(\bar{x}(T,\alpha'),\alpha') = g(\bar{x}(T,\alpha),\alpha)\}$ and

$$\phi(S,x_0,\alpha) = g(\bar{x}(T,\alpha),\alpha) \quad \Lambda\text{-a.e. } \alpha \in \mathscr{A}.$$

(iii) For each $(t,z) \in \operatorname{int} \bigcap_{\alpha \in \mathscr{A}} T(\bar{x}(\cdot,\alpha),\varepsilon)$ and for all $(\xi,\eta) \in \partial^L \varphi(t,z)$, we have

$$\xi + h(t,z,\eta,\alpha) = \int_E \xi_\alpha + \min_u \eta_\alpha \cdot f(t,z,u,\alpha) \Lambda d(\alpha) \geqslant 0 \quad \forall E \in \mathscr{B}^{\mathscr{A}} \tag{3.7}$$

for some family $\{\xi_\alpha,\eta_\alpha\}_{\alpha \in \mathscr{A}}$ such that $(\xi_\alpha,\eta_\alpha) \in \partial_0 \phi(t,z,\alpha)$ for Λ-a.e. $\alpha \in \mathscr{A}$.

(iv) Defining $\varphi(t,z) := \max_{\alpha\in\mathscr{A}} \phi(t,z,\alpha)$, the following statements hold:

$$\varphi(T,z) \leqslant \max_{\alpha\in\mathscr{A}} g(z,\alpha), \tag{3.8}$$

$$\varphi(S,x_0) = \max_{\alpha\in\mathscr{A}} g(\bar{x}(T,\alpha),\alpha). \tag{3.9}$$

The set $\partial_0\phi(t,z,\alpha)$ can be empty unless it is active in the sense of our previous comments. As $\mathscr{A}$ is a finite set, the application $\alpha \mapsto \phi(t,z,\alpha)$ is continuous for all $(t,z) \in [S,T]\times\mathbb{R}^n$.

3.2. **Case where the Set $\mathscr{A}$ is Infinite.** To obtain results for this case, we first considereded the scenario where the set $\mathscr{A}$ of parameters is finite in the previous subsection. The proof concept is as follows: we approximate the compact metric space $\mathscr{A}$ by increasing finite sets $\mathscr{A}_i$, and through convergence analysis, we extend the results to the general problem. However, when the set of parameters is any compact metric space, the situation becomes much more complex as certain techniques and properties break when $\mathscr{A}$ is infinite. The multifunction $\partial_0\phi(t,z,\cdot)$ may not be convex, which is necessary to obtain the limit. Therefore, it is convenient to define a new type of gradient that accounts for variations in the parameters.

We need to replace $\partial_0\phi(t,z,\cdot)$ with a larger set. We incorporate $\partial_0\phi(t,z,\cdot)$ with a family of multifunctions $\{\partial_\delta\phi(t,z,\cdot) \mid \delta \geqslant 0\}$ defined as follows. For any $\delta \geqslant 0$ and $\alpha \in \mathscr{A}$, let's define

$$\partial_\delta\phi(t,z,\alpha) = \begin{cases} \partial^L\phi(t,z,\alpha) & \text{if } \phi(t,z,\alpha) \geqslant \max_{\alpha'\in\mathscr{A}}\phi(t,z,\alpha') - \delta \\ \varnothing & \text{otherwise.} \end{cases}$$

We denote by $\partial_{[\mathscr{A}]}\phi(t,z,\alpha)$ the following set

$$\partial_{[\mathscr{A}]}\phi(t,z,\alpha) := \bigcap_{\delta>0} \overline{\text{co}} \bigcup_{\alpha'\in B(\alpha,\delta)} \partial_\delta\phi(t,z,\alpha'), \tag{3.10}$$

where $\overline{\text{co}}$ means the strong*-closed convex hull and $B(\alpha,\delta)$ is the open ball of radius δ centered at α. This ensures that the new subdifferential has closed graph and convex values. Set

$$\mathbf{T}(\bar{x}(\cdot),\varepsilon) := \bigcap_{\alpha\in\mathscr{A}} T(\bar{x}(\cdot,\alpha),\varepsilon).$$

We are now prepared to state the main result.

Theorem 3.3. *Let $(\bar{u},\{\bar{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$ be a strong local minimum for the general minimax optimal control problem (P). Assume the hypotheses $H1$ to $H6$ are satisfied for some $\delta > 0$. Suppose also that g is bounded. Then there exists a family of locally Lipschitz functions (parameterized by $\varepsilon > 0$) $\{\phi(\cdot,\cdot,\alpha) : \mathbf{T}(\bar{x}(\cdot),\varepsilon) \to \mathbb{R} \mid \alpha \in \mathscr{A}\}$ and Radon probability measure $\Lambda \in C^*(\mathscr{A})$ satisfying:*

(i) The application $\alpha \mapsto \phi(t,z,\alpha)$ is continuous.

(ii) $\operatorname{supp}\Lambda \subseteq \{\alpha \in \mathscr{A} \mid \max_{\alpha'\in\mathscr{A}} g(\bar{x}(T,\alpha'),\alpha') = g(\bar{x}(T,\alpha),\alpha)\}$ *and*

$$\phi(S,x_0,\alpha) = g(\bar{x}(T,\alpha),\alpha) \quad \Lambda\text{-a.e. } \alpha \in \mathscr{A}.$$

(iii) For each $(t,z) \in intT(\bar{x}(\cdot,\alpha);\varepsilon)$ and family $\{(\xi_\alpha,\eta_\alpha) \mid \alpha \in \mathscr{A}\}$ such that $(\xi_\alpha,\eta_\alpha) \in \partial_{[\mathscr{A}]}\phi(t,z,\alpha) \quad \Lambda - a.e\ \alpha \in \mathscr{A}$, we have

$$\int_{\mathscr{A}} (\xi_\alpha + h(t,z,\eta_\alpha,\alpha))d\Lambda(\alpha) \geqslant 0. \tag{3.11}$$

(iv) Defining

$$\varphi(t,z) := \max_{\alpha\in\mathscr{A}} \phi(t,z,\alpha), \tag{3.12}$$

one has

$$\varphi(T,z) \leqslant \max_{\alpha\in\mathscr{A}} g(z,\alpha) \ \ \textit{and} \tag{3.13}$$

$$\varphi(S,x_0) = \max_{\alpha\in\mathscr{A}} g(\bar{x}(T,\alpha),\alpha). \tag{3.14}$$

Given the hypotheses of the theorem, the application

$$(t,\alpha) \to \min_{u\in\Omega(t)} p\cdot f(t,x,u,\alpha)$$

is $\mathscr{L}\times\mathscr{B}^{\mathscr{A}}$ measurable. Additionally, due to $H2)$, the function $\alpha\to h(t,x,p,\alpha)$ is Λ-integrable. Since $|\xi_\alpha|\leqslant k$ and $|\eta_\alpha|\leqslant k$, where k is the Lipschitz constant of $\phi(\cdot,\cdot,\alpha)$ for each $\alpha\in\mathscr{A}$, it follows that ξ_α and η_α are Λ-integrable. Thus, the Hamiltonian condition (3.11) in the theorem is well defined.

4. Proof of the Theorem 3.3

Without loss of generality, we can replace $H2), H3)$ and $H5)$ with stronger hypotheses, where $\delta = +\infty$. This means the hypotheses hold for all $x, x'\in\mathbb{R}^n$, not just within the ball. This can be achieved by transforming f and g as follows:

$$(t,x,u,\alpha)\to f(t,tr_\delta(x-\bar{x}(t,\alpha)),u,\alpha), \quad \text{and} \quad (x,\alpha)\to g(tr_\delta(x-\bar{x}(t,\alpha)),\alpha).$$

Recall that the truncation function $tr_\delta(\xi)$ is defined in Section 2. The property that $\bar{x}$ is a strong local minimizer is preserved with these data modifications. As a result of the strengthened hypotheses, for each $u\in\mathscr{U}$ and $\alpha\in\mathscr{A}$, there corresponds a unique trajectory on $[S,T]$ with the initial state x_0, which we denote as $x(\cdot;\alpha,u)$.

Let $\Delta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be a metric on $\mathscr{U}$, defined as:

$$\Delta(u_1,u_2) := \int_S^T |u_1(t)-u_2(t)|\,dt.$$

Since $(\bar{u},\{\bar{x}(.;\alpha)\mid\alpha\in\mathscr{A}\})$ is a strong local minimum, there exists $\varepsilon>0$ such that

$$\max_{\alpha\in\mathscr{A}} g(x(T;\alpha),\alpha) \geqslant \max_{\alpha\in\mathscr{A}} g(\bar{x}(T;\alpha),\alpha),$$

for all feasible processes $(u,\{x(.;\alpha)\mid\alpha\in\mathscr{A}\})$ satisfying $\|x(\cdot,\alpha)-\bar{x}(t,\alpha)\|\leqslant\varepsilon$ for all $\alpha\in\mathscr{A}$. Consider a sequence $\varepsilon_i\downarrow 0$, $\varepsilon_i\in(0,\varepsilon)$. Without loss of generality, assume that $\max_{\alpha\in\mathscr{A}} g(\bar{x}(T;\alpha),\alpha) = 0$. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a countable everywhere dense set contained in $\mathscr{A}$. Since $\mathscr{A}$ is compact, for every i there exists $N=N(i)$ such that

$$\max_{\alpha\in\mathscr{A}_N} g(x(T;\alpha),\alpha) \geqslant \max_{\alpha\in\mathscr{A}} g(\bar{x}(T;\alpha),\alpha) - \varepsilon_i^2, \tag{4.1}$$

where $\mathscr{A}_N = \{\alpha_1,\cdots,\alpha_N\}$ can be chosen such that $\mathscr{A}_N\subseteq\mathscr{A}_{N+1}$.

We define, for each i, $J_i:\mathscr{U}\to\mathbb{R}$, $J_i(u) = \max_{\alpha\in\mathscr{A}_N} g(x(T;\alpha,u),\alpha)+\varepsilon_i^2$. Thus defined, $J_i(u)\geqslant 0$ for all u and $J_i(\bar{u}) = \max_{\alpha\in\mathscr{A}_N} g(\bar{x}(T;\alpha),\alpha)+\varepsilon_i^2 = \varepsilon_i^2$. We also have

$$J_i(\bar{u}) \leqslant \inf J_i(u) + \varepsilon_i^2 \quad \forall i.$$

Applying Ekeland's Theorem (see [5, 9]), for each i, there exists $u_i \in \mathscr{U}$ such that $\Delta(u_i, \bar{u}) \leqslant \varepsilon_i$ and

$$J_i(u) + \varepsilon_i \Delta(u, u_i)\Big|_{u=u_i} = \inf_{u \in \mathscr{U}} \{J_i(u) + \varepsilon_i \Delta(u, u_i)\}.$$

So we have the following conditions:

i): $J_i(u) + \varepsilon_i \Delta(u_i, u)\Big|_{u=u_i} = \inf_{u \in \mathscr{U}} \{J_i(u) + \varepsilon_i \Delta(u, u_i)\}$,

ii): $J_i(u_i) > 0$,

iii): $\Delta(u_i, \bar{u}) \to 0$ as $i \to \infty$.

We denote the trajectories corresponding to u_i as $\{x_i(\cdot;\alpha) \mid \alpha \in \mathscr{A}_N\}$. The above properties can be expressed as follows: For each i, $(u_i, \{x_i(\cdot;\alpha_k) \mid k = 1, 2, \ldots, N\})$ is a minimum for the optimal control problem.

$$(P_i) \quad \begin{cases} \text{Minimize} \quad \max_{\alpha \in \mathscr{A}_N} g(x(T;\alpha), \alpha) + \varepsilon_i \int_S^T |u(t) - u_i(t)|\, dt \\ \text{subject to measurable functions} \quad u : [S,T] \to \mathbb{R}^m \quad \text{such that} \\ u(t) \in \Omega(t) \quad \text{a.e.} \quad t \in [S,T], \\ \text{and arcs} \quad \{x(.;\alpha_1), x(.;\alpha_2), \ldots, x(.;\alpha_N)\} \quad \text{such that for all} \\ k = 1, 2, \ldots, N, \\ \dot{x}(t;\alpha_k) = f(t, x(t,\alpha_k), u(t), \alpha_k) \quad \text{a.e.} \quad t \in [S,T], \\ x(S, \alpha_k) = x_0. \end{cases}$$

We also have $u_i \to \bar{u}$ with respect to the Δ-metric, and $x(\cdot; u_i, \alpha) \to \bar{x}(\cdot, \alpha)$ for all $\alpha \in \mathscr{A}$ as $i \to \infty$, by Lemma 2.3. It follows from Proposition 3.2, of the finite case, that there exists a family $\{\phi_i(\cdot,\cdot,\alpha) : \mathbf{T}(x_i(\cdot), \delta_i) \to \mathbb{R}/\alpha \in \mathscr{A}_N\}$ of locally Lipschitz functions (with parameter $\delta_i \downarrow 0$) such that

$$\varphi_i(t,z) := \max_{\alpha \in \mathscr{A}_N} \phi_i(t,z,\alpha) \tag{4.2}$$

satisfies

$$\varphi_i(T,z) \leqslant \max_{\alpha \in \mathscr{A}_N} g(z,\alpha) + \varepsilon_i \int_S^T |u(t) - u_i(t)| dt; \tag{4.3}$$

$$\varphi_i(S,x_0) = \max_{\alpha \in \mathscr{A}_N} g(x_i(T,\alpha), \alpha); \tag{4.4}$$

and for, all $(\xi^i, \eta^i) \in \partial^L \varphi_i(t,z)$,

$$\xi^i + h(t,z,\eta^i,u,\alpha) = \int_{\mathscr{A}_N} (\xi^i_\alpha + h(t,z,\eta^i_\alpha,\alpha)) d\Lambda_i(\alpha) \geqslant 0,$$

which can be rewritten as

$$\int_{\mathscr{A}_N} (\xi^i_\alpha + \min_u \{\eta^i_\alpha \cdot f(t,x,u,\alpha) - \varepsilon_i |u - u_i(t)|\}) d\Lambda_i(\alpha) \geqslant 0, \tag{4.5}$$

for some Radon probability measure Λ_i with $\text{supp}\Lambda_i \subset \mathscr{B}^i(t,z)$ and a family $\{(\xi^i_\alpha, \eta^i_\alpha)/\alpha \in \mathscr{A}\}$ such that

$$(\xi^i_\alpha, \eta^i_\alpha) \in \partial_0 \phi_i(t,z,\alpha) \quad \Lambda_i - \text{a.e. } \alpha \in \mathscr{A}_N,$$

where

$$\partial_0 \phi_i(t,z,\alpha) = \begin{cases} \partial^L \phi_i(t,z,\alpha), & \text{if} \quad \phi_i(t,z,\alpha) = \max_{\alpha' \in \mathscr{A}_N} \phi_i(t,z,\alpha') \\ \emptyset, & \text{otherwise.} \end{cases}$$

Additionally, we have

$$\operatorname{supp} \Lambda_i \subseteq \{\alpha \in \mathscr{A}_N \mid \max_{\alpha' \in \mathscr{A}_N} g(x_i(T,\alpha'),\alpha') = g(x_i(T,\alpha),\alpha)\}, \tag{4.6}$$

$$\phi_i(S,x_0,\alpha) = g(x_i(T,\alpha),\alpha) \quad \Lambda_i\text{-a.e. } \alpha \in \mathscr{A}_N \quad \text{and} \tag{4.7}$$

$$\alpha \mapsto \phi_i(t,z,\alpha) \text{ is continuous for } \alpha \in \mathscr{A}_N. \tag{4.8}$$

It follows from (4.4) and the increasing density of $\mathscr{A}_N$ in $\mathscr{A}$ that

$$\lim_{i\to\infty} \varphi_i(S,x_0) = \max_{\alpha\in\mathscr{A}} g(\bar{x}(T,\alpha),\alpha). \tag{4.9}$$

Taking the limit at (4.3), as $i \to \infty$, we obtain

$$\lim_{i\to\infty} \varphi_i(T,z) \leqslant \max_{\alpha\in\mathscr{A}} g(z,\alpha). \tag{4.10}$$

Note that if $(t,z) \in \bigcap_{\alpha\in\mathscr{A}_N} T(x_i(\cdot,\alpha),\delta_i)$, then $(t,z) \in \bigcap_{\alpha\in\mathscr{A}} T(\bar{x}(\cdot,\alpha),\varepsilon)$.

We know that $\alpha \mapsto \phi_i(t,z,\alpha)$ is continuous for $\alpha \in \mathscr{A}_N$. By Tietze's theorem, for each $\mathscr{A}_N$ there is a continuous extension $\alpha \mapsto \widetilde{\phi}_i(t,z,\alpha)$ for the whole set $\mathscr{A}$ such that $\widetilde{\phi}_i(t,z,\alpha) = \phi_i(t,z,\alpha)$ for all $\alpha \in \mathscr{A}_N$.

Claim 4.1. $\widetilde{\phi}_i(\cdot,\cdot,\alpha)$ is Lipschitz for each $\alpha \in \mathscr{A}$.

Proof. In fact, let $(t,z),(t',z') \in T(\bar{x}(\cdot,\alpha);\varepsilon) \subset [S,T]\times\mathbb{R}^n$. It follows from the compactness of A that, for each i, there exists $\alpha_i \in \mathscr{A}_N$ such that $\alpha_i \to \alpha$. Thus

$$\begin{aligned} |\widetilde{\phi}_i(t,z,\alpha) - \widetilde{\phi}_i(t',z',\alpha)| &= \lim_{i\to\infty} |\widetilde{\phi}_i(t,z,\alpha_i) - \widetilde{\phi}_i(t',z',\alpha_i)| \\ &= \lim_{i\to\infty} |\phi_i(t,z,\alpha_i) - \phi_i(t',z',\alpha_i)| \\ &\leqslant \lim_{i\to\infty} k|(t,z)-(t',z')| \\ &= k|(t,z)-(t',z')|, \end{aligned}$$

concluding the claim. □

As a consequence of the claim proof, the family $\{\widetilde{\phi}_i(\cdot,\cdot,\alpha) \mid \alpha\in\mathscr{A}\}$ is Lipschitz with the same Lipschitz constant. Additionally, they are bounded by the same constant that bounds g. By the Arzela-Ascoli theorem, in each compact subset of $[S,T]\times\mathbb{R}^n$, the family $\{\widetilde{\phi}_i(\cdot,\cdot,\alpha) \mid \alpha\in\mathscr{A}\}$ is a relatively compact set. This means that the sequence $\{\widetilde{\phi}_i(\cdot,\cdot,\alpha) \mid \alpha\in\mathscr{A}\}_{i\in\mathbb{N}}$ has a convergent subsequence to some function $\phi(\cdot,\cdot,\alpha)$, i.e., $\widetilde{\phi}_i(\cdot,\cdot,\alpha) \to \phi(\cdot,\cdot,\alpha)$ for all $\alpha\in\mathscr{A}$.

Claim 4.2. $\phi(\cdot,\cdot,\alpha)$ is a family of Lipschitz functions, and the application $\alpha \mapsto \phi(t,z,\alpha)$ is continuous for all $(t,z) \in T(\bar{x}(t,\alpha),\varepsilon)$.

Proof. In fact, let $(t,z),(t',z') \in T(\bar{x}(t,\varepsilon),\delta)$ and $\alpha\in\mathscr{A}$. Then,

$$
\begin{aligned}
|\phi(t,z,\alpha) - \phi(t',z',\alpha)| &= \lim_{i\to\infty} |\widetilde{\phi}_i(t,z,\alpha) - \widetilde{\phi}_i(t',z',\alpha)| \\
&\leqslant \lim_{i\to\infty} k|(t,z) - (t',z')| \\
&\leqslant k|(t,z) - (t',z')|,
\end{aligned}
$$

verifying the Lipschitz condition of $\phi(\cdot,\cdot,\alpha)$. Since the function $\alpha \to \phi(t,z,\alpha)$ is a limit of continuous functions, it must also be continuous, completing the proof of Claim 4.2. □

Using the same argument used for validating (4.1), we see that

$$
\max_{\alpha\in\mathscr{A}} \widetilde{\phi}_i(t,z,\alpha) \geqslant \max_{\alpha\in\mathscr{A}_N} \phi_i(t,z,\alpha) \geqslant \max_{\alpha\in\mathscr{A}} \widetilde{\phi}_i(t,z,\alpha) - \delta_i.
$$

Taking the limit as $i\to\infty$, we have

$$
\lim_{i\to\infty} \max_{\alpha\in\mathscr{A}_N} \phi_i(t,z,\alpha) = \max_{\alpha\in\mathscr{A}} \phi(t,z,\alpha). \tag{4.11}
$$

Since the limit exists, let us denote $\varphi(t,z) := \lim_{i\to\infty} \varphi_i(t,z)$. So, it follows from (4.2), (4.9) and (4.10) that the conditions (3.12)-(3.14) of the theorem are verified.

It remains to prove (3.11). Consider a family of control functions $\{v_j\}_{j=1,2,\ldots}$, which is countably everywhere dense in $\mathscr{U}$. It follows from (4.5) that, for each j,

$$
\int_S^T \int_{\mathscr{A}} \left(\xi_\alpha^i + \eta_\alpha^i \cdot f(t,x,v_j,\alpha) - \varepsilon_i |v_j - u_i(t)|\right) d\Lambda_i(\alpha)\, dt \geqslant 0,
$$

for some $(\xi_\alpha^i, \eta_\alpha^i) \in \partial_{\delta_i} \widetilde{\phi}_i(t,z,\alpha)$, Λ_i-a.e. $\alpha\in\mathscr{A}_N$. Since Λ_i is a Radon probability measure, the sequence of measures $\{\Lambda_i\}$ has a weakly* convergent subsequence

$$
\Lambda_i \to \Lambda \quad \text{weakly* as} \quad i\to\infty,
$$

for some Radon probability measure Λ defined on the σ-algebra of the Borel sets of $\mathscr{A}$.

Take an integer M and define $B_M^i = (\gamma_0^i(\alpha), \gamma_1^i(\alpha), \ldots, \gamma_M^i(\alpha))$, a sequence of multifunctions

$$
\begin{aligned}
\mathfrak{T}_i(\alpha) = \Big\{ (\gamma_0(\alpha), \gamma_1(\alpha), \ldots, \gamma_M(\alpha)) / \exists (\xi_\alpha, \eta_\alpha) \in \partial_{\delta_i} \widetilde{\phi}_i(t,z,\alpha); \gamma_0(\alpha) = \max_{\alpha'\in\mathscr{A}_N} \phi_i(t,z,\alpha') \quad \text{and} \\
\gamma_j(\alpha) = \int_S^T \xi_\alpha + \eta_\alpha \cdot f(t,z,v_j(t),\alpha) dt, \quad j = 1,2,\ldots,M \Big\},
\end{aligned}
$$

$i = 1,2,\ldots$ and a multifunction

$$
\begin{aligned}
\mathfrak{T}(\alpha) = \Big\{ (\gamma_0(\alpha), \gamma_1(\alpha), \ldots, \gamma_M(\alpha)) / \exists (\xi_\alpha, \eta_\alpha) \in \partial_{[\mathscr{A}]} \phi(t,z,\alpha); \gamma_0(\alpha) = \max_{\alpha'\in\mathscr{A}} \phi(t,z,\alpha') \quad \text{and} \\
\gamma_j(\alpha) = \int_S^T \xi_\alpha + \eta_\alpha \cdot f(t,z,v_j(t),\alpha) dt, \quad j = 1,2,\ldots,M \Big\}.
\end{aligned}
$$

We affirm that this sequence of multifunctions satisfies the following:

Claim 4.3. $\limsup_i Gr\mathfrak{T}_i \subset Gr\mathfrak{T}$.

Proof. Let $(\alpha,x) \in \limsup_i Gr\mathfrak{T}_i$. Then, there exists a subsequence $\{Gr\mathfrak{T}_{i_k}\}$ of the graph sequence $\{Gr\mathfrak{T}_i\}$ and $(\alpha_k,x_k) \in Gr\mathfrak{T}_{i_k}$ such that $(\alpha_k,x_k) \to (\alpha,x)$, where $x = (\gamma_0(\alpha),\gamma_1(\alpha),\cdots,\gamma_M(\alpha))$. Since $x_k \in \mathfrak{T}_{i_k}(\alpha_k)$, there exists $(\xi_{\alpha_k},\eta_{\alpha_k}) \in \partial_{\delta_k}\widetilde{\phi}_k(t,z,\alpha_k)$ such that

$$\gamma_j^k(\alpha_k) = \int_S^T \xi_{\alpha_k} + \eta_{\alpha_k}\cdot f(t,z_k,v_j(t),\alpha_k)dt, \quad j = 1,...,M, \tag{4.12}$$

where

$$\partial_{\delta_k}\widetilde{\phi}_k(t,z,\alpha k) = \begin{cases} \partial^L\widetilde{\phi}_k(t,z,\alpha_k) & \text{if } \widetilde{\phi}_k(t,z,\alpha_k) \geqslant \max_{\alpha\in\mathscr{A}}\widetilde{\phi}_k(t,z,\alpha)-\delta_k \\ \varnothing & \text{otherwise.} \end{cases}$$

Fixing k, by the definition of subdifferential, there exist $(\xi_{\alpha_{k_m}},\eta_{\alpha_{k_m}}) \to (\xi_{\alpha_k},\eta_{\alpha_k})$ and $(t_m,z_m) \to (t,z)$ such that $(\xi_{\alpha_{k_m}},\eta_{\alpha_{k_m}}) \in \partial^P\widetilde{\phi}_k(tm,z_m,\alpha_k)$. By Proposition 4.4.2 in [5], we have

$$\begin{aligned}\widetilde{\phi}_k(t',z',\alpha k) \geqslant& \widetilde{\phi}_k(tm,z_m,\alpha_k) + (\xi_{\alpha_{k_m}},\eta_{\alpha_{k_m}})\cdot((t',z',\alpha_k)-(t_m,z_m,\alpha_k)) \\ &- M|(t',z',\alpha_k)-(t_m,z_m,\alpha_k)|^2.\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\begin{aligned}\phi(t',z',\alpha) \geqslant& \phi(t_m,z_m,\alpha) + (\xi_{\alpha_{k_m}},\eta_{\alpha_{k_m}})\cdot((t',z',\alpha)-(t_m,z_m,\alpha)) \\ &- M|(t',z',\alpha)-(t_m,z_m,\alpha)|^2.\end{aligned}$$

Consequently, $(\xi_{\alpha_{k_m}},\eta_{\alpha_{k_m}}) \in \partial^P\phi(t_m,z_m,\alpha)$, which implies that $(\xi_{\alpha_k},\eta_{\alpha_k}) \in \partial\phi^L(t,z,\alpha)$.

Since the set $\{\phi(\cdot,\cdot,\alpha)/\alpha\in\mathscr{A}\}$ is uniformly Lipschitz, the sequence $(\xi_{\alpha_k},\eta_{\alpha_k})$ has a convergent subsequence. Without loss of generality, assume $(\xi_{\alpha_k},\eta_{\alpha_k}) \to (\xi_\alpha,\eta_\alpha)$. By Proposition 4.3.5 in [5], we have $(\xi_\alpha,\eta_\alpha) \in \partial\phi^L(t,z,\alpha)$, which implies that $(\xi_\alpha,\eta_\alpha) \in \partial_{[\mathscr{A}]}\phi(t,z,\alpha)$. Taking $k \to \infty$ in (4.12), it follows that

$$\gamma_j(\alpha) = \int_S^T \xi_\alpha + \eta_\alpha\cdot f(t,z,v_j(t),\alpha)dt, \quad j = 1,...,M.$$

Therefore, $(\alpha,x) \in Gr\mathfrak{T}$, concluding the proof of Claim 4.3. □

Being $Gr\mathfrak{T}(\alpha)$ compact, $\mathfrak{T}(\alpha)$ convex, $\limsup_i Gr\mathfrak{T}_i \subset Gr\mathfrak{T}$ and $B_M^i(\alpha) \in \mathfrak{T}_i(\alpha)$, it follows from Proposition 2.1 that there exists $B_M(\alpha) \in \mathfrak{T}(\alpha)$ such that

$$\int_{\mathscr{A}} B_M^i(\alpha)\Lambda_i d(\alpha) \to \int_{\mathscr{A}} B_M(\alpha)\Lambda d(\alpha) \geqslant 0.$$

By the definition of B_M, there exists some $(\xi_\alpha^M,\eta_\alpha^M) \in \partial_{[\mathscr{A}]}\phi(t,z,\alpha)$ that satisfies

$$\int_{\mathscr{A}}\int_S^T \xi_\alpha^M + \eta_\alpha^M\cdot f(t,z,v_j(t),\alpha)dt\Lambda d(\alpha) \geqslant 0 \quad \forall j = 1,2,...,M. \tag{4.13}$$

For each M, consider $\alpha \to h^M(\alpha) := (\xi_\alpha^M,\eta_\alpha^M)$ as a representative of an equivalence class of $\Lambda - a.e.$ equal elements in the Hilbert space

$$\mathscr{H} := L^2_\Lambda\big(\mathscr{A};L^2([S,T];\mathbb{R}\times\mathbb{R}^n)\big)$$

with inner produt

$$\langle h,h'\rangle_\Lambda = \int_{\mathscr{A}}\int_S^T \Big(\xi(s),\eta(s))(\xi'(s),\eta'(s)\Big)ds\Lambda d(\alpha) = \int_{\mathscr{A}}\int_S^T \Big(\xi(s)\xi'(s)+\eta(s)\eta'(s)\Big)ds\Lambda d(\alpha).$$

The correspondence mentioned above can be viewed as

$$\alpha \mapsto h^M(\alpha) = (\xi^M_\alpha, \eta^M_\alpha) : \begin{array}{ccccc} [S,T] & \to & \mathbb{R}\times\mathbb{R}^n & & \\ s & \mapsto & h^M(\alpha)(s) & = & \Big(\xi^M_\alpha(s), \eta^M_\alpha(s)\Big). \end{array}$$

The sequence $\{\alpha \mapsto h^M(\alpha)\}_{M=1}^\infty$ is norm bounded. In fact, since the family $\{\phi(\cdot,\cdot,\alpha)/\alpha \in \mathscr{A}\}$ is uniformly Lipschitz, then the sequence $(\xi^M_\alpha, \eta^M_\alpha)_{M=1}^\infty$ is norm bounded

$$\begin{aligned} \|h^M\|^2_\Lambda &= \int_{\mathscr{A}} |h^M(\alpha)|^2 \Lambda d(\alpha) = \int_{\mathscr{A}}\int_S^T |h^M(\alpha)(s)|^2 dt \Lambda d(\alpha) \\ &= \int_{\mathscr{A}}\int_S^T |(\xi^M_\alpha(s), \eta^M_\alpha(s))|^2 dt\Lambda d(\alpha) < \infty. \end{aligned}$$

It implies that the sequence $\{\alpha \mapsto h^M(\alpha)\}_{M=1}^\infty$ has a subsequence with a weak limit given by some $\{\alpha \to (\xi_\alpha, \eta_\alpha)\}$. But

$$B := \Big\{h \in \mathscr{H} \ : \ h(\alpha) \in \partial_{[\mathscr{A}]}\phi(t,z,\alpha), \quad \alpha \in \mathscr{A}\Big\}$$

is strongly closed and convex, so it is weakly closed. It follows that $(\xi_\alpha, \eta_\alpha) \in \partial_{[\mathscr{A}]}\phi(t,z,\alpha)$.

Therefore, taking the limit when $M \to \infty$ in the inequality of the integral (4.13) and taking into account the weak convergence and the fact that v_j is everywhere dense in $\mathscr{U}$, we deduce that

$$\int_{\mathscr{A}}\int_S^T (\xi + \min_{u\in\mathscr{U}} \eta \cdot f(t,z,u,\alpha)) dt d\Lambda(\alpha) \geqslant 0.$$

This is equivalent to the corresponding condition stated in the theorem, concluding the proof. ▫

5. Sufficient Optimality Conditions for Problem (P) and the Value Function

In this section, we provide sufficient conditions for MOCP (P) and its relationship with the value function. We suppose that hypotheses $H1) - H6)$ are satisfied throughout this section.

5.1. **Sufficient optimality conditions.** We have the following Proposition.

Proposition 5.1. *Let $(\bar{u}, \{\bar{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$ be a feasible process for the problem (P). Then $(\bar{u}, \{\bar{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$ is a strong local minimizer if, for some $\varepsilon > 0$, there exist an equicontinuous family of functions $\{\phi(\cdot,\cdot,\alpha) : T(\bar{x}(\cdot,\alpha),\varepsilon) \to \mathbb{R}/\alpha \in \mathscr{A}\}$ and a Radon probability measure $\Lambda \in C^*(\mathscr{A})$ satisfying:*

- *(i) The application $\alpha \mapsto \phi(t,z,\alpha)$ is continuous.*
- *(ii) $\operatorname{supp}\Lambda \subseteq \{\alpha \in \mathscr{A} \mid \max_{\alpha'\in\mathscr{A}} g(\bar{x}(T,\alpha'),\alpha') = g(\bar{x}(T,\alpha),\alpha)\}$.*
- *(iii) For each $(t,z) \in intT(\bar{x}(\cdot,\alpha);\varepsilon)$ and all $(\xi_\alpha, \eta_\alpha) \in \partial_{[\mathscr{A}]}\phi(t,z,\alpha)$ Λ-almost everywhere $\alpha \in \mathscr{A}$, we have*

$$\int_E \xi_\alpha + \min_u \eta_\alpha \cdot f(t,z,u,\alpha)\Lambda d(\alpha) \geqslant 0 \quad \forall E \in \mathscr{B}^{\mathscr{A}}. \tag{5.1}$$

(iv) Defining $\varphi(t,z) := \max_{\alpha\in\mathscr{A}} \phi(t,z,\alpha)$*, we have*

$$\phi(S,x_0,\alpha) = g(\bar{x}(T,\alpha),\alpha) \quad \Lambda\text{-a.e. } \alpha\in\mathscr{A}, \tag{5.2}$$

$$\varphi(T,z) \leqslant \max_{\alpha\in\mathscr{A}} g(z,\alpha), \tag{5.3}$$

$$\varphi(S,x_0) = \max_{\alpha\in\mathscr{A}} g(\bar{x}(T,\alpha),\alpha). \tag{5.4}$$

Given a feasible process $(\bar{u},\{\bar{x}(.;\alpha) \mid \alpha\in\mathscr{A}\})$ of problem (P), an equicontinuous family of functions $\{\phi(\cdot,\cdot,\alpha) : T(\bar{x}(\cdot,\alpha),\varepsilon)\to\mathbb{R}/\alpha\in\mathscr{A}\}$ is called verification function for the local optimality of this process if there exists a Radon probability measure $\Lambda\in C^*(\mathscr{A})$, in which the pair $(\phi(\cdot,\cdot,\alpha),(\bar{u},\{\bar{x}(.;\alpha) \mid \alpha\in\mathscr{A}\}))$ satisfies conditions (i)-(iv) of Proposition 5.1

Proof. Taking $(\bar{u},\{\bar{x}(.;\alpha) \mid \alpha\in\mathscr{A}\})$, the feasible process for problem (P), we want to check the optimality. Let $(u,\{x(.;\alpha) \mid \alpha\in\mathscr{A}\})$ be an arbitrary feasible process of the problem such that

$$\|x(\cdot,\alpha)-\bar{x}(\cdot;\alpha)\|_{L^\infty} \leqslant \varepsilon \quad \text{for all} \quad \alpha\in\mathscr{A}.$$

Suppose that the statements of the proposition are valid. It follows from condition (5.1) that, for $(\xi_\alpha,\eta_\alpha)\in\partial_{[\mathscr{A}]}\phi(t,z,\alpha)$,

$$\xi_\alpha + \min_u \eta_\alpha\cdot f(t,z,u,\alpha) \geqslant 0, \quad \Lambda\text{-a.e. } \alpha\in\mathscr{A}.$$

By Proposition 6.5 in [9], we have $\phi(S,x_0,\alpha)\leqslant\phi(T,x(T,\alpha),\alpha)$, Λ-a.e. $\alpha\in\mathscr{A}$. Then, it follows from the definition of φ, (5.2) and (5.3) that:

$$\max_{\alpha\in\mathscr{A}} g(\bar{x}(T,\alpha),\alpha) \leqslant \max_{\alpha\in\mathscr{A}} g(x(T,\alpha),\alpha).$$

Therefore, the feasible process $(\bar{u},\{\bar{x}(.;\alpha) \mid \alpha\in\mathscr{A}\})$ is a strong local minimum for problem (P). □

5.2. **The value function.** Consider the following family of control problems, parameterized in $(t,z,\alpha)\in[S,T]\times\mathbb{R}^n\times\mathscr{A}$.

$$(Q^\alpha_{t,z}) \quad \begin{cases} \text{Minimize} \quad g(x(T;\alpha),\alpha) \quad \text{over} \\ \qquad\qquad \dot{x}(s;\alpha) = f(s,x(s,\alpha),u(s),\alpha), \quad \text{a.e.} \quad t\in[S,T], \\ \qquad\qquad x(t,\alpha) = z \text{ and } u\in\mathscr{U}. \end{cases}$$

For each $\alpha\in\mathscr{A}$, the value function $W(\cdot,\cdot,\alpha):[S,T]\times\mathbb{R}^n\to\mathbb{R}$ is defined by:

$$W(\alpha,t,z) =: \inf_{u\in\mathscr{U}(t)} \{g(x(T,\alpha),\alpha) \ / \ \dot{x}(s;\alpha) = f(s,x(s,\alpha),u(s),\alpha) \ \text{ and } \ x(t,\alpha)=z\}.$$

Proposition 5.2. *Assume the hypotheses* $H1)-H5)$ *are satisfied. Then,*

$$V(t,z) = \max_{\alpha\in\mathscr{A}} W(t,z,\alpha). \tag{5.5}$$

Proof. We need to prove (5.5), which is equivalent to

$$\inf_{u\in\mathscr{U}} \max_{\alpha\in\mathscr{A}} g(x(T,\alpha),\alpha) = \max_{\alpha\in\mathscr{A}} \inf_{u\in\mathscr{U}} g(x(T,\alpha),\alpha).$$

Let $(\bar{u},\{\bar{x}(.;\alpha) \mid \alpha\in\mathscr{A}\})$ be an arbitrary feasible process of the problem $(P_{t,z})$. For any fixed $\alpha'\in\mathscr{A}$, we have the following inequality:

$$\max_{\alpha\in\mathscr{A}} g(\bar{x}(T,\alpha),\alpha) \geqslant g(\bar{x}(T,\alpha'),\alpha') \geqslant \inf_{u\in\mathscr{U}} g(x(T,\alpha'),\alpha').$$

Given the arbitrariness of $(\bar{u}, \{\bar{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$, it follows that

$$\inf_{u\in\mathscr{U}} \max_{\alpha\in\mathscr{A}} g(x(T,\alpha),\alpha) \geqslant \inf_{u\in\mathscr{U}} g(x(T,\alpha'),\alpha'),$$

and consequently,

$$\inf_{u\in\mathscr{U}} \max_{\alpha\in\mathscr{A}} g(x(T,\alpha),\alpha) \geqslant \max_{\alpha\in\mathscr{A}} \inf_{u\in\mathscr{U}} g(x(T,\alpha),\alpha). \tag{5.6}$$

Note that this last inequality does not need hypothesis $H5$), which will be necessary next. It follows from Proposition 2.62 in [5] that the problem $(P_{t,z})$ has an optimal process. Let $(\hat{u}, \{\hat{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$ be an optimal process and $(u, \{x(.;\alpha) \mid \alpha \in \mathscr{A}\})$ be an arbitrary feasible process of $(P_{t,z})$. Given $\varepsilon > 0$, take $\delta' = \frac{\varepsilon}{L}$, where $L := \exp\left(\int_S^T k_f(t)dt\right)$. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a countable everywhere dense subset of $\mathscr{A}$. Due to hypothesis H3, there exists $\delta'' > 0$ such that $s \in [0,\infty), |s| < \delta''$ implies $|\theta(s)| < \delta'$. As the application $\alpha \mapsto h(\alpha) = g(x(T,\alpha),\alpha)$ is continuous, there exists $\delta''' > 0$ such that $\rho(\alpha_1,\alpha_2) < \delta'''$ implies $|h(\alpha_1) - h(\alpha_2)| < \varepsilon$. Let $\delta = \min\{\delta'',\delta'''\}$. As $\mathscr{A}$ is a compact metric space, there exists a finite subset $\widetilde{\mathscr{A}} = \{\alpha_1,...,\alpha_N\}$ of $\mathscr{A}$ such that

$$\mathscr{A} = \bigcup_{j=1}^{N} B(\alpha_j,\delta).$$

So, for all $\alpha \in \mathscr{A}$, there exists $\alpha' \in \widetilde{\mathscr{A}}$ such that $\rho(\alpha,\alpha') < \delta$. It follows from Lemma 2.2(ii) that

$$\begin{aligned} &|x(s,\alpha) - x(s,\alpha')| \leqslant \theta(\rho(\alpha,\alpha'))L < \varepsilon, \\ &|g(x(T,\alpha),\alpha) - g(x(T,\alpha'),\alpha')| \leqslant \varepsilon. \end{aligned} \tag{5.7}$$

As a consequence of the latter, we have

$$\max_{\alpha\in\mathscr{A}} g(x(T,\alpha),\alpha) \leqslant g(x(T,\alpha'),\alpha') + \varepsilon. \tag{5.8}$$

Since $(\hat{u}, \{\hat{x}(.;\alpha) \mid \alpha \in \mathscr{A}\})$ is a minimum process for $(P_{t,z})$, there exists some $\alpha' \in \widetilde{\mathscr{A}}$ such that

$$g(\hat{x}(T,\alpha'),\alpha') \leqslant \max_{\alpha\in\mathscr{A}} g(\hat{x}(T,\alpha),\alpha) \leqslant \max_{\alpha\in\mathscr{A}} g(x(T,\alpha),\alpha) \leqslant g(x(T,\alpha'),\alpha') + \varepsilon.$$

Fix this $\alpha' \in \widetilde{\mathscr{A}}$. Thus, by the arbitrariness of $\varepsilon > 0$ and the process $(u, \{x(.;\alpha) \mid \alpha \in \mathscr{A}\})$, it follows that $(\hat{u}, \hat{x}(\cdot,\alpha'))$ is minimum of the problem $(Q_{t,z}^{\alpha'})$. Thus, for this $\alpha' \in \widetilde{\mathscr{A}}$, we have

$$W(t,z,\alpha') = g(\hat{x}(T,\alpha'),\alpha').$$

We know that $V(t,z) = \max_{\alpha\in\mathscr{A}} g(\hat{x}(T,\alpha),\alpha)$, which implies that

$$V(t,z) \leqslant g(x(T,\alpha'),\alpha') + \varepsilon \leqslant g(\hat{x}(T,\alpha'),\alpha') + 2\varepsilon = W(t,z,\alpha') + 2\varepsilon \leqslant \max_{\alpha\in\mathscr{A}} W(t,z,\alpha) + 2\varepsilon.$$

Combining this last inequality with (5.6), we obtain (5.5), concluding the proof. □

As an immediate consequence, we get the following Corollary.

Corollary 5.3. *Assume that the hypotheses* $H1) - H4)$ *are valid,* g *is continuous and* $g(\cdot,\alpha)$ *is Lipschitz continuous with Lipschitz constant* k_g *for each* $\alpha \in \mathscr{A}$*. Then the value function* V *is Lipschitz.*

We also have the usual constancy property of the Value function for the minimax control problem.

Proposition 5.4. *Let $(t,z) \in [S,T] \times \mathbb{R}^n$ and $(u, \{x(.;\alpha) \mid \alpha \in \mathscr{A}\})$ be a feasible control process that satisfies $x(t,\alpha) = z$. Assume that the hypotheses $(H1)-(H4)$ are satisfied. Then the following properties are verified:*

a): *for all $s_1, s_2 \in [t,T]$, with $s_1 \leqslant s_2$, we have*

$$V(s_1, x(s_1,\alpha)) \leqslant V(s_2, x(s_2,\alpha));$$

b): *if the process $(\{x(\cdot,\alpha)|\alpha \in \mathscr{A}\}, u)$ is optimal for the problem $(P_{t,z})$, then, for all $s_1, s_2 \in [t,T]$, with $s_1 \leqslant s_2$, we have*

$$V(s_1, x(s_1,\alpha)) = V(s_2, x(s_2,\alpha)).$$

Proof. **(a)** Suppose that $V(s_1, x(s_1,\alpha)) \leqslant V(s_2, x(s_2,\alpha))$ is false. Then there exists $\varepsilon > 0$ such that $V(s_1, x(s_1,\alpha)) > V(s_2, x(s_2,\alpha)) + \varepsilon$. Note that there exists $\tilde{u} : [s_2, T] \to \mathbb{R}^m$ such that

$$V(s_2, x(s_2,\alpha)) \geqslant \max_{\alpha \in \mathscr{A}} g(\tilde{x}(T,\alpha),\alpha) - \frac{\varepsilon}{2},$$

where $\tilde{x}(\cdot,\alpha) : [s_2,T] \to \mathbb{R}^n$ for each $\alpha \in \mathscr{A}$ satisfies

$$\begin{aligned} \dot{\tilde{x}}(t,\alpha) &= f(t, \tilde{x}(t,\alpha), \tilde{u}(t), \alpha) \\ \tilde{x}(s_2,\alpha) &= x(s_2,\alpha). \end{aligned}$$

Define the control

$$\bar{u}(t) = \begin{cases} u(t), & t \in [s_1, s_2) \\ \tilde{u}(t), & t \in [s_2, T]. \end{cases}$$

By the uniqueness of the trajectories, for every $\alpha \in \mathscr{A}$, there exist

$$\bar{z}(t,\alpha) = \begin{cases} x(t,\alpha), & t \in [s_1, s_2) \\ \tilde{x}(t,\alpha), & t \in [s_2, T]. \end{cases}$$

such that the process $(\{\bar{z}(\cdot,\alpha)|\alpha \in \mathscr{A}\}, \bar{u})$ is feasible for the problem $P_{s_1, x(s_1,\alpha)}$. So

$$\begin{aligned} V(s_1, x(s_1,\alpha)) &\leqslant \max_{\alpha \in \mathscr{A}} g(\bar{z}(T,\alpha),\alpha) = \max_{\alpha \in \mathscr{A}} g(\tilde{x}(T,\alpha),\alpha) \\ &\leqslant V(s_2, x(s_2,\alpha)) + \frac{\varepsilon}{2} < V(s_1, x(s_1,\alpha)) - \frac{\varepsilon}{2}, \end{aligned}$$

which it is absurd. Therefore $V(s_1, x(s_1,\alpha)) \leqslant V(s_2, x(s_2,\alpha))$. Thus the value function is increasing over feasible processes.

(b) Since that the process $(u, \{x(.;\alpha) \mid \alpha \in \mathscr{A}\})$ is optimal for the problem $(P_{t,z})$ then for item $a)$

$$V(t,z) \leqslant V(s_1, x(s_1,\alpha)) \leqslant V(s_2, x(s_2,\alpha)) \leqslant \max_{\alpha \in \mathscr{A}} g(x(T,\alpha),\alpha) = V(t,z).$$

Thus we have that the value function is constant over optimal processes. □

6. Closing Remarks and Future Perspectives

This work addresses the evolution problem for optimal control of Ordinary Differential Equations (ODEs) under parameter uncertainty, modeling the uncertainty as a worst-case scenario and then minimizing over controls. The primary novelty lies in deriving optimality conditions through a dynamic programming approach for the minimax control problem under consideration. Notably, this approach yields sufficient conditions for optimality without requiring convexity assumptions, which are often difficult to verify in practice. This demonstrates the significance of our findings for a broader class of optimal control problems for ODEs.

Furthermore, the techniques developed in this work offer a promising foundation for extending the analysis to optimal control problems governed by Partial Differential Equations (PDEs) under parameter uncertainty. By adapting the dynamic programming principle to the context of PDEs, researchers can potentially derive novel optimality conditions and develop more efficient numerical algorithms for solving these challenging problems.

Potential avenues for future research include:

- Developing a generalized dynamic programming principle for Hamilton-Jacobi-Bellman equations associated with PDE-constrained optimal control problems with parameter uncertainties.
- Investigating the role of viscosity solutions in establishing sufficient conditions for optimality in the PDE setting.
- Exploring numerical methods for approximating solutions to the Hamilton-Jacobi-Bellman equations associated with PDEs, leveraging insights gained from the ODE case.

These extensions for minimax control problems have the potential to significantly advance the field of optimal control of PDEs, with applications in areas such as fluid dynamics, materials science, and biological systems.

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