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A DYNAMIC PROGRAMMING APPROACH TO THE FIRE BLOCKING PROBLEM

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Dedicated to Richard Vinter in the occasion of his 75th birthday

Abstract. This short note is concerned with a dynamic blocking problem for a model of fire propagation. The region burned by the fire is described as the set reached by trajectories of a differential inclusion, and can be reduced by constructing barriers, in real time. Optimal strategies are sought, which minimize the area destroyed by the fire together with the cost of the barriers. A dynamic programming approach is developed, providing necessary conditions for optimal strategies. In this general setting, we also introduce a notion of "instantaneous value of time", and prove that it is non-increasing along optimal strategies. The paper is concluded by a discussion of various open problems.

Keywords. Dynamic programming; Fire blocking problem; Necessary conditions; Optimal strategy.

2020 Mathematics Subject Classification. 49Q10, 49Q20, 49L20.

1. Introduction

The aim of this paper is to analyze optimal strategies, for the dynamic blocking problems introduced in [5]. These problems were originally motivated by the control of wild fires [19, 20] or the spatial spreading of a contaminating agent. At each time $t \geq 0$, we denote by $R(t) \subset \mathbb{R}^2$ the region burned by the fire. In absence of control, for each $t \geq 0$ the set $R(t)$ is described as the set reached by trajectories of a differential inclusion:

$$\dot{x} \in F(x), \qquad x(0) \in R_0\,, \tag{1.1}$$

where the upper dot denotes a derivative w.r.t. time. In other words,

$$R(t) \doteq \Big\{ x(t);\ x(\cdot) \text{ is absolutely continuous, } x(0) \in R_0\,,\ \dot{x}(\tau) \in F\big(x(\tau)\big) \text{ for a.e. } \tau \in [0,t] \Big\}. \tag{1.2}$$

We assume that the initial set $R_0 \subset \mathbb{R}^2$ is open and bounded. Moreover, we assume that $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a Lipschitz continuous multifunction with compact, convex values, and satisfies $0 \in F(x)$ for all $x \in \mathbb{R}^2$. This clearly implies $R(t_1) \subseteq R(t_2)$ whenever $0 < t_1 < t_2$. In our model, the growth

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of the reachable set (i.e., the spreading of the fire) can be controlled by constructing barriers, in real time. More precisely, consider a continuous, strictly positive function $\psi : \mathbb{R}^2 \mapsto \mathbb{R}_+$.

Definition 1.1. *Calling $\gamma(t) \subset \mathbb{R}^2$ the portion of the wall constructed within time $t \geq 0$, we say that the strategy $t \mapsto \gamma(t)$ is* **admissible** *if*

- **(H1)** *Every set $\gamma(t)$ is rectifiable. For any $0 < t_1 < t_2$, one has $\gamma(t_1) \subseteq \gamma(t_2)$.*
- **(H2)** *For every $t \geq 0$, the total length of the wall satisfies*

$$\int_{\gamma(t)} \psi \, dm_1 \leq t\,. \tag{1.3}$$

For the theory of rectifiable sets we refer to [1, 18]. Here m_1 denotes the one-dimensional Hausdorff measure, normalized so that $m_1(\gamma)$ yields the usual length of a smooth curve γ. In the above formula, $1/\psi(x)$ is the speed at which the wall can be constructed, at the location x. In particular, if $\psi(x) \equiv \sigma^{-1}$ is constant, then (1.3) simply means that at any time $t > 0$ the length of the curve $\gamma(t)$ must be $\leq \sigma t$.

When a barrier is constructed, the set reached by the fire is reduced. Namely, we define

$$\begin{aligned} R^\gamma(t) \doteq \; & \Big\{ x(t)\,;\; x(\cdot) \text{ absolutely continuous}\,,\; x(0) \in R_0\,, \\ & \qquad \dot x(\tau) \in F\big(x(\tau)\big) \text{ for a.e. } \tau \in [0,t]\,, \qquad x(\tau) \notin \gamma(\tau) \text{ for all } \tau \in [0,t] \Big\}. \end{aligned} \tag{1.4}$$

Compared with (1.2), notice that in (1.4) one has the additional requirement that, at any time $\tau > 0$, a trajectory of the fire cannot cross the portion of barrier already constructed.

To define an optimization problem, we need to introduce a cost functional. This should take into account:

- The value of the land burned by the fire.
- The cost of building the barrier.

Consider the sets

$$R^\gamma_\infty \doteq \bigcup_{t \geq 0} R^\gamma(t)\,, \qquad\qquad \gamma_\infty \doteq \bigcup_{t \geq 0} \gamma(t)\,, \tag{1.5}$$

describing the whole region eventually burned by the fire, and the entire barrier. As in [5], we also consider two continuous, non-negative functions $\alpha, \beta : \mathbb{R}^2 \mapsto \mathbb{R}_+$ and define the cost functional

$$J(\gamma) \doteq \int_{R^\gamma_\infty} \alpha \, dm_2 + \int_{\gamma_\infty} \beta \, dm_1\,. \tag{1.6}$$

In (1.6), m_2 denotes the two-dimensional Lebesgue measure, while m_1 is the one-dimensional Hausdorff measure. We think of $\alpha(x)$ as the value of a unit area of land at the point x, while $\beta(x)$ is the cost of building a unit length of wall at the point x. This leads to

(OP1) Optimization Problem 1. *Find an admissible strategy $t \mapsto \gamma(t)$ for which the corresponding functional $J(\gamma)$ at (1.6) attains its minimum value.*

In its original formulation, a strategy is a mapping $t \mapsto \gamma(t) \subset \mathbb{R}^2$ describing the portion of the wall which has been constructed up to time t. The subsequent paper [12] showed that the above problem can be restated in a simpler way, where a strategy is entirely determined by assigning one single rectifiable set $\Gamma \subset \mathbb{R}^2$.

Consider a rectifiable set $\Gamma \subset \mathbb{R}^2$ which is **complete**, in the sense that it contains all of its points of positive upper density. Otherwise stated, one has the implication

$$\limsup_{r\to 0+} \frac{m_1\Big(B(x,r)\cap\Gamma\Big)}{r} > 0 \qquad \Longrightarrow \qquad x\in\Gamma. \tag{1.7}$$

The reachable set for the differential inclusion (1.1) restricted to $\mathbb{R}^2\setminus\Gamma$ is then defined as

$$\begin{aligned} R^\Gamma(t) \doteq \ & \Big\{x(t)\,;\ x(\cdot) \text{ absolutely continuous}\,,\ x(0)\in R_0\,, \\ & \dot x(\tau)\in F\big(x(\tau)\big) \text{ for a.e. } \tau\in[0,t]\,,\ x(\tau)\notin\Gamma \text{ for all } \tau\in[0,t]\Big\}. \end{aligned} \tag{1.8}$$

Throughout the following, $\overline{S}$ denotes the closure of a set S. We say that the rectifiable set Γ is **admissible** in connection with the differential inclusion (1.1) and the bound on the construction speed (1.3) if $\int_{\Gamma\cap\overline{R^\Gamma(t)}}\psi\,dm_1 \ \le\ t$ for all $t\ge 0$. Of course, this means that the strategy

$$t\ \mapsto\ \gamma(t)\ \doteq\ \Gamma\cap\overline{R^\Gamma(t)} \tag{1.9}$$

is admissible according to Definition 1.1. Notice that, for each $t\ge 0$, the set $\gamma(t)$ in (1.9) is the part of the wall Γ touched by the fire at time t. This is the portion that actually needs to be put in place within time t, in order to constrain the fire. The remaining portion $\Gamma\setminus\gamma(t)$ can be constructed at a later time. We then introduce:

(OP2) Optimization Problem 2. *Find an admissible rectifiable set* $\Gamma\subset\mathbb{R}^2$ *such that, calling* R^Γ_∞ *the union of all connected components of* $\mathbb{R}^2\setminus\Gamma$ *which intersect* R_0*, the cost*

$$J(\Gamma) \doteq \int_{R^\Gamma_\infty}\alpha\,dm_2 + \int_\Gamma \beta\,dm_1 \tag{1.10}$$

attains the minimum possible value.

As proved in [12], the two formulations are equivalent.

A general theorem on the existence of optimal blocking strategies was proved in [9]. See also [15] for an alternative proof, based on a compactness property of the set of SBV functions [1].

We remark that the optimal barrier Γ^* constructed in [9, 15], is a complete rectifiable set, which can be decomposed as

$$\Gamma^*\ =\ \left(\bigcup_{i\ge 1}\Gamma_i\right)\cup\Gamma_0\,. \tag{1.11}$$

Here the countably many sets Γ_i are disjoint, compact, and connected, while Γ_0 is a set whose 1-dimensional Hausdorff measure is zero. On the other hand, all the necessary conditions for optimality derived in [5, 13, 22] require that Γ^* is a finite union of Lipschitz curves. This leaves a huge gap in the theory, between the regularity provided by the existence theorems and the regularity required to prove the necessary conditions for optimality. The only additional regularity result, in this general setting, was recently proved in [8]. Namely, the optimal barrier Γ^* in (1.11) is nowhere dense.

The present paper aims at establishing some additional properties of optimal strategies. In Section 2 we introduce a dynamic programming framework for a general class of fire blocking problems, and prove a corresponding dynamic programming principle. In Section 3 we show that, in connection with an optimal strategy, one can define a non-increasing scalar function $W(t)$ describing the "instantaneous value of time". Roughly speaking, the function W can be regarded as a Lagrange multiplier related to the constraint (1.3). Finally, in Section 4 we discuss open problems and formulate various conjectures.

Additional results on the existence of blocking strategies, and properties of optimal barriers, can be found in [7, 11, 14, 17, 23, 24, 25]. See also [6] for an earlier survey. For the basic theory of multifunctions and differential inclusions we refer to [2].

2. A Dynamic Programming Approach

Consider an optimal strategy $\gamma^*(\cdot)$ for the optimization problem **(OP1)**. Then, for any $\tau > 0$, the restriction of γ^* to the set of times $t \in [\tau, +\infty[$ should be optimal for a subproblem where all the data are assigned at time τ. Before we state more precisely this dynamic programming principle, we notice that at time τ the admissibility constraint $\int_{\gamma^*(\tau)} \psi\, dm_1 \leq \tau$ may well be satisfied as a strict inequality. To completely describe the optimization problem on the remaining time interval $[\tau, +\infty[$, one should thus take into account:

- The portion $\mathscr{B} = \gamma(\tau) \subset \mathbb{R}^2$ of barrier already constructed at time τ.
- The set $S = R^\gamma(\tau)$ reached by the fire at time τ.
- The amount by which the constraint (1.3) at time τ is not saturated: $\delta \doteq \tau - \int_{\gamma(\tau)} \psi\, dm_1$

Performing a shift in the time variable: $t \mapsto t - \tau$, we thus consider a somewhat more general problem, where a portion of barrier $\mathscr{B}$ is already in place.

Definition 2.1. *Let $\delta \geq 0$ be given, together with a complete rectifiable set $\mathscr{B} \subset \mathbb{R}^2$, and a measurable set $S \subset \mathbb{R}^2$. We say that a strategy $t \mapsto \gamma(t)$ is δ-**admissible**, and write $\gamma \in \mathscr{A}_\delta$, if it satisfies* **(H1)** *together with $\int_{\gamma^*(t)} \psi\, dm_1 \leq t + \delta$ for all $t \geq 0$. The corresponding set reached by the fire at time $t > 0$ is then defined as*

$$R^\gamma(t, S, \mathscr{B}) \doteq \Big\{ x(t)\,;\ x(\cdot) \text{ absolutely continuous},\ x(0) \in S,\\ \dot x(\tau) \in F\big(x(\tau)\big) \text{ for a.e. } \tau \in [0,t], \quad x(\tau) \notin \mathscr{B} \cup \gamma(\tau) \text{ for all } \tau \in [0,t] \Big\}.$$

(OP3) **Optimization Problem 3.** *Given $(S, \mathscr{B}, \delta)$, find a δ-admissible strategy $\gamma^* \in \mathscr{A}_\delta$ that minimizes the functional*

$$J^{S,\mathscr{B}}(\gamma) = \sup_{t>0} \left\{ \int_{R^\gamma(t,S,\mathscr{B})} \alpha\, dm_2 + \int_{\gamma(t)} \beta\, dm_1 \right\}.$$

Let us denote by $V(S, \mathscr{B}, \delta)$ the corresponding value function, namely

$$V(S, \mathscr{B}, \delta) \doteq \inf_{\gamma \in \mathscr{A}_\delta} J^{S,\mathscr{B}}(\gamma).$$

In analogy with the well known principle in optimal control [3, 4, 10, 16, 21], we now prove

Theorem 2.2. (Dynamic Programming Principle). *For every triple $(S, \mathscr{B}, \delta)$ and every $\tau > 0$ one has*

$$V(S, \mathscr{B}, \delta) = \inf_{\gamma \in \mathscr{A}_\delta} \left\{ \int_{\gamma(\tau)} \beta\, dm_1 + V\Big(R^\gamma(\tau, S, \mathscr{B}),\ \mathscr{B} \cup \gamma(\tau),\ \delta(\tau) \Big) \right\}, \tag{2.1}$$

with

$$\delta(\tau) \doteq \tau + \delta - \int_{\gamma(\tau)} \psi\, dm_1\,. \tag{2.2}$$

Moreover, if $t \mapsto \gamma^*(t)$ is an optimal strategy for the original problem **(OP1)**, *then, for every $\tau \geq 0$, there holds*

$$V(R_0,\emptyset,0) = \int_{\gamma^*(\tau)} \beta\, dm_1 + V\big(R^{\gamma^*}(\tau),\, \gamma^*(\tau),\, \delta^*(\tau)\big),$$

where $\delta^*(\tau) \doteq \tau - \int_{\gamma^*(\tau)} \psi\, dm_1$.

Proof. 1. Let a triple of initial data $(S,\mathscr{B},\delta)$ and $\tau > 0$ be given.

Fix any $\varepsilon > 0$. Chose a δ-admissible strategy γ such that $J^{S,\mathscr{B}}(\gamma) < V(S,\mathscr{B},\delta)+\varepsilon$. Call $\widetilde{\gamma}(s) \doteq \gamma(\tau+s)\setminus\gamma(\tau)$. Observing that

$$R^\gamma(\tau+s;S,\mathscr{B}) = R^{\widetilde{\gamma}}\big(s;\, R^\gamma(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma(\tau)\big),$$

we conclude

$$\begin{aligned}\int_{\gamma(\tau)} \beta\, dm_1 + V\Big(R^\gamma(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma(\tau),\, \delta(\tau)\Big) \;&\leq\; \int_{\gamma(\tau)} \beta\, dm_1 + J^{R^\gamma(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma(\tau)}(\widetilde{\gamma})\\ &=\; J^{S,\mathscr{B}}(\gamma) \;<\; V(S,\mathscr{B},\delta)+\varepsilon\,.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this establishes the "$\geq$" inequality in (2.1).

2. To establish the converse inequality, we choose a δ-admissible strategy $\gamma^\sharp \in \mathscr{A}_\delta$ such that

$$\begin{aligned}&\int_{\gamma^\sharp(\tau)} \beta\, dm_1 + V\Big(R^{\gamma^\sharp}(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma^\sharp(\tau),\, \delta^\sharp(t)\Big)\\ &\qquad <\; \varepsilon + \inf_{\gamma\in\mathscr{A}_\delta}\left\{\int_{\gamma(\tau)} \beta\, dm_1 + V\Big(R^\gamma(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma(t),\, \delta(\tau)\Big)\right\},\end{aligned} \tag{2.3}$$

with $\delta(\tau)$ given at (2.2). Moreover, choose a $\delta^\sharp(\tau)$-admissible strategy $\widetilde{\gamma}$ such that

$$J^{R^{\gamma^\sharp}(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma^\sharp(\tau)}(\widetilde{\gamma}) \;<\; \varepsilon + V\Big(R^{\gamma^\sharp}(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma^\sharp(\tau),\, \delta^\sharp(\tau)\Big). \tag{2.4}$$

Define the δ-admissible strategy $\gamma(\cdot)$ by setting

$$\gamma(t) = \begin{cases} \gamma^\sharp(t) & \text{if} \quad t\in[0,\tau]\,,\\ \gamma^\sharp(\tau)\cup\widetilde{\gamma}(t-\tau) & \text{if} \quad t>\tau. \end{cases}$$

Using (2.4) and (2.3), the total cost achieved by this strategy can be bounded by

$$\begin{aligned}J^{S,\mathscr{B}}(\gamma) \;&=\; \int_{\gamma^\sharp(\tau)} \beta\, dm_1 + J^{R^{\gamma^\sharp}(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma^\sharp(\tau)}(\widetilde{\gamma})\\ &<\; \int_{\gamma^\sharp(\tau)} \beta\, dm_1 + \varepsilon + V\Big(R^{\gamma^\sharp}(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma^\sharp(\tau),\, \delta^\sharp(\tau)\Big)\\ &<\; 2\varepsilon + \inf_{\gamma\in\mathscr{A}_\delta}\left\{\int_{\gamma(\tau)} \beta\, dm_1 + V\Big(R^\gamma(\tau;\, S,\mathscr{B}),\, \mathscr{B}\cup\gamma(\tau),\, \delta(\tau)\Big)\right\}.\end{aligned}$$

Since γ is δ-admissible, we have $V(S,\mathscr{B},\delta) \leq J^{S,\mathscr{B}}(\gamma)$. Since $\varepsilon > 0$ is arbitrary, the above estimate yields the "$\leq$" inequality in (2.1).

3. To prove the last statement, let γ^* be an optimal strategy for **(OP1)** and fix a time $\tau > 0$. Consider the strategy $\widetilde{\gamma}(s) \doteq \gamma^*(\tau+s) \setminus \gamma^*(\tau)$. Observe that this strategy $\widetilde{\gamma}$ is $\delta^*(\tau)$-admissible. Using (2.1), we now obtain

$$\begin{aligned} V(R_0,\emptyset,0) \ &\leq\ \int_{\gamma^*(\tau)} \beta\, dm_1 + V\big(R^{\gamma^*}(\tau),\, \gamma^*(\tau),\, \delta^*(\tau)\big) \\ &\leq\ \int_{\gamma^*(\tau)} \beta\, dm_1 + J^{R^{\gamma^*}(\tau),\,\gamma^*(\tau)}(\widetilde{\gamma}) \ =\ J^{R_0,\emptyset}(\gamma^*) \ =\ V(R_0,\emptyset,0), \end{aligned}$$

completing the proof. □

3. The Value of Time

In connection with an optimal strategy γ, we now introduce a scalar function $t \mapsto W(t)$ accounting for the **instantaneous value of time**.

As in the previous section, let a triple $(S,\mathscr{B},\delta)$ be given. Define

$$W(S,\mathscr{B},\delta) \doteq \limsup_{\varepsilon\to 0+}\ \sup\left\{\frac{1}{\varepsilon}\Big(V(S,\mathscr{B},\delta) - V(S,\mathscr{B}\cup\gamma_\varepsilon,\delta) - \int_{\gamma_\varepsilon}\beta\, dm_1\Big);\ \int_{\gamma_\varepsilon}\psi\, dm_1 \leq \varepsilon\right\}. \tag{3.1}$$

In other words, assume that we are allowed an additional small interval of time $[0,\varepsilon]$ during which the region burned by the fire does not expand. To make the best use of it, we construct an additional piece of wall γ_ε satisfying the constraint $\int_{\gamma_\varepsilon}\psi\, dm_1 \leq \varepsilon$, in such a way to minimize the cost

$$V(S,W\cup\gamma_\varepsilon,\delta) + \int_{\gamma_\varepsilon}\beta\, dm_1.$$

The rate of decrease in this cost, defined at (3.1), measures the instantaneous value of time.

If $t \mapsto \gamma(t)$ is an optimal strategy for optimization problem (**OP1**), one can regard the function

$$t \ \mapsto\ W(t) \ \doteq\ W\big(R^\gamma(t),\, \gamma(t),\, \delta(t)\big) \tag{3.2}$$

as a Lagrange multiplier associated to the constraint (1.3). Here

$$\delta(t) = t - \int_{\gamma(t)}\psi\, dm_1\,. \tag{3.3}$$

Our main result in this direction is as follows.

Theorem 3.1. *Let $t \mapsto \gamma(t)$ be an optimal strategy for the optimization problem* **(OP1)**. *Then the corresponding function $t \mapsto W(t) \in [0,+\infty]$ at (3.2) is non-increasing.*

Proof. Define $\delta(t)$ as in (3.3). Given any two times $0 \leq s < t$, the optimality of $\gamma(\cdot)$ implies

$$V\big(R^\gamma(s),\, \gamma(s),\, \delta(s)\big) = \int_{\gamma(t)\setminus\gamma(s)}\beta\, dm_1 + V\big(R^\gamma(t),\, \gamma(t),\, \delta(t)\big). \tag{3.4}$$

Let γ_ε be any curve such that

$$\int_{\gamma_\varepsilon}\psi\, dm_1 \ \leq\ \varepsilon. \tag{3.5}$$

For $\tau \geq 0$, define the strategy $\widetilde{\gamma}(\tau) = \gamma(s+\tau)\setminus\gamma(s)$. Notice that this strategy is $\delta(s)$-admissible. Applying the dynamic programming principle (2.1) to the strategy $\widetilde{\gamma}$ at time $t-s$, with $\mathscr{B} =$

$\gamma(s)\cup\gamma_\varepsilon$, one obtains

$$\begin{aligned} V\big(R^\gamma(s),\,\gamma(s)\cup\gamma_\varepsilon,\,\delta(s)\big) \;\leq\; & \int_{\widetilde\gamma(t-s)} \beta\,dm_1 \\ & + V\Big(R^{\widetilde\gamma}(t-s;\,R^\gamma(s),\,\gamma(s)\cup\gamma_\varepsilon)\,,\;\gamma(s)\cup\gamma_\varepsilon\cup\widetilde\gamma(t-s)\,,\;\delta(t)\Big). \end{aligned} \tag{3.6}$$

Since

$$R^{\widetilde\gamma}(t-s;\,R^\gamma(s),\,\gamma(s)\cup\gamma_\varepsilon)\;\subseteq\;R^{\widetilde\gamma}(t-s;\,R^\gamma(s),\,\gamma(s))\;=\;R^\gamma(t)\,,$$

and $\gamma(s)\cup\widetilde\gamma(t-s)=\gamma(t)$, we have

$$V\Big(R^{\widetilde\gamma}(t-s;\,R^\gamma(s),\,\gamma(s)\cup\gamma_\varepsilon)\,,\;\gamma(s)\cup\gamma_\varepsilon\cup\widetilde\gamma(t-s)\,,\;\delta(t)\Big)\;\leq\;V\big(R^\gamma(t)\,,\;\gamma(t)\cup\gamma_\varepsilon\,,\;\delta(t)\big). \tag{3.7}$$

Using (3.4) and (3.6)-(3.7), one obtains

$$\begin{aligned} & V\big(R^\gamma(s),\,\gamma(s),\,\delta(s)\big)-V\big(R^\gamma(s),\,\gamma(s)\cup\gamma_\varepsilon\,,\delta(s)\big)-\int_{\gamma_\varepsilon}\beta\,dm_1 \\ & \quad\geq\;V\big(R^\gamma(t),\,\gamma(t),\,\delta(t)\big)-V\big(R^\gamma(t),\,\gamma(t)\cup\gamma_\varepsilon\,,\;\delta(t)\big)-\int_{\gamma_\varepsilon}\beta\,dm_1\,. \end{aligned}$$

Since this is true for any curve γ_ε satisfying (3.5), we conclude

$$W\big(R^\gamma(t),\,\gamma(t),\,\delta(t)\big)\;\leq\;W\big(R^\gamma(s),\,\gamma(s),\,\delta(s)\big).$$

□

4. Some Open Problems

As remarked in the Introduction, at the present time the regularity theory for optimal barriers remains largely incomplete. It would be interesting to establish further properties of the instantaneous value of time $W(t)$, and understand how this function is related to the necessary conditions for optimality proved in [5, 14], and to the regularity of the curves constructed by an optimal strategy. More precisely, we formulate:

Conjecture 4.1. *Let Γ be an optimal barrier for the minimization problem* **(OP2)***, and let $t\mapsto\gamma(t)$ be as in (1.9). If $[a,b]$ is a time interval where the constraint is not saturated:*

$$\int_{\gamma(t)}\psi\,dm_1\,<\,t\qquad\qquad \textit{for all } t\in[a,b], \tag{4.1}$$

then the corresponding function $t\mapsto W(t)$ is constant on $[a,b]$.

One may now consider:

Conjecture 4.2. *For an optimal strategy $\gamma(\cdot)$, the curvature of optimal arcs constructed during a time interval $[a,b]$ where (4.1) holds can be bounded in terms of the the instantaneous value of time $W(t)$.*

Remark 4.3. If true, Conjecture 4.2 would yield a strong regularity property of optimal barriers. In view of some examples in [5, 13], the above conjecture appears to be true for **blocking arcs**, i.e. portions of the barrier γ_∞ in (1.5) that remain on the boundary of the set $\overline{R^\gamma_\infty}$ (see Fig. 1, left).

However, it is not clear whether the value of time $W(t)$ can provide a bound on the curvature of **delaying arcs**. As defined in [13], these arcs are portions of the barrier γ_∞ which eventually fall in the interior of $\overline{R^\gamma_\infty}$, being surrounded by the fire on both sides. In an optimal strategy, such walls are constructed only to slow down the advancement of the fire, not to block it. See for example the arc CD in Fig. 1, right.

Next, to simplify our discussion, we focus on the basic case where

- the fire expands in all directions with unit speed,
- the barrier can be constructed at rate $\sigma > 0$,
- the total burned area has to be minimized.

In (1.1) and (1.3), this corresponds to the case where

$$F(x) = \overline{B}_1 \doteq \{y \in \mathbb{R}^2;\ |y| \leq 1\} \tag{4.2}$$

is the closed unit disc, while $\psi(x) \equiv \sigma^{-1}$. In particular, the portion of the barrier constructed within time $t > 0$ should satisfy

$$m_1\big(\gamma(t)\big) \leq \sigma t. \tag{4.3}$$

Taking $\alpha(x) \equiv 1$ and $\beta(x) \equiv 0$, the optimization problem reduces to

$$\text{minimize:} \quad m_2\big(R_\infty^\gamma\big). \tag{4.4}$$

For any open and bounded initial set $R_0 \subset \mathbb{R}^2$, when $\sigma > 2$, it is known that an optimal strategy exists [5, 9]. However, little is known about the qualitative structure of the optimal barrier. In this direction, a reasonable guess is:

Conjecture 4.4. *If the initial set R_0 is open, bounded and convex, and an optimal barrier Γ exists, then Γ must be a simple closed curve surrounding R_0.*

In the case where R_0 is a disc, assuming that Conjecture 4.4 is true, the optimal barrier was explicitly determined in [14]. Namely, it is the union of an arc of a circle and two arcs of logarithmic spirals.

Remark 4.5. By proving Conjecture 4.4 when R_0 is the unit disc, one would already solve a long standing open problem on the minimal construction speed $\sigma > 0$ needed to build a barrier that completely surrounds the fire [5, 6, 11, 25]. Indeed, it is known that an admissible barrier, consisting of simple closed curve that blocks the fire, can be constructed only if $\sigma > 2$. Therefore, if $\sigma \leq 2$, no admissible barrier of any kind can entirely surround the fire.

Ultimately, the relevance of the necessary conditions derived in [5, 13, 22] rests on the regularity of optimal barriers. This leads to:

Conjecture 4.6. *Assume that the initial burned set $R_0 \subset \mathbb{R}^2$ is bounded, open, connected, with smooth boundary. Then the optimal barrier Γ is connected, and consists of finitely many Lipschitz curves.*

Needless to say, the above conjecture can also be considered in the case of a more general multifunction F in (1.1), and more general smooth functions ψ, α, β in (1.3) and (1.6). However, already in the basic setting (4.2)–(4.4), proving (or disproving) Conjecture 4.6 appears to be a challenging task, which will require substantially new ideas.

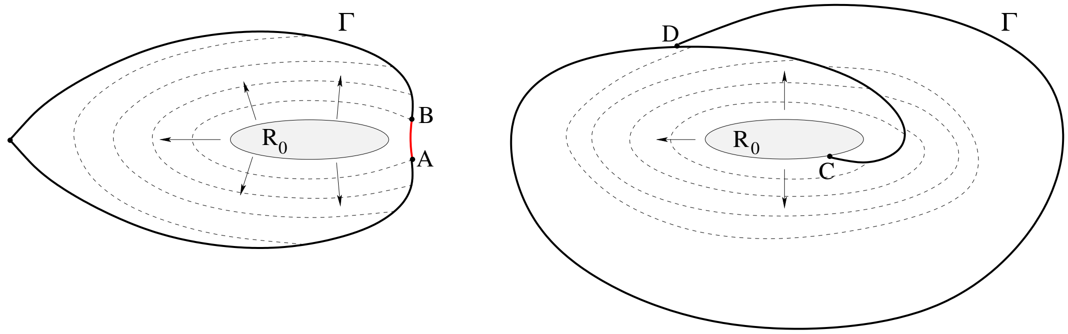

FIGURE 1. Left: a (presumably) optimal barrier Γ, blocking a fire initially burning on the region R_0. Here Γ is a simple closed curve. The arc AB, constructed during an initial time interval when the constraint (4.3) is not saturated, has constant curvature, i.e., it is an arc of circumference. Right: a spiral-like admissible barrier Γ. Here the arc CD, constructed during an initial time interval $[0,\tau]$, is a delaying arc. It is eventually surrounded by the fire on both sides. In both figures, the dotted lines represent the regions reached by the expanding fire front at various times.

REFERENCES

[1] L. Ambrosio, N. Fusco, and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, Oxford, 2000.

[2] J. P. Aubin and A. Cellina, Differential Inclusions. Springer-Verlag, Berlin, 1984.

[3] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser, Boston, 1997.

[4] V. G. Boltyanskii, Sufficient conditions for optimality and the justification of the dynamic programming principle, SIAM J. Control Optim. 4 (1966) 326–361.

[5] A. Bressan, Differential inclusions and the control of forest fires, J. Differential Equations 243 (2007) 179–207.

[6] A. Bressan, Dynamic blocking problems for a model of fire propagation, In: Advances in Applied Mathematics, Modeling, and Computational Science, pp. 11–40. R. Melnik and I. Kotsireas (ed.), Fields Institute Communications, Springer, New York, 2013.

[7] A. Bressan, M. Burago, A. Friend, and J. Jou, Blocking strategies for a fire control problem, Anal. Appl. 6 (2008) 229–246.

[8] A. Bressan and M. T. Chiri, On the regularity of optimal dynamic blocking strategies, Calc. Var. Part. Diff. Equat. 61 (2022) 36.

[9] A. Bressan and C. De Lellis, Existence of optimal strategies for a fire confinement problem, Comm. Pure Appl. Math. 62 (2009) 789–830.

[10] A. Bressan and B. Piccoli, Introduction to the Mathematical Theory of Control, AIMS Series in Applied Mathematics, Springfield 2007.

[11] A. Bressan and T. Wang, The minimum speed for a blocking problem on the half plane, J. Math. Anal. Appl. 356 (2009) 133–144.

[12] A. Bressan and T. Wang, Equivalent formulation and numerical analysis of a fire confinement problem, ESAIM; Control Optim. Calc. Var 16 (2010) 974–1001.

[13] A. Bressan and T. Wang, Global necessary conditions for a dynamic blocking problem, ESAIM; Control Optim. Calc. Var. 18 (2012) 124–156.

[14] A. Bressan and T. Wang, On the optimal strategy for an isotropic blocking problem. Calculus of Variations & PDE 45 (2012) 125–145.

[15] C. De Lellis and R. Robyr, Hamilton-Jacobi equations with obstacles, Arch. Rational Mech. Anal. 200 (2011) 1051–1073.

[16] W. H. Fleming and R. W. Rishel, Deterministic and Stochastic Optimal Control, Springer-Verlag, New York, 1975.
[17] S. Kim, R. Klein, D. Kübel, E. Langetepe, and B. Schwarzwald, Geometric firefighting in the half-plane, Comput. Geometry 95 (2021) 101728.
[18] F. Maggi, Sets of Finite Perimeter and Geometric Variational Problems, Cambridge University Press, Cambridge, 2012.
[19] G. D. Richards, An elliptical growth model of forest fire fronts and its numerical solution, Internat. J. Numer. Meth. Eng. 30 (1990) 1163–1179.
[20] A. L. Sullivan, Wildland surface fire spread modeling, 1990-2007. Internat. J. Wildland Fire 18 (2009) 349–403.
[21] R. Vinter, Optimal Control, Birkhäuser, Boston, 2000.
[22] T. Wang, Optimality conditions for a blocking strategy involving delaying arcs, J. Optim. Theory Appl. 152 (2012) 307–333.
[23] T. Wang, Dynamic sheltering problem and an existence theorem for admissible strategies. Set-Valued Var. Anal. 22 (2014) 391–446.
[24] T. Wang, An equivalence theorem on minimum sheltering speed for non-convex habitats. Set-Valued Var. Anal. 26 (2018), 531–560.
[25] M. Zizza, Some results on mixing flows and on the blocking fire problem. Ph.D. thesis, SISSA, Trieste, 2023. https://iris.sissa.it/handle/20.500.11767/134270