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ON TOPOLOGICAL PROPERTIES OF SOLUTION SETS FOR SEMILINEAR DIFFERENTIAL INCLUSIONS OF FRACTIONAL ORDER $\alpha \in (1,2)$ WITH NON-CONVEX RIGHT-HAND SIDES

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Dedicated to Professor Vladimir Mikhailovich Tikhomirov on the occasion of his jubilee

Abstract. In this paper, we study the Cauchy problem for a semilinear differential inclusion of a fractional order $\alpha \in (1,2)$ with a nonconvex-valued almost lower semicontinuous nonlinearity and a linear closed operator generating a family of cosine operator functions. By using the theory of measure of noncompactness and condensing operators theory we study topological properties of the solution set of this problem. We prove that the solution set of the Cauchy problem possesses the classical Kneser connectedness property.
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1. INTRODUCTION

The theory of differential equations and inclusions of fractional order attracts the attention of a large number of researchers thanks to its numerous applications in mathematical control theory, mathematical physics, biology, economics, engineering, ecology and other branches of natural sciences (see, e.g., [10, 15] and the references therein). Up to the present moment, various approaches to the solvability of differential equations and inclusions of a fractional order $q \in (0,1)$ have been developed.

In the present paper, we consider the Cauchy problem for a semilinear fractional differential inclusion in a separable Banach space E of the following form:

$$^{C}D_0^{\alpha}x(t) \in Ax(t) + F(t,x(t)),\ t \in [0,T], \tag{1.1}$$

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$$x(0) = x_0,\ x'(0) = x_1, \tag{1.2}$$

where ${}^{C}D_0^{\alpha},\ 1 < \alpha < 2$, is the Caputo fractional derivative and $F : [0,T] \times E \multimap E$ is an almost lower semicontinuous multivalued map with compact values, $A : D(A) \subset E \to E$ is a closed linear (not necessarily bounded) operator in E and $x_0, x_1 \in E$. By using the theory of measures of noncompactness and condensing operators theory we study the topological properties of the solution set of problem (1.1)-(1.2).

2. Preliminaries

2.1. **Fractional integral and derivative.**

Definition 2.1. *(See, e.g.,* [10, 15]*).* The fractional integral of order $\alpha > 0$ of a function $g : [0,T] \to E$ is the function $I_0^{\alpha} g$ of the following form

$$I_0^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds,$$

where Γ is Euler's gamma-function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

Definition 2.2. The Caputo fractional derivative of the order $\alpha \in (N-1, N]$ of a function $g \in C^N([0,T];E)$ is the function ${}^{C}D_0^{\alpha} g$ of the following form

$${}^{C}D_0^{\alpha} g(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} g^{(N)}(s)\, ds.$$

2.2. **Multivalued maps.** Let us recall some concepts (see, e.g., [8, 12]).

We denote by $\mathscr{E}$ is a Banach space and introduce the following notation:

- $P(\mathscr{E}) = \{A \subseteq \mathscr{E} : A \neq \varnothing\}$ is the collection of all non-empty subsets of $\mathscr{E}$;
- $Pb(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is bounded}\}$;
- $Pv(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is convex}\}$;
- $C(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is closed}\}$;
- $K(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is compact}\}$;
- $Kv(\mathscr{E}) = \{Pv(\mathscr{E}) \cap K(\mathscr{E})\}$ is the collection of all convex and non-empty compact subsets of $\mathscr{E}$;
- $Cv(\mathscr{E}) = \{Pv(\mathscr{E}) \cap C(\mathscr{E})\}$ is the collection of all convex and non-empty closed subsets of $\mathscr{E}$.

Definition 2.3. (see [1]) Let $(\mathscr{A}, \geq)$ be a partially ordered set. A function $\beta : Pb(\mathscr{E}) \to \mathscr{A}$ is called the measure of noncompactness (MNC) in $\mathscr{E}$ if, for each set $\Omega \in Pb(\mathscr{E})$, $\beta(\overline{\text{co}}\,\Omega) = \beta(\Omega)$, where $\overline{\text{co}}\,\Omega$ is the closure of the convex hull of Ω.

A measure of noncompactness β is called:

1) *monotone*, if for all $\Omega_0, \Omega_1 \in Pb(\mathscr{E})$, from $\Omega_0 \subseteq \Omega_1$ follows $\beta(\Omega_0) \leq \beta(\Omega_1)$.
2) *nonsingular*, if for all $a \in E$ and each $\Omega \in Pb(\mathscr{E})$ we have $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

If $\mathscr{A}$ is a cone in a Banach space, then the MNC β is called:

3) *regular*, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in Pb(\mathscr{E})$;
4) *real*, if $\mathscr{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.

The example of a real MNC obeying all above properties is the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathscr{E}\ \}.$$

Notice that the Hausdorff MNC satisfies the semi-homogeneity condition, i.e.,

$$\chi(\lambda\Omega) = |\lambda|\chi(\Omega),$$

for each $\lambda \in \mathbb{R}$ and each $\Omega \in Pb(\mathscr{E})$. For a bounded set $M \subset \mathscr{E}$, $\|M\| = \sup_{x\in M}\|x\|_{\mathscr{E}}$.

Let X be a metric space and Y be a normed space.

Definition 2.4. A multivalued map (multimap) $\mathscr{F} : X \to P(Y)$ is said to be lower semicontinuous (l.s.c.) at a point $x \in X$, if for each open set $V \subset Y$ such that $\mathscr{F}(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of the point x such that $\mathscr{F}(x') \cap V \neq \emptyset$ for all $x' \in U(x)$.

A multimap is lower semicontinuous if it is lower semicontinuous at every point $x \in X$.

Definition 2.5. A multimap $\mathscr{F} : [0,T] \times X \to K(Y)$ is said to be almost lower semicontinuous (a.l.s.c.) if there exists a sequence of disjoint compact sets $I_n \subseteq [0,T]$ such that

(i) $meas([0,T] \setminus I) = 0$, where $I = \cup_n I_n$;
(ii) the restriction of $\mathscr{F}$ on each set $J_n = I_n \times X$ is l.s.c.

Definition 2.6. A multivalued map $\mathscr{F} : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) at a point $x \in X$, if for every open set $V \subset Y$ such that $\mathscr{F}(x) \subset V$, there exists a neighborhood $U(x)$ of x such that $\mathscr{F}(U(x)) \subset V$.

A multimap is upper semicontinuous if it is upper semicontinuous at every point $x \in X$.

Proposition 2.7. *(see, e.g.,* [12, Theorem 1.2.37]*) Let $\mathscr{F} : X \to P(Y)$ be a u.s.c. multimap. If $A \subset X$ is a connected set and $\mathscr{F}(x)$ is connected for every $x \in A$ then the image $\mathscr{F}(A)$ is a connected subset of Y.*

Definition 2.8. A multimap $\mathscr{F} : X \to P(Y)$ is called closed if its graph $G_{\mathscr{F}} = \{(x,y) : x \in X, y \in \mathscr{F}(x)\}$ is a closed subset of $X \times Y$.

Let us recall the following notions.

Definition 2.9. A metric space X is called contractible if there exist a point $x_0 \in X$ and a continuous map (homotopy) $h : [0,1] \times X \to X$ such that $h(0,x) = x$ and $h(1,x) = x_0$ for all $x \in X$.

It is obvious that convex and, more generally, star-shaped sets are contractible.

Definition 2.10. (see [7]). A compact metric space A is called an R_δ-set if there exists a decreasing sequence $\{A_n\}$ of compact contractible sets such that $A = \bigcap_{n\geq 1} A_n$.

Notice that an R_δ-set need not be contractible (see an example in [5]).

Definition 2.11. A continuous map $f : X \subseteq \mathscr{E} \to \mathscr{E}$ is called condensing with respect to a MNC β (or β–condensing) if, for each bounded set $\Omega \subseteq X$ which is not relatively compact, $\beta(f(\Omega)) \ngeq \beta(\Omega)$.

2.3. **Measurable Multifunctions.** Recall some notions (see, e.g., [8, 12]). Let E be a separable Banach space.

Definition 2.12. Let $p \geq 1$, a multifunction $G : [0,T] \to K(E)$ is called:

- L^p–integrable if G admits an L^p–Bochner integrable selection, i.e., there exists a function $g \in L^p([0,T];E)$ such that $g(t) \in G(t)$ for a.e. $t \in [0,T]$;
- L^p–integrably bounded if there exists a function $\xi \in L^p([0,T])$ such that $\|G(t)\| \leq \xi(t)$ for a.e. $t \in [0,T]$.

The set of all L^p–integrable selections of a multifunction $G : [0,T] \to K(E)$ is denoted by $\mathscr{S}^p_G$.

A multifunction $G : [0,T] \to K(E)$ is called measurable if for every open subset $V \subset E$ the set $G^{-1}(V)$ is Lebesgue measurable. Every multimap $\mathscr{F} : [0,T] \times E \to K(E)$ generates a correspondence assigning to every function $q : [0,T] \to E$ the multifunction $\Phi : [0,T] \to P(E)$ defined by $\Phi(t) = \mathscr{F}(t, q(t))$. If for every measurable function q the multifunction Φ is measurable, then a multimap $\mathscr{F}$ is called superpositionally measurable.

Lemma 2.13. *(see* [8, Proposition 1.3.1] *and* [12, Theorem 1.5.19]*) If a multimap $\mathscr{F} : [0,T] \times E \to P(E)$ is a.l.s.c., then it is superpositionally measurable.*

In the sequel, we need the following important property on the χ-estimation of the integral of a multifunction.

Lemma 2.14. *(see* [8, Theorem 4.2.3]*). Let E be a separable Banach space and $G : [0,T] \to K(E)$ an integrable, integrably bounded multifunction such that $\chi(G(t)) \leq \nu(t)$ for a.e. $t \in [0,T]$, where χ is the Hausdorff MNC in E and $\nu(\cdot) \in L^1_+(0,T)$. Then*

$$\chi\Big(\int_0^t G(s)\,ds\Big) \leq \int_0^t \nu(s)\,ds.$$

Definition 2.15. A sequence of functions $\{\xi_n\} \subset L^p([0,T];E)$, $p \geq 1$, is called L^p–semicompact if it is L^p–integrably bounded and the set $\{\xi_n(t)\}$ is relatively compact in E for a.e. $t \in [0,T]$.

2.4. **Family of cosine operator functions.**

Definition 2.16. (See [6, 11, 16]) A family of bounded operators $\{C(t)\}_{t\in\mathbb{R}}$ in a Banach space E is called a strongly continuous family of cosine operator functions if:

(1) $C(0) = I$;
(2) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $t, s \in \mathbb{R}$;
(3) $t \to C(t)x$ is continuous for all $x \in E$.

The family of strongly continuous sine operator functions associated with the family of cosine operator functions $\{C(t)\}_{t\in\mathbb{R}}$ is the family of operators $\{S(t)\}_{t\in\mathbb{R}}$ such that

$$S(t)x = \int_0^t C(s)x ds,\ x \in E, t \in \mathbb{R}.$$

The operator A generates a family of cosine operator functions $\{C(t)\}_{t\in\mathbb{R}}$ if:

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0},$$

for all $x \in D(A)$ for which the last expression is well defined.

3. Auxiliary Results

Consider Cauchy problem (1.1) - (1.2) in separable Banach space E.

We assume that the multimap $F : [0,T] \times E \to K(E)$ obeys the following conditions:

$(F1)$ $F : [0,T] \times E \to K(E)$ is a.l.s.c.;

$(F2)$ for each $r > 0$, there exists a function $\omega_r \in L^\infty([0,T])$ such that, for each $x \in E$, $\|x\| \leq r$, $\|F(t,x)\| \leq \omega_r(t)$ for a.e. $t \in [0,T]$;

$(F3)$ there exists a function $\mu \in L^\infty([0,T])$ such that for each nonempty bounded set $Q \subset E$

$$\chi(F(t,Q)) \leq \mu(t)\chi(Q),$$

for a.e. $t \in [0,T]$, where χ is the Hausdorff MNC in$E$.

(A) $A : D(A) \subset E \to E$ be a closed linear operator in E generating an uniformly bounded family of strongly continuous cosine operator functions $\{C(t)\}_{t\geq 0}$. Denote

$$M = \sup\{\|C(t)\|_{t\geq 0}\}.$$

For $x \in C([0,\tau];E), 0 < \tau \leq T$, consider the multifunction

$$\Phi_F : [0,\tau] \to K(E), \qquad \Phi_F(t) = F(t,x(t)).$$

From conditions $(F1)-(F2)$ and Lemma 2.13, it follows that Φ_F is measurable and L^p–integrable for $p \geq 1$. Then, the superposition multimap $\mathscr{P}_F^\infty : C([0,\tau];E) \multimap L^\infty([0,\tau];E)$ defined as $\mathscr{P}_F^\infty(x) = \mathscr{S}_{\Phi_F}^\infty$, is well defined.

Definition 3.1. A mild solution of Cauchy problem (1.1) - (1.2) on an interval $[0,\tau]$, $\tau \in (0,T]$, is a function $x \in C([0,\tau];E)$ such that:

$$x(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\phi(s)ds, \qquad t \in [0,\tau],$$

where $\phi \in \mathscr{P}_F^\infty(x)$, $q = \alpha/2$, and

$$\mathscr{G}(t) = \int_0^\infty \xi_q(\theta)C(t^q\theta)d\theta, \qquad \mathscr{K}(t) = \int_0^t \mathscr{G}(s)ds, \qquad \mathscr{T}(t) = q\int_0^\infty \theta\xi_q(\theta)S(t^q\theta)d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-1/q}),$$

$$\Psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in (0,+\infty).$$

Remark 3.2. (See, e.g., [18]) $\xi_q(\theta) \geq 0$, $\int_0^\infty \xi_q(\theta)d\theta = 1$, $\int_0^\infty \theta\xi_q(\theta)d\theta = \frac{1}{\Gamma(q+1)}$.

Lemma 3.3. *(See [17]) The operator functions $\mathscr{G}$, $\mathscr{K}$ and $\mathscr{T}$ possess the following properties:*

1) *for all $t \in [0,T]$, $\mathscr{G}(t)$, $\mathscr{K}(t)$ and $\mathscr{T}(t)$ are bounded linear operators, more precisely, for every $x \in E$:*

$$\begin{aligned} &\|\mathscr{G}(t)x\|_E \leq M\|x\|_E; \\ &\|\mathscr{K}(t)x\|_E \leq M\|x\|_E t; \\ &\|\mathscr{T}(t)x\|_E \leq \frac{M}{\Gamma(2q)}\|x\|_E t^q; \end{aligned} \tag{3.1}$$

2) *the operator functions* $\mathscr{G}(\cdot)$, $\mathscr{K}(\cdot)$ *and* $\mathscr{T}(\cdot)$ *are strongly continuous, i.e., functions* $t \in [0,T] \to \mathscr{G}(t)x$, $t \in [0,T] \to \mathscr{K}(t)x$ *and* $t \in [0,T] \to t^{q-1}\mathscr{T}(t)x$ *are continuous for all* $x \in E$.

Consider the operator $\mathscr{S} : L^\infty([0,T];E) \to C([0,T];E)$ defined as

$$\mathscr{S}(\phi)(t) = \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\phi(s)ds.$$

Lemma 3.4. *(see* [13, Lemma 4]*) Let* $\{\xi_n\}$ *be an* L^∞*-semicompact sequence in* $L^\infty([0,\tau];E)$*. Then the sequence* $\{S\xi_n\}$ *is compact in* $C([0,\tau];E)$*.*

Now, we consider the multioperator $\Upsilon_F : C([0,\tau];E) \multimap C([0,\tau];E)$, given as

$$\Upsilon_F = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \mathscr{S} \circ \mathscr{P}_F^\infty(x), \qquad t \in [0,\tau].$$

In [13], the following existence theorems for problem (1.1) - (1.2) were proved.

Theorem 3.5. *Under conditions* (A), $(F1)-(F3)$ *and*

$$c := \frac{MT^{2q}}{\Gamma(1+2q)}\,\|\mu\|_\infty < 1, \tag{3.2}$$

there exists $\tau \in (0,T]$ *such that the set* $\Sigma^F_{x_0,x_1}[0,\tau]$ *of all mild solutions to Cauchy problem* (1.1) - (1.2) *on the interval* $[0,\tau]$ *is a non-empty subset of the space* $C([0,\tau];E)$*.*

Theorem 3.6. *Under conditions* (A), $(F1)$, $(F3)$, *suppose that condition* $(F2)$ *has the following form:*

$(F'2)$ *there exists* $l \in L^\infty_+([0,T])$ *such that* $\|F(t,x)\|_E \le l(t)(1+\|x\|_E)$ *for a.e.* $t \in [0,T]$*.*

If $\frac{MT^{2q}}{\Gamma(1+2q)}k < 1$, *where* $k = \max\{\|l\|_\infty, \|\mu\|_\infty\}$ *and functions* l *and* μ *are from conditions* $(F'2)$ *and* $(F3)$ *respectively, then problem* (1.1) - (1.2) *has a mild solution.*

4. Main Results

Consider the question on the topological structure of the solutions set for problem (1.1) - (1.2). We prove that this set possesses the classical Kneser connectedness property.

Suppose that the linear part of inclusion (1.1) satisfies condition (A), and the multivalued nonlinearity F obeys condition $(F1)$. Moreover, assume for F the following slighly more strict conditions of boundedness and χ-regularity:

$(F2_L)$ there exists a constant $K > 0$ such that $\|F(t,x)\|_E \le K(1+\|x\|_E)$ for a.e. $t \in [0,T]$;

$(F3_L)$ there exists a function $\mu \in L^\infty([0,T])$ such that, for each bounded set $Q \subset E$,

$$\lim_{\tau\to+0}\chi(F(J_{t,\tau}\times Q)) \le \mu(t)\chi(Q), \text{ for a.e. } t \in [0,T],$$

where χ is the Hausdorff MNC in E, $J_{t,\tau} = [t-\tau, t+\tau] \cap [0,T]$.

Consider a multivalued map $\tilde{F} : [0,T]\times E \to K(E)$, $\tilde{F}(t,x) = \cap_{\varepsilon>0}F^\varepsilon(t,x)$, where $F^\varepsilon(t,x) = \overline{co}\{F(s,y) : |s-t| < \varepsilon, \|y-x\| < \varepsilon\}$.

Let us show that $\tilde{F}$ satisfies the χ–regularity condition:

$$\chi(\tilde{F}(t,Q)) \le \mu(t)\chi(Q), \tag{4.1}$$

for a.e. $t \in [0,T]$ and for each bounded set $Q \subset E$.

In fact, for any $t \in [0,T]$, for which estimation $(F3_L)$ holds, we take arbitrary $\delta > 0$ and choose τ, $0 < \tau < \delta$ such that

$$\chi(F(J_{t,\tau} \times W_\delta(Q))) \le \mu(t)\chi(W_\delta(Q)) + \delta \le \mu(t)(\chi(Q)+\delta)+\delta,$$

where $W_\delta(Q)$ is a δ-neighbourhood of set Q. Now

$$\chi(\tilde{F}(t,Q)) \le \chi(F^\tau(t,Q)) \le \chi(F(J_{t,\tau} \times W_\delta(Q))) \le \mu(t)(\chi(Q)+\delta)+\delta.$$

Estimate (4.1) follows from the arbitrariness of δ.

Lemma 4.1. *Under above conditions, there exists a non-empty compact convex set $X \subset C([0,T];E)$ such that:*

$$x(0) = x_0, \quad x'(0) = x_1 \quad \textit{for all} \quad x \in X; \tag{4.2}$$

$$\mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\overline{co}\tilde{F}(s,X(s))ds \subseteq X(t) \quad \textit{for all} \quad t \in [0,T]; \tag{4.3}$$

and

$$\Sigma^F_{x_0,x_1} \subset X. \tag{4.4}$$

Proof. Assume without loss of generality that $\tilde{F}$ satisfies the following condition of global integral boundedness:

$(F''2)$: $\left\|\tilde{F}(t,x)\right\|_E \le \gamma(t)$ for a.e. $t \in [0,T], x \in E$ for a given function $\gamma(\cdot) \in L^\infty_+([0,T])$.

Notice that in this case a solution x to problem (1.1) - (1.2) satisfies $\|x\|_{C([0,T];E)} \le N$, where

$$N = M\Big(\|x_0\|_E + T\|x_1\|_E + \frac{\|\gamma\|_\infty T^{2q}}{\Gamma(2q+1)}\Big).$$

Construct a decreasing sequence of closed convex sets $\{X^n\}_{n=1}^\infty \subset C([0,T];E)$ by the following inductive process. Set

$$X^0 = \left\{x \in C([0,T];E) : x(0) = x_0,\ x'(0) = x_1,\ \|x\|_{C([0,T];E)} \le N\right\}. \tag{4.5}$$

Further, let $X^n = \overline{Y^n}, n \ge 1$, where Y^n is

$$\left\{y \in C([0,T];E) : y(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)f(s)ds, f \in \mathscr{P}^\infty_{\overline{\mathrm{co}}\tilde{F}(\cdot,X^{n-1}(\cdot))}\right\}.$$

Notice that $X^n, n \ge 1$ are non-empty sets since $\Sigma^F_{x_0,x_1}[0,T] \subset X^n$ for every $n \ge 0$. In the space $C([0,T];E)$, introduce the following MNC:

$$\psi(\Omega) = \sup_{t\in[0,T]} \chi(\{z(t) : z \in \Omega\}).$$

Notice that the MNC ψ is monotone and non-singular. From condition $(F3)$, we have, for $0 \le s \le t \le T$,

$$\begin{aligned}
\chi\left(\mathscr{T}(t-s)\,\overline{\mathrm{co}}F\left(s,X^{n-1}(s)\right)\right) &\le \|\mathscr{T}(t-s)\|\,\chi\left(\overline{\mathrm{co}}F\left(s,X^{n-1}(s)\right)\right)\\
&\le \frac{(t-s)^q M}{\Gamma(2q)}\chi\left(F\left(s,X^{n-1}(s)\right)\right)\\
&\le \frac{(t-s)^q M}{\Gamma(2q)}\mu(s)\chi\left(X^{n-1}(s)\right)\\
&\le \frac{(t-s)^q M}{\Gamma(2q)}\mu(s)\psi\left(X^{n-1}\right).
\end{aligned}$$

From (3.1) and $(F''2)$, it follows that, for each $t \in [0,T]$, the multifunction

$$s \multimap \mathscr{T}(t-s)\,\overline{\mathrm{co}}\,F\left(s, X^{n-1}(s)\right),\ s \in [0,t],$$

is integrable and a.e. bounded. By pplying Lemma 2.14, for an arbitrary $t \in [0,T]$, we have

$$\begin{aligned}
\chi\left(Y^n(t)\right) &= \chi\left(\mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\,\overline{\mathrm{co}}\,F\left(s, X^{n-1}(s)\right)ds\right)\\
&= \chi\left(\int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\,\overline{\mathrm{co}}\,F\left(s, X^{n-1}(s)\right)ds\right)\\
&\le \frac{M\|\mu\|_\infty}{\Gamma(2q)}\int_0^t (t-s)^{2q-1}\psi\left(X^{n-1}\right)ds\\
&= \frac{M\|\mu\|_\infty}{\Gamma(2q)}\psi\left(X^{n-1}\right)\int_0^t (t-s)^{2q-1}ds\\
&= \frac{M\|\mu\|_\infty}{\Gamma(2q)}\psi\left(X^{n-1}\right)\frac{t^{2q}}{2q}\\
&\le \frac{T^{2q}M\|\mu\|_\infty}{\Gamma(2q+1)}\psi\left(X^{n-1}\right).
\end{aligned}$$

Therefore, we obtain the estimate $\psi(Y^n) \le c\psi\left(X^{n-1}\right)$, where $c < 1$ from (3.2). It follows that $\psi(X^n) \le c\psi\left(X^{n-1}\right)$, implying $\psi(X^n) \to 0 \quad \text{as} \quad n \to \infty$. Consider the set $\widetilde{X} = \cap_{n\ge 1} X^n$. From the monotonicity of the MNC ψ, it follows that $\psi(\widetilde{X}) = 0$. Hence, $\chi(\widetilde{X}(t)) = 0$ for all $t \in [0,T]$. Moreover, inequality (4.1) implies $\chi\left(\tilde{F}\left(t, \widetilde{X}(t)\right)\right) = 0$ for each $t \in [0,T]$. B applying $(F''2)$, we get that every sequence $\{f_n\} \subset \mathscr{P}^\infty_{\overline{\mathrm{co}}\tilde{F}(\cdot,\widetilde{X}(\cdot))}$ is semicompact in $L^\infty([0,T];E)$.

Now, we define the set $X \subseteq \widetilde{X}$ as

$$\left\{y \in C([0,T];E) : y(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)f(s)ds, f \in \mathscr{P}^\infty_{\overline{\mathrm{co}}\tilde{F}(\cdot,\widetilde{X}(\cdot))}\right\}. \tag{4.6}$$

Applying Lemma 3.4, we obtain that X is a compact set and hence it is a desirable one. To verify (4.4), it is sufficient to remember that by construction

$$F(t,x) \subseteq \tilde{F}(t,x) \quad \text{for all} \quad t \in [0,T] \times E. \tag{4.7}$$

□

Consider a compact set $D \subset [0,T] \times E$, $D = \{(s,y) : s \in [0,T], y = x(s), x \in X\}$. Let $r > 0$ be the radius of the ball X^0, defined by formula (4.5). For $(t,x) \in D, \varepsilon > 0$ and $W > \sigma = MK(1+r)$, consider the set

$$V(t,x,\varepsilon) = \left\{(s,y) \in D : \quad t \le s < t + \varepsilon, \quad \|y - x\| \le \frac{W(s-t)^{2q}}{\Gamma(1+2q)}\right\}.$$

We need the following assertions (see [2, 3]).

Lemma 4.2. *The family of sets $\{V(t,x,\varepsilon) : (t,x) \in D, \varepsilon > 0\}$ forms a basis of closed-open nieghborhoods for a topology $\mathfrak{T}^+$ on D, stronger that usual metric topology.*

Lemma 4.3. *The multifunction $F : D \to K(E)$ admits an almost $\mathfrak{T}^+$–continuous selection $\gamma : D \to E$ in the sense that γ is $\mathfrak{T}^+$–continuous on every set $D_n = J_n \cap D$.*

For $(t,x)\in D_n$, set

$$G_n(t,x)=\cap_{\varepsilon>0}G_n^{\varepsilon}(t,x)=\cap_{\varepsilon>0}\overline{co}\{\gamma(s,y):(s,y)\in D_j,|s-t|<\varepsilon,\|y-x\|<\varepsilon\},$$

and, for $(t,x)\in D$ define

$$G(t,x)=\begin{cases} G_n(t,x), & (t,x)\in D_n,\\ \{\gamma(t,x)\}, & (t,x)\notin\cap_n D_n.\end{cases}$$

It follows from the construction that

$$\gamma(t,x)\in G(t,x),\ G(t,x)\subseteq\tilde{F}(t,x) \tag{4.8}$$

for all $(t,x)\in D$. From condition $(F3_L)$ and the compactness of sets D_n, we see that for every sequence $\varepsilon_k\to+0$ the corresponding sequence $\chi(F_n^{\varepsilon_k}(t,x))$ tends to zero and hence every $G(t,x)$ is a nonempty compact convex set.

The following assertion holds (see, e.g., [8, Lemma 5.5.6]).

Lemma 4.4. *The multimap G is u.s.c. on $\cup_n D_n$.*

Consider the following problem on the compact set D

$$^{C}D_0^q x(t)\in Ax(t)+G(t,x(t)),\quad (t,x)\in D, \tag{4.9}$$

$$x(0)=x_0,\quad x'(0)=x_1. \tag{4.10}$$

It is clear that G saisfies the following conditions

- $(G1)$ for each $x\in E$, the multifunction $G:[0,T]\to K(E)$ admits a strongly continious selection;
- $(G2)$ for a.e. $t\in[0,T]$, the multimap $G:E\to K(E)$ is u.s.c.;
- $(G3)$ $\|G(t,x)\|_E\le\gamma(t)$ for a.e. $t\in[0,T]$, $x\in E$ and a given $\gamma(\cdot)\in L_+^{\infty}([0,T])$;
- $(G4)$ there exists a function $\mu\in L^{\infty}([0,T])$ such that, for every bounded subset $Q\subset E$, $\chi(G(t,Q))\le\mu(t)\chi(Q)$, for a.e. $t\in[0,T]$, where χ denote the Hausdorff MNC in E.

Since the multimap G is also obviously integrably bounded, the integral multioperator Υ_G is defined and u.s.c. on X.

Definition 4.5. We say that problem (4.9), (4.10) satisfies condition (Q) provided:

- $(Q1)$ $\Sigma_{x_0,x_1}^G[0,T]$ is a non-empty compact subset of $C([0,T];E)$;
- $(Q1)$ $\Sigma_{x_0,x_1}^G[0,\tau]=\Sigma_{x_0,x_1}^G[0,T]|_{[0,\tau]}$ holds for every $\tau\in(0,T]$.

Choose an arbitrary function $x\in X$. From the construction of X, it is easy to see that, for each $\eta>0$, the function x can be uniformly η-approximated by the function

$$\tilde{x}(t)=\mathscr{G}(t)x_0+\mathscr{K}(t)x_1+\int_0^t(t-s)^{q-1}\mathscr{T}(t-s)\vartheta(s)ds,$$

where $\vartheta(\cdot)\in L^{\infty}([0,T];E)$ and $\|\vartheta(t)\|\le K(1+r)$ for a.e. $t\in[0,T]$.

Lemma 4.6. *The function x satisfies the following condition:*

$$\left\{(s,y)\in D:\ t\le s<t+\varepsilon;\ y\in\overline{B}_{\frac{(s-t)^{2q}(W-\sigma)}{\Gamma(1+2q)}}(x(s))\}\subset V(t,x(t),\varepsilon\right\}.$$

In other words, for points s, close enough to t, the set $V(t,x(t),\varepsilon)$ is a metric neighborhood of the point $(s,x(s))$ (in the relative topology of the space D).

Proof. Let $(s,y) \in D$ and $y \in \overline{B}_{\frac{(s-t)^{2q}(W-\sigma)}{\Gamma(1+2q)}}(x(s))$. Then

$$\|y - x(t)\| \leq \|y - \tilde{x}(t)\| + \|\tilde{x}(t) - x(t)\| \leq \frac{(s-t)^{2q}(W-\sigma)}{\Gamma(1+2q)} + 2\eta.$$

Since η is arbitrary, it follows that

$$\|y - x(t)\| \leq \frac{(s-t)^{2q}(W-\sigma)}{\Gamma(1+2q)}.$$

□

Consider now an arbitrary function $z \in \Upsilon_G(x)$ with the form

$$z(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)g(s)ds,$$

where $g \in \mathscr{P}_G(x)$.

Lemma 4.7. *The following equality holds:*

$$g(t) = \gamma(t, x(t)) \quad \textit{for a.e. } t \in [0,t].$$

Proof. For each index value n, consider the set $I_n^* \subseteq I_n$ consisting of all $t \in I_n$ with the properties that:

(*i*) $g(t) \in G_n(t, x(t))$,
(*ii*) there exists a sequence $\{t_k \subset I_n\}$ strictly decreasing to t such that $g(t_k) \in G_n(t, x(t_k))$ for all k and $g(t_k) \to g(t)$.

It is known (see [3, Lemma 2.3]) that $meas\,(I_n \backslash I_n^*) = 0$. Following [2], we prove that $g(t) = \gamma(t, x(t))$ for all $t \in I_n^*$. Using a contradiction argument, we suppose that there exist a point $t \in I_n^*$ and $\varepsilon > 0$ such that

$$\|g(t) - \gamma(t, x(t))\| = \varepsilon. \tag{4.11}$$

Since γ is $\mathfrak{T}^*$–continuous on D_n, there exists $\delta > 0$ such that $\|\gamma(s,y) - \gamma(t,x(t))\| < \frac{\varepsilon}{2}$, for all $(s,y) \in V(t, x(t), \delta)$, $s \in I_n$. Now, we let $\{t_k \subset I_n\}$ be a sequence with the properties described in (*ii*). Choose k_0 such that, for $k \geq k_0$, $0 < t_k - t < \delta$, and

$$\|g(t_k) - g(t)\| < \frac{\varepsilon}{2}. \tag{4.12}$$

From Lemma 4.6, we know that the set $V(t, x(t), \delta)$ is a neighborhood of $(t_k, x(t_k))$ in the usual relative metric topology of D for all $k \geq k_0$. So, let the ω–neighborhood W_ω of $(t_k, x(t_k))$ in D be contained in $V(t, x(t), \delta)$. Then , for $k \geq k_0$,

$$g(t_k) \in G_n(t_k, x(t_k)) \subseteq \overline{co}\{\gamma(s,y) : (s,y) \in W_\omega, s \in I_n\}$$

$$\subset \overline{co}\{\gamma(s,y) : (s,y) \in V(t, x(t), \delta), s \in I_n\} \subset B_{\frac{\varepsilon}{2}}(\gamma(t, x(t))),$$

i.e.,

$$\|\gamma(t, x(t)) - g(t_k)\| < \frac{\varepsilon}{2}. \tag{4.13}$$

It is clear that (4.12) and (4.13) give the contradiction to (4.11) that proves the lemma. □

From the above lemma, it follows that the integral multioperator Υ_G is single-valued on X, and moreover, since it has the form

$$\Upsilon_G = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\gamma(s,x(s))ds,$$

it is a continuous selection of the integral multioperator Υ_F. Consequently, Υ_G transforms X into itself since Υ_F has the same property (see (4.2) and (4.7)). Moreover, every mild solution of problem (4.9) - (4.10) is a mild solution to (1.1) - (1.2).

Now we pass from problem (4.9) - (4.10) to the Cauchy problem for a differential inclusion whose right-hand side nonlinearity is defined on the whole $[0,T]\times E$. To this aim, we consider a metric projection $P:[0,T]\times E \to Kv(E)$,

$$P(t,x) = \{y \in X(t), \|x-y\| = dist(x,X(t))\}$$

and the multimap $\tilde{G}:[0,T]\times E \to Kv(E)$ defined by $\tilde{G}(t,x) = \overline{co}\, G(t,P(t,x))$.

Lemma 4.8. *(*[8, Lemma 5.3.2]*). The multimap $\tilde{G}$ satisfies conditions $(G1)-(G4)$.*

Following the lines of [8, Lemma 5.5.9], we can prove the following statement.

Lemma 4.9. *The multimap $\tilde{G}$ satisfies the following conditions:*

(i) *the multifunction $\tilde{G}(\cdot,x):[0,T]\to Kv(E)$ admits a measurable selection for every $x\in E$;*
(ii) *the multimap $\tilde{G}(t,\cdot):E\to Kv(E)$ is u.s.c. for a.e. $t\in[0,T]$;*
(iii) *there exists a constant $R>0$ such that $\|\tilde{G}(t,x)\| \leqslant R$ for a.e $t\in[0,T]$ and $x\in E$;*
(iv) *the multimap $\tilde{G}(t,\cdot):E\to Kv(E)$ is compact for a.e $t\in[0,T]$.*

We can consider now the Cauchy problem on $[0,T]\times E$:

$$^{C}D_0^q x(t) \in Ax(t) + \tilde{G}(t,x(t)),\ t\in[0,T], \tag{4.14}$$

$$x(0) = x_0,\ x'(0) = x_1. \tag{4.15}$$

From Lemma 4.9 and corresponding existence result ([6, Theorem 2]), we conclude that the set $\Sigma^{\tilde{G}}_{x_0,x_1}$ of all mild solutions to (4.14) - (4.15) is a nonempty compact subset of $C([0,d];E)$.

Now, we can prove that the solution sets of problem (4.9) - (4.10) and (4.14) - (4.15) coincide.

Lemma 4.10.

$$\Sigma^{\tilde{G}}_{x_0,x_1} = \Sigma^{G}_{x_0,x_1}.$$

Proof. In fact, let $x\in\Sigma^{\tilde{G}}_{x_0,x_1}$. Then

$$x(t) \in \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\tilde{G}(s,x(s))ds =$$

$$\mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\overline{co}G(s,P(s,x(s)))ds \subseteq$$

$$\mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\overline{co}G(s,X(s))ds \subset X(t).$$

Hence $P(t,x(t)) = \{x(t)\}$. It follows that

$$x(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)f(s)ds,$$

where $f \in \mathscr{S}_{\tilde{G}(\cdot,x(\cdot))} = \mathscr{S}_{G(\cdot,x(\cdot))}$, and therefore $x \in \Sigma^G_{x_0,x_1}$. The inclusion $\Sigma^G_{x_0,x_1} \subseteq \Sigma^{\tilde{G}}_{x_0,x_1}$ follows easily from the observation that $\Sigma^G_{x_0,x_1} \subset X$. □

Following the lines of [8, Lemma 5.3.4], we have the following statement.

Lemma 4.11. *There exists a sequence of multimaps $\{G_n\}_{n=1}^{\infty}, G_n : [0,T] \times E \to Kv(E)$ such that:*

- (Gi) *every multimap $G_n(t,\cdot) : E \to Kv(E), n \geq 1$, is continuous for a.e. $t \in [0,T]$;*
- (Gii) $\tilde{G}(t,x) \subset \ldots \subset G_{n+1}(t,x) \subset G_n(t,x) \subset \overline{\mathrm{co}}\, G(t,X(t)), n \geq 1$;
- $(Giii)$ $\tilde{G}(t,x) = \cap_{n\geq 1} G_n(t,x)$;
- (Giv) *for every $n \geq 1$, there exists a selection $g_n(t,x) \in G_n(t,x)$ such that $g_n(\cdot,x)$ is measurable for each $x \in E$ and $g_n(t,\cdot)$ for a.e. $t \in [0,T]$ satisfies local Lipshitz condition, i.e., for every $x \in E$, there exist an open neighbourhood $U_x \subset E$ and a constant $k_x > 0$ such that, for every $x', x'' \in U_x$, $\|g_n(t,x') - g_n(t,x'')\| \leq \gamma(t) k_x \|x' - x''\|$, where $\gamma(\cdot)$ is the function from condition $(G3)$.*

Theorem 4.12. *(see* [14]*) Under conditions (A), $(G1)$, $(G2)$, $(G4)$, (Q), $\Sigma^G_{x_0,x_1}[0,T]$ is an R_δ-set in the space $C([0,T];E)$.*

Proof. From items $(Gi), (Gii)$, and (Giv) of Lemma 4.11, it follows that each multimap G_n satisfies conditions $(G1)-(G4)$, and hence every set $\Sigma^{G_n}_{x_0,x_1}[0,T], n \geq 1$, is non-empty and compact. Let us show that every set $\Sigma^{G_n}_{x_0,x_1}[0,T], n \geq 1$, is contractible. In fact, take an arbitrary $x_n \in \Sigma^{G_n}_{x_0,x_1}[0,T]$ and let $f_n \in \mathscr{P}^{\infty}_{G_n}(x_n)$ be such that

$$x_n(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1} \mathscr{T}(t-s) f_n(s)\, ds.$$

For a given $0 \leq \lambda \leq 1$, define an operator $\mathscr{B}^\lambda_n : C([0,T];E) \to C([0,T];E)$ as

$$\mathscr{B}^\lambda_n z(t) = \begin{cases} \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1} \mathscr{T}(t-s) f_n(s)\, ds, & 0 \leq t \leq \lambda T; \\ x_n(\lambda T) + \int_{\lambda T}^t (t-s)^{q-1} \mathscr{T}(t-s) g_n(s,z(s))\, ds, & \lambda T \leq t \leq T. \end{cases}$$

Notice that $\mathscr{B}^\lambda_n$ transforms the set X into itself. Indeed, taking $z(\cdot) \in X$, we see that $g_n(t,z(t))$ is a selection of $G_n(t,z(t)) \subset \overline{co}G(t,X(t))$. Hence,

$$\phi_n(t) = \left\{ \begin{array}{ll} f_n(t), & 0 \leq t \leq \lambda T; \\ g_n(t,z(t)), & \lambda T \leq t \leq T. \end{array} \right\}$$

is also a selection of $\overline{co}G(t,X(t))$. This means that the value $\mathscr{B}^\lambda_n z$ which may be defined through ϕ_n by the formula

$$\mathscr{B}^\lambda_n z(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1} \mathscr{T}(t-s) \phi_n(s)\, ds,$$

lays in X (see (4.6)). Since the set X is compact, $\mathscr{X} \subset E$, $\mathscr{X} = \cup_{t\in[0,T]} X(t)$ is also compact and hence we may assume that the function $g_n(t,x)$ is Lipshitz for a.e. $t \in [0,T]$ in the second argument on $\mathscr{X}$ with the coefficient $\gamma(t)k_n > 0$. Introduce in the space $C([0,T];E)$ the equivalent norm

$$\|x\|^* = \sup_{t\in[0,T]} e^{-\xi t} \|x(t)\|,$$

where a constant $\xi > 0$ is chosen so that

$$L := \frac{Mk_n\|\gamma\|_\infty}{\Gamma(1+2q)} \sup_{t\in[0,T]} \left(\int_0^t (t-s)^{2q-1} e^{-\xi(t-s)} ds \right) < 1.$$

Now, let us show that $\mathscr{B}_n^\lambda$ is contractive on X with respect to the norm $\|\cdot\|^*$. By using estimate (3.1) and item (iv) of Lemma 4.11, we have for arbitrary $z, \bar{z}$ from X and $t \in [0,T]$:

$$\begin{aligned} e^{-\xi t} \left\| \mathscr{B}_n^\lambda z(t) - \mathscr{B}_n^\lambda \bar{z}(t) \right\|_E &\leq e^{-\xi t} \left\| \int_{\lambda T}^t (t-s)^{q-1} \mathscr{T}(t-s)(g_n(s,z(s)) - g_n(s,z'(s)))ds \right\|_E \\ &\leq \frac{M}{\Gamma(2q)} e^{-\xi t} \int_{\lambda T}^t (t-s)^{2q-1} \|g_n(s,z(s)) - g_n(s,\bar{z}(s))\|_E \, ds \\ &\leq \frac{M}{\Gamma(2q)} e^{-\xi t} \int_{\lambda T}^t (t-s)^{2q-1} \|\gamma\|_\infty k_n e^{\xi s} e^{-\xi s} \|z(s) - \bar{z}(s)\|_E \, ds \\ &\leq \frac{Mk_n\|\gamma\|_\infty}{\Gamma(1+2q)} \|z-\bar{z}\|^* \sup_{t\in[0,T]} \left(\int_0^t (t-s)^{2q-1} e^{-\xi(t-s)} ds \right) < L\|z-\bar{z}\|^*. \end{aligned}$$

Hence $\mathscr{B}_n^\lambda$ has a unique fixed point $z_n^\lambda \in X$ which obviously belongs to $\Sigma_{x_0,x_1}^{G_n}[0,T]$. Since $\mathscr{B}_n^\lambda$ continuously depends on x_n and λ, function z_n^λ also continuously depends on x_n and λ. Therefore the map $h : [0,1] \times \Sigma_{x_0,x_1}^{G_n}[0,T] \to \Sigma_{x_0,x_1}^{G_n}[0,T]$ defined by $h(\lambda, x_n) = z_n^\lambda$ is continuous. Notice that $z_n^1 = x_n$, i.e., h is the homotopy of the identity map $h(1,\cdot)$ of the set $\Sigma_{x_0,x_1}^{G_n}[0,T]$ to the constant map $h(0,x_n) \equiv z_n^0$. This means that $\Sigma_{x_0,x_1}^{G_n}[0,T]$ is contractible.

Now, let us show that

$$\Sigma_{x_0,x_1}^{G}[0,T] = \cap_{n\geq 1} \Sigma_{x_0,x_1}^{G_n}[0,T]. \tag{4.16}$$

From Lemma 4.10, it follows that it is sufficient to show that

$$\Sigma_{x_0,x_1}^{\tilde{G}}[0,T] = \cap_{n\geq 1} \Sigma_{x_0,x_1}^{G_n}[0,T]. \tag{4.17}$$

It is clear that $\Sigma_{x_0,x_1}^{\tilde{G}}[0,T] \subset \cap_{n\geq 1} \Sigma_{x_0,x_1}^{G_n}[0,T]$. Take $x \in \cap_{n\geq 1} \Sigma_{x_0,x_1}^{G_n}[0,T]$, then, for every $n \geq 1$ we have

$$x(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1} \mathscr{T}(t-s) f_n(s) ds, f_n \in \mathscr{P}_{G_n}^\infty(x). \tag{4.18}$$

Since the sequence $\{f_n\}$ is L^∞-semicompact, by Lemma 3.4, we can assume, without loss of generality, that $f_n \rightharpoonup f_0$ in $L^1((0,T);E)$. Then, by the Mazur lemma (see, e.g., [4]) there exists a double sequence of non-negative numbers $\{\lambda_{ik}\}_{i=1}^\infty{}_{k=1}^\infty$ such that, for each $i \geq 1$, $\sum_{k=i}^\infty \lambda_{ik} = 1$, $\lambda_{ik} = 0$ for all $k \geq k_0(i)$ and the sequence of convex combinations $\overline{f}_i = \sum_{k=i}^\infty \lambda_{ik} f_k$, converges in $L^1((0,T);E)$ to a function f_0. From the sequence $\overline{f}_i$, we may select a subsequence convergent to f_0 for a.e. $t \in [0,T]$. Since for each $\overline{f}_i$ we have the estimate

$$\|\overline{f}_i(t)\| \leq v(t) \quad \text{a.e.} \quad t \in [0,T],$$

we have the same estimate for the limit function f_0 and hence it belongs to $L^\infty((0,T);E)$. Since the sets $G_i(t,x)$ are convex and compact, taking into account inclusions (Gii) of Lemma 4.11, we get, for each $i \geq 1$, $\overline{f}_i(t) \in G_i(t,x(t))$ a.e. $t \in [0,T]$, and hence $\overline{f}_i(t) \in G_n(t,x(t))$ a.e. $t \in [0,T]$, for all $i \geq n$. For the limiting function f_0, we have

$$f_0(t) \in \cap_{n\geq 1} G_n(t,x(t)) = \tilde{G}(t,x(t)) \quad \text{a.e.} \quad t \in [0,T],$$

i.e., $f_0 \in \mathscr{P}^{\infty}_{\widehat{F}}[0,T]$. By [13, Lemma 4], we have $Sf_n \rightharpoonup Sf_0$ in $C([0,T];E)$. By Lemma 3.4, we have $Sf_n \to Sf_0$ in $C([0,T];E)$. Therefore, we may pass to the limit in the equality (4.18) that yields

$$x(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)f_0(s)ds,$$

which means $x \in \Sigma^{\tilde{G}}_{x_0,x_1}[0,T]$, implying (4.17) and hence (4.16). □

Moreover, set $\Sigma^{\tilde{G}}_{x_0,x_1}$ is connected. Problem (4.14) - (4.15) with u.s.c. nonlinear part is said to be associated with Cauchy problem (1.1) - (1.2).

Theorem 4.13. *Under conditions* (A), $(F1)$, $(F2_L)$, $(F3_L)$*, the set* $\Sigma^F_{x_0,x_1}$ *of all mild solutions to problem* (1.1) - (1.2) *is connected. In particular, each set* $P(t) = \{x(t) : x \in \Sigma^F_{x_0,x_1}, t \in [0,T]\}$ *is also connected.*

Proof. In fact, let $x^1, x^2 \in \Sigma^F_{x_0}$ be mild solutions. They have the form

$$x^i(t) = \mathscr{G}(t)x_0 + \mathscr{K}(t)x_1 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)\phi^i(s)ds,$$

where $\phi^i \in \mathscr{P}_F(x^i)$, $i = 1,2$. Consider the multimaps $F^i(t,x) \subset F(t,x)$, $i = 1,2$ defined as

$$F^i(t,x) = \begin{cases} \{\phi^i(t)\}, & x \in x^i(t), \\ F(t,x), & x \notin x^i(t). \end{cases}$$

Since each function ϕ^i is measurable, then there exists a sequence of disjoint compact sets $\{I_k\}, I_k \subset [0,T]$ such that $meas([0,T] \setminus \cup_k I_k) = 0$ and the restriction of ϕ^i on each I_k is continuous. Hence each multimap F^i, $i = 1,2$ is a.l.s.c. and satisfies properties $(F2_L)$, $(F3_L)$. In accordance to Lemma 4.1, we construct a nonempty compact convex subset $X \subset C([0,T];E)$ containing mild solutions x^1, x^2 and invariant with respect to the action of Υ_F and, consequently Υ_{F^i}, $i = 1,2$. For each of semilinear inclusions with nonlinearities F^1, F^2, let us pass to the associated differential inclusions with u.s.c. nonlinear parts $\tilde{G}^i$, $i = 1,2$. From the construction ((4.8)), it follows that

$$\phi^i(t) = F^i(t,x^i(t)) = \gamma^i(t,x^i(t)) \in \tilde{G}^i(t,x^i(t)),$$

for a.e. $t \in [0,T]$ and $i = 1,2$. Hence each of x^i is the solution of the associated problem with $\tilde{G}^i$, $i = 1,2$.

Consider now the parametrized family of semilinear differential inclusions

$$^C D^q_0 x(t) \in Ax(t) + \tilde{G}_\lambda(t,x(t)), \quad t \in [0,T], \lambda \in [0,1], \tag{4.19}$$

$$x(0) = x_0, \quad x'(0) = x_1, \tag{4.20}$$

where the one-parameter family $\tilde{G}_\lambda$ is defined as

$$\tilde{G}_\lambda(t,x) = \begin{cases} \tilde{G}^1(t,x), & t \in [0,\lambda T]; \\ \overline{co}(\tilde{G}^1(t,x) \cup \tilde{G}^2(t,x)), & t = \lambda T; \\ \tilde{G}^2(t,x), & t \in [\lambda T, T]. \end{cases}$$

From [6, Theorem 1] and Theorem 4.12, it follows that, for every $\lambda \in [0,1]$, the set $\Sigma^{\tilde{G}_\lambda}_{x_0,x_1}$ of mild solutions of (4.19) - (4.20) is a nonempty compact connected subset of $C([0,T];E)$ and it is easy to see that $\Sigma^{\tilde{G}_\lambda}_{x_0,x_1} \subset \Sigma^F_{x_0,x_1}$ for all $\lambda \in [0,1]$. Moreover, family (4.19) - (4.20) satisfies conditions of the theorem on continuous dependence of the solutions set on a parameter (see [9, Theorem 2]) so the multimap

$$\lambda \multimap \Sigma_{x_0,x_1}^{\tilde{G}_\lambda}, \quad \lambda \in [0,1],$$

is u.s.c. and the set $\cup_{\lambda\in[0,1]}\Sigma_{x_0,x_1}^{\tilde{G}_\lambda}$ is connected (Proposition 2.7). It remains to observe only that $x^1 \in \Sigma_{x_0,x_1}^{\tilde{G}_1}$ and $x^2 \in \Sigma_{x_0,x_1}^{\tilde{G}_0}$. □

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