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MARGINAL ANALYSIS OF CONVEX OPTIMIZATION PROBLEMS WITH SET-VALUED INCLUSION CONSTRAINTS

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Abstract. In this paper, stability and sensitivity properties of a class of parametric constrained optimization problem, whose feasible region is defined by a set-valued inclusion, are investigated through the associated optimal value function. Set-valued inclusions are a kind of constraint system, which naturally emerges in contexts requiring the robust fulfilment of traditional cone constraints, where data are affected by uncertain elements having a non stochastic nature, or in (MPEC) as a vector equilibrium constraint, where feasible solutions are intended as equilibrium point in a strong sense. Under proper convexity assumptions on the objective function and the constraining set-valued term, combined with a global qualification condition, a class of parametric optimization problems is singled out, which displays a global Lipschitz behaviour. By employing recent results of variational analysis, elements for a sensitivity analysis of this class of problems are provided via an exact subgradient formula for the optimal value function. Further consequences of the stability behaviour are explored in terms of problem calmness and viability of penalization techniques.
Keywords. C-concave multi-valued mappings; Metric C-increase; Parametric optimization; Problem calmness; Penalty function; Optimal value function; Set-valued inclusions.

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1. INTRODUCTION

The present paper aims at bringing new insights into the study of the stability and sensitivity properties of parametric optimization problems with set-valued inclusion constraints. Stability and sensitivity issues for constrained optimization problems subject to parameter perturbations lie at the core of a well-recognized and very active area of variational analysis, often indicated as perturbation analysis of optimization problems (see [3, 7, 10, 18, 20]). According to a deep-rooted approach to this topic, the two main objects of study are the *optimal value* (a.k.a. *marginal* or *performance*) *function* associated with a class of parametric optimization problems and the (generally) multi-valued mapping collecting the optimal solutions (if any) of these problems, when the value of the parameter varies. Various qualitative and quantitative information

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describing how problems are conditioned by and react to changes in the parameter (e.g. solvability near reference values, closeness to known values, rate of changes and so on) can be grasped by analyzing these two objects. The investigations reported in the present paper focus on the analysis of the optimal value function, whereas the analysis of the solution mapping will be left for a future, specifically devoted, project of research. According to [20, Chapter 4.6]

> "*it would not be an exaggeration to say that marginal functions manifest the essence of modern techniques in variational analysis involving perturbation and approximation procedures with the subsequent passing to the limit.*"

The distinguishing feature of the optimization problems considered in the present paper consists in the fact that their constraints are expressed by parameterized set-valued inclusions. These lead to a constraint system format different from the generalized equation format typically occurring in the variational analysis literature, with its own geometry needing specifically devised tools to be adequately handled. Historically, to the best of the author's knowledge, the appearance of such a constraint format can be traced back to [23], where in order to address inexact linear programming problems feasible regions were considered, which are defined via "set-containment", so constraints are called there "set-inclusive". Subsequently, set-valued inclusions revealed to be a natural language to formalize the fulfilment of traditional cone constraints in the context of robust optimization, as it was introduced by A. Ben-Tal and A. Nemirovsky (see [5, 6]). Besides, they found further contexts of relevant application in vector optimization, in expressing ideal efficiency (see [27]), and, more in general, in vector equilibrium theory, in formalizing the concept of strong solution to vector Ky-Fan inequalities (see [26]).

Let us consider the following class of parametric constrained optimization problems

$$(\mathsf{P}_p) \qquad \begin{array}{ll} \displaystyle\min_{x\in\mathbb{X}} & \varphi(p,x) \\ \text{sub} & F(p,x)\subseteq C, \end{array}$$

where $\varphi:\mathbb{P}\times\mathbb{X}\longrightarrow\mathbb{R}$ represents the objective function, $\{\mathbf{0}\}\subsetneqq C\subsetneqq\mathbb{Y}$ is a (nontrivial) closed, convex cone, and $F:\mathbb{P}\times\mathbb{X}\rightrightarrows\mathbb{Y}$ is a multi-valued mapping defining the set-valued inclusion problem, which formalizes the constraint system of the optimization problems. Throughout the paper, $(\mathbb{P},\|\cdot\|)$, $(\mathbb{X},\|\cdot\|)$, and $(\mathbb{Y},\|\cdot\|)$ denote Banach spaces over the real field $\mathbb{R}$, with null vector $\mathbf{0}$. As it should be clear, the variable $p\in\mathbb{P}$ indicates the parameter subject to perturbation, while the variable $x\in\mathbb{X}$ stands for the problem unknown. In the above setting, the (generally) multi-valued mapping $\mathsf{R}:\mathbb{P}\rightrightarrows\mathbb{X}$ that describes the changes of the feasible region of (P_p), when the parameter p varies, is given by

$$\mathsf{R}(p)=F^{+1}(p,\cdot)(C)=\{x\in X\mid F(p,x)\subseteq C\}, \tag{1.1}$$

where the notation F^{+1} (borrowed from [2]) indicates one of the two possible manners to define the inverse image through the set-valued mapping F of a given subset of the range space, namely the core through F of that subset. The optimal value function $\mathsf{val}:\mathbb{P}\longrightarrow\mathbb{R}\cup\{\pm\infty\}$ associated with the class of parametric constrained optimization problems (P_p) takes the form

$$\mathsf{val}(p)=\inf_{x\in\mathsf{R}(p)}\varphi(p,x), \tag{1.2}$$

while the problem solution set-valued mapping $\mathsf{Argmin}:\mathbb{P}\rightrightarrows\mathbb{X}$ is given by

$$\mathsf{Argmin}(p)=\{x\in\mathsf{R}(p)\mid\varphi(p,x)=\mathsf{val}(p)\}.$$

The investigations exposed in the present paper carry on the value analysis started in [24], where several calmness properties of the marginal function val have been established by employing various sufficient conditions for the local Lipschitz semicontinuity behaviour of R. Although conducted within the same framework, the present analysis aims nonetheless at enlightening a different aspect of this topic. The idea triggering the investigations here contained relies on the expectation that much stronger global properties of val can be obtained under suitable convexity assumptions on the problem data, which in [24] are completely disregarded. As one expects, convexity alone is not able to ensure those properties of val, so its effects are studied in synergy with qualification conditions for set-valued inclusions, which are expressed in terms of C-increase behaviour of the set-valued term F. More precisely, it can be shown that convexity and Lipschitz continuity of val can be achieved by applying a recently established result about the existence of continuous selection of R. This leads to single out a class of well-behaved problems (P_p), that will be called "qualified convex optimization problems with set-valued inclusion constraints", for which either $\mathsf{val} \equiv -\infty$ or val is convex and locally Lipschitz. The reader should notice that, in a finite-dimensional setting, this means that val turns out to be smooth on $\mathbb{P}$, up to a (Lebesgue) residual subset. Furthermore, elements for the sensitivity analysis of (P_p) are proposed by providing sharp estimates of the subgradients of val, which are established by means of calculus rules for the Moreau-Rockafellar subdifferential and coderivative calculus for multifunctions with convex graph.

The contents of the paper are arranged according to the following scheme. Section 2 collects the notations, standing assumptions and all the technical tools, mainly coming from convex and variational analysis and from the theory of set-valued inclusions, needed in the subsequent analysis. For the sake of clarity, these elements are presented in several subsections. Section 3 exposes the main results of the paper, namely Lipschitzian properties of val and exact formulae for its subgradients, then illustrates them through several examples. In Section 4 some remarkable consequences of the results presented in the previous section are discussed, which deal with the property of problem calmness and the viability of penalization techniques for solving problems (P_p).

2. Technical Preliminaries

The notations in use throughout the paper are standard. Whenever $(\mathbb{X}, \|\cdot\|)$ is a Banach space, $\mathbb{X}^*$ denotes its topological dual, with $\langle \cdot, \cdot \rangle$ indicating their duality pairing. Given $x \in \mathbb{X}$ and $r \geq 0$, the closed ball centered at x, with radius r is denoted by $\mathrm{B}[x; r]$. In particular, $\mathrm{B}[\mathbf{0}; 1] = \mathbb{B}$, while the unit ball in a dual space is denoted by $\mathbb{B}^*$.

Given a subset S of a Banach space, $\mathrm{int}\, S$ denotes the topological interior of S, whereas $\mathrm{cone}\, S$ and $\mathrm{clco}\, S$ denote the conical hull and the convex closure of S, respectively. Given an element x in the same space, by $\mathrm{dist}(x; S) = \inf_{z \in S} \|x - z\|$ the distance of x from S is indicated. Given two subsets A and B of the same Banach space, the excess of A beyond B is denoted by $\mathrm{exc}(A; B) = \sup_{a \in A} \mathrm{dist}(a; B)$, with $\mathrm{haus}(A; B) = \max\{\mathrm{exc}(A; B), \mathrm{exc}(B; A)\}$ denoting the Pompeiu-Hausdorff distance of A and B. $\mathscr{L}(\mathbb{X}, \mathbb{Y})$ stands for the Banach space of all linear bounded operators from $\mathbb{X}$ to $\mathbb{Y}$. If $\varphi : \mathbb{X} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ is a given function, $\mathrm{dom}\, \varphi = \varphi^{-1}(\mathbb{R})$ denotes the domain of φ, whereas by $\mathrm{dom}\, G = \{x \in \mathbb{X} \mid G(x) \neq \varnothing\}$ the domain of G is denoted whenever $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ is a set-valued mapping. The acronyms l.s.c. and u.s.c. stand for lower semicontinuous and upper semicontinuous, respectively, when referred to both, scalar functions as well as to set-valued mappings.

Throughout the paper the following standing assumptions are maintained:

(a_1) F takes nonempty and closed values (in particular, $\mathrm{dom}\,F = \mathbb{P}\times\mathbb{X}$);
(a_2) $\mathrm{dom}\,\varphi = \mathbb{P}\times\mathbb{X}$;

Moreover, whenever considered, the space $\mathbb{R}^n$ is assumed to be equipped with its usual Euclidean space structure.

2.1. **Elements of convex analysis.** Whenever $\Omega \subseteq \mathbb{X}$ is a nonempty convex subset of a normed space and $\bar{x}\in\Omega$, the set

$$\mathrm{N}(\Omega;\bar{x}) = \{x^*\in\mathbb{X}^* \mid \langle x^*, x-\bar{x}\rangle \le 0, \quad \forall x\in\Omega\}$$

is called the *normal cone* (*in the sense of convex analysis*) to Ω at $\bar{x}$.

Given a convex function $\psi:\mathbb{X}\longrightarrow \cup\{\pm\infty\}$ and $\bar{x}\in\mathrm{dom}\,\psi$, the *subdifferential in the sense of convex analysis* (a.k.a. *Moreau-Rockafellar subdifferential*) of ψ at $\bar{x}$ is defined by

$$\partial\psi(\bar{x}) = \{x^*\in\mathbb{X}^* \mid \langle x^*, x-\bar{x}\rangle \le \psi(x)-\psi(\bar{x}), \quad \forall x\in\mathbb{X}\}.$$

It is well known that if Ω is a closed and convex set, then the distance function $x\mapsto \mathrm{dist}\,(x;\Omega)$ from Ω is a (Lipschitz continuous) convex function. In this special case, the two above dual constructions are linked by the following relationships, which will be useful in the sequel.

Proposition 2.1 ([21]). *Let Ω be a closed and convex set and let $\bar{x}\in\Omega$. The following equalities hold:*

(i) $\partial\mathrm{dist}\,(\cdot;\Omega)\,(\bar{x}) = \mathrm{N}(\Omega;\bar{x})\cap\mathbb{B}^*$*;*
(ii) $\mathrm{N}(\Omega;\bar{x}) = \bigcup_{t\ge 0} t\,\partial\mathrm{dist}\,(\cdot;\Omega)\,(\bar{x})$.

Their are also linked by the indicator function $x\mapsto\iota(x;\Omega)$, which is convex iff $\Omega\subseteq\mathbb{X}$ is so, through the formula

$$\partial\iota(\cdot;\Omega)(\bar{x}) = \mathrm{N}(\Omega;\bar{x}), \tag{2.1}$$

with $\bar{x}\in\Omega$ (see, for instance, [21, Example 3.55]).

Among the various and well developed calculus rules for subdifferential, the one concerning the composition with linear mappings, recalled below, will be of use in the next section: let $\Lambda\in\mathscr{L}(\mathbb{X},\mathbb{Y})$, let $\bar{x}\in\mathbb{X}$ and let $\psi:\mathbb{Y}\longrightarrow\mathbb{R}\cup\{\pm\infty\}$ be continuous at $\Lambda\bar{x}$. Then, it holds

$$\partial\,(\psi\circ\Lambda)\,(\bar{x}) = \Lambda^*\partial\psi(\Lambda\bar{x}). \tag{2.2}$$

A proof of the above composition rule can be found, for instance, in [21, Theorem 3.55].

The further tool needed for the subsequent analysis is a kind of generalized derivative for convex set-valued mappings, i.e. mappings whose graph is convex, which can be built graphically via the normal cone. Let $G:\mathbb{X}\rightrightarrows\mathbb{Y}$ be a convex set-valued mapping between normed spaces and let $(\bar{x},\bar{y})\in\mathrm{gph}\,G$. The *coderivative* of G at $(\bar{x},\bar{y})$ (*in the sense of convex analysis*) is the set-valued mapping $\mathrm{D}^*G(\bar{x},\bar{y}):\mathbb{Y}^*\rightrightarrows\mathbb{X}^*$

$$\mathrm{D}^*G(\bar{x},\bar{y})(y^*) = \{x^*\in\mathbb{X}^* \mid (x^*,-y^*)\in\mathrm{N}(\mathrm{gph}\,G;(\bar{x},\bar{y}))\}.$$

2.2. **Subtransversality of sets.** Several advanced calculus rules in convex analysis, as well as in other areas of nonsmooth analysis, can be performed provided that proper qualification conditions are satisfied. Fur the purposes of the present analysis, the following one plays a crucial role: a given pair of nonempty subsets A and B of a normed space is said to be *subtransversal* at $\bar{x} \in A \cap B \neq \varnothing$ if there exist positive κ and δ such that

$$\mathrm{dist}(x; A \cap B) \leq \kappa \max\{\mathrm{dist}(x; A), \mathrm{dist}(x; B)\}, \quad \forall x \in \mathrm{B}[\bar{x}; \delta]. \tag{2.3}$$

Such a property has a long and articulated history, as its origins can be traced back to the well-known concept of transversality from differential topology (see [12, Chapter 3]). The latter is usually formulated as a surjectivity condition on the sum of the tangent spaces to each of a pair of smooth manifolds, which can guarantee that also their intersection is locally a smooth manifold. The condition postulated by (2.3) can be regarded as a generalization of the concept of transversality in terms of a metric inequality, a language more adequate to the specific needs of nonsmooth analysis. With such formulation, it has emerged as a key notion in the convergence analysis of iterative methods for solving feasibility problems (see [4]). As shown in [16, Theorem 3.1], it can be equivalently reformulated as the existence of positive constants η and δ such that

$$[A + (\eta r)\mathbb{B}] \cap [B + (\eta r)\mathbb{B}] \cap \mathrm{B}[\bar{x}; \delta] \subseteq (A \cap B) + r\mathbb{B}, \quad \forall r \in (0, \delta).$$

Notice that, since it holds

$$\max\{\alpha, \beta\} \leq \alpha + \beta \leq 2\max\{\alpha, \beta\}, \quad \forall \alpha, \beta \geq 0,$$

the maximum of distances appearing in (2.3) can be obviously replaced (what is done pretty often in the variational analysis literature) by the sum of distances (see, for instance, [15, Definition 7.5]). The geometric idea behind the above metric inequality seems to be transparent: "if you are close to both the sets of the pair, then the intersection cannot be too far away" to quote [4]. In one or another of the above reformulations, subtransversality has been employed in different contexts under various names, such as linear regularity, linear coherence, metric inequality and metric qualification condition (see [17] for details about the evolution of the terminology during the last two decades). The reason behind such a frequent occurrence within various topics of variational analysis rests upon the fact it can describe a certain "good mutual arrangement" of several sets in space, with seemingly independent, yet deeply intertwined, far-reaching consequences. Indeed, it has been well understood that the condition in (2.3) is connected with such themes as error bounds, weak sharp minimality, linear subopeness/metric subregularity of set-valued mappings and hence, through the inverse mapping, with calmness (see [15, 17]). The regularity behaviour behind the manifestation of the aforementioned properties works as a key assumption for deriving subdifferential, coderivative and normal cone calculus formulae. Furthermore, it enables to establish linear convergence in alternating projection algorithms.

For the purposes of the present analysis, it is worth observing that, whenever A and B are closed convex sets, a sufficient condition for the pair A and B to be subtransversal at $\bar{x} \in A \cap B$ is that $\mathbf{0} \in \mathrm{int}(A - B)$. Pairs of polyhedral convex sets are known to be (globally) subtransversal at any intersection point (see [15, Theorem 8.35]). For a systematic study of this property, including several characterizations and sufficient conditions in various settings, the reader is referred to [4, 15, 16, 17] and the bibliography therein.

2.3. **C-Increasing multi-valued mappings.** The property recalled below is employed to guarantee solvability of set-valued inclusion problems and related error bounds, thereby propitiating stability behaviours of their solution mappings.

Definition 2.2. Let $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping between vector normed spaces and let $C \subset \mathbb{Y}$ be a closed convex cone. G is said to be *metrically C-increasing* at $x_0 \in \mathbb{X}$ if there exist $\alpha > 1$ and $\delta > 0$ such that

$$\forall r \in (0, \delta]\ \exists u \in \mathrm{B}[x_0; r] \mid \mathrm{B}[G(u); \alpha r] \subseteq \mathrm{B}[G(x_0) + C; r]. \tag{2.4}$$

The value

$$\mathrm{inc}(G; x_0) = \sup\{\alpha > 1 \mid \exists \delta > 0 \text{ for which (2.4) holds}\}$$

is called *exact bound of C-increase* of G at x_0.

Connections of the metric C-increase property and the decrease principle (in the sense of Borwein-Clarke-Ledyaev) as well as with Caristi type conditions are discussed in [25] and [27], respectively. Some examples of classes of metrically C-increasing mappings are provided below.

Example 2.3 (Rescaled rotations with Lipschitz additive perturbations)**.** For any $n \geq 2$, let $\mathbb{X} = \mathbb{Y} = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$. Denote by $(\mathbf{SO}(n), \circ)$ the special orthogonal group, consisting of all rotations acting on $\mathbb{R}^n$. It has been proven that any rescaled rotation $\lambda O \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, defined by $\lambda > n$ and $O \in \mathbf{SO}(n)$, is $\mathbb{R}^n_+$-increasing at each point $x \in \mathbb{R}^n$, with

$$\mathrm{inc}(\lambda O; x) \geq \sqrt{n}, \quad \forall x \in \mathbb{R}^n$$

(see, for more details, [27, Example 2.2]). By recalling that the C-increase property is preserved under small additive Lipschitz perturbations (see [27, Proposition 2.4]), it is possible to see that, if $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a Lipschitz continuous set-valued mapping, with constant ℓ such that

$$\ell < 1 - \frac{1}{\sqrt{n}},$$

then the set-valued mapping $\lambda O + H$ turns out to be $\mathbb{R}^n_+$-increasing at each point $x \in \mathbb{R}^n$ with

$$\mathrm{inc}(\lambda O + H; x) \geq (1 - \ell)\sqrt{n}.$$

The next example points out connections of the metric C-increase property with openness (a.k.a. covering) at a linear rate, whose phenomenology has been the subject of intensive investigations in variational analysis.

Example 2.4 (Compactly generated fans)**.** Given a nonempty, convex and compact subset of linear mappings $\mathscr{G} \subset \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, the set-valued mapping $H_{\mathscr{G}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined as

$$H_{\mathscr{G}}(x) = \{\Lambda x \mid \Lambda \in \mathscr{G}\}$$

is known to fulfil the following properties:

(i) $H_{\mathscr{G}}(tx) = tH_{\mathscr{G}}(x), \quad \forall t > 0,\ \forall x \in \mathbb{R}^n$;
(ii) $H_{\mathscr{G}}(x)$ is nonempty, closed and convex for every $x \in \mathbb{R}^n$;
(iii) $H_{\mathscr{G}}(x_1 + x_2) \subseteq H_{\mathscr{G}}(x_1) + H_{\mathscr{G}}(x_2), \quad \forall x_1, x_2 \in \mathbb{R}^n$.

Therefore $H_{\mathscr{G}}$ falls in the class of those set-valued mappings which are called *fans* after [14]. Let $C \subseteq \mathbb{R}^m$ be a closed, convex and pointed cone. In [25, Proposition 2.15 + Corollary 2.18] it has been proven that if

$$\inf_{\Lambda\in\mathscr{G}} \mathrm{sur}(\Lambda) = \eta_{\mathscr{G}} > 0$$

and

$$\mathrm{int}\left(\bigcap_{\Lambda\in\mathscr{G}} \Lambda^{-1}(C)\right) \neq \varnothing,$$

then $H_{\mathscr{G}}$ is metrically C-increasing at each point $x \in \mathbb{R}^n$ with $\mathrm{inc}(H_{\mathscr{G}};x) \geq \eta_{\mathscr{G}} + 1$. Here $\mathrm{sur}(\Lambda)$ denotes the *exact bound of open covering* of the linear mapping Λ, which provides a sharp estimate of its surjectivity behaviour as follows

$$\mathrm{sur}(\Lambda) = \sup\{\eta > 0 \mid \Lambda\mathbb{B} \supseteq \eta\mathbb{B}\}.$$

Such kind of bound was fully investigated at the early stage of the theory of metric regularity and the following characterization in terms of the Banach constant of the adjoint mapping Λ^* to Λ was soon understood to hold true

$$\mathrm{sur}(\Lambda) = \mathrm{dist}\left(\mathbf{0};\Lambda^*\mathbb{B}^*\right) = \inf_{\|y^*\|=1} \|\Lambda^* y^*\|$$

(see, for instance, [19, Corollary 1.58]). The reader should also notice that any set-valued mapping such as $H_{\mathscr{G}}$ takes compact values and turns out to be Lipschitz continuous with respect to the Pompeiu-Hausdorff distance, i.e. there exists $\ell \geq 0$ such that

$$\mathrm{haus}(H_{\mathscr{G}}(x_1);H_{\mathscr{G}}(x_2)) \leq \ell\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{X}$$

(see, for more details, [25, Remark 2.14(iii)]).

2.4. ***C*-concave multi-valued mappings.** Set-valued mappings may exhibit different forms of convexity property, the mostly employed in variational analysis being the convexity with reference to the graph. Nevertheless, in connection with the theory of set-valued inclusions, the following concept of concavity (which can not be of graphical nature) reveals to be useful.

Definition 2.5 (C-concavity)**.** Let $C \subseteq \mathbb{Y}$ be a convex cone. A set-valued mapping $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be C-concave on $A \subseteq \mathbb{X}$ is for every $x_1, x_2 \in A$ and $t \in [0,1]$ it holds

$$G(tx_1 + (1-t)x_2) \subseteq tG(x_1) + (1-t)G(x_2) + C.$$

Example 2.6. Let $C \subseteq \mathbb{Y}$ be a convex cone. Let us recall that, following [11], a single-valued mapping $g : \mathbb{X} \longrightarrow \mathbb{Y}$ between vector spaces is said to be C-concave if

$$g(tx_1 + (1-t)x_2) \in tg(x_1) + (1-t)g(x_2) + C, \quad \forall x_1, x_2 \in \mathbb{X},\ \forall t \in [0,1].$$

For instance, it is well known that, whenever $\mathbb{Y} = \mathbb{R}^m$, $C = \mathbb{R}^m_+$, and $g = (g_1, \ldots, g_m)$ is defined by scalar functions $g_i : \mathbb{X} \longrightarrow \mathbb{R}$ which are concave for every $i = 1, \ldots, m$, then g turns out to be $\mathbb{R}^m_+$-concave.

Now, assume that Ω is a nonempty set and $g : \mathbb{X} \times \Omega \longrightarrow \mathbb{Y}$ is a given mapping such that $g(\cdot, \omega) : \mathbb{X} \longrightarrow \mathbb{Y}$ is C-concave for every $\omega \in \Omega$. Then, according to [24, Example 4.4], the set-valued mapping $G_{g,\Omega} : \mathbb{X} \rightrightarrows \mathbb{Y}$ defined by

$$G_{g,\Omega}(x) = g(x,\Omega) = \{g(x,\omega) \mid \omega \in \Omega\}$$

is C-concave. In particular, if taking $\Omega = C$ and $g : \mathbb{X} \times C \longrightarrow \mathbb{Y}$ given by

$$g(x,y) = h(x) + y,$$

where $h : \mathbb{X} \longrightarrow \mathbb{Y}$ is C-concave, then the resulting set-valued mapping $G_{h,C} = h + C$ is C-concave. It is also worth noticing that, if taking $\Omega = \mathscr{G} \subseteq \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ as in Example 2.4 and $g : \mathbb{R}^n \times \mathscr{G} \longrightarrow \mathbb{R}^m$ given by

$$g(x,\Lambda) = \Lambda x,$$

by linearity of Λ one obtains that any compactly generated fan is C-concave with respect to any cone $C \subseteq \mathbb{R}^m$.

Example 2.7 (Separable fans)**.** Let $\mathscr{A} = \{A_1, \dots, A_n\}$, where $A_i \subseteq \mathbb{R}^n$ is a nonempty, convex and compact set for every $i = 1, \dots, n$. Then, consider the set-valued mapping $H_{\mathscr{A}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as being

$$H_{\mathscr{A}}(x) = \sum_{i=1}^{n} A_i x_i.$$

It is readily seen that $H_{\mathscr{A}}$ takes nonempty, convex compact values and it holds

$$H_{\mathscr{A}}(tx) = \sum_{i=1}^{n} A_i t x_i = t H_{\mathscr{A}}(x), \quad \forall t > 0, x \in \mathbb{R}^n,$$

and

$$\begin{aligned} H_{\mathscr{A}}(x+z) &= \sum_{i=1}^{n} A_i(x_i + z_i) \subseteq \sum_{i=1}^{n} (A_i x_i + A_i z_i) = \sum_{i=1}^{n} A_i x_i + \sum_{i=1}^{n} A_i z_i \\ &= H_{\mathscr{A}}(x) + H_{\mathscr{A}}(z), \quad \forall x, z \in \mathbb{R}^n. \end{aligned}$$

This means that $H_{\mathscr{A}}$ is a fan, so it is C-concave with respect to any convex cone $C \subseteq \mathbb{R}^n$.

To the best of the author's knowledge, such kind of multi-valued mappings was introduced in [23] in the context of convex optimization, when the terminology of "fan" was not yet widespread in the related literature.

The next proposition makes clear the role of C-concavity in ensuring convexity properties for the solution mapping associated with set-valued inclusion problems.

Proposition 2.8. *With reference to the constraint system of problems* (P_p)*, if* $F : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ *is* C*-concave, then* $\mathsf{R} : \mathbb{P} \rightrightarrows \mathbb{X}$ *is a convex set-valued mapping.*

Proof. Let $p_1, p_2 \in \operatorname{dom} \mathsf{R}$ and $t \in [0,1]$. For arbitrary $x_1 \in \mathsf{R}(p_1)$ and $x_2 \in \mathsf{R}(p_2)$, it holds

$$F(p_1, x_1) \subseteq C \quad \text{and} \quad F(p_2, x_2) \subseteq C,$$

whence, by C-concavity of F it follows

$$\begin{aligned} F(tp_1 + (1-t)p_2, tx_1 + (1-t)x_2) &= F(t(p_1,x_1) + (1-t)(p_2,x_2)) \\ &\subseteq tF(p_1,x_1) + (1-t)F(p_2,x_2) + C \\ &\subseteq tC + (1-t)C + C = C. \end{aligned}$$

By arbitrariness of $x_1 \in \mathsf{R}(p_1)$ and $x_2 \in \mathsf{R}(p_2)$, the above inclusion shows that

$$t\mathsf{R}(p_1) + (1-t)\mathsf{R}(p_2) \subseteq \mathsf{R}(tp_1 + (1-t)p_2).$$

If p_1 or p_2 is not in $\operatorname{dom} \mathsf{R}$, the convention $S + \varnothing = \varnothing$ for every set S makes the last inclusion still valid. $\square$

In what follows, a subset $S \subseteq \mathbb{Y}$ is said to be C-bounded if the set $S \backslash C$ is (metrically) bounded. The combination of global metric C-increase and C-concavity of F, along with some technical assumptions, yields global solvability and continuous parameter dependence for the solutions to parameterized set-valued inclusion problems. This fact is stated in the next result, whose formulation requires to define the following problem constant

$$\alpha_F = \inf\left\{\mathrm{inc}(F(p,\cdot);x) \mid (p,x) \in \mathbb{P} \times \mathbb{X},\ F(p,x) \not\subseteq C\right\} \tag{2.5}$$

and to recall that a set-valued mapping $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ between normed spaces is said to be:

- l.s.c. at $x_0 \in \mathrm{dom}\, G$ if for every open set $O \subseteq \mathbb{Y}$, such that $G(x_0) \cap O \neq \varnothing$, there exists $\delta_O > 0$ such that $G(x) \cap O \neq \varnothing$, for every $x \in \mathrm{B}[x_0;\delta_O]$;
- Hausdorff C-u.s.c. at x_0, where $C \subseteq \mathbb{Y}$ is a cone, if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $G(x) \subseteq \mathrm{B}[G(x_0)+C;\varepsilon]$, for every $x \in \mathrm{B}[x_0;\delta_\varepsilon]$.

Theorem 2.9 (Continuous selection and global error bound)**.** *With reference to the constraint system of problems* (P_p)*, suppose that:*

(i) $\forall p \in \mathbb{P}\ \exists \hat{x}_p \in \mathbb{X}$ *such that* $F(p,\hat{x}_p)$ *is C-bounded;*
(ii) $F(p,\cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ *is l.s.c. on* $\mathbb{X}$, $\forall p \in \mathbb{P}$ *;*
(iii) $F(p,\cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ *is C-concave on* $\mathbb{X}$, $\forall p \in \mathbb{P}$ *;*
(iv) $F(\cdot,x) : \mathbb{X} \rightrightarrows \mathbb{Y}$ *is Hausdorff C-u.s.c. on* $\mathbb{P}$, $\forall x \in \mathbb{X}$*;*
(v) *it holds* $\alpha_F > 1$.

Then, $\mathrm{dom}\,\mathsf{R} = \mathbb{P}$ *and* $\mathsf{R} : \mathbb{P} \rightrightarrows \mathbb{X}$ *admits a continuous selection. Moreover, the following estimate holds for any* $\alpha \in (1,\alpha_F)$

$$\mathrm{dist}\,(x;\mathsf{R}(p)) \leq \frac{\mathrm{exc}(F(p,x);C)}{\alpha - 1}, \quad \forall (p,x) \in \mathbb{P} \times \mathbb{X}. \tag{2.6}$$

Proof. It suffices to observe that, as a metric space, $(\mathbb{P}, \|\cdot\|)$ is a paracompact topological space, and then to apply [27, Theorem 3.4]. The estimate in (2.6) is actually valid under more general assumptions by virtue of [27, Proposition 3.1]. □

3. Stability and Sensitivity Results

By exploiting the analysis tools recalled in the previous section, it is possible to establish several properties of val of both qualitative and quantitative interest in studying stability and sensitivity issues with reference to (P_p). Let us start with the convexity behaviour of val, which is a straightforward consequence of a well-known phenomenon in parametric constrained optimization, occurring whenever a convex objective function is to be minimized over a parameter dependent feasible region, which is a convex multifunction of the parameter (see, for instance, [21, Theorem 2.129]). The proof is presented here in full details for the sake of completeness.

Proposition 3.1 (Convexity of val)**.** *Given a family of problems* (P_p)*, suppose that:*

(i) $\varphi : \mathbb{P} \times \mathbb{X} \longrightarrow \mathbb{R}$ *is convex;*
(ii) $F : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ *is C-concave.*

Then, $\mathsf{val} : \mathbb{P} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ *is convex.*

Proof. Take arbitrary $p_1, p_2 \in \mathbb{P}$ and $t \in [0,1]$ and suppose first that $tp_1 + (1-t)p_2 \in \mathrm{dom}\,\mathsf{val}$. Then, it must be $\mathsf{R}(tp_1 + (1-t)p_2) \neq \varnothing$. By combining the convexity of the set-valued mapping

R established in Proposition 2.8 with the convexity of φ, one finds

$$\begin{aligned}\mathsf{val}(tp_1+(1-t)p_2) &= \inf_{x\in\mathsf{R}(tp_1+(1-t)p_2)} \varphi(tp_1+(1-t)p_2,x) \\ &\le \inf_{x\in[t\mathsf{R}(p_1)+(1-t)\mathsf{R}(p_2)]} \varphi(tp_1+(1-t)p_2,x) \\ &= \inf_{\substack{x_1\in\mathsf{R}(p_1)\\ x_2\in\mathsf{R}(p_2)}} \varphi(tp_1+(1-t)p_2,tx_1+(1-t)x_2) \\ &\le \inf_{\substack{x_1\in\mathsf{R}(p_1)\\ x_2\in\mathsf{R}(p_2)}} [t\varphi(p_1,x_1)+(1-t)\varphi(p_2,x_2)] \\ &= t\mathsf{val}(p_1)+(1-t)\mathsf{val}(p_2).\end{aligned}$$

In the case $\mathsf{val}(tp_1+(1-t)p_2)=+\infty$, by virtue of ($\mathsf{a}_2$) it must $\mathsf{R}(tp_1+(1-t)p_2)=\varnothing$. This fact by convexity of R implies that $\mathsf{R}(p_1)=\mathsf{R}(p_2)=\varnothing$ and hence $\mathsf{val}(p_1)=\mathsf{val}(p_2)=+\infty$, so one obtains

$$\mathsf{val}(tp_1+(1-t)p_2)=+\infty=t\mathsf{val}(p_1)+(1-t)\mathsf{val}(p_2).$$

In the last case, in which $\mathsf{val}(tp_1+(1-t)p_2)=-\infty$, nothing is left to prove, so the proof is complete. □

Under proper qualification conditions on the set-valued inclusion, which formalizes the constraint system of problems (P_p), the convexity of val entails the local Lipschitz continuity property, which is much stronger than mere continuity appearing in Berge's type theorems (see [1]), as well as than those forms of calmness established in [24].

Theorem 3.2 (Local Lipschitz continuity of val)**.** *With reference to the family of problems (P_p), suppose that:*

(i) $\varphi:\mathbb{P}\times\mathbb{X}\longrightarrow\mathbb{R}$ *is convex and continuous;*
(ii) $\exists\hat{x}_p\in\mathbb{X}:\ F(p,\hat{x}_p)$ *is C-bounded,* $\forall p\in\mathbb{P}$ *;*
(iii) $F(p,\cdot):\mathbb{X}\rightrightarrows\mathbb{Y}$ *is l.s.c. on* $\mathbb{X}$*,* $\forall p\in\mathbb{P}$*;*
(iv) $F:\mathbb{P}\times\mathbb{X}\rightrightarrows\mathbb{Y}$ *is C-concave;*
(v) $F(\cdot,x):\mathbb{P}\rightrightarrows\mathbb{Y}$ *is Hausdorff C-u.s.c. on* $\mathbb{P}$*,* $\forall x\in\mathbb{X}$*;*
(vi) *it holds* $\alpha_F>1$.

Then, either $\mathsf{val}=-\infty$ *or* $\operatorname{dom}\mathsf{val}=\mathbb{P}$ *and* val *is locally Lipschitz around each point* $p\in\mathbb{P}$.

Proof. By Proposition 3.1, $\mathsf{val}:\mathbb{P}\longrightarrow\mathbb{R}\cup\{\pm\infty\}$ is convex. Notice that, as a consequence of assumption (iv), each set-valued mapping $F(p,\cdot)$ is C-concave, for every $p\in\mathbb{P}$. Thus, all hypotheses being satisfied, Theorem 2.9 applies, so $\operatorname{dom}\mathsf{R}=\mathbb{P}$ and $\mathsf{val}(p)<+\infty$ for every $p\in\mathbb{P}$. Now, if there exists $p_0\in\mathbb{P}$ such that $\mathsf{val}(p_0)=-\infty$ (namely, problem (P_{p_0}) does not admit any solution), then by a well-known result in convex analysis (see, for instance, [28, Proposition 2.1.4]) it must be $\mathsf{val}(p)=-\infty$ for every $p\in\operatorname{int}\{p\in\mathbb{P}\mid\mathsf{val}(p)<+\infty\}=\operatorname{int}\mathbb{P}=\mathbb{P}$. Otherwise, $\operatorname{dom}\mathsf{val}=\mathbb{P}$. In such an event, Theorem 2.9 ensures the existence of a continuous selection $\mathsf{s}_\mathsf{R}:\mathbb{P}\longrightarrow\mathbb{X}$ of R. Thus, according to the definition of val one has

$$\mathsf{val}(p)\le\varphi(p,\mathsf{s}_\mathsf{R}(p)),\quad\forall p\in\mathbb{P}.$$

Since φ is continuous on $\operatorname{int}\operatorname{dom}\varphi=\mathbb{P}\times\mathbb{X}$, the composition $p\mapsto\varphi(p,\mathsf{s}_\mathsf{R}(p))$ turns out to be continuous on $\mathbb{P}$. Therefore, it is bounded from above in a neighbourhood of each point $p\in\mathbb{P}$, and so is val by the above inequality. In the light of [28, Them 2.2.9] this suffices to guarantee the local Lipschitz continuity of val around each point, and a fortiori its continuity. □

The assumptions of Theorem 3.2 lead to single-out a class of parametric constrained optimization problems with a good marginal behaviour. Henceforth, any parametric family of optimization problems such as (P_p) satisfying all the assumptions (i)-(vi) in Theorem 3.2 will be called *qualified convex* problems *with set-valued inclusion constraints* (for short, *q.c.s.v.i.* problems).

Remark 3.3. (i) A first notable consequence of Theorem 3.2 is the subdifferentiability of the value function, whenever $\mathsf{val} \neq -\infty$, associated with any family of q.c.s.v.i. problems. Indeed, according to [28, Theorem 2.4.9], the continuity and the convexity of val result in

$$\partial \mathsf{val}(p) \neq \varnothing, \quad \forall p \in \mathbb{P}.$$

(ii) Further consequences of the joint convexity and continuity of val can be derived when the space $\mathbb{P}$ is a Banach space enjoying special additional properties. In particular, whenever $\mathbb{P}$ is a separable Banach space (more generally, a weak Asplund space), then according to [21, Theorem 5.61] val is Gâteaux differentiable on a G_δ subset of $\mathbb{P}$. Whenever $\mathbb{P}$ is a space with a separable dual (more generally, an Asplund space), then according to [21, Theorem 5.66] val is even Fréchet differentiable on a G_δ subset of $\mathbb{P}$. If, more in particular, $\mathbb{P}$ is a finite-dimensional Euclidean space, then the well-known Rademacher theorem ensures that the points of differentiability of val form a full (Lebesgue) measure set. As remarked in [20, Chapter 4.6], the reader should take into account that the lack of smoothness of val was one of the major concerns in considering such a fundamental mathematical object in classical calculus of variations, in contrast stimulating the development of meaningful constructions in nonsmooth analysis.

Example 3.4. Let $\mathbb{P} = \mathbb{X} = \mathbb{Y} = \mathbb{R}$ and let $C = [0,+\infty)$. Let us consider the family of problems (P_p), which are defined by $\varphi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ and $F : \mathbb{R}\times\mathbb{R} \rightrightarrows \mathbb{R}$ as follows

$$\varphi(p,x) = p + x, \qquad F(p,x) = [p-x,+\infty),$$

respectively. It is then clear that $\mathsf{R} : \mathbb{R} \rightrightarrows \mathbb{R}$ is given by

$$\mathsf{R}(p) = (-\infty, p), \quad \forall p \in \mathbb{R},$$

and consequently

$$\mathsf{val}(p) = \inf_{x\in(-\infty,p]} (p+x) = -\infty, \quad \forall p \in \mathbb{R}.$$

Let us check that this family of problems actually falls in the class q.c.s.v.i..

As φ is linear, assumption (i) is trivially fulfilled.

As it is $F(p,x)\backslash[0,+\infty) \subseteq [-|p-x|,0]$ for every $(p,x)\in\mathbb{R}\times\mathbb{R}$, each set $F(p,x)$ is $[0,+\infty)$-bounded, which amounts to assumption (ii) being fulfilled.

As for assumptions (iii) and (v), their fulfillment follows at once from the continuity property of the linear function $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$, being $h(p,x) = p - x$.

Assumption (iv) about the $[0,+\infty)$-concavity of F is satisfied because it is $F(p,x) = h(p,x) + [0,+\infty)$ and, as a linear mapping, h is $[0,+\infty)$-concave (remember Example 2.6).

As for assumption (vi), observe that $F(p,\cdot)$ is $[0,+\infty)$-increasing at each point $x_0 \in \mathbb{R}$, with $\mathrm{inc}(F(p,\cdot);x_0) \geq 2$. Indeed, for any $\delta > 0$ and $r \in (0,\delta]$, by taking $u = x_0 - r \in \mathrm{B}[x_0;r]$, one finds

$$\mathrm{B}[F(p,u);2r] = [p-(x_0-r)-2r,+\infty) \subseteq [p-x_0-r,+\infty) = \mathrm{B}[F(p,x_0)+[0,+\infty);r].$$

The above inclusion remains true for every $p \in \mathbb{P}$. Thus, for the set-valued mapping under consideration one obtains

$$\alpha_F \geq \inf\{\mathrm{inc}(F(p,\cdot);x) \mid (p,x) \in \mathbb{R}\times\mathbb{R}\} \geq 2 > 1,$$

which shows that the condition in assumption (vi) happens to be satisfied.

The next lemma points out a further property of convexity stemming from the C-concavity for set-valued mappings, which will be employed for estimating subgradients of val.

Lemma 3.5. *Let $C \subseteq \mathbb{Y}$ be a convex cone. If $F : \mathbb{P}\times\mathbb{X} \rightrightarrows \mathbb{Y}$ is C-concave then function $x \mapsto \mathrm{exc}(F(p,x);C)$ is convex.*

Proof. Observe first that, as C is a convex cone, the function $y \mapsto \mathrm{dist}(y;C)$ is sublinear. As a consequence, for any $A, B \subseteq \mathbb{Y}$ and $t \in (0,+\infty)$, one has

$$\mathrm{exc}(A+B;C) \leq \mathrm{exc}(A;C) + \mathrm{exc}(B;C) \quad \text{and} \quad \mathrm{exc}(tA;C) = t\,\mathrm{exc}(A;C).$$

Moreover, observe that for any $A \subseteq \mathbb{Y}$ it holds $\mathrm{exc}(A+C;C) = \mathrm{exc}(A;C)$. On the account of these observations, taken arbitrary $(p_1,x_1), (p_2,x_2) \in \mathbb{P}\times\mathbb{X}$ and $t \in [0,1]$, by exploiting the C-concavity of F one can write

$$\begin{aligned}
\mathrm{exc}(F(t(p_1,x_1)+(1-t)(p_1,x_1));C) &\leq \mathrm{exc}(tF(p_1,x_1)+(1-t)F(p_2,x_2)+C;C) \\
&= \mathrm{exc}(tF(p_1,x_1)+(1-t)F(p_2,x_2);C) \\
&\leq t\,\mathrm{exc}(F(p_1,x_1);C)+(1-t)\mathrm{exc}(F(p_2,x_2);C).
\end{aligned}$$

The above inequalities complete the proof. □

The next result establishes an exact formula for calculating the subgradients of val, which is expressed in terms of problem data (φ, F and C), thereby providing relevant elements for the sensitivity analysis of (P_p).

Theorem 3.6. *Let (P_p) be a family of q.c.s.v.i. problems, with $\mathsf{val} \neq -\infty$. Let $\bar{p} \in \mathbb{P}$ and $\bar{x} \in \mathsf{Argmin}(\bar{p})$. If $\mathrm{epi}\,\varphi$ and $\mathrm{gph}\,\mathsf{R}\times\mathbb{R}$ are subtransversal at $(\bar{p},\bar{x},\varphi(\bar{p},\bar{x}))$, then it holds*

$$\partial\mathsf{val}(\bar{p}) = \{p^*+q^* \mid (p^*,x^*) \in \partial\varphi(\bar{p},\bar{x}) \text{ and } (q^*,-x^*) \in \mathrm{cone}\,\partial\mathrm{exc}(F(\cdot);C)(\bar{p},\bar{x})\}. \quad (3.1)$$

Proof. Under the subtransversality qualification condition it is possible to employ the exact representation of $\partial\mathsf{val}$ provided by [13, Theorem 4.5], which is valid in any normed space setting. Indeed, this theorem can be applied because, upon the q.c.s.v.i. assumption on (P_p), φ and R are convex (remember Proposition 2.8) while, as in the present case it is $\mathrm{dom}\,\mathsf{val} = \mathbb{P}$, then $\mathsf{val}(\bar{p})$ is finite and $\bar{x} \in \mathsf{Argmin}(\bar{p})$. Moreover, the metric qualification condition (Q_1) invoked in Theorem 4.5 corresponds to the subtranversality of the pair $\mathrm{epi}\,\varphi$ and $\mathrm{gph}\,\mathsf{R}\times\mathbb{R}$ at $(\bar{p},\bar{x},\varphi(\bar{p},\bar{x}))$ (recall what has been remarked in Section 2.2 about equivalent reformulations of such property). According to the mentioned representation, one has

$$\partial\mathsf{val}(\bar{p}) = \bigcup_{(p^*,x^*)\in\partial\varphi(\bar{p},\bar{x})} \{p^*+\mathrm{D}^*\mathsf{R}(\bar{p},\bar{x})(x^*)\}.$$

Then, what remains to do is to express the set $\mathrm{D}^*\mathsf{R}(\bar{p},\bar{x})(x^*)$ in terms of the problem data F and C. By recalling the definition of coderivative of a convex set-valued mapping, this can be done via the normal cone representation

$$(q^*,-x^*) \in \mathrm{N}(\mathrm{gph}\,\mathsf{R};(\bar{p},\bar{x})) = \bigcup_{t\ge 0} t\partial \mathrm{dist}\,(\cdot;\mathrm{gph}\,\mathsf{R})\,(\bar{p},\bar{x}),$$

which has been formulated in Proposition 2.1(iii). Indeed, if $(q^*,-x^*) \in \mathrm{N}(\mathrm{gph}\,\mathsf{R};(\bar{p},\bar{x}))$, then for some $t\ge 0$, as $\bar{x}\in\mathsf{R}(\bar{p})$, it must be

$$\begin{aligned} \langle (q^*,-x^*),(p,x)-(\bar{p},\bar{x})\rangle &= t\mathrm{dist}\,((p,x);\mathrm{gph}\,\mathsf{R}) \\ &\le t\mathrm{dist}\,(x;\mathsf{R}(p)), \quad \forall (p,x)\in\mathbb{P}\times\mathbb{X}. \end{aligned}$$

Thus, since for q.c.s.v.i. problems all the assertions of Theorem 2.9 hold true, by recalling the global error bound estimate in (2.6), if $\alpha\in(1,\alpha_F)$ one obtains

$$\langle (q^*,-x^*),(p,x)-(\bar{p},\bar{x})\rangle \le \frac{t}{\alpha-1}\mathrm{exc}(F(p,x);C), \quad \forall (p,x)\in\mathbb{P}\times\mathbb{X},$$

which clearly shows that $(q^*,-x^*)\in \mathrm{cone}\,\partial\mathrm{exc}(F(\cdot);C)(\bar{p},\bar{x})$.

On the other hand, it suffices to observe that for every $t\ge 0$ it holds

$$t\mathrm{exc}(F(p,x);C) \le \iota((p,x);\mathrm{gph}\,\mathsf{R}), \quad \forall (p,x)\in\mathbb{P}\times\mathbb{X},$$

whence, by passing to the respective subdifferential at $(\bar{p},\bar{x})$ in both the sides, in the light of (2.1), one finds

$$t\partial\mathrm{exc}(F(\cdot);C)(\bar{p},\bar{x}) \subseteq \partial\iota(\cdot;\mathrm{gph}\,\mathsf{R})(\bar{p},\bar{x}) = \mathrm{N}(\mathrm{gph}\,\mathsf{R};(\bar{p},\bar{x})).$$

This completes the proof. □

Example 3.7. Consider a parametric class of problems (P_p) defined by $\mathbb{P}=\mathbb{R}^s$, $\mathbb{X}=\mathbb{R}^n$, $\mathbb{Y}=\mathbb{R}^m$, a convex function $\varphi:\mathbb{R}^s\times\mathbb{R}^n\longrightarrow\mathbb{R}$ and by the following constraint system

$$H_{\mathscr{G}}(p,x)\subseteq C,$$

where C is a nontrivial, closed, pointed, convex cone and $H_{\mathscr{G}}$ is a fan compactly generated by the nonempty, convex compact set $\mathscr{G}\subseteq\mathscr{L}(\mathbb{R}^s\times\mathbb{R}^n,\mathbb{R}^m)\cong\mathscr{L}(\mathbb{R}^s,\mathbb{R}^m)\times\mathscr{L}(\mathbb{R}^n,\mathbb{R}^m)$. Notice that if taking an element $(M,\Lambda)\in\mathscr{L}(\mathbb{R}^s\times\mathbb{R}^n,\mathbb{R}^m)$, where $M\in\mathscr{L}(\mathbb{R}^s,\mathbb{R}^m)$ and $\Lambda\in\mathscr{L}(\mathbb{R}^n,\mathbb{R}^m)$, its representation matrix is formed by juxtaposing the representation matrices of M and Λ, respectively. As remarked in Example 2.4, $H_{\mathscr{G}}$ is a Lipschitz continuous C-concave set-valued mapping, taking compact values, so hypotheses (ii)-(v) of Theorem 3.2 are satisfied. Assume that

$$\inf_{(M,\Lambda)\in\mathscr{G}} \mathrm{sur}((M,\Lambda)) = \eta_{\mathscr{G}} > 0$$

and

$$\mathrm{int}\left(\bigcap_{(M,\Lambda)\in\mathscr{G}} (M,\Lambda)^{-1}(C)\right)\ne\varnothing,$$

so $\mathrm{inc}(H_{\mathscr{G}};p,x)\ge\eta_{\mathscr{G}}+1$ for every $(p,x)\in\mathbb{R}^s\times\mathbb{R}^n$. This implies

$$\alpha_{H_{\mathscr{G}}} = \inf\{\mathrm{inc}(H_{\mathscr{G}}(p,\cdot);x) \mid (p,x)\in\mathbb{P}\times\mathbb{X},\ H_{\mathscr{G}}(p,x)\not\subseteq C\} \ge \eta_{\mathscr{G}}+1>1.$$

Thus also hypothesis (vi) of Theorem 3.2 is fulfilled. This shows that this kind of problem is actually q.c.s.v.i.. According to Theorem 3.2, if $\mathsf{val} \neq -\infty$, then val is a convex locally Lipschitz function and, upon the subtransversality condition on $\operatorname{epi}\varphi$ and $\operatorname{gph}\mathsf{R}\times\mathbb{R}$, the following characterization of its subgradients holds

$$\partial\mathsf{val}(\bar p) = \{p^* + q^* \mid (p^*, x^*) \in \partial\varphi(\bar p, \bar x) \text{ and } (q^*, -x^*) \in \operatorname{cone}\partial\operatorname{exc}(H_{\mathscr{G}}(\cdot); C)(\bar p, \bar x)\}, \quad (3.2)$$

for every pair $(\bar p, \bar x)$, with $\bar p \in \mathbb{R}^s$ and $\bar x \in \mathsf{Argmin}(\bar p)$. Denote by

$$\begin{aligned}\mathscr{G}(\bar p, \bar x) = \{(M_0, \Lambda_0) \in \mathscr{G} \mid \operatorname{exc}(H_{\mathscr{G}}(\bar p, \bar x); C) &= \max_{(M,\Lambda)\in\mathscr{G}} \operatorname{dist}(M\bar p + \Lambda\bar x; C) \\ &= \operatorname{dist}(M_0\bar p + \Lambda_0\bar x; C)\}.\end{aligned}$$

Notice that, since function $y \mapsto \operatorname{dist}(y; C)$ is (Lipschitz) continuous and the evaluation function $\mathrm{v}_{(\bar p, \bar x)} : \mathscr{L}(\mathbb{R}^s \times \mathbb{R}^n, \mathbb{R}^m) \longrightarrow \mathbb{R}^m$, i.e. $\mathrm{v}_{(\bar p, \bar x)}(M, \Lambda) = M\bar p + \Lambda\bar x$, is continuous, also their composition is continuous. Therefore, by compactness of $\mathscr{G}$ it must be $\mathscr{G}(\bar p, \bar x) \neq \varnothing$. According to the subdifferential calculus rule for max functions (see, for instance, [22, Proposition 3.42]), by taking into account of Proposition 2.1(i) and the formula in (2.2), one obtains

$$\begin{aligned}\partial\operatorname{exc}(H_{\mathscr{G}}(\cdot); C)(\bar p, \bar x) &= \partial \max_{(M,\Lambda)\in\mathscr{G}} \operatorname{dist}(M\cdot + \Lambda\cdot; C)(\bar p, \bar x) \\ &= \operatorname{clco} \bigcup_{(M,\Lambda)\in\mathscr{G}(\bar p, \bar x)} \partial\operatorname{dist}(M\cdot + \Lambda\cdot; C)(\bar p, \bar x) \\ &= \operatorname{clco} \bigcup_{(M,\Lambda)\in\mathscr{G}(\bar p, \bar x)} (M^*, \Lambda^*)[\mathrm{N}(C; M\bar p + \Lambda\bar x) \cap \mathbb{B}].\end{aligned}$$

Thus, from the subgradient representation in (3.2) it is possible to derive the following exact estimate, which is fully expressed in terms of problem data

$$\begin{aligned}\partial\mathsf{val}(\bar p) = \Big\{p^* + q^* \quad | \quad & (p^*, x^*) \in \partial\varphi(\bar p, \bar x) \text{ and} \qquad (3.3) \\ & (q^*, -x^*) \in \operatorname{cone}\Big(\operatorname{clco} \bigcup_{(M,\Lambda)\in\mathscr{G}(\bar p, \bar x)} (M^*, \Lambda^*)[\mathrm{N}(C; M\bar p + \Lambda\bar x) \cap \mathbb{B}]\Big)\Big\}.\end{aligned}$$

Let us test formula (3.3) in a specific case, where calculations are easy to be checked. Let $\mathbb{P} = \mathbb{X} = \mathbb{Y} = \mathbb{R}$, $C = [0, +\infty)$, $\mathscr{G} = \{-1\} \times [1, 2]$ (with linear mappings being identified with their representation matrix) and $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\varphi(p, x) = |p| + x,$$

so the class (P_p) becomes

$$\begin{array}{ll}\displaystyle\min_{x\in\mathbb{R}} & |p| + x \\ \text{sub} & H_{\{-1\}\times[1,2]}(p, x) = \{-p + \lambda x \mid \lambda \in [1, 2]\} \subseteq [0, +\infty).\end{array}$$

Since it is

$$-p + \lambda x \geq 0, \quad \forall \lambda \in [1, 2]$$

iff it holds

$$x \geq \frac{p}{\lambda}, \quad \forall \lambda \in [1, 2],$$

the feasible region mapping $\mathsf{R}:\mathbb{R}\rightrightarrows\mathbb{R}$ turns out to be

$$\mathsf{R}(p)=\begin{cases}[p,+\infty) & \text{if } p\geq 0,\\ [p/2,+\infty) & \text{if } p<0.\end{cases}$$

Notice that $\operatorname{gph}\mathsf{R}$ is a polyhedral convex cone with vertex at the origin (so R is what is called after Rockafellar a convex process). Therefore, with the given problem data, one readily deduces that $\mathsf{Argmin}:\mathbb{R}\rightrightarrows\mathbb{R}$ takes the form

$$\mathsf{Argmin}(p)=\begin{cases}\{p\} & \text{if } p\geq 0,\\ \{p/2\} & \text{if } p<0.\end{cases}$$

Consequently, the optimal value function $\mathsf{val}:\mathbb{R}\longrightarrow\mathbb{R}$ associated with the present class of problems results in

$$\mathsf{val}(p)=\begin{cases}2p & \text{if } p\geq 0,\\ -p/2 & \text{if } p<0.\end{cases}$$

As it should be, val is convex and locally Lipschitz. By taking into account that, in the present setting, it is $\mathbb{P}^*=\mathbb{X}^*=\mathbb{R}$, one readily sees that

$$\partial\mathsf{val}(\bar p)=\begin{cases}\{2\} & \text{if } \bar p>0\\ \left[-\frac{1}{2},2\right] & \text{if } \bar p=0\\ \left\{-\frac{1}{2}\right\} & \text{if } \bar p<0.\end{cases}\tag{3.4}$$

Before checking the validity of formula (3.3), it should be noticed that $\operatorname{epi}\varphi$ is a polyhedral cone (with vertex at the origin) in $\mathbb{R}^3$ as well as $\operatorname{gph}\mathsf{R}\times\mathbb{R}$, so that the subtransversality condition on $\operatorname{epi}\varphi$ and $\operatorname{gph}\mathsf{R}\times\mathbb{R}$ is satisfied at each point $(\bar p,\bar x,\varphi(\bar p,\bar x))$ in as much both the sets are polyhedral convex sets. Besides, it is useful to note that, by well-known subdifferential calculus rules, it holds

$$\partial\varphi(p,x)=\begin{cases}\{(1,1)\} & \forall(p,x)\in(0,+\infty)\times\mathbb{R}\\ [-1,1]\times\{1\} & \forall(p,x)\in\{0\}\times\mathbb{R}\\ \{(-1,1)\} & \forall(p,x)\in(-\infty,0)\times\mathbb{R}.\end{cases}$$

• **Case $\bar p>0$.** In such an event, it must be $\bar x=\bar p$ and hence

$$\begin{aligned}\mathscr{G}(\bar p,\bar p) &= \{\,(-1,\lambda_0)\mid \quad \lambda_0\in[1,2] \text{ and}\\ &\qquad \operatorname{dist}(-\bar p+\lambda_0\bar p;[0,+\infty))=\max_{\lambda\in[1,2]}\operatorname{dist}(-\bar p+\lambda\bar p;[0,+\infty))\,\}\\ &=\{-1\}\times[1,2].\end{aligned}$$

If $(-1,\lambda_0)\in\mathscr{G}(\bar p,\bar p)$ it is

$$-1\cdot\bar p+\lambda_0\bar x=(\lambda_0-1)\bar p\begin{cases}\in(0,+\infty) & \forall\lambda_0\in(1,2]\\ =0 & \text{if } \lambda_0=1.\end{cases}$$

Then, one has

$$\mathrm{N}([0,+\infty);-\bar p+\lambda_0\bar x)\cap\mathbb{B}=\begin{cases}\{0\} & \forall\lambda_0\in(1,2]\\ [-1,0] & \text{if } \lambda_0=1.\end{cases}$$

This implies

$$\{(-t,\lambda_0 t)\mid t\in\mathrm{N}([0,+\infty);-\bar p+\lambda_0\bar x)\cap\mathbb{B}\}=\begin{cases}\{(0,0)\} & \forall\lambda_0\in(1,2]\\ \mathrm{co}\{(0,0),\,(1,-1)\} & \text{if } \lambda_0=1,\end{cases}$$

whence it follows

$$\mathrm{cone}\left(\mathrm{clco}\bigcup_{(-1,\lambda_0)\in\mathscr{G}(\bar p,\bar p)}\{(-t,\lambda_0 t)\mid t\in\mathrm{N}([0,+\infty);-\bar p+\lambda_0\bar x)\cap\mathbb{B}\}\right)=\{t(1,-1)\mid t\in[0,+\infty)\}.$$

Since if $(p^*,x^*)\in\partial\varphi(\bar x,\bar p)$ it must be $p^*=1$ and $x^*=1$, then $(q^*,-1)\in\{t(1,-1)\mid t\in[0,+\infty)\}$ only if $t=1$ and hence $q^*=1$. Thus, according to formula (3.3) one obtains

$$\partial\mathsf{val}(\bar p)=\{p^*+q^*\}=\{2\},$$

consistently with the value in (3.4).

• **Case** $\bar p=0$**.** In such an event, it must be $\bar x=\bar p=0$ and hence

$$\begin{aligned}\mathscr{G}(0,0)\quad &=\{\,(-1,\lambda_0)\mid\quad \lambda_0\in[1,2]\text{ and}\\ &\qquad\qquad \mathrm{dist}\,(0;[0,+\infty))=\max_{\lambda\in[1,2]}\mathrm{dist}\,(-1\cdot 0+\lambda\cdot 0;[0,+\infty))\,\}\\ &=\{-1\}\times[1,2].\end{aligned}$$

If $(-1,\lambda_0)\in\mathscr{G}(0,0)$ it is $-1\cdot 0+\lambda_0\cdot 0=0$ which yields

$$\mathrm{N}([0,+\infty);0)\cap\mathbb{B}=[-1,0],\quad\forall\lambda_0\in[1,2].$$

This implies

$$\begin{aligned}\{(-t,\lambda_0 t)\mid t\in\mathrm{N}([0,+\infty);0)\cap\mathbb{B}\}\quad &=\quad\bigcup_{\lambda_0\in[1,2]}\mathrm{co}\{(0,0),\,(1,-\lambda_0)\}\\ &=\quad\mathrm{co}\{(0,0),\,(1,-1),\,(1,-2)\},\end{aligned}$$

whence it follows

$$\mathrm{cone}\left(\mathrm{clco}\bigcup_{(-1,\lambda_0)\in\mathscr{G}(0,0)}\{(-t,\lambda_0 t)\mid t\in\mathrm{N}([0,+\infty);0)\cap\mathbb{B}\}\right)=\mathrm{cone}\,(\mathrm{co}\{(0,0),\,(1,-1),\,(1,-2)\})\,.$$

Since if $(p^*,x^*) \in \partial\varphi(\bar{x},\bar{p})$ one has $p^* \in [-1,1]$ and $x^* = 1$, then $(q^*,-1)$ belongs to the set cone $(\mathrm{co}\{(0,0),\,(1,-1),\,(1,-2)\})$ only if $q^* \in \left[\frac{1}{2},1\right]$. Thus, according to formula (3.3) one obtains

$$\partial\mathsf{val}(\bar{p}) = [-1,1] + \left[\frac{1}{2},1\right] = \left[-\frac{1}{2},2\right],$$

which is consistent with the set found in (3.4).

• **Case $\bar{p} < 0$.** In such an event, it must be $\bar{x} = \bar{p}/2$, and hence

$$\begin{aligned}\mathscr{G}(\bar{p},\bar{p}/2) &= \Big\{(-1,\lambda_0) \mid \quad \lambda_0 \in [1,2] \text{ and } \\ &\qquad \mathrm{dist}\left(-\bar{p}+\lambda_0\frac{\bar{p}}{2};[0,+\infty)\right) = \max_{\lambda\in[1,2]} \mathrm{dist}\left(-\bar{p}+\lambda\frac{\bar{p}}{2};[0,+\infty)\right)\Big\} \\ &= \{-1\}\times[1,2].\end{aligned}$$

If $(-1,\lambda_0) \in \mathscr{G}(\bar{p},\bar{p}/2)$ it is

$$-1\cdot\bar{p}+\lambda_0\bar{x} = \left(\frac{\lambda_0}{2}-1\right)\bar{p}\begin{cases}\in(0,+\infty) & \forall\lambda_0\in[1,2)\\ =0 & \text{if } \lambda_0=2.\end{cases}$$

Then, one has

$$\mathrm{N}([0,+\infty);-\bar{p}+\lambda_0\bar{x})\cap\mathbb{B} = \begin{cases}\{0\} & \forall\lambda_0\in[1,2)\\ [-1,0] & \text{if } \lambda_0=2.\end{cases}$$

This implies

$$\{(-t,\lambda_0 t) \mid t\in\mathrm{N}([0,+\infty);-\bar{p}+\lambda_0\bar{x})\cap\mathbb{B}\} = \begin{cases}\{(0,0)\} & \forall\lambda_0\in[1,2)\\ \mathrm{co}\{(0,0),\,(1,-2)\} & \text{if } \lambda_0=2,\end{cases}$$

which gives

$$\mathrm{cone}\left(\mathrm{clco}\bigcup_{(-1,\lambda_0)\in\mathscr{G}(\bar{p},\bar{p}/2)}\{(-t,\lambda_0 t) \mid t\in\mathrm{N}([0,+\infty);-\bar{p}+\lambda_0\bar{x})\cap\mathbb{B}\}\right) = \{t(1,-2) \mid t\in[0,+\infty)\}.$$

It is clear that if $(p^*,x^*) \in \partial\varphi(\bar{x},\bar{p}/2)$ it must be $p^* = -1$ and $x^* = 1$, then $(q^*,-1) \in \{t(1,-2) \mid t \in [0,+\infty)\}$ only if $t = \frac{1}{2}$ and hence $q^* = \frac{1}{2}$. Thus, according to formula (3.3), one obtains

$$\partial\mathsf{val}(\bar{p}) = \{p^*+q^*\} = \left\{-1+\frac{1}{2}\right\} = \left\{-\frac{1}{2}\right\},$$

again consistently with the value in (3.4).

4. Problem Calmness

The stability and sensitivity analysis conducted in the previous section is complemented here with a related issue, dealing with a certain stability behaviour that parametric constrained optimization problems may exhibit. Proposed by R.T. Rockafellar, such behaviour was studied in [9] and then employed in connection with penalization reduction procedures within perturbed nonlinear programming in [8] and subsequent works.

Definition 4.1 (Problem calmness)**.** Given a family of problems (P_p), let $\bar{p} \in \mathbb{P}$ and let $\bar{x} \in \mathsf{Argmin}(\bar{p})$. Problem $(\mathsf{P}_{\bar{p}})$ is said to be *calm* at $\bar{x}$ if $\exists r,\ \lambda > 0$:

$$\inf_{p\in \mathrm{B}[\bar{p};r]\setminus\{\bar{p}\}} \ \inf_{x\in \mathrm{B}[\bar{x};r]\cap \mathsf{R}(p)} \frac{\varphi(p,x)-\varphi(\bar{p},\bar{x})}{\|p-\bar{p}\|} \geq -\lambda. \tag{4.1}$$

As a straightforward consequence of Theorem 3.2, for the class of parametric optimization problems under investigations problem calmness comes under natural qualified convexity assumptions.

Corollary 4.2. *If the problems in* (P_p) *are q.c.s.v.i., then for any* $\bar{p} \in \mathrm{dom}\,\mathsf{val} = \mathbb{P}$*, problem* $(\mathsf{P}_{\bar{p}})$ *is calm at any* $\bar{x} \in \mathsf{Argmin}(\bar{p})$.

Proof. According to Theorem 3.2, there exist positive r and ℓ such that

$$|\mathsf{val}(p_1)-\mathsf{val}(p_2)| \leq \ell\|p_1-p_2\|, \quad \forall p_1,\, p_2 \in \mathrm{B}\left[\bar{p};r\right].$$

On the account of this inequality it follows

$$\inf_{p\in \mathrm{B}[\bar{p};r]\setminus\{\bar{p}\}} \ \inf_{x\in \mathrm{B}[\bar{x};r]\cap \mathsf{R}(p)} \frac{\varphi(p,x)-\varphi(\bar{p},\bar{x})}{\|p-\bar{p}\|} \geq \inf_{p\in \mathrm{B}[\bar{p};r]\setminus\{\bar{p}\}} \ \frac{\mathsf{val}(p)-\mathsf{val}(\bar{p})}{\|p-\bar{p}\|} \geq -\ell.$$

□

Definition 4.3. Let $\bar{x} \in F^{+1}(C)$ be a local solution of the problem

$$(\mathsf{P}) \qquad \begin{array}{ll} \min\limits_{x\in X} & \varphi(x) \\ \text{sub} & F(x) \subseteq C. \end{array}$$

with a set-valued inclusion constraint. (P) is said to admit a *penalty function* at $\bar{x}$ if there exists $\lambda_* \geq 0$ such that for every $\lambda \in (\lambda_*, +\infty)$ $\bar{x}$ is an unconstrained local minimizer of

$$\varphi_\lambda(x) = \varphi(x) + \lambda \operatorname{exc}(F(x);C).$$

In the next result, a sufficient condition for the existence of a penalty function for general parametric optimization problems with set-valued inclusion constraints is formulated, where problem calmness plays a crucial role.

Theorem 4.4. *With reference to the family of problems* (P_p)*, let* $\bar{p} \in \mathbb{P}$ *and let* $\bar{x} \in \mathsf{Argmin}(\bar{p})$*. Suppose that:*

(i) $\varphi(\bar{p},\cdot)$ *is l.s.c. at* $\bar{x}$*;*

(ii) $\varphi(\cdot,\bar{x})$ *is calm from above at* $\bar{p}$*, uniformly in x, i.e. there exist positive* γ *and* r_γ *such that*

$$\varphi(p,x)-\varphi(\bar{p},x) \leq \gamma\|p-\bar{p}\|, \quad \forall p \in \mathrm{B}\left[\bar{p};r_\gamma\right],\ \forall x \in \mathrm{B}\left[\bar{x};r_\gamma\right];$$

(iii) *there exist positive* β *and* r_β *such that*

$$\operatorname{dist}\left(\bar{p};F^{+1}(\cdot,x)(C)\right) \leq \beta \operatorname{exc}(F(\bar{p},x);C), \quad \forall x \in \mathrm{B}\left[\bar{x};r_\beta\right]; \tag{4.2}$$

(iv) $(\mathsf{P}_{\bar{p}})$ *is calm at* $\bar{x}$.

Then, $(\mathsf{P}_{\bar{p}})$ *admits a penalty function at* $\bar{x}$.

Proof. Let us argue by contradiction. Assume that $(\mathsf{P}_{\bar{p}})$ fails to admit a penalty function at $\bar{x}$. This means that for each $\lambda_* \geq 0$ there exist $k \in \mathbb{N}$, with $k > \lambda_*$, and $x_k \in \mathrm{B}[\bar{x}; 1/k]$, such that

$$\varphi(\bar{p}, x_k) + k\mathrm{exc}(F(\bar{p}, x_k); C) < \varphi(\bar{p}, \bar{x}). \tag{4.3}$$

Since $\bar{x}$ is a local solution to $(\mathsf{P}_{\bar{p}})$ the inequality in (4.3) implies the existence of $k_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, with $k \geq k_0$, it is $x_k \notin \mathsf{R}(\bar{p})$, so

$$\mathrm{exc}(F(\bar{p}, x_k); C) > 0.$$

Because of $x_k \longrightarrow \bar{x}$ as $k \to \infty$, then by taking into account hypothesis (i), from (4.3) one deduces

$$\limsup_{k\to\infty} k\mathrm{exc}(F(\bar{p}, x_k); C) \leq \limsup_{k\to\infty} [\varphi(\bar{p}, \bar{x}) - \varphi(\bar{p}, x_k)] = \varphi(\bar{p}, \bar{x}) - \liminf_{k\to\infty} \varphi(\bar{p}, x_k) \leq 0.$$

The last inequality entails

$$\exists \lim_{k\to\infty} \mathrm{exc}(F(\bar{p}, x_k); C) = 0^+. \tag{4.4}$$

On the other hand, again because of $x_k \longrightarrow \bar{x}$ as $k \to \infty$, up to an increase of the value of k_0 if needed, one has $x_k \in \mathrm{B}\left[\bar{x}; r_\beta\right]$, where $r_\beta > 0$ is as in hypothesis (iii). Consequently, one obtains

$$\mathrm{dist}\left(\bar{p}; F^{+1}(\cdot, x_k)(C)\right) \leq \beta \mathrm{exc}(F(\bar{p}, x_k); C), \quad \forall k \in \mathbb{N},\ k \geq k_0.$$

Thus, by taking $\tilde{\beta} > \beta$ and recalling that $\mathrm{dist}\left(\bar{p}; F^{+1}(\cdot, x_k)(C)\right) > 0$ as $x_k \notin \mathsf{R}(\bar{p})$, for each $k \geq k_0$ it is possible to get the existence of $p_k \in F^{+1}(\cdot, x_k)(C)$ with the property that

$$\tilde{\beta}^{-1} d(\bar{p}, p_k) < \mathrm{exc}(F(\bar{p}, x_k); C) \tag{4.5}$$

and $F(p_k, x_k) \subseteq C$, so $x_k \in \mathsf{R}(p_k)$. Notice that it must be $p_k \neq \bar{p}$ for every $k \geq k_0$, otherwise it would result in $x_k \in \mathsf{R}(\bar{p})$, what has been already excluded above. Moreover, on the account of (4.4), the estimate in (4.5) entails that the sequence $(p_k)_k$ converges to $\bar{p}$ as $k \to \infty$. By combining the inequalities in (4.3) and in (4.5), one obtains

$$\frac{\varphi(\bar{p}, x_k) - \varphi(\bar{p}, \bar{x})}{\tilde{\beta}^{-1} d(\bar{p}, p_k)} \leq \frac{\varphi(\bar{p}, x_k) - \varphi(\bar{p}, \bar{x})}{\mathrm{exc}(F(\bar{p}, x_k); C)} < -k, \quad \forall k \geq k_0,$$

which yields

$$\frac{\varphi(\bar{p}, x_k) - \varphi(\bar{p}, \bar{x})}{d(\bar{p}, p_k)} \leq -\frac{k}{\tilde{\beta}}, \quad \forall k \geq k_0. \tag{4.6}$$

Since by hypothesis (ii) and the convergence of $(p_k)_k$, up to a further increase of the value of k_0, if needed, so that $x_k \in \mathrm{B}\left[\bar{x}; r_\gamma\right]$ and $p_k \in \mathrm{B}\left[\bar{p}; r_\gamma\right]$, for $\gamma > 0$ one has

$$\varphi(p_k, x_k) - \gamma \|p_k - \bar{p}\| \leq \varphi(\bar{p}, x_k), \quad \forall k \geq k_0,$$

from the inequality in (4.6) it follows

$$\frac{\varphi(p_k, x_k) - \varphi(\bar{p}, \bar{x})}{\|\bar{p} - p_k\|} \leq -\frac{k}{\tilde{\beta}} + \gamma, \quad \forall k \geq k_0.$$

This amounts to say that for every $k \geq k_0$ there exists $p_k \in \mathrm{B}\left[\bar{p}; r_\gamma\right] \setminus \{\bar{p}\}$ such that

$$\inf_{x \in \mathrm{B}\left[\bar{x}; r_\gamma\right] \cap \mathsf{R}(p_k)} \frac{\varphi(p_k, x) - \varphi(\bar{p}, \bar{x})}{\|\bar{p} - p_k\|} \leq -\frac{k}{\tilde{\beta}} + \ell.$$

The last inequality shows that the condition (4.1) in Definition 4.1 is violated, what contradicts the assumption (iv) on the calmness of $(\mathsf{P}_{\bar{p}})$ at $\bar{x}$. This completes the proof. □

Remark 4.5. (i) It is readily seen that both hypotheses (i) and (ii) of Theorem 4.4 happen to be satisfied, in particular, by any function φ, which is locally Lipschitz around $(\bar{p},\bar{x})$.

(ii) Hypothesis (iii) can be regarded as an error bound condition leading to a kind of metric subregularity for the set-valued mapping $F(\cdot,x)$ at $\bar{p}$, which is uniform with respect to x around $\bar{x}$.

Corollary 4.6. *If the problems in* (P_p) *are q.c.s.v.i. and for any* $\bar{p}\in \mathrm{dom}\,\mathsf{val}=\mathbb{P}$ *there exist positive* β *and* r_β *for which the inequality in (4.2) is true, then problem* ($\mathsf{P}_{\bar{p}}$) *admits a penalty function at any* $\bar{x}\in \mathsf{Argmin}(\bar{p})$.

Proof. As a continuous convex function, φ is also locally Lipschitz around $(\bar{p},\bar{x})$. On the basis of what was observed in Remark 4.5(i), both hypotheses (i) and (ii) of Theorem 4.4 are then fulfilled. Since problems in in (P_p) are q.c.s.v.i., then by Corollary 4.2 ($\mathsf{P}_{\bar{p}}$) is a calm problem at any $\bar{x}\in \mathsf{Argmin}(\bar{p})$. Thus, the thesis follows from Theorem 4.4. □

It is clear that the existence of a penalty function paves the way to deriving optimality conditions for constrained optimization problems from those valid in the unconstrained case. In the specific case of q.c.s.v.i. problems, it must be noticed that, since R is a convex multifunction, so it takes convex values, each problem in (P_p) falls in the realm of convex optimization. Consequently, any solution $\bar{x}\in \mathsf{Argmin}(\bar{p})$ is global. Moreover, in the light of Lemma 3.5, each penalized function $x\mapsto \varphi_\lambda(\bar{p},x)=\varphi(\bar{p},x)+\lambda \mathrm{exc}(F(\bar{p},x);C)$ is convex, so many theoretical and computational tools are at disposal for its minimization. In particular, on the basis of the specific form taken by F, various constructions of generalized differentiation and subdifferential calculus rules could be exploited in order to formulate first-order optimality conditions for the unconstrained minimization problem

$$\min_{x\in\mathbb{X}}[\varphi(\bar{p},x)+\lambda \mathrm{exc}(F(\bar{p},x);C)].$$

The exploration and the assessment of such kind of research perspectives will be the theme of future investigations.

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