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WEIGHTED HYPERBOLIC CROSS POLYNOMIAL APPROXIMATION

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Dedicated to the 90th birthday of Professor Vladimir Tikhomirov

Abstract. We study linear polynomial approximation of functions in weighted Sobolev spaces $W^r_{p,w}(\mathbb{R}^d)$ of mixed smoothness $r \in \mathbb{N}$, and their optimality in terms of Kolmogorov and linear n-widths of the unit ball $\mathbf{W}^r_{p,w}(\mathbb{R}^d)$ in these spaces. The approximation error is measured by the norm of the weighted Lebesgue space $L_{q,w}(\mathbb{R}^d)$. The weight w is a tensor-product Freud weight. For $1 \le p,q \le \infty$ and $d = 1$, we prove that the polynomial approximation by de la Vallée Poussin sums of the orthonormal polynomial expansion of functions with respect to the weight w^2, is asymptotically optimal in terms of relevant linear n-widths $\lambda_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ and Kolmogorov n-widths $d_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ for $1 \le q \le p < \infty$. For $1 \le p,q \le \infty$ and $d \ge 2$, we construct linear methods of hyperbolic cross polynomial approximation based on tensor product of successive differences of dyadic-scaled de la Vallée Poussin sums, which are counterparts of hyperbolic cross trigonometric linear polynomial approximation, and give some upper bounds of the error of these approximations for various pair p,q with $1 \le p,q \le \infty$. For some particular weights w and $d \ge 2$, we prove the right convergence rate of $\lambda_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ and $d_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ which is performed by a constructive hyperbolic cross polynomial approximation.

Keywords. Convergence rate; Hyperbolic cross polynomial approximation; Kolmogorov widths; Linear widths; Weighted approximation; Weighted Sobolev space of mixed smoothness.

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1. Introduction

We investigate weighted linear hyperbolic cross polynomial approximations of functions on $\mathbb{R}^d$ from weighted Sobolev spaces of mixed smoothness and their optimalities in terms of Kolmogorov and linear n-widths.

We begin with a notion of weighted Sobolev spaces of mixed smoothness. Let

$$w(\mathbf{x}) := w_{\lambda,a,b}(\mathbf{x}) := \bigotimes_{i=1}^d w(x_i), \ \ \mathbf{x} \in \mathbb{R}^d,$$

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be the tensor product of d copies of a generating univariate Freud weight of the form

$$w(x) := \exp\left(-a|x|^\lambda + b\right), \ \ \lambda > 1, \ a > 0, \ b \in \mathbb{R}. \tag{1.1}$$

The most important parameter in the weight w is λ. The parameter b which produces only a positive constant in the weight w is introduced for a certain normalization for instance, for the standard Gaussian weight which is one of the most important weights. In what follows, we fix the parameters λ, a, b in the weight w.

Let $1 \le p \le \infty$ and Ω be a Lebesgue measurable set on $\mathbb{R}^d$. We denote by $L_{p,w}(\Omega)$ the weighted Lebesgue space of all measurable functions f on Ω such that the norm

$$\|f\|_{L_{p,w}(\Omega)} := \begin{cases} \left(\int_\Omega |f(\mathbf{x})w(\mathbf{x})|^p \mathrm{d}\mathbf{x} \right)^{1/p}, & 1 \le p < \infty; \\ \operatorname{ess\,sup}_{\mathbf{x}\in\Omega} |f(\mathbf{x})w(\mathbf{x})|, & p = \infty, \end{cases} \tag{1.2}$$

is finite.

For $r \in \mathbb{N}$ and $1 \le p \le \infty$, we define the weighted Sobolev space $W^r_{p,w}(\Omega)$ of mixed smoothness r as the normed space of all functions $f \in L_{p,w}(\Omega)$ such that the weak partial derivative $D^{\mathbf{k}} f$ belongs to $L_{p,w}(\Omega)$ for every $\mathbf{k} \in \mathbb{N}^d_0$ satisfying the inequality $|\mathbf{k}|_\infty \le r$. The norm of a function f in this space is defined by

$$\|f\|_{W^r_{p,w}(\Omega)} := \left(\sum_{|\mathbf{k}|_\infty \le r} \|D^{\mathbf{k}} f\|^p_{L_{p,w}(\Omega)} \right)^{1/p}.$$

Let γ be the standard d-dimensional Gaussian measure with the density function

$$v_{\mathrm{g}}(\mathbf{x}) := (2\pi)^{-d/2} \exp(-|\mathbf{x}|_2^2/2).$$

The well-known spaces $L_p(\Omega;\gamma)$ and $W^r_p(\Omega;\gamma)$ which are used in many applications, are defined in the same way by replacing the norm (1.2) with the norm

$$\|f\|_{L_p(\Omega;\gamma)} := \left(\int_\Omega |f(\mathbf{x})|^p \gamma(\mathrm{d}\mathbf{x}) \right)^{1/p} = \left(\int_\Omega |f(\mathbf{x}) \left(v_{\mathrm{g}}\right)^{1/p} (\mathbf{x})|^p \mathrm{d}\mathbf{x} \right)^{1/p}.$$

Thus, the spaces $L_p(\Omega;\gamma)$ and $W^r_p(\Omega;\gamma)$ coincide with $L_{p,w}(\Omega)$ and $W^r_{p,w}(\Omega)$, where $w := \left(v_{\mathrm{g}}\right)^{1/p}$ for a fixed $1 \le p < \infty$.

Let X be a Banach space and F a central symmetric compact set in X. By linear approximation we understand an approximation of elements in F by elements from a fixed finite-dimensional subspace L. For a given number $n \in \mathbb{N}_0$, a natural question arising is how to choose an optimal subspace of dimension at most n for this approximation. This leads to the concept of the Kolmogorov n-width introduced in 1936 [7]. The Kolmogorov n-width of F is defined by

$$d_n(F,X) := \inf_{L_n} \sup_{f\in F} \inf_{g\in L_n} \|f-g\|_X,$$

where the left-most infimum is taken over all subspaces L_n of dimension $\le n$ in X.

The Kolmogorov n-width provides a way to determine optimal approximation n-dimensional subspaces. Clearly, we would like to use as simple approximation operators as possible. In particular, the restriction by linear operators leads to the linear n-width of F in X which was introduced by V.M. Tikhomirov [13] in 1960. This n-width is defined by

$$\lambda_n(F,X) := \inf_{A_n} \sup_{f\in F} \|f - A_n(f)\|_X,$$

where the infimum is taken over all linear operators A_n in X with $\operatorname{rank} A_n \le n$. In general, the Kolmogorov n-width and the linear n-width are different approximation characterizations. However, if X is a Hilbert space, then $\lambda_n(F,X) = d_n(F,X)$. In what follows, for a normed space X of functions on Ω, the boldface $\mathbf{X}$ denotes the unit ball in X.

There is a large number of works devoted to the problem of (unweighted) linear hyperbolic cross approximations of functions having a mixed smoothness on a compact domain, and their optimalities in of terms Kolmogorov and linear n-widths, see for survey and bibliography in [4, 10, 12]. Here by linear hyperbolic cross approximation we understand approximation of multivariate periodic functions by trigonometric polynomials with frequencies from so-called hyperbolic crosses, or their counterpart for multivariate non-periodic functions.

The weighted polynomial approximation is a classical branch of approximation theory. There is a huge body of works on different aspects of the univariate weighted polynomial approximation. We refer the reader to the books [6, 8, 9] for relevant results and bibliography. In the recent paper [3], we have studied the linear approximation of functions from $W^r_p(\mathbb{R}^d;\gamma)$ with the error measured in $L_q(\mathbb{R}^d;\gamma)$ for $1 \le q \le p \le \infty$. In particular, we proved in the last paper the right convergence rate

$$\lambda_n(\mathbf{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) = d_n(\mathbf{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)) \asymp n^{-r/2}(\log n)^{r(d-1)/2}. \tag{1.3}$$

In the present paper, we investigate linear hyperbolic cross polynomial approximation of functions with a mixed smoothness on $\mathbb{R}^d$. Functions to be approximated are in weighted Sobolev spaces $W^r_{p,w}(\mathbb{R}^d)$. The approximation error is measured by the norm of the weighted Lebesgue spaces $L_{q,w}(\mathbb{R}^d)$. The values of p,q may vary satisfying the inequalities $1 \le p,q \le \infty$. The results on this approximation will imply upper bounds of the high dimensional linear n-widths $\lambda_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)\big)$ and Kolmogorov n-widths $d_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)\big)$ ($d \ge 2$), the right convergence rate of these linear n-widths in one-dimensional case ($d = 1$). We also study the right convergence rate of $\lambda_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ and $d_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ for particular weights w.

We briefly describe the main results of the present paper. Throughout the present paper, for given $1 \le p,q \le \infty$ and the parameter $\lambda > 1$ in the definition (1.1) of the generating weight w, we make use of the notations

$$r_\lambda := (1 - 1/\lambda)r; \tag{1.4}$$

$$\delta_{\lambda,p,q} := \begin{cases} (1-1/\lambda)(1/p - 1/q) & \text{if } p \le q, \\ (1/\lambda)(1/q - 1/p) & \text{if } p > q; \end{cases}$$

and

$$r_{\lambda,p,q} := r_\lambda - \delta_{\lambda,p,q}.$$

We also use the abbreviations:

$$d_n := d_n(\mathbf{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)), \ \ \lambda_n := \lambda_n(\mathbf{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)).$$

Let $1 \le p,q \le \infty$, $r_{\lambda,p,q} > 0$ and $\mathscr{V}_\xi$ be the de la Vallée Poussin hyperbolic cross operator (see (3.5) for the definition). Then we prove that, for $\xi > 1$,

$$\|f - \mathscr{V}_\xi f\|_{L_{q,w}(\mathbb{R}^d)} \ll \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \begin{cases} 2^{-r_\lambda \xi} \xi^{d-1} & \text{if } p = q, \\ 2^{-r_{\lambda,p,q}\xi} \xi^{(d-1)/q} & \text{if } p \ne q < \infty, \\ 2^{-r_{\lambda,p,q}\xi} \xi^{(d-1)} & \text{if } q = \infty, \end{cases} \quad \xi > 1, \ f \in W^r_{p,w}(\mathbb{R}^d).$$

If ξ_n is the largest number such that $\operatorname{rank}\big(\mathscr{V}_{\xi_n}\big) \le n$ for $n \in \mathbb{N}$, as a consequence, we have that for $n \ge 2$,

$$d_n \le \lambda_n \le \sup_{f \in \mathbf{W}^r_{p,w}(\mathbb{R}^d)} \|f - \mathscr{V}_{\xi_n} f\|_{L_{q,w}(\mathbb{R}^d)} \ll \begin{cases} n^{-r_\lambda} (\log n)^{(r_\lambda + 1)(d-1)} & \text{if } p = q, \\ n^{-r_{\lambda,p,q}} (\log n)^{(r_{\lambda,p,q} + 1/q)(d-1)} & \text{if } p \ne q < \infty, \\ n^{-r_{\lambda,p,q}} (\log n)^{(r_{\lambda,p,q} + 1)(d-1)} & \text{if } q = \infty. \end{cases} \tag{1.5}$$

In the one-dimensional case when $d = 1$, for $1 \le q \le p < \infty$ we prove the right convergence rate

$$d_n \asymp \lambda_n \asymp \sup_{f \in \mathbf{W}^r_{p,w}(\mathbb{R})} \|f - V_n f\|_{L_{q,w}(\mathbb{R}^d)} \asymp n^{-r_{\lambda,p,q}},$$

where $V_n f$ is the nth de la Vallée Poussin sum of the orthonormal polynomial expansion of f with respect to the multivariate weight w^2.

The linear polynomial approximation method $\mathscr{V}_{\xi_n}$ performing the upper bounds (1.5) – a counterpart of hyperbolic cross trigonometric approximation method – is based on tensor product of successive differences of dyadic-scaled de la Vallée Poussin sums of the orthonormal polynomial expansion of f with respect to the multivariate weight w^2.

For $\lambda = 2, 4$, we prove the right convergence rate for $n \ge 2$,

$$\lambda_n(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)) = d_n(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)) \asymp n^{-r_\lambda} (\log n)^{r_\lambda (d-1)},$$

which is a generalization of (1.3).

The paper is organized as follows. In Section 2, we study linear polynomial approximations in the norm $L_{q,w}(\mathbb{R})$ of univariate functions from $\mathbf{W}^r_{p,w}(\mathbb{R})$ by de la Vallée Poussin and Fourier sums of the orthonormal polynomial expansion of functions with respect to the univariate weight w^2. We give some upper bounds of the error of these approximations for $1 \le p,q \le \infty$, and prove their asymptotic optimality in terms of linear n-widths $\lambda_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ and Kolmogorov n-widths $d_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ for $1 \le q \le p < \infty$. In Section 3, we study linear approximations of multivariate functions $f \in \mathbf{W}^r_{p,w}(\mathbb{R}^d)$. We construct linear methods of hyperbolic cross polynomial approximation. We give some upper bounds of the error of these approximations for various pair p,q with $1 \le p,q \le \infty$. In Section 4, for the particular weights w with $\lambda = 2, 4$, we prove the right convergence rate of n-widths $\lambda_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ and $d_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$.

Notation. Denote $\mathbf{x} =: (x_1, \dots, x_d)$ for $\mathbf{x} \in \mathbb{R}^d$; $|\mathbf{x}|_p := \Big(\sum_{j=1}^d |x_j|^p\Big)^{1/p}$ $(1 \le p < \infty)$ and $|\mathbf{x}|_\infty := \max_{1 \le j \le d} |x_j|$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, the inequality $\mathbf{x} \le \mathbf{y}$ ($\mathbf{x} < \mathbf{y}$) means $x_i \le y_i$ ($x_i < y_i$) for every $i = 1, \dots, d$. We use letters C and K to denote general positive constants which may take different values. For the quantities $A_n(f, \mathbf{k})$ and $B_n(f, \mathbf{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\mathbf{k} \in \mathbb{Z}^d$, we write $A_n(f, \mathbf{k}) \ll B_n(f, \mathbf{k})$, $f \in W$, $\mathbf{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant

$C > 0$ independent of $n, f, \mathbf{k}$ such that $A_n(f,\mathbf{k}) \le CB_n(f,\mathbf{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\mathbf{k} \in \mathbb{Z}^d$ (the notation $A_n(f,\mathbf{k}) \gg B_n(f,\mathbf{k})$ has the obvious opposite meaning), and $A_n(f,\mathbf{k}) \asymp B_n(f,\mathbf{k})$ if $A_n(f,\mathbf{k}) \ll B_n(f,\mathbf{k})$ and $B_n(f,\mathbf{k}) \ll A_n(f,\mathbf{k})$. Denote by $|G|$ the cardinality of the set G. For a Banach space X, denote by the boldface $\mathbf{X}$ the unit ball in X.

2. Approximation by de la Vallée Poussin sums

In this section, we study linear approximations of univariate functions $f \in W^r_{p,w}(\mathbb{R})$ by de la Vallée Poussin and Fourier sums of the orthonormal polynomial expansion with respect to the univariate weight w^2. The approximation error is measured in the norm of $L_{q,w}(\mathbb{R})$. We give some upper bounds of the error of these approximations and prove their asymptotic optimality in terms of linear n-widths $\lambda_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ and Kolmogorov n-widths $d_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ for $1 \le q \le p < \infty$.

Let $(p_k)_{k\in\mathbb{N}_0}$ be the sequence of orthonormal polynomials with respect to the univariate weight

$$w^2(x) = \exp\Big(-2a|x|^\lambda + 2b\Big). \tag{2.1}$$

The polynomials $(p_k)_{k\in\mathbb{N}_0}$ constitute an orthonormal basis of the Hilbert space $L_{2,w}(\mathbb{R})$, and every $f \in L_{2,w}(\mathbb{R})$ can be represented by the polynomial series

$$f = \sum_{k\in\mathbb{N}_0} \hat{f}(k) p_k \ \text{ with } \ \hat{f}(k) := \int_{\mathbb{R}} f(x)\, p_k(x) w^2(x) \mathrm{d}x$$

converging in the norm of $L_{2,w}(\mathbb{R})$. Moreover, there holds Parseval's identity

$$\|f\|^2_{L_{2,w}(\mathbb{R})} = \sum_{k\in\mathbb{N}_0} |\hat{f}(k)|^2.$$

Since every polynomial belongs to the space $L_{q,w}(\mathbb{R})$ with $1 \le q \le \infty$, we can define for any $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $f \in L_{p,w}(\mathbb{R})$ the kth Fourier coefficient

$$\hat{f}(k) := \int_{\mathbb{R}} f(x)\, p_k(x) w^2(x) \mathrm{d}x;$$

the mth Fourier sum

$$S_m f := \sum_{k=0}^{m-1} \hat{f}(k) p_k;$$

and the mth de la Vallée Poussin sum

$$V_m := \frac{1}{m} \sum_{k=m+1}^{2m} S_k.$$

Let $\mathscr{P}_m$ denote the space of polynomials of degree at most m. From the definition we have the following properties of the operator V_m for $1 \le p \le \infty$ and $m \in \mathbb{N}$, $V_m f \in \mathscr{P}_{2m-1}$, $f \in L_{p,w}(\mathbb{R})$, and $V_m \varphi = \varphi$, $\varphi \in \mathscr{P}_m$. For $m \in \mathbb{N}$, let q_m be the Freud number defined by

$$q_m := (m/a\lambda)^{1/\lambda} \asymp m^{1/\lambda},$$

and a_m the Mhaskar-Rakhmanov-Saff number defined by

$$a_m := (\nu_\lambda m)^{1/\lambda} \asymp m^{1/\lambda}, \ \ \nu_\lambda := \frac{2^{\lambda-1}\Gamma(\lambda/2)^2}{\Gamma(\lambda)},$$

and Γ is the gamma function. From the definitions, one can see that

$$q_m \asymp a_m \asymp m^{1/\lambda}. \tag{2.2}$$

The numbers q_m and a_m are relevant to convergence rates of weighted polynomial approximation (see, e.g., [8, 9]). For $1 \le p \le \infty$ and $f \in L_{p,w}(\mathbb{R})$, we define

$$E_m(f)_{p,w} := \inf_{\varphi \in \mathscr{P}_m} \|f - \varphi\|_{L_{p,w}(\mathbb{R})}$$

as the quantity of best approximation of f by polynomials of degree at most m. Then there holds the inequalities for $1 \le p \le \infty$ [9, Proposition 4.1.2, Lemma 4.1.5]

$$\|V_m f\|_{L_{p,w}(\mathbb{R})} \ll \|f\|_{L_{p,w}(\mathbb{R})}, \ \ f \in L_{p,w}(\mathbb{R}), \tag{2.3}$$

$$E_{2m}(f)_{p,w} \le \|f - V_m f\|_{L_{p,w}(\mathbb{R})} \ll E_m(f)_{p,w}, \ \ f \in L_{p,w}(\mathbb{R}), \tag{2.4}$$

and [9, Theorem 4.1.1] taking account (2.2)

$$E_m(f)_{p,w} \le m^{-r_\lambda} \|f\|_{W^r_{p,w}(\mathbb{R})}, \ \ f \in W^r_{p,w}(\mathbb{R}). \tag{2.5}$$

From (2.2), (2.4), and (2.5), it follows that if $1 \le p \le \infty$, then, for every $m \in \mathbb{N}$,

$$\|f - V_m f\|_{L_{p,w}(\mathbb{R})} \le C m^{-r_\lambda} \|f\|_{W^r_{p,w}(\mathbb{R})}, \ \ f \in W^r_{p,w}(\mathbb{R}). \tag{2.6}$$

For the operators S_m, we have [5]

$$\|S_m f\|_{L_{p,w}(\mathbb{R})} \ll \|f\|_{L_{p,w}(\mathbb{R})}, \ \ f \in L_{p,w}(\mathbb{R}), \ \ \text{if and only if} \ \ 4/3 < p < 4, \tag{2.7}$$

or, equivalently,

$$\|f - S_m f\|_{L_{p,w}(\mathbb{R})} \ll E_m(f)_{p,w}, \ \ f \in L_{p,w}(\mathbb{R}), \ \ \text{if and only if} \ \ 4/3 < p < 4. \tag{2.8}$$

For proofs of the following lemmata see [9, Theorem 3.4.2, Theorem 4.2.4], providing (2.2).

Lemma 2.1. *Let $1 \le p, q \le \infty$. Then we have the following.*

(i) *There holds the Markov-Bernstein-type inequality*

$$\|\varphi'\|_{L_{p,w}(\mathbb{R})} \ll m^{1-1/\lambda} \|\varphi\|_{L_{p,w}(\mathbb{R})} \quad \forall \varphi \in \mathscr{P}_m, \ \ \forall m \in \mathbb{N}.$$

(ii) *For $1 \le p < q \le \infty$, there holds the Nikol'skii-type inequality*

$$\|\varphi\|_{L_{q,w}(\mathbb{R})} \ll m^{(1-1/\lambda)(1/p-1/q)} \|\varphi\|_{L_{p,w}(\mathbb{R})} \quad \forall \varphi \in \mathscr{P}_m, \ \ \forall m \in \mathbb{N}.$$

(iii) *For $1 \le q < p \le \infty$, there holds the Nikol'skii-type inequality*

$$\|\varphi\|_{L_{q,w}(\mathbb{R})} \ll m^{(1/\lambda)(1/q-1/p)} \|\varphi\|_{L_{p,w}(\mathbb{R})}, \ \ \forall \varphi \in \mathscr{P}_m, \ \forall m \in \mathbb{N}.$$

We define the one-dimensional operators for $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$

$$v_{m,k} := V_{m2^k} - V_{m2^{k-1}}, \ k > 0, \ \ v_{m,0} := V_m, \tag{2.9}$$

and

$$s_{m,k} := S_{m2^k} - S_{m2^{k-1}}, \ k > 0, \ \ s_{m,0} := S_m.$$

We also use the abbreviations: $v_k := v_{1,k}$ and $s_k := s_{1,k}$.

Lemma 2.2. *Let $1 \le p,q \le \infty$, $r_{\lambda,p,q} > 0$ and $m \in \mathbb{N}$. Then we have that for every $f \in W^r_{p,w}(\mathbb{R})$, there hold the series representation*

$$f = \sum_{k \in \mathbb{N}_0} v_{m,k} f \tag{2.10}$$

with absolute convergence in the space $L_{q,w}(\mathbb{R})$ of the series, and the norm estimates

$$\left\| v_{m,k} f \right\|_{L_{q,w}(\mathbb{R})} \ll (m2^k)^{-r_{\lambda,p,q}} \|f\|_{W^r_{p,w}(\mathbb{R})}, \ \ f \in W^r_{p,w}(\mathbb{R}), \ \ m \in \mathbb{N}, \ k \in \mathbb{N}_0. \tag{2.11}$$

Proof. Let $f \in W^r_{p,w}(\mathbb{R})$. Since $v_{m,k} f \in \mathscr{P}_{m2^{k+1}-1}$ by the claims (iii) and (iv) of Lemma 2.1 we have that

$$\left\| v_{m,k} f \right\|_{L_{q,w}(\mathbb{R})} \ll (m2^k)^{\delta_{\lambda,p,q}} \left\| v_{m,k} f \right\|_{L_{p,w}(\mathbb{R})}, \ \ m \in \mathbb{N}, \ k \in \mathbb{N}_0. \tag{2.12}$$

By Lemma 2.2 we have that for every $f \in W^r_{p,w}(\mathbb{R})$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} \left\| v_{m,k} f \right\|_{L_{p,w}(\mathbb{R})} &\le \left\| f - V_{m2^k} \right\|_{L_{p,w}(\mathbb{R})} + \left\| f - V_{m2^{k-1}} \right\|_{L_{p,w}(\mathbb{R})} \\ &\ll (m2^k)^{-r_\lambda k} \|f\|_{W^r_{p,w}(\mathbb{R})}, \end{aligned}$$

which together with (2.12) proves (2.14) and hence the absolute convergence of the series in (2.10) follows. The equality in (2.10) is implied from (2.6) and the equality

$$V_{m2^k} = \sum_{s \le k} v_{m,s}.$$

$\square$

Theorem 2.3. *Let $1 \le p,q \le \infty$ and $r_{\lambda,p,q} > 0$. Then*

$$\sup_{f \in \mathbf{W}^r_{p,w}(\mathbb{R})} \left\| f - V_n f \right\|_{L_{q,w}(\mathbb{R})} \ll n^{-r_{\lambda,p,q}}.$$

Proof. By using Lemma 2.2, we derive, for every $f \in \mathbf{W}^r_{p,w}(\mathbb{R})$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} \left\| f - V_n f \right\|_{L_{q,w}(\mathbb{R})} &= \left\| \sum_{k \in \mathbb{N}} v_{n,k} f \right\|_{L_{q,w}(\mathbb{R})} \\ &\le \sum_{k \in \mathbb{N}} \left\| v_{n,k} f \right\|_{L_{q,w}(\mathbb{R})} \ll \sum_{k \in \mathbb{N}} (n2^k)^{-r_{\lambda,p,q}} \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \\ &\le n^{-r_{\lambda,p,q}} \sum_{k \in \mathbb{N}} 2^{-r_{\lambda,p,q} k} \asymp n^{-r_{\lambda,p,q}}. \end{aligned}$$

$\square$

Corollary 2.4. *Let $1 \le q \le p < \infty$ and $r_{\lambda,p,q} > 0$. Then*

$$\lambda_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \asymp d_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \asymp n^{-r_{\lambda,p,q}}. \tag{2.13}$$

Proof. The upper bound of (2.13) can be easily derived from Theorem 2.3. The lower bound was proven in [2, (2.32)]. $\square$

Similarly, from (2.8) and (2.5), we deduce the following results for the approximation by Fourier sums.

Lemma 2.5. *Let $4/3 < p < 4$, $1 \le q \le \infty$, $r_{\lambda,p,q} > 0$ and $m \in \mathbb{N}$. Then, for every $f \in W^r_{p,w}(\mathbb{R})$, there hold the series representation $f = \sum_{k \in \mathbb{N}_0} s_{m,k} f$ with absolute convergence in the space $L_{q,w}(\mathbb{R})$ of the series, and the norm estimates*

$$\left\| s_{m,k} f \right\|_{L_{q,w}(\mathbb{R})} \le C (m 2^k)^{-r_{\lambda,p,q}} \|f\|_{W^r_{p,w}(\mathbb{R})}, \;\; m \in \mathbb{N}, \; k \in \mathbb{N}_0. \tag{2.14}$$

Theorem 2.6. *Let $4/3 < p < 4$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$. Then*

$$\sup_{f \in \mathbf{W}^r_{p,w}(\mathbb{R})} \left\| f - S_n f \right\|_{L_{q,w}(\mathbb{R})} \ll n^{-r_{\lambda,p,q}}.$$

3. Hyperbolic Cross Polynomial Approximation

In this section, we consider weighted hyperbolic cross linear polynomial approximations of multivariate functions $f \in \mathbf{W}^r_{p,w}(\mathbb{R}^d)$. The approximation error is measured in the norm of $L_{q,w}(\mathbb{R}^d)$. We construct linear methods of polynomial approximation which are counterparts of linear hyperbolic cross trigonometric approximation for periodic multivariate functions. We give some upper bounds of the error of these approximations and of linear n-widths $\lambda_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)\big)$ and Kolmogorov n-widths $d_n\big(\mathbf{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)\big)$ for various pair p,q with $1 \le p,q \le \infty$. For the weights w with $\lambda = 2,4$, we establish the right convergence rate of n-widths $\lambda_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ and $d_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$.

Recall that $(p_m)_{m \in \mathbb{N}_0}$ is the sequence of orthonormal polynomials with respect to the univariate Freud-type weight w^2 as in (2.1). For every multi-index $\mathbf{k} \in \mathbb{N}^d_0$, the d-variate polynomial $p_{\mathbf{k}}$, we define

$$p_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^d p_{k_j}(x_j), \;\; \mathbf{x} \in \mathbb{R}^d.$$

The polynomials $\{p_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}^d_0}$ constitute an orthonormal basis of the Hilbert space $L_{2,w}(\mathbb{R}^d)$, and every $f \in L_{2,w}(\mathbb{R}^d)$ can be represented by the polynomial series

$$f = \sum_{\mathbf{k} \in \mathbb{N}^d_0} \hat{f}(\mathbf{k}) p_{\mathbf{k}} \;\text{ with }\; \hat{f}(\mathbf{k}) := \int_{\mathbb{R}^d} f(\mathbf{x})\, p_{\mathbf{k}}(\mathbf{x}) w(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{3.1}$$

converging in the norm of $L_{2,w}(\mathbb{R}^d)$. Moreover, there holds Parseval's identity

$$\|f\|^2_{L_{2,w}(\mathbb{R}^d)} = \sum_{\mathbf{k} \in \mathbb{N}^d_0} |\hat{f}(\mathbf{k})|^2. \tag{3.2}$$

For $\mathbf{x} \in \mathbb{R}^d$ and $e \subset \{1,\ldots,d\}$, let $\mathbf{x}^e \in \mathbb{R}^{|e|}$ be defined by $(x^e)_i := x_i$, and $\bar{\mathbf{x}}^e \in \mathbb{R}^{d-|e|}$ by $(\bar{x}^e)_i := x_i$, $i \in \{1,\ldots,d\} \setminus e$. With an abuse we write $(\mathbf{x}^e, \bar{\mathbf{x}}^e) = \mathbf{x}$.

For the proof of the following lemma, see [1, Lemma 3.2].

Lemma 3.1. *Let $1 \le p \le \infty$, $e \subset \{1,\ldots,d\}$ and $\mathbf{r} \in \mathbb{N}^d_0$. Assume that f is a function on $\mathbb{R}^d$ such that for every $\mathbf{k} \le \mathbf{r}$, $D^{\mathbf{k}} f \in L_{p,w}(\mathbb{R}^d)$. Put for $\mathbf{k} \le \mathbf{r}$ and $\bar{\mathbf{x}}^e \in \mathbb{R}^{d-|e|}$, $g(\mathbf{x}^e) := D^{\bar{\mathbf{k}}^e} f(\mathbf{x}^e, \bar{\mathbf{x}}^e)$. Then $D^{\mathbf{s}} g \in L_{p,w}(\mathbb{R}^{|e|})$ for every $\mathbf{s} \le \mathbf{k}^e$ and almost every $\bar{\mathbf{x}}^e \in \mathbb{R}^{d-|e|}$.*

Based on the operators $v_k := v_{1,k}$ $k \in \mathbb{N}_0$, defined in (2.9), we construct approximation operators for functions in $L_{p,w}(\mathbb{R}^d)$ by using the well-known Smolyak algorithm. We define for $k \in \mathbb{N}_0$, the one-dimensional operators

$$E_k f := f - V_{2^k} f, \quad k \in \mathbb{N}_0.$$

For $\mathbf{k} \in \mathbb{N}^d$, the d-dimensional operators $V_{2^\mathbf{k}}$, $v_\mathbf{k}$ and $E_\mathbf{k}$ are defined as the tensor product of one-dimensional operators:

$$V_{2^\mathbf{k}} := \bigotimes_{i=1}^{d} V_{2^{k_i}}, \quad v_\mathbf{k} := \bigotimes_{i=1}^{d} v_{k_i}, \quad E_\mathbf{k} := \bigotimes_{i=1}^{d} E_{k_i},$$

where $2^\mathbf{k} := (2^{k_1}, \cdots, 2^{k_d})$ and the univariate operators $V_{2^{k_j}}$, v_{k_j} and E_{k_j} are successively applied to the univariate functions $\bigotimes_{i<j} V_{2^{k_i}}(f)$, $\bigotimes_{i<j} v_{k_i}(f)$ and $\bigotimes_{i<j} E_{k_i}$, respectively, by considering them as functions of variable x_j with the other variables held fixed. The operators $V_{2^\mathbf{k}}$, $v_\mathbf{k}$ and $E_\mathbf{k}$ are well-defined for functions from $L_{p,w}(\mathbb{R}^d)$ for $1 \le p \le \infty$.

Observe that

$$v_\mathbf{k} f = \sum_{e \subset \{1,\dots,d\}} (-1)^{d-|e|} V_{2^{\mathbf{k}(e)}} f,$$

where $\mathbf{k}(e) \in \mathbb{N}_0^d$ is defined by $k(e)_i = k_i$, $i \in e$, and $k(e)_i = \max(k_i - 1, 0)$, $i \notin e$. We also have

$$\big(E_\mathbf{k} f\big)(\mathbf{x}) = \sum_{e \subset \{1,\dots,d\}} (-1)^{|e|} \big(V_{2^{\mathbf{k}^e}} f(\cdot, \bar{\mathbf{x}}^e)\big)(\mathbf{x}^e).$$

Lemma 3.2. *Let $1 \le p, q \le \infty$ and $r_{\lambda,p,q} > 0$. Then we have that*

$$\big\|E_\mathbf{k} f\big\|_{L_{q,w}(\mathbb{R}^d)} \le 2^{-r_{\lambda,p,q}|\mathbf{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)}, \quad \mathbf{k} \in \mathbb{N}_0^d, \ \ f \in W^r_{p,w}(\mathbb{R}^d).$$

Proof. The case $d = 1$ of the lemma follows from Theorem 2.3. For simplicity we prove the lemma for the case $d = 2$ and $q < \infty$. The general case can be proven in the same way by induction on d. Indeed, by applying successively the case $d = 1$ of the lemma with respect to variables x_2 and x_1 we obtain

$$\begin{aligned}
\big\|E_{(k_1,k_2)} f\big\|^q_{L_{q,w}(\mathbb{R}^2)} &= \int_\mathbb{R} \int_\mathbb{R} \big|E_{k_2}(E_{k_1} f(x_1,x_2))\big|^q w(\mathbf{x})^q \mathrm{d}x_2 \mathrm{d}x_1 \\
&\le 2^{-qr_{\lambda,p,q}k_2} \int_\mathbb{R} \sum_{s_2=0}^{r} \int_\mathbb{R} \big|D^{(0,s_2)}(E_{k_1} f(x_1,x_2))\big|^q w(\mathbf{x})^q \mathrm{d}x_2 \mathrm{d}x_1 \\
&= 2^{-qr_{\lambda,p,q}k_2} \int_\mathbb{R} \sum_{s_2=0}^{r} \int_\mathbb{R} \big|(E_{k_1} D^{(0,s_2)} f(x_1,x_2))\big|^q w(\mathbf{x})^q \mathrm{d}x_2 \mathrm{d}x_1 \\
&\le 2^{-qr_{\lambda,p,q}k_2} \int_\mathbb{R} \sum_{s_2=0}^{r} \int_\mathbb{R} 2^{-qr_{\lambda,p,q}k_1} \sum_{s_1=0}^{r} \int_\mathbb{R} \big|D^{(s_1,s_2)} f(x_1,x_2)\big|^q w(\mathbf{x})^q \mathrm{d}x_1 \mathrm{d}x_2 \\
&= 2^{-qr_{\lambda,p,q}|\mathbf{k}|_1} \sum_{|\mathbf{s}|_\infty \le r} \int_{\mathbb{R}^2} \big|D^{(s)} f(\mathbf{x})\big|^q w(\mathbf{x})^q \mathrm{d}\mathbf{x} \\
&= 2^{-qr_{\lambda,p,q}|\mathbf{k}|_1} \|f\|^q_{W^r_{p,w}(\mathbb{R}^2)}.
\end{aligned}$$

□

We say that $\mathbf{k} \to \infty$, $\mathbf{k} \in \mathbb{N}_0^d$, if and only if $k_i \to \infty$ for every $i = 1, \dots, d$.

Lemma 3.3. *Let $1 \le p, q \le \infty$ and $r_{\lambda,p,q} > 0$. Then, for every $f \in W^r_{p,w}(\mathbb{R}^d)$,*

$$f = \sum_{\mathbf{k} \in \mathbb{N}_0^d} v_\mathbf{k} f \tag{3.3}$$

with absolute convergence in the space $L_{q,w}(\mathbb{R}^d)$ of the series, and

$$\left\|v_{\mathbf{k}} f\right\|_{L_{q,w}(\mathbb{R}^d)} \le C 2^{-r_{\lambda,p,q}|\mathbf{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)}, \ \ \mathbf{k} \in \mathbb{N}_0^d. \tag{3.4}$$

Proof. The operator $v_{\mathbf{k}}$ can be represented in the form

$$v_{\mathbf{k}} f = \sum_{e \subset \{1,\dots,d\}} (-1)^{|e|} E_{\mathbf{k}(e)} f.$$

Therefore, by using Lemma 3.2, we derive that, for every $f \in W^r_{p,w}(\mathbb{R}^d)$ and $\mathbf{k} \in \mathbb{N}_0^d$,

$$\begin{aligned}\left\|v_{\mathbf{k}} f\right\|_{L_{q,w}(\mathbb{R}^d)} &\le \sum_{e \subset \{1,\dots,d\}} \left\|E_{\mathbf{k}(e)} f\right\|_{L_{q,w}(\mathbb{R}^d)} \\ &\le \sum_{e \subset \{1,\dots,d\}} C 2^{-r_{\lambda,p,q}|\mathbf{k}(e)|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \le C 2^{-r_{\lambda,p,q}|\mathbf{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)},\end{aligned}$$

which proves (3.4) and hence the absolute convergence of the series in (3.8) follows. Notice that

$$f - V_{2^{\mathbf{k}}} f = \sum_{e \subset \{1,\dots,d\},\ e \ne \varnothing} (-1)^{|e|} E_{\mathbf{k}^e} f,$$

where recall $\mathbf{k}^e \in \mathbb{N}_0^d$ is defined by $k_i^e = k_i$, $i \in e$, and $k_i^e = 0$, $i \notin e$. By using Lemma 3.2 we derive for $\mathbf{k} \in \mathbb{N}_0^d$ and $f \in W^r_{p,w}(\mathbb{R}^d)$,

$$\begin{aligned}\left\|f - V_{2^{\mathbf{k}}} f\right\|_{L_{q,w}(\mathbb{R}^d)} &\le \sum_{e \subset \{1,\dots,d\},\ e \ne \varnothing} \left\|E_{\mathbf{k}^e} f\right\|_{L_{q,w}(\mathbb{R}^d)} \\ &\le C \max_{e \subset \{1,\dots,d\},\ e \ne \varnothing} \ \max_{1 \le i \le d} 2^{-r_{\lambda,p,q}|k_i^e|} \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \\ &\le C \max_{1 \le i \le d} 2^{-r_{\lambda,p,q}|k_i|} \|f\|_{W^r_{p,w}(\mathbb{R}^d)},\end{aligned}$$

which is going to 0 when $\mathbf{k} \to \infty$. This together $V_{2^{\mathbf{k}}} f = \sum_{\mathbf{k} \in \mathbb{N}_0^d : \mathbf{s} \le \mathbf{k}} v_{\mathbf{s}} f$ proves (3.8). ◻

For $\xi > 0$, we define the de la Vallée Poussin hyperbolic cross linear operator $\mathcal{V}_\xi$ for functions $f \in L_{q,w}(\mathbb{R}^d)$ by

$$\mathcal{V}_\xi f := \sum_{\mathbf{k} \in \mathbb{N}_0^d :\, |\mathbf{k}|_1 \le \xi} v_{\mathbf{k}} f. \tag{3.5}$$

Notice that $\mathcal{V}_\xi f$ belongs to $\mathcal{P}(\xi) := \operatorname{span}\{p_{\mathbf{s}} : \mathbf{s} \in H(\xi)\}$, where

$$H(\xi) := \bigcup_{\mathbf{k} \in \mathbb{N}_0^d :\, |\mathbf{k}|_1 \le \xi} \left\{\mathbf{s} \in \mathbb{N}_0^d : \mathbf{s} < 2.2^{2\mathbf{k}}\right\}.$$

From (2.3), it follows that $\mathcal{V}_\xi$ is a linear bounded operator in $L_{q,w}(\mathbb{R}^d)$ for $1 \le q \le \infty$, and

$$\operatorname{rank}\left(\mathcal{V}_\xi\right) = |H(\xi)| = \sum_{|\mathbf{k}|_1 \le \xi} \prod_{j=1}^d (2^{k_j+1} - 1) \asymp 2^\xi \xi^{d-1}. \tag{3.6}$$

The multi-index set $H(\xi)$ consists of the non-negative elements of the step hyperbolic cross

$$\tilde{H}(\xi) := \bigcup_{\mathbf{k} \in \mathbb{N}_0^d :\, |\mathbf{k}|_1 \le \xi} \left\{\mathbf{s} \in \mathbb{Z}^d : |s_i| < 2^{2k_i}, i = 1,\dots,d\right\},$$

which is similar by the form to the frequency set of trigonometric polynomials used in the classical hyperbolic cross approximation (see [4] for details). Hence with an abuse, we call an approximation by elements from subspaces $H(\xi)$ weighted hyperbolic cross polynomial approximation, and $\mathscr{V}_\xi f$ de la Vallée Poussin hyperbolic cross sum of the orthonormal polynomial expansion of f with respect to the multivariate weight w^2.

In what follows, for short we write $|\mathbf{k}|_1 \le \xi$ as $\mathbf{k} \in \mathbb{N}_0^d : |\mathbf{k}|_1 \le \xi$ and etc., if there is not misunderstanding. Let

$$\mathscr{P}_{2^{\mathbf{k}}} := \operatorname{span}\left\{ p_{\mathbf{s}} : s_i \le 2^{k_i}, i = 1, ..., d \right\}.$$

For the proof of the following lemma, see [2, Lemma A.2].

Lemma 3.4. *Let $1 \le p, q < \infty$, $p \ne q$ and $f \in L_{q,w}(\mathbb{R}^d)$ be represented by the series*

$$f = \sum_{\mathbf{k} \in \mathbb{N}_0^d} \varphi_{\mathbf{k}}, \ \varphi_{\mathbf{k}} \in \mathscr{P}_{2^{\mathbf{k}}},$$

converging in $L_{q,w}(\mathbb{R}^d)$. Then there holds the inequality

$$\|f\|_{L_{q,w}(\mathbb{R}^d)} \le C \left(\sum_{\mathbf{k} \in \mathbb{N}_0^d} \|2^{\delta_{\lambda,p,q} |\mathbf{k}|_1} \varphi_{\mathbf{k}}\|_{L_{p,w}(\mathbb{R}^d)}^q \right)^{1/q},$$

with some constant C depending at most on λ, p, q, d, whenever the right side is finite.

Theorem 3.5. *Let $1 \le p, q \le \infty$ and $r_{\lambda,p,q} > 0$. Then, for $\xi > 1$,*

$$\left\|f - \mathscr{V}_\xi f\right\|_{L_{q,w}(\mathbb{R}^d)} \ll \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \begin{cases} 2^{-r_\lambda \xi} \xi^{d-1} & \text{if } p = q, \\ 2^{-r_{\lambda,p,q} \xi} \xi^{(d-1)/q} & \text{if } p \ne q < \infty, \quad \xi > 1, \ f \in W^r_{p,w}(\mathbb{R}^d). \\ 2^{-r_{\lambda,p,q} \xi} \xi^{d-1} & \text{if } q = \infty, \end{cases} \tag{3.7}$$

Proof. From Lemma 3.3, we derive that, for $\xi > 1$ and $f \in W^r_{p,w}(\mathbb{R}^d)$,

$$f - \mathscr{V}_\xi f = \sum_{|\mathbf{k}|_1 > \xi} v_{\mathbf{k}} f, \ \ v_{\mathbf{k}} f \in \mathscr{P}_{2^{\mathbf{k}}}, \tag{3.8}$$

with absolute convergence in the space $L_{q,w}(\mathbb{R}^d)$ of the series, and there holds (3.4). If $p \ne q$, applying Lemma 3.4 and (3.4), we obtain (3.7):

$$\begin{aligned} \left\|f - \mathscr{V}_\xi f\right\|_{L_{q,w}(\mathbb{R}^d)}^q &\ll \sum_{|\mathbf{k}|_1 > \xi} \left\|2^{\delta_{\lambda,p,q} |\mathbf{k}|_1} v_{\mathbf{k}} f\right\|_{L_{p,w}(\mathbb{R}^d)}^q \ll \sum_{|\mathbf{k}|_1 > \xi} 2^{-q r_{\lambda,p,q} |\mathbf{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)}^q \\ &= \|f\|_{W^r_{p,w}(\mathbb{R}^d)}^q \sum_{|\mathbf{k}|_1 > \xi} 2^{-q r_{\lambda,p,q} |\mathbf{k}|_1} \ll 2^{-q r_{\lambda,p,q} \xi} \xi^{d-1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)}^q. \end{aligned}$$

If $p = q$ or $q = \infty$, upper bound (3.7) can be derived similarly by using (3.8), (3.4) and the inequality

$$\left\|f - \mathscr{V}_\xi f\right\|_{L_{q,w}(\mathbb{R}^d)} \le \sum_{|\mathbf{k}|_1 > \xi} \left\|v_{\mathbf{k}} f\right\|_{L_{q,w}(\mathbb{R}^d)}.$$

□

For given $1 \leq p,q \leq \infty$ and $r \in \mathbb{N}$, we make use of the abbreviations:

$$\lambda_n := \lambda_n(\mathbf{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)), \ \ d_n := d_n(\mathbf{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)).$$

Theorem 3.6. *Let $1 \leq p,q \leq \infty$ and $r_{\lambda,p,q} > 0$. For every $n \in \mathbb{N}$, let ξ_n be the largest number such that* rank $(\mathscr{V}_{\xi_n}) \leq n$. *Then, for $n \geq 2$,*

$$d_n \leq \lambda_n \leq \sup_{f \in \mathbf{W}^r_{p,w}(\mathbb{R}^d)} \left\| f - \mathscr{V}_{\xi_n} f \right\|_{L_{q,w}(\mathbb{R}^d)} \ll \begin{cases} n^{-r_\lambda} (\log n)^{(r_\lambda + 1)(d-1)} & \text{if } \ p = q, \\ n^{-r_{\lambda,p,q}} (\log n)^{(r_{\lambda,p,q} + 1/q)(d-1)} & \text{if } \ p \neq q < \infty, \\ n^{-r_{\lambda,p,q}} (\log n)^{(r_{\lambda,p,q} + 1)(d-1)} & \text{if } \ q = \infty. \end{cases} \tag{3.9}$$

Proof. To prove upper bound (3.9), we approximate a function $f \in \mathbf{W}^r_{p,w}(\mathbb{R}^d)$ by using the linear operator $\mathscr{V}_\xi$. Let us prove the case $p \neq q < \infty$ of (3.9). The cases $p = q$ and $q = \infty$ can be proven in a similar manner. From (3.10), it follows $2^{\xi_n} \xi_n^{d-1} \asymp$ rank $(\mathscr{V}_{\xi_n}) \asymp n$. Hence we deduce the asymptotic equivalences

$$2^{-\xi_n} \asymp n^{-1} (\log n)^{d-1}, \ \ \xi_n \asymp \log n,$$

which together with Theorem 3.5 yields that

$$\begin{aligned} d_n \leq \lambda_n \leq & \sup_{f \in \mathbf{W}^r_{p,w}(\mathbb{R}^d)} \left\| f - \mathscr{V}_{\xi_n} f \right\|_{L_{q,w}(\mathbb{R}^d)} \\ & \leq C 2^{-r_\lambda \xi_n} \xi_n^{(d-1)/q} \asymp n^{-r_{\lambda,p,q}} (\log n)^{(r_{\lambda,p,q} + 1/q)(d-1)}. \end{aligned}$$

The upper bound in (3.9) for the case $p \neq q < \infty$ is proven. ☐

For $\mathbf{k} \in \mathbb{N}^d$, the d-dimensional operators $s_\mathbf{k}$ are defined as the tensor product of one-dimensional operators:

$$s_\mathbf{k} := \bigotimes_{i=1}^d s_{k_i}.$$

For $\xi > 0$, we define the linear operator $\mathscr{S}_\xi$ for functions $f \in L_{2,w}(\mathbb{R}^d)$ by

$$\mathscr{S}_\xi f := \sum_{|\mathbf{k}|_1 \leq \xi} s_\mathbf{k} f.$$

Notice that $\mathscr{S}_\xi f$ belongs to $\mathscr{P}_1(\xi) :=$ span $\{p_\mathbf{s} : \mathbf{s} \in H(\xi)\}$, where

$$H_1(\xi) := \bigcup_{\mathbf{k} \in \mathbb{N}_0^d : |\mathbf{k}|_1 \leq \xi} \left\{ \mathbf{s} \in \mathbb{N}_0^d : \mathbf{s} \leq 2^\mathbf{k} \right\}.$$

Notice by (2.7) that $\mathscr{S}_\xi$ is a linear bounded operator in $L_{q,w}(\mathbb{R}^d)$ for $4/3 < q < 4$, and

$$\text{rank} \left(\mathscr{S}_\xi\right) = |H_1(\xi)| = \sum_{|\mathbf{k}|_1 \leq \xi} \prod_{j=1}^d (2^{k_j} - 1) \asymp 2^\xi \xi^{d-1}. \tag{3.10}$$

In a way similar to the proofs of Lemma 3.3 and Theorem 3.6, we can prove the following results.

Lemma 3.7. *Let $4/3 < p < 4$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$. Then, for every $f \in W^r_{p,w}(\mathbb{R}^d)$, $f = \sum_{\mathbf{k}\in\mathbb{N}_0^d} s_{\mathbf{k}} f$ with absolute convergence in the space $L_{q,w}(\mathbb{R}^d)$ of the series, and*

$$\left\|s_{\mathbf{k}} f\right\|_{L_{q,w}(\mathbb{R}^d)} \le C 2^{-r_{\lambda,p,q}|\mathbf{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)}, \ \ \mathbf{k} \in \mathbb{N}_0^d.$$

Theorem 3.8. *Let $4/3 < p < 4$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$. For every $n \in \mathbb{N}$, let ξ_n be the largest number such that* rank $\left(\mathscr{S}_{\xi_n}\right) \le n$. *Then, for $n \ge 2$,*

$$d_n \le \lambda_n \le \sup_{f\in\mathbf{W}^r_{p,w}(\mathbb{R}^d)} \left\|f - \mathscr{S}_{\xi_n} f\right\|_{L_{q,w}(\mathbb{R}^d)} \ll \begin{cases} n^{-r_\lambda}(\log n)^{(r_\lambda+1)(d-1)} & \text{if } \ p = q, \\ n^{-r_{\lambda,p,q}}(\log n)^{(r_{\lambda,p,q}+1/q)(d-1)} & \text{if } \ p \ne q < \infty, \\ n^{-r_{\lambda,p,q}}(\log n)^{(r_{\lambda,p,q}+1)(d-1)} & \text{if } \ q = \infty. \end{cases}$$

4. Right Convergence Rate of n-widths

In this section, we investigate the right convergence rate of $\lambda_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ and $d_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ in the case when the generating weight w is given as in (1.1) with $\lambda = 2, 4$.

For $r \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}_0^d$, we define

$$\rho_{\lambda,r,\mathbf{k}} := \prod_{j=1}^d \left(k_j + 1\right)^{r_\lambda},$$

where recall, r_λ is given as in (1.4) and λ as in (1.1). Denote by $H^{r_\lambda}_w(\mathbb{R}^d)$ the space of all functions $f \in L_{2,w}(\mathbb{R}^d)$ represented by the series (3.1) for which the norm

$$\|f\|_{H^{r_\lambda}_w(\mathbb{R}^d)} := \left(\sum_{\mathbf{k}\in\mathbb{N}_0^d} |\rho_{\lambda,r,\mathbf{k}} \hat{f}(\mathbf{k})|^2\right)^{1/2}$$

is finite.

For functions $f \in H^{r_\lambda}_w(\mathbb{R}^d)$, we construct a hyperbolic cross polynomial approximation based on truncations of the orthonormal polynomial series (3.1). For the hyperbolic cross

$$G(\xi) := \left\{\mathbf{k} \in \mathbb{N}_0^d : \rho_{\lambda,r,\mathbf{k}} \le \xi\right\}, \ \ \xi \ge 1,$$

the truncation $S^*_\xi(f)$ of the series (3.1) on this set is defined by

$$S^*_\xi(f) := \sum_{\mathbf{k}\in G(\xi)} \hat{f}(\mathbf{k}) p_{\mathbf{k}}.$$

Notice that S^*_ξ is a linear projection from $L_2(\mathbb{R}^d, \gamma)$ onto the linear subspace $L(\xi)$ spanned by the orthonormal polynomials $p_{\mathbf{k}}$, $\mathbf{k} \in G(\xi)$, and $\dim L(\xi) = |G(\xi)|$.

We will need the following Tikhomirov lemma which is often used for lower estimation of Kolmogorov n-widths [13, Theorem 1].

Lemma 4.1. *If X is a Banach space and U the ball of radius $\rho > 0$ in a linear $n+1$-dimensional subspace of X, then $d_n(U, X) = \rho$.*

Theorem 4.2. *We can construct a sequence* $\{\xi_n\}_{n=2}^\infty$ *with* $|G(\xi_n)| \le n$ *so that, for* $n \ge 2$,

$$\sup_{f \in \mathbf{H}^r_w(\mathbb{R}^d)} \left\| f - S^*_{\xi_n}(f) \right\|_{L_{2,w}(\mathbb{R}^d)} \asymp \lambda_n(\mathbf{H}^{r_\lambda}_w(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)) \asymp d_n(\mathbf{H}^{r_\lambda}_w(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)) \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{4.1}$$

Proof. Since $L_{2,w}(\mathbb{R}^d)$ is a Hilbert space, we have the equality

$$\lambda_n(\mathbf{H}^{r_\lambda}_w(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)) = d_n(\mathbf{H}^{r_\lambda}_w(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)).$$

To prove the upper bounds in (4.1), it is sufficient to construct a sequence $\{\xi_n\}_{n=2}^\infty$ so that $|G(\xi_n)| \le n$ and

$$\sup_{f \in \mathbf{H}^{r_\lambda}_w(\mathbb{R}^d)} \left\| f - S^*_{\xi_n}(f) \right\|_{L_{2,w}(\mathbb{R}^d)} \ll n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{4.2}$$

From Parseval's identity (3.2), we have that, for every $f \in \mathbf{H}^{r_\lambda}_w(\mathbb{R}^d)$ and $\xi > 1$,

$$\left\| f - S^*_{\xi}(f) \right\|_{L_{2,w}(\mathbb{R}^d)} \ll \xi^{-r_\lambda}. \tag{4.3}$$

Indeed,

$$\left\| f - S^*_{\xi}(f) \right\|^2_{L_{2,w}(\mathbb{R}^d)} = \sum_{\mathbf{k} \notin G(\xi)} \hat{f}(\mathbf{k})^2 \ll \xi^{-2r_\lambda} \sum_{\mathbf{k} \notin G(\xi)} |\rho_{\lambda,r,\mathbf{k}} \hat{f}(\mathbf{k})|^2 \ll \xi^{-2r_\lambda} \|f\|_{H^{r_\lambda}_w(\mathbb{R}^d)} \le \xi^{-2r_\lambda}.$$

Let $\{\xi_n\}_{n=2}^\infty$ be the sequence of ξ_n defined as the largest number satisfying the condition $|G(\xi_n)| \le n$. From $|G(\xi_n)| \asymp \xi_n(\log \xi_n)^{d-1}$ (see, e.g., [11, page 130]), we derive that

$$\xi_n^{-r_\lambda} \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}$$

which together with (4.3) yields (4.2).

To show the lower bounds of (4.1) we will apply Lemma 4.1. If

$$U(\xi) := \left\{ f \in L(\xi) : \|f\|_{L_{2,w}(\mathbb{R}^d)} \le 1 \right\}$$

and $f \in U(\xi)$, then by Parseval's identity (3.2) and the definition of $\mathbf{H}^{r_\lambda}_w(\mathbb{R}^d)$, similarly to (4.3), we deduce that $\|f\|_{H^{r_\lambda}_w(\mathbb{R}^d)} \ll \xi^{r_\lambda}$, which means that

$$C\xi^{-r_\lambda} U(\xi) \subset \mathbf{H}^{r_\lambda}_w(\mathbb{R}^d)$$

for some $C > 0$. Let $\{\xi'_n\}_{n=2}^\infty$ be the sequence of ξ'_n defined as the smallest number satisfying the condition $|G(\xi'_n)| \ge n+1$. Then $\dim L(\xi'_n) = |G(\xi'_n)| \ge n+1$, and similarly as in the upper estimation, $(\xi'_n)^{-r_\lambda} \asymp n^{-r_\lambda}(\log n)^{(d-1)r_\lambda}$. By Lemma 4.1 for the smallest quantity d_n in (4.1), we have that

$$d_n(\mathbf{H}^{r_\lambda}_w(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)) \ge d_n(C(\xi'_n)^{-r_\lambda} U(\xi'_{n+1}), L_{2,w}(\mathbb{R}^d)) = C(\xi'_n)^{-r_\lambda} \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}.$$

□

Theorem 4.3. *Let $\lambda = 2,4$ for the generating univariate weight w as in (1.1). Then we have the norm equivalence*

$$\|f\|_{W^r_{2,w}(\mathbb{R}^d)} \asymp \|f\|_{H^{r_\lambda}_w(\mathbb{R}^d)}, \quad f \in W^r_{2,w}(\mathbb{R}^d). \tag{4.4}$$

This theorem was proven in [3, Lemma 3.4] for $\lambda = 2$, and in [2] for $\lambda = 4$.

Due to the norm equivalence (4.4) in Theorem 4.3, we identify $W^r_{2,w}(\mathbb{R}^d)$ with $H^{r_\lambda}_w(\mathbb{R}^d)$ for the cases $\lambda = 2,4$ and $r \in \mathbb{N}$. In these cases, Theorem 4.2 gives the following result on right asymptotic order of linear n-widths $\lambda_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$ and Kolmogorov n-widths $d_n\big(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)\big)$.

Theorem 4.4. *Let $\lambda = 2,4$. Then, for $n \ge 2$,*

$$\lambda_n(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)) = d_n(\mathbf{W}^r_{2,w}(\mathbb{R}^d), L_{2,w}(\mathbb{R}^d)) \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{4.5}$$

We conjecture that the right convergence rate (4.5) is still holds true at least for every even λ.

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