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# APPROXIMATE COINCIDENCE POINTS OF SET-VALUED MAPPINGS

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Dedicated to Professor Stojan Radenovic on the occasion of his 75th birthday

**Abstract.** In the present paper, we study the existence of approximate coincidence points of set-valued mappings in complete metric spaces and iterative schemes for their generation.
**Keywords.** Approximation; Coincidence point; Complete metric space; Set-valued mapping
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## 1. Introduction

Since the seminal result of Banach [2] the fixed point theory of nonexpansive mappings has been a rapidly growing field of research; see, e.g., [3, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 23, 24] and the reference mentioned therein. A significant progress has been done, in particular, in studies of common fixed point problems, which have important applications in engineering and medical sciences [6, 7, 22, 23, 24]. The study of coincidence points of nonlinear mappings is an important topic of the fixed point theory [1, 4, 5, 13, 14, 15].

In this paper, we study the existence of approximate coincidence points of set-valued mappings in complete metric spaces and iterative schemes for their approximation.

Assume that $(X,\rho)$ is a complete metric space. For each $x \in X$ and each $r > 0$, set

$$B(x,r) = \{y \in X :\ \rho(x,y) \leq r\}.$$

For each $x \in X$ and each set $A \subset X$, put $\rho(x,A) = \inf\{\rho(x,y) :\ y \in A\}$. Fix, for each pair of sets $A,B \subset X$, set

$$H(A,B) = \max\{\sup\{\rho(x,B) :\ x \in A\},\ \sup\{\rho(y,A) :\ y \in B\}\}.$$

For each $z \in R^1$, set $z_+ = \max\{z,0\}$. Fix $\theta \in X$.

Assume that $g : X \to X$ and $T : X \to 2^X \setminus \{\emptyset\}$. If $x \in X$ and $g(x) \in T(x)$, then the point $x$ is called a coincidence point, while the point $y = g(x)$ is called a point of coincidence. Usually in

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the literature it considered the case when $T$ is a single-valued map. In this paper $T$ is a general set-valued map.

## 2. The First Main Result

Assume that

$$T(X) \subset g(X), \tag{1}$$

$\phi : [0,\infty) \to [0,1]$ is a decreasing function,

$$\phi(t) < 1,\ t \in (0,\infty) \tag{2}$$

and that, for each $x, y \in X$,

$$H(T(x),T(y)) \le \phi(\rho(g(x),g(y)))\rho(g(x),g(y)). \tag{3}$$

In other words, $T$ is the Rakotch type mapping [19, 20].

In this paper, we prove the following result.

**Theorem 2.1.** *Let $\varepsilon \in (0,1)$, $M_1 > M > 0$,*

$$M_2 > \max\{M_1, 2M+1\},$$

$$g^{-1}(B(\theta, 2M+1)) \subset B(\theta, M_1), \tag{4}$$

$$g(B(\theta, M_1)) \subset B(\theta, M_2), \tag{5}$$

$$0 < \delta < 40^{-1}\varepsilon(1-\phi(\varepsilon/8)) \tag{6}$$

*and a natural number $n_0$ satisfy*

$$n_0 > 8(3+6M_2)\varepsilon^{-1}(1-\phi(\varepsilon/8))^{-1}. \tag{7}$$

*Assume that $\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty} \subset X$,*

$$\rho(\theta, x_0) \le M,\ \rho(x_0, y_0) < M \tag{8}$$

*and that for each integer $n \ge 0$,*

$$B(y_n, \delta) \cap T(x_n) \ne \emptyset,\ \rho(g(x_{n+1}), y) \le \delta \tag{9}$$

*and*

$$B(y_{n+1}, \delta) \cap \{\xi \in T(x_{n+1}) :\ \rho(\xi, y_n) \le \rho(y_n, T(x_{n+1})) + \delta\} \ne \emptyset. \tag{10}$$

*Then, for each integer $n \ge n_0+1$, $\rho(g(x_n), T(x_n)) < \varepsilon$.*

## 3. Auxiliary Results

**Lemma 3.1.** *Let the assumptions of Theorem 2.1 hold. Then $\rho(y_0, y_1) \le 3 + \rho(g(x_0), g(x_1))$, for each integer $n \ge 1$, $\rho(y_n, y_{n+1}) \le \phi((\rho(y_{n-1}, y_n) - 2\delta)_+)\rho(y_{n-1}, y_n)$ and if $\rho(y_n, y_{n-1})) \ge \varepsilon/4$, then $\rho(y_{n-1}, y_n)) - \rho(y_n, y_{n+1}) \ge 8^{-1}\varepsilon(1-\phi(\varepsilon/8))$.*

*Proof.* By (9), there exists

$$\xi_0 \in T(x_0) \cap B(y_0, \delta). \tag{11}$$

In view of (11) and (8),

$$\rho(x_0, \xi_0) \le \rho(x_0, y_0) + \delta < M + 1. \tag{12}$$

Let $n \ge 0$ be an integer. By (10), there exists

$$\xi_{n+1} \in T(x_{n+1}) \tag{13}$$

such that

$$\rho(y_{n+1}, \xi_{n+1}) \leq \delta \tag{14}$$

and

$$\rho(y_n, \xi_{n+1}) \leq \rho(y_n, T(x_{n+1})) + \delta. \tag{15}$$

Equations (3), (9), and (15) imply that

$$\begin{aligned}\rho(\xi_1, y_0) &\leq 1 + \rho(y_0, T(x_1))\\ &\leq 1 + \rho(y_0, T(x_0)) + H(T(x_0), T(x_1))\\ &\leq 2 + \rho(g(x_0), g(x_1)).\end{aligned}$$

Combined with (14) this implies that $\rho(y_0, y_1) \leq 3 + \rho(g(x_0), g(x_1))$. Let $n \geq 0$ be an integer. By (3), (11), (13), and (14), one has

$$\begin{aligned}\rho(y_n, T(x_{n+1})) &\leq \rho(y_n, \xi_n) + \rho(\xi_n, T(x_{n+1}))\\ &\leq \delta + H(T(x_n), T(x_{n+1}))\\ &\leq \delta + \phi(\rho(g(x_n), g(x_{n+1})))\rho(g(x_n), g(x_{n+1})).\end{aligned}$$

Let $n \geq 1$ be an integer. It follows from (2), (9), and the relation above that

$$\begin{aligned}\rho(y_n, T(x_{n+1})) &\leq \delta + \phi(\rho(g(x_n), g(x_{n+1})))\rho(g(x_n), g(x_{n+1}))\\ &\leq \delta + \phi(\rho(g(x_n), g(x_{n+1})))(\rho(g(x_n), y_{n-1}) + \rho(y_{n-1}, y_n) + \rho(y_n, g(x_{n+1})))\\ &\leq 3\delta + \phi(\rho(g(x_n), g(x_{n+1})))\rho(y_{n-1}, y_n).\end{aligned} \tag{16}$$

In view of (9), we have

$$\begin{aligned}\rho(g(x_n), g(x_{n+1})) &\geq \rho(y_{n-1}, y_n) - \rho(y_{n-1}, g(x_n)) - \rho(y_n, g(x_{n+1}))\\ &\geq \rho(y_{n-1}, y_n) - 2\delta.\end{aligned} \tag{17}$$

Equations (16) and (17) imply that

$$\rho(y_n, T(x_{n+1})) \leq 3\delta + \phi(\rho(y_{n-1}, y_n) - 2\delta)_+)\rho(y_{n-1}, y_n). \tag{18}$$

By (13), (15), and (18), we obtain

$$\begin{aligned}\rho(y_n, y_{n+1}) &\leq \rho(y_{n+1}, \xi_{n+1}) + \rho(\xi_{n+1}, y_n)\\ &\leq \delta + \rho(y_n, T(x_{n+1})) + \delta\\ &\leq 5\delta + \phi((\rho(y_{n-1}, y_n) - 2\delta)_+)\rho(y_{n-1}, y_n).\end{aligned} \tag{19}$$

If

$$\rho(y_{n-1}, y_n) \geq \varepsilon/4, \tag{20}$$

then it follows from (6), (19), and (20) that $\rho(y_n, y_{n+1}) \leq 5\delta + \phi(\varepsilon/8)\rho(y_{n-1}, y_n)$ and

$$\begin{aligned}\rho(y_{n-1}, y_n) - \rho(y_n, y_{n+1}) &\geq \rho(y_n, y_{n-1})(1 - \phi(\varepsilon/8)) - 5\delta\\ &\geq 4^{-1}\varepsilon(1 - \phi(\varepsilon/8)) - 5\delta\\ &\geq 8^{-1}\varepsilon(1 - \phi(\varepsilon/8)).\end{aligned}$$

Lemma 3.1 is proved. □

**Lemma 3.2.** *Assume that the assumptions of Theorem 2.1 hold. Then* $\rho(g(x_0), g(x_1)) \leq 2M_2$.

*Proof.* By (8) and (9), we have $\rho(g(x_1), y_0) \le \delta$ and

$$\rho(g(x_1),\theta) \le \rho(g(x_1),y_0) + \rho(y_0,\theta) \le 2M+1.$$

Together with (4) and (5), this implies $\rho(x_1,\theta) \le M_1,\ \rho(g(x_1),\theta) \le M_2$, and $\rho(g(x_1), g(x_0)) \le 2M_2$. Lemma 3.2 is proved. □

## 4. Proof of Theorem 2.1

We show that there exists $j \in \{1,\dots,n_0\}$ such that $\rho(y_{j-1}, y_j) \le \varepsilon/4$. Assume the contrary. Then, for each $i \in \{1,\dots .n_0\}$,

$$\rho(y_{i-1}, y_i) > \varepsilon/4. \tag{21}$$

Lemma 3.1 and (21) imply that, for each $i \in \{1,\dots,n_0\}$,

$$\rho(y_{i-1}, y_i) - \rho(y_i, y_{i+1}) \ge 8^{-1}\varepsilon(1-\phi(\varepsilon/8)). \tag{22}$$

Lemmas 3.1 and 3.2 imply that

$$\rho(y_0, y_1) \le 3 + 3\rho(g(x_0), g(x_1)) \le 3 + 6M_2. \tag{23}$$

In view of (22) and (23), we obtain

$$\begin{aligned} 3+6M_2 \ge \rho(y_0,y_1) &\ge \rho(y_0,y_1) - \rho(y_{n_0}, y_{n_0+1}) \\ &= \sum_{i=0}^{n_0-1} (\rho(y_i, y_{i+1}) - \rho(y_{i+1}, y_{i+2})) \\ &\ge 8^{-1} n_0 \varepsilon (1-\phi(\varepsilon/8)), \end{aligned}$$

$$n_0 \le 8\varepsilon^{-1}(3+6M_2)(1-\phi(\varepsilon/8))^{-1}.$$

This contradicts (7). The contradiction that we have reached proves that there exists

$$j \in \{1,\dots,n_0\} \tag{24}$$

for which

$$\rho(y_{j-1}, y_j) \le \varepsilon/4. \tag{25}$$

We show that, for each integer $i \ge j$, $\rho(y_i, y_{i+1}) \le \varepsilon/2$. Assume the contrary, Then there exists an integer $k > j$ for which

$$\rho(y_k, y_{k+1}) > \varepsilon/2. \tag{26}$$

By (25), we may assume without loss of generality that $\rho(y_i, y_{i+1}) \le \varepsilon/2,\ i = j,\dots,k-1$. In particular,

$$\rho(y_k, y_{k-1}) \le \varepsilon/2. \tag{27}$$

There are two cases: $\rho(y_k, y_{k-1}) > \varepsilon/4$ and $\rho(y_k, y_{k-1}) \le \varepsilon/4$. Assume $\rho(y_k, y_{k-1}) > \varepsilon/4$. Lemma 3.1 and (27) imply that $\rho(y_k, y_{k+1}) \le \rho(y_k, y_{k-1}) \le \varepsilon/2$. This contradicts (26). Therefore $\rho(y_k, y_{k-1}) \le \varepsilon/4$. Lemma 3.1 and (6) imply that

$$\rho(y_k, y_{k+1}) \le \rho(y_k, y_{k-1}) + 5\delta \le \varepsilon/4 + 5\delta \le \varepsilon/2.$$

This contradicts (26). The contradiction we have reached proves $\rho(y_i, y_{i+1}) \le \varepsilon/2$ for each integer $i \ge n_0$. Let $n \ge n_0$ be an integer. In view of the relation above, we have

$$\rho(y_n, y_{n+1}) \le \varepsilon/2. \tag{28}$$

It follows from (9) that

$$\rho(g(x_{n+1},y_n) \le \delta \tag{29}$$

and there exists

$$\xi \in T(x_{n+1}) \cap B(y_{n+1},\delta). \tag{30}$$

It follows from (6) and (28)-(30) that

$$\rho(g(x_{n+1}),T(x_{n+1})) \le \rho(g(x_{n+1}),y_n)+\rho(y_{n+1},y_n)+\rho(y_{n+1},\xi) \le \delta+\varepsilon/2+\delta < \varepsilon.$$

Theorem 2.1 is proved.

## 5. The Second Main Result

Assume that $g : X \to X$, $T : X \to 2^X \setminus \{\emptyset\}$,

$$T(X) \subset g(X), \tag{31}$$

$\phi : [0,\infty) \to [0,1]$ is a decreasing function,

$$\phi(t) < 1,\ t \in (0,\infty) \tag{32}$$

$$x_*, y_* \in X,\ \ y_* = g(x_*) \in T(x_*) \tag{33}$$

and that for each $x \in X$,

$$H(y_*,T(x)) \le \phi(\rho(g(x),y_*))\rho(g(x),y_*). \tag{34}$$

Fix $\theta \in X$. In this paper, we prove the following result.

**Theorem 5.1.** *Let* $\{\varepsilon_i\}_{i=0}^{\infty} \in (0,\infty)$,

$$\sum_{i=0}^{\infty} \varepsilon_i < \infty, \tag{35}$$

$\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset X$, *for each integer* $n \ge 0$,

$$B(y_n,\varepsilon_n) \cap T(x_n) \ne \emptyset, \tag{36}$$

$$\rho(g(x_{n+1}),y_n) \le \varepsilon_n \tag{37}$$

*and*

$$B(y_{n+1},\varepsilon_{n+1}) \cap \{\xi \in T(x_{n+1}) :\ \rho(\xi,y_*) \le \rho(y_*,T(x_{n+1}))+\varepsilon_{n+1}\} \ne \emptyset. \tag{58}$$

*Then*

$$y_* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} g(x_n),\ \lim_{n\to\infty} \rho(y_*,T(x_n)) = 0.$$

**Lemma 5.2.** *Assume that the assumptions of Theorem 5.1 hold. Then, for each integer* $n \ge 0$,

$$\rho(y_n,y_*) - \rho(y_*,y_{n+1}) \ge (1-\phi((\rho(y_*,y_n)-\varepsilon_n)_+))\rho(y_*,y_n) - 3\varepsilon_n.$$

*Proof.* By (36), there exists

$$\xi_0 \in T(x_0) \cap B(y_0,\varepsilon_0). \tag{39}$$

Let $n \ge 0$ be an integer. By (38), there exists

$$\xi_{n+1} \in T(x_{n+1}) \cap B(y_{n+1},\varepsilon_{n+1}) \tag{40}$$

such that

$$\rho(y_*,\xi_{n+1}) \le \rho(y_*,T(x_{n+1}))+\varepsilon_{n+1}. \tag{41}$$

By (34), we have

$$\rho(y_*, T(x_{n+1})) \le \phi(\rho(y_*, g(x_{n+1})))\rho(y_*, g(x_{n+1})).$$

It follows from (37) and (40)-(42) that

$$\begin{aligned}
\rho(y_{n+1}, y_*) &\le \rho(\xi_{n+1}, y_n) + \rho(y_{n+1}, \xi_{n+1}) \\
&\le 2\varepsilon_{n+1} + \rho(y_*, T(x_{n+1})) \\
&\le 2\varepsilon_{n+1} + \phi(\rho(y_*, g(x_{n+1})))\rho(y_*, g(x_{n+1})) \\
&\le 2\varepsilon_{n+1} + \phi((\rho(y_*, y_n) - \varepsilon_n)_+)\rho(y_*, y_n) + \varepsilon_{n+1} \\
&\le 3\varepsilon_{n+1} + \phi((\rho(y_*, y_n) - \varepsilon_n)_+)\rho(y_*, y_n)
\end{aligned}$$

and

$$\rho(y_n, y_*) - \rho(y_*, y_{n+1}) \ge (1 - \phi((\rho(y_*, y_n) - \varepsilon_n)_+))\rho(y_*, y_n) - 3\varepsilon_n.$$

Lemma 5.2 is proved. □

*Proof of Theorem 5.1.* We show that $\lim_{n\to\infty} \rho(y_n, y_*) = 0$. Let $\varepsilon > 0$. By (35), there exists a natural number $n_0$ such that

$$\sum_{n=n_0}^{\infty} \varepsilon_n < 24^{-1}\varepsilon(1 - \phi(\varepsilon/8)). \tag{43}$$

We show that there exists an integer $n_1 > n_0$ such that $\rho(y_*, y_n) \le \varepsilon/4$. Assume the contrary. Then, for each $n \ge n_0$, $\rho(y_n, y_*) > \varepsilon/4$. It follows from Lemma 5.2 that

$$\begin{aligned}
\rho(y_*, y_n) - \rho(y_*, y_{n+1}) &\ge \rho(y_*, y_n)(1 - \phi(\varepsilon/8)) - 3\varepsilon_n \\
&\ge 4^{-1}\varepsilon(1 - \phi(\varepsilon/8)) - 3\varepsilon_n.
\end{aligned}$$

This implies that, for each integer $p > n_0$,

$$\begin{aligned}
\rho(y_*, y_{n_0}) &\ge \rho(y_*, y_{n_0}) - \rho(y_*, y_p) \\
&= \sum_{n=n_0}^{p-1} (\rho(y_*, y_n) - \rho(y_*, y_{n+1})) \\
&\ge 4^{-1}(p - n_0)\varepsilon(1 - \phi(\varepsilon/8)) - 3\sum_{n=n_0}^{\infty} \varepsilon_n \to \infty
\end{aligned}$$

as $p \to \infty$. The contradiction we have reached proves that there exists an integer $n_1 > n_0$ for which $\rho(y_*, y_{n_1}) \le \varepsilon/4$. We show that, for each integer $i > n_1$, $\rho(y_*, y_i) \le \varepsilon/2$. Assume the contrary, Then there exists an integer $k > n_1$ for which

$$\rho(y_*, y_k) > \varepsilon/2. \tag{44}$$

We may assume without loss of generality that $\rho(y_*, y_i) \le \varepsilon/2$, $i = n_1, \dots, k-1$. In particular, $\rho(y_*, y_{k-1}) \le \varepsilon/2$. If $\rho(y_*, y_{k-1}) \ge \varepsilon/4$ then by Lemma 5.2 and (43)

$$\rho(y_*, y_k) \le \rho(y_*, y_{k-1}) \le \varepsilon/2,$$

which contradicts (44). Therefore $\rho(y_*, y_{k-1}) < \varepsilon/4$, which together with Lemma 5.2 implies that $\rho(y_*, y_k) \le \rho(y_*, y_{k-1}) + 3\varepsilon_k < \varepsilon/2$. This contradicts (44). The contradiction that we have reached proves Theorem 5.1.

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