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BOUNDARY CONTROL FOR OPTIMAL DATA TRANSPORT

DANIEL OWUSU ADU[1], ALEXANDER JAY BROWER[2], WEIWEI HU[2,*], ZHAIMING SHEN[3]

[1]Department of Mathematics and Statistics, Villanova University, Villanova, PA 19085, USA
[2]Department of Mathematics, University of Georgia, Athens, GA 30602, USA
[3]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

To the memory of Professor Sanjoy Kumar Mitter

Abstract. This paper discusses an optimal control design for data transport and exchange via domain boundaries. The mathematical model is governed by the transport equations which are driven by the incompressible velocity fields. The objective is to optimize the density distributions of the data to the desired ones through active control of the velocity for transporting data on a portion of the domain boundaries. We provide a rigorous proof of existence of an optimal control and establish the Gâteaux differentiability of the objective functional with respect to the boundary control inputs. Finally, we derive the first-order optimality conditions for solving such an optimal solution using a variational inequality.

Keywords. Boundary control; Data transport; First-order optimality conditions; Gâteaux differentiability; Variational inequality.

2020 Mathematics Subject Classification. 35Q49, 35Q93, 49J20, 49K20.

1. INTRODUCTION

In this work, we consider a basic mathematical model for control of data transport and exchange via the adjacent boundaries of data sets or the regions of interest. The problem is motivated by the application of dynamic clustering (e.g. [6, 12, 11]), power system (e.g. [27, 23]), traffic flow on road networks (e.g. [8, 16, 9]) as well as gas flow in pipelines (e.g. [4, 15]), etc. There is an extensive literature on boundary control of transport equations, where the trace of the density or mass distributions of the data is very often employed as the control input for steering the system behavior (e.g. [8, 13, 3, 5, 10]). However, our current work addresses the scenario where the velocity fields are controlled to optimally transport data across the domain boundaries.

To focus our discussion, we model data transport between two datasets, where the evolution of the density distributions is governed by transport equations driven by incompressible flows.

*Corresponding author.

E-mail address: Weiwei.Hu@uga.edu (W. Hu).

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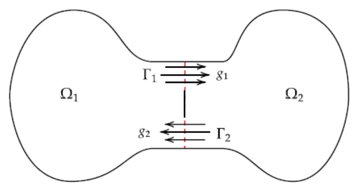
Specifically, we let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two open, bounded and connected domains with sufficiently regular boundaries $\partial\Omega_i, i = 1,2$ (say, $\mathscr{C}^2$), which may allow a finite number of corners. Assume that $\Omega_1 \cap \Omega_2$ has no interior points. Let $\Gamma_i \subset \partial\Omega_1 \cap \partial\Omega_2, i = 1,2$, be the open subsets of the intersection of the domain boundaries and $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. We further assume that $\partial(\Omega_1 \cup \Omega_2)$ is sufficiently regular. One configuration of our domains is demonstrated in Fig. 1.



Consider the distributions of two data sets $\theta_i, i = 1,2$, governed by the transport equations

$$\begin{cases} \frac{\partial \theta_1}{\partial t} + v_1 \cdot \nabla \theta_1 = 0, \\ \frac{\partial \theta_2}{\partial t} + v_2 \cdot \nabla \theta_2 = 0, \end{cases} \tag{1.1}$$

where the velocity fields $v_i, i = 1,2$ are assumed to be incompressible, i.e.

$$\nabla \cdot v_i = 0. \tag{1.2}$$

The boundary conditions are given by

$$\begin{cases} v_1 \cdot n_1|_{\Gamma_1} = g_1 \geq 0, \\ \theta_1|_{\Gamma_2} = \theta_1^I, \quad \text{where} \quad v_1 \cdot n_1|_{\Gamma_2} = -v_2 \cdot n_2|_{\Gamma_2} = -g_2 \leq 0, \\ v_1 \cdot n_1|_{\partial\Omega_1 \setminus (\Gamma_1 \cup \Gamma_2)} = 0, \\ v_1 \cdot \tau_1|_{\partial\Omega_1} = 0; \end{cases} \tag{1.3}$$

$$\begin{cases} v_2 \cdot n_2|_{\Gamma_2} = g_2 \geq 0, \\ \theta_2|_{\Gamma_1} = \theta_2^I, \quad \text{where} \quad v_2 \cdot n_2|_{\Gamma_1} = -v_1 \cdot n_1|_{\Gamma_1} = -g_1 \leq 0, \\ v_2 \cdot n_2|_{\partial\Omega_2 \setminus (\Gamma_1 \cup \Gamma_2)} = 0, \\ v_2 \cdot \tau_2|_{\partial\Omega_2} = 0, \end{cases} \tag{1.4}$$

and the initial conditions are given by

$$\theta_1(x,0) = \theta_{10}(x), \quad \theta_2(x,0) = \theta_{20}(x). \tag{1.5}$$

Here $g_i, i = 1,2$, stand for the controls of data transport and exchange between these data sets via the subsets of the boundaries $\Gamma_i \subset \partial\Omega_i$ and g_i has a compact support in Γ_i, i.e., $\text{supp} g_i \subset \overline{\Gamma_i}$ and $g_i = 0$ on $\partial\Omega_i \setminus \Gamma_i$; n_i and $\tau_i, i = 1,2$, stand for the unit outward normal and tangential vectors to the domain boundaries $\partial\Omega_i, i = 1,2$, respectively; and $\theta_i^I, i = 1,2$ are given inlet boundary data. In the current work, since we are mainly interested in control in the normal direction, we assume that $v_i \cdot \tau_i|_{\partial\Omega_i} = 0, i = 1,2$. Otherwise, we can always consider the general non-homogeneous Dirichlet boundary conditions for the velocity fields, i.e., $v_i(x,t)|_{\Gamma_i} = G_i(x,t)$. In this case, $g_i = G_i \cdot n_i|_{\Gamma_i}$.

In the application of clustering problems (e.g. [6, 12, 11]), one can group the spatially distributed data based on their movement or flow characteristics through the boundary control g_i at $\Gamma_i, i = 1,2$. On the other hand, in the corporate network problems g_i represents the firewall at Γ_i that regulates data flow to maintain security and prevent unauthorized access (e.g. [2, 24, 1]), while in traffic flow control problems (e.g. [14, 8, 16]), g_i ensures that traffic can only enter the region at certain interactions between different sections of the road networks.

1.1. **The sets of admissible controls.** Since it is not realistic to control the data distribution arbitrarily in space, in our current work we assume that $g_i(x,t), i=1,2$, is of the form

$$g_i(x,t) = \vec{u}_i(t)^T \vec{b}_i(x), \tag{1.6}$$

where the boundary input profiles $\vec{b}_i(x) \geq 0, i=1,2$, are prescribed spatial vector valued functions, which are sufficiently regular and localized in Γ_i. Without loss of generality, we assume that they are of the same dimension, i.e.,

$$\vec{b}_i(x) = (b_{i1}(x), \ldots, b_{iM}(x))^T, \quad i=1,2.$$

The vectors

$$\vec{u}_i(t) = (u_{i1}(t), \ldots, u_{iM}(t))^T, \quad i=1,2,$$

are our control input functions in time, which regulate the intensity of data transport via boundary.

This work aims to determine an optimal time-dependent control for data transport and exchange across the boundaries $\Gamma_i, i=1,2$. For a given final time $t_f > 0$, we seek optimal controls $\vec{u}_i(t) \geq 0, i=1,2$, that are sparse and steer the data toward two sets of desired distributions: $\Theta_i^d(x) \in L^2(\Omega_i)$ over the time interval $[0,t_f]$ and $\theta_i^d(x) \in L^2(\Omega_i)$ at the final time $t=t_f$. Set

$$U_{ad}^i = \{\vec{u}_i \in (L^2(0,t_f))^M \colon 0 \leq u_{ij} \leq \overline{u}_i, j=1,\ldots,M\}, \quad i=1,2, \tag{1.7}$$

for some $0 < \overline{u}_i < \infty, i=1,2$. The objective is to find $(\vec{u}_1, \vec{u}_2) \in U_{ad}^1 \times U_{ad}^2$ such that the following cost functional is minimized

$$\begin{aligned} J(\vec{u}_1,\vec{u}_2) =& \frac{1}{2}\sum_{i=1}^{2} \alpha_i \|\theta_i(t_f) - \theta_i^d\|_{L^2}^2 + \frac{1}{2}\sum_{i=1}^{2} \beta_i \int_0^{t_f} \|\theta_i(t) - \Theta_i^d\|_{L^2}^2 \, dt \\ &+ \sum_{i=1}^{2}\sum_{j=1}^{M} \gamma_{ij} \int_0^{t_f} u_{ij}(t)\, dt + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{M} \zeta_{ij} \int_0^{t_f} u_{ij}^2(t)\, dt, \end{aligned} \tag{P}$$

subject to (1.1)–(1.5) and (1.9), where $\alpha_i, \beta_i, \gamma_{ij}, \zeta_{ij} \geq 0, i=1,2; j=1,\ldots,M$, are the state and control weight parameters. Moreover, the pair of state weights α_i and β_i and the pair of control weights γ_{ij} and ζ_{ij} do not vanish simultaneously. In the case that $\zeta_{ij} = 0$ for all $i=1,2$ and $j=1,\ldots,M$, the L^1-penalization will promote the sparsity of the controls in time.

Note that problem (P) is non-convex due to the nonlinearity of the control-to-state map. The non-vanishing trace of velocity fields injects extra complication in characterizing the optimal control. To establish the well-posedness of the transport system (1.1)–(1.5) and the existence of an optimal solution to problem (P), the velocity fields must be regular enough. To this end, it is key to have regular boundary input profiles $\vec{b}_i(x)$ as well as the appropriate time control vectors $\vec{u}_i(t), i=1,2$, to ensure the needed regularity properties. Moreover, because of non-convexity, the optimal solution to problem (P) may not be unique in general.

In the following discussion, we shall first address the existence of an optimal solution to problem (P) and then derive the first-order necessary conditions for solving such a solution based on a variational inequality.

1.2. **Preliminaries.** We first introduce the notations and spaces used in the rest of this work. Let

$$V_n^0(\Omega_i) = \{v_i \in L^2(\Omega_i) : \text{div } v_i = 0,\ v_i \cdot n_i|_{\partial\Omega_i \setminus (\Gamma_1 \cup \Gamma_2)} = 0\}, \quad i = 1,2,$$

and $H^s_{00}(\Gamma_i)$ be the set of functions belonging to $H^s(\Gamma_i), s \geq 0$, whose extension by 0 to the entire boundary $\partial\Omega_i$ belongs to $H^s(\partial\Omega_i)$ (e.g. [22, p. 66]). The notations $(\cdot,\cdot)_{\Omega_i}$ and $\langle \cdot,\cdot \rangle_{\Gamma_i}$, $i = 1,2$, denote the inner product or the duality as indicated over the domain Ω_i and the boundary Γ_i, respectively. We also introduce following measures on $(0,t_f) \times \Gamma_i$:

$$d\mu_{v_i} = (v_i \cdot n_i)\, dxdt, \quad i = 1,2.$$

Let $d\mu^+_{v_i} = (v_i \cdot n_i)^+\, dxdt$ and $d\mu^-_{v_i} = (v_i \cdot n_i)^-\, dxdt$ are positive and negative parts of $d\mu_{v_i}$. Then $d\mu_{v_i} = d\mu^+_{v_i} - d\mu^-_{v_i}$. Furthermore, due to the divergence free condition (1.2), we have

$$\int_{\Gamma_1 \cup \Gamma_2} v_i \cdot n_i\, dx = 0, \quad i = 1,2,$$

which implies

$$\int_{\Gamma_1} g_1\, dx - \int_{\Gamma_2} g_2\, dx = 0, \tag{1.8}$$

or

$$\int_{\Gamma_1} \vec{u}_1^T \vec{b}_1\, dx - \int_{\Gamma_2} \vec{u}_2^T \vec{b}_2\, dx = 0. \tag{1.9}$$

This property, however induces an additional constraint on the boundary control.

Let $v = v_1 + v_2$ (i.e., $v(x) = v_i(x)$ for $x \in \Omega_i$), $\theta = \theta_1 + \theta_2$ (i.e., $\theta(x) = \theta_i(x)$ for $x \in \Omega_i$), be defined over the entire domain $\Omega = \Omega_1 \cup \Omega_2$, and n be the unit outward normal vector to $\partial\Omega$. It is easy to verify that the total mass of these two data sets is conserved using the divergence free conditions (1.2) and $(v_1 + v_2) \cdot n|_{\partial\Omega} = 0$. In fact,

$$\begin{aligned}
\frac{d \int_\Omega \theta\, dx}{dt} &= \frac{d \int_{\Omega_1} \theta_1\, dx}{dt} + \frac{d \int_{\Omega_2} \theta_2\, dx}{dt} \\
&= -\int_{\Omega_1} \nabla \cdot (v_1 \theta_1)\, dx - \int_{\Omega_2} \nabla \cdot (v_2 \theta_2)\, dx \qquad (1.10)\\
&= -\int_{\Omega_1 \cup \Omega_2} \nabla \cdot (v\theta)\, dx = -\int_{\partial\Omega} (v\theta) \cdot n\, dx = 0, \qquad (1.11)
\end{aligned}$$

which follows

$$\int_\Omega \theta\, dx = \int_{\Omega_1} \theta_1\, dx + \int_{\Omega_2} \theta_2\, dx = \int_{\Omega_1} \theta_{10}\, dx + \int_{\Omega_2} \theta_{20}\, dx, \quad \forall t > 0.$$

Moreover, one can show that $\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}$ for any $p \in [1,\infty]$ and $t > 0$. Due to the advection term $v_i \cdot \nabla\theta_i$, the control-to-state map

$$(\vec{u}_1, \vec{u}_2) \mapsto (\theta_1, \theta_2) \tag{1.12}$$

is nonlinear. As a result, problem (P) is non-convex. As discussed in (e.g. [17, 18, 20, 21]), to guarantee the Gâteaux differentiability of the control-to-state map (1.12) we need velocity fields regular enough such that

$$\sup_{0 \leq t \leq t_f} \|\nabla\theta_i\|_{L^2} < \infty, i = 1,2. \tag{1.13}$$

In fact, it is known that if $\theta_0 \in H^1(\Omega)$ and $\int_0^{t_f} \|\nabla v\|_{L^\infty}\, dt < \infty$, then $\sup_{t\in[0,t_f]} \|\nabla\theta\|_{L^2} < \infty$ (e.g. [19] and the references cited therein), and consequently (1.13) holds. With the help of Agmon's inequality and trace theorem, we know that if $\vec{b}_i \in (H^2_{00}(\Gamma_i))^M$ and $\vec{u}_i \in (L^2(0,t_f))^M$, then $v_i \in L^2(0,t_f;H^{5/2}(\Omega_i))$, and hence $\int_0^{t_f} \|\nabla v\|_{L^\infty}\, dt < \infty$ is satisfied. More precisely, to understand the relation between the boundary data and the velocity fields, we define the lifting operators (e.g. [22, p. 42], [25, (1.67), p. 173]),

$$D_i\colon H^s_{00}(\Gamma_1)\times H^s_{00}(\Gamma_2) \to H^{s+1/2}(\Omega_i), \quad s \geq 0,$$

such that

$$V_i = D_i(\phi_{i1},\phi_{i2}) = D_i(\phi_{i1},0) + D_i(0,\phi_{i2})$$

satisfies

$$\nabla\cdot V_i = 0 \quad \text{in} \quad \Omega_i, \tag{1.14}$$

$$V_i\cdot n_i|_{\Gamma_1} = \phi_{i1}, \quad V_i\cdot n_i|_{\Gamma_2} = \phi_{i2}, \tag{1.15}$$

for any $(\phi_{i1},\phi_{i2}) \in H^s_{00}(\Gamma_1)\times H^s_{00}(\Gamma_2), i = 1,2$. If $b_{ij} \in H^2_{00}(\Gamma_i)$ and $\vec{u}_i \in U^i_{ad}$, $i = 1,2; j = 1,2,\ldots,M$, then the velocity defined via (1.2)–(1.4) satisfying

$$v_1 = D_1(g_1,-g_2) = D_1(g_1,0) - D_1(0,g_2) \in L^2(0,T;H^{5/2}(\Omega_1)), \tag{1.16}$$

$$v_2 = D_2(-g_1,g_2) = -D_2(g_1,0) + D_2(0,g_2) \in L^2(0,T;H^{5/2}(\Omega_2)). \tag{1.17}$$

In this case, our velocity fields are regular enough to establish the Gâteaux differentiability of the control-to-state map (1.12). Therefore, the cost functional J is Gâteaux differentiable with respect to $(\vec{u}_1,\vec{u}_2) \in U^1_{ad}\times U^2_{ad}$.

2. Existence of an Optimal Solution

In this section, we prove the existence of an optimal solution to problem (P). To this end, we first introduce the weak solution to the scalar transport system (1.1)–(1.5). It is natural to assume that the initial data $\theta_{i0} \in L^\infty(\Omega_i)$ and the boundary data $\theta^I_i \in L^\infty((0,t_f)\times\Gamma_i, d\mu^-_{v_i})$, $i = 1,2$. According to [7, Theorem 4.1], we have the following result for the existence of a unique weak solution to (1.1)–(1.5).

Proposition 2.1. *Let $t_f > 0$. Assume that $g_i \in L^2(0,t_f;H^2_{00}(\Gamma_i)), i = 1,2,$ satisfy* (1.8) *and the velocities v_1 and v_2 are given by* (1.16)–(1.17). *For any initial datum $(\theta_{10},\theta_{20}) \in L^\infty(\Omega_1)\times L^\infty(\Omega_2)$ and any boundary datum $(\theta^I_1,\theta^I_2) \in L^\infty((0,t_f)\times\Gamma_1,d\mu^-_{v_1})\times L^\infty((0,t_f)\times\Gamma_2,d\mu^-_{v_2})$, there exists a unique pair of $(\theta_1,\theta_2) \in L^\infty((0,t_f)\times\Omega_1))\times L^\infty((0,t_f)\times\Omega_2))$ and $(\theta^O_1,\theta^O_2) \in L^\infty((0,t_f)\times\Gamma_1;d\mu^+_{v_1})\times L^\infty((0,t_f)\times\Gamma_2;d\mu^+_{v_2})$ such that*

$$\begin{aligned}\int_0^{t_f}\int_{\Omega_1} \theta_1\left(\frac{\partial\rho_1}{\partial t} + v_1\cdot\nabla\rho_1\right) dx\,dt + \int_{\Omega_1}\theta_{10}\rho_1(0)\,dx \\ - \int_0^{t_f}\int_{\Gamma_1}\theta^O_1 g_1\rho_1\,dx\,dt + \int_0^{t_f}\int_{\Gamma_2}\theta^I_1 g_2\rho_1\,dx\,dt = 0,\end{aligned} \tag{2.1}$$

$$\begin{aligned}\int_0^{t_f}\int_{\Omega_2} \theta_2\left(\frac{\partial\rho_2}{\partial t} + v_2\cdot\nabla\rho_2\right) dx\,dt + \int_{\Omega_2}\theta_{20}\rho_2(0)\,dx \\ - \int_0^{t_f}\int_{\Gamma_2}\theta^O_2 g_2\rho_2\,dx\,dt + \int_0^{t_f}\int_{\Gamma_1}\theta^I_2 g_1\rho_2\,dx\,dt = 0,\end{aligned} \tag{2.2}$$

for any $(\rho_1,\rho_2)\in\mathscr{C}^1([0,t_f]\times\overline{\Omega}_1)\times\mathscr{C}^1([0,t_f]\times\overline{\Omega}_2)$ *with* $(\rho_1(t_f),\rho_2(t_f))=(0,0)$*. In particular,*

$$\theta_1^O=\theta_2^I \quad \text{on} \quad \Gamma_1 \quad \text{and} \quad \theta_2^O=\theta_1^I \quad \text{on} \quad \Gamma_2. \tag{2.3}$$

Furthermore, (θ_1,θ_2) *is continuous in time with values in* L^q *for any* $1\le q<\infty$*.*

To see (2.3), from (1.10)–(1.11) and (1.3)–(1.4), we have

$$\int_{\Gamma_1\cup\Gamma_2}(v_1\theta_1)\cdot n_1\,dx+\int_{\Gamma_1\cup\Gamma_2}(v_2\theta_2)\cdot n_2\,dx=\int_{\Omega_1}\nabla\cdot(v_1\theta_1)\,dx+\int_{\Omega_2}\nabla\cdot(v_2\theta_2)\,dx=0,$$

which implies

$$\begin{aligned}&\int_{\Gamma_1}\theta_1^O g_1-\int_{\Gamma_2}\theta_1^I g_2\,dx+\int_{\Gamma_2}\theta_2^O g_2\,dx-\int_{\Gamma_1}\theta_2^I g_1\,dx\\&=\int_{\Gamma_1}(\theta_1^O-\theta_2^I)g_1-\int_{\Gamma_2}(\theta_1^I-\theta_2^O)g_2\,dx=0,\end{aligned}\tag{2.4}$$

independent of $g_i, i=1,2$. Thus (2.3) holds.

With Proposition 2.1 at our disposal, we are in a position to establish the existence.

Theorem 2.2. *For* $t_f>0$*,* $(\theta_{10},\theta_{20})\in L^\infty(\Omega_1)\times L^\infty(\Omega_2)$*,* $(\vec{b}_1,\vec{b}_2)\in(H^2_{00}(\Gamma_1))^M\times(H^2_{00}(\Gamma_2))^M$ *and* $(\theta_1^I,\theta_2^I)\in L^\infty((0,t_f)\times\Gamma_1,d\mu^-_{v_1})\times L^\infty((0,t_f)\times\Gamma_2,d\mu^-_{v_2})$*, there exists at least one optimal solution* $(\vec{u}_1,\vec{u}_2)\in U^1_{ad}\times U^2_{ad}$ *to problem (P).*

Proof. Since J is bounded from below, we can choose a minimizing sequence $\{\vec{u}_{1m},\vec{u}_{2m}\}\subset U^1_{\text{ad}}\times U^2_{\text{ad}}$ such that

$$\lim_{m\to\infty}J(\vec{u}_{1m},\vec{u}_{2m})=\inf_{(\vec{u}_1,\vec{u}_2)\in U^1_{\text{ad}}\times U^2_{\text{ad}}}J(\vec{u}_1,\vec{u}_2). \tag{2.5}$$

By the definition of J, the sequence $\{(\vec{u}_{1m},\vec{u}_{2m})\}$ is uniformly bounded in $U^1_{\text{ad}}\times U^2_{\text{ad}}$, and hence there exists a weakly convergent subsequence, still denoted by $\{(\vec{u}_{1m},\vec{u}_{2m})\}$, such that

$$(\vec{u}_{1m},\vec{u}_{2m})\to(\vec{u}_1^*,\vec{u}_2^*) \quad \text{weakly in} \quad L^2(0,t_f). \tag{2.6}$$

Correspondingly,

$$\begin{aligned}(v_{1m},v_{2m})&=\Big(D_1(\vec{u}_{1m}^T\vec{b}_1,-\vec{u}_{2m}^T\vec{b}_2),D_2(\vec{u}_{2m}^T\vec{b}_2,-\vec{u}_{1m}^T\vec{b}_1\Big)\\&\to\Big(D_1(\vec{u}_1^{*T}\vec{b}_1,-\vec{u}_2^{*T}\vec{b}_2),D_2(\vec{u}_2^{*T}\vec{b}_2,-\vec{u}_1^{*T}\vec{b}_1\Big)\\&=(v_1^*,v_2^*) \quad \text{weakly in} \quad L^2(0,t_f;H^{5/2}(\Omega)).\end{aligned}\tag{2.7}$$

Let $\{(\theta_{1m},\theta_{2m})\}$ be the sequence of solutions corresponding to $\{(\vec{u}_{1m},\vec{u}_{2m})\}$ with $\theta_{1m}^I=\theta_1^I$ on Γ_2 and $\theta_{2m}^I=\theta_2^I$ on Γ_1, and $(\theta_{1m}(0),\theta_{2m}(0))=(\theta_{10},\theta_{20})\in L^\infty(\Omega_1)\times L^\infty(\Omega_2)$. Then by Proposition 2.1 we have $(\theta_{1m},\theta_{2m})\in L^\infty((0,t_f)\times\Omega_1)\times L^\infty((0,t_f)\times\Omega_2)$, and thus $v_{im}\theta_{im}\in L^2(0,t_f;L^2(\Omega_i)), i=1,2$. In fact,

$$\int_0^{t_f}\|v_{im}\theta_{im}\|^2_{L^2}\,dt\le\int_0^{t_f}\|v_{im}\|^2_{L^2}\,dt\sup_{t\in[0,t_f]}\|\theta_{im}\|^2_{L^\infty}\le Ct_f\max\{\overline{u}_1^2,\overline{u}_2^2\}\max\{\|\theta_{10}\|^2_{L^\infty},\|\theta_{20}\|^2_{L^\infty}\}$$

for some constant $C>0$. Moreover, using Stokes formula and trace theorem it is easy to verify that $v_{im}\cdot\nabla\theta_{im}\in L^2(0,t_f;(H^1(\Omega_i))')$ uniformly in m, and hence $\frac{\partial\theta_{im}}{\partial t}\in L^2(0,t_f;(H^1(\Omega_i))')$,

$i=1,2$, uniformly in m. We may extract subsequences, still denoted by $\{(\theta_{1m},\theta_{2m})\}$ and $\{(v_{1m},v_{2m})\}$ such that

$$(v_{1m}\theta_{1m},v_{2m}\theta_{2m})\to(v_1^*\theta_1^*,v_2^*\theta_2^*)\quad\text{weakly in } L^2((0,t_f)\times\Omega_1)\times L^2((0,t_f)\times\Omega_2),\tag{2.8}$$

$$(\theta_{1m},\theta_{2m})\to(\theta_1^*,\theta_2^*)\quad\text{weakly in } L^2((0,t_f)\times\Omega_1)\times L^2((0,t_f)\times\Omega_2),\tag{2.9}$$

$$(\theta_{1m},\theta_{2m})\to(\theta_1^*,\theta_2^*)\quad\text{weak}^*\text{ in } L^\infty((0,t_f)\times\Omega_1)\times L^\infty((0,t_f)\times\Omega_2),\tag{2.10}$$

and

$$\left(\frac{\partial\theta_{1m}}{\partial t},\frac{\partial\theta_{2m}}{\partial t}\right)\to\left(\frac{\partial\theta_1^*}{\partial t},\frac{\partial\theta_2^*}{\partial t}\right)\quad\text{weakly in } L^2(0,t_f;(H^1(\Omega_1))')\times L^2(0,t_f;(H^1(\Omega_2))').\tag{2.11}$$

By Aubin-Lions lemma, we know that

$$(\theta_{1m},\theta_{2m})\to(\theta_1^*,\theta_2^*)\quad\text{strongly in } (H^1(\Omega_1))'\times(H^1(\Omega_2))',\text{continuous in } t\in[0,t_f],\tag{2.12}$$

thus

$$(\theta_{1m}(0),\theta_{2m}(0))=(\theta_{10},\theta_{20})\to(\theta_{10}^*,\theta_{20}^*)\quad\text{strongly in } (H^1(\Omega_1))'\times(H^1(\Omega_2))'.\tag{2.13}$$

Next we verify that (θ_1^*,θ_2^*) is a pair of solution corresponding to $(\vec{u}_1^*,\vec{u}_2^*)$ based on (2.1)–(2.2). Note that $(\vec{u}_{1m},\vec{u}_{2m})$ and $(\theta_{1m},\theta_{2m})$ satisfy

$$\begin{aligned}&\int_0^{t_f}\int_{\Omega_1}\theta_{1m}\frac{\partial\rho_1}{\partial t}\,dx\,dt+\int_0^{t_f}\int_{\Omega_1}\theta_{1m}(v_{1m}\cdot\nabla\rho_1)\,dx\,dt+\int_{\Omega_1}\theta_{10}\rho_1(0)\,dx\\&\qquad-\int_0^{t_f}\vec{u}_{1m}\int_{\Gamma_1}\theta_1^O\vec{b}_1\rho_1\,dx\,dt+\int_0^{t_f}\vec{u}_{2m}\int_{\Gamma_2}\theta_1^I\vec{b}_2\rho_1\,dx\,dt=0,\end{aligned}\tag{2.14}$$

$$\begin{aligned}&\int_0^{t_f}\int_{\Omega_2}\theta_{2m}\frac{\partial\rho_2}{\partial t}\,dx\,dt+\int_0^{t_f}\int_{\Omega_2}\theta_{2m}(v_{2m}\cdot\nabla\rho_2)\,dx\,dt+\int_{\Omega_2}\theta_{20}\rho_2(0)\,dx\\&\qquad-\int_0^{t_f}\vec{u}_{2m}\int_{\Gamma_2}\theta_2^O\vec{b}_2\rho_2\,dx\,dt+\int_0^{t_f}\vec{u}_{1m}\int_{\Gamma_1}\theta_2^I\vec{b}_1\rho_2\,dx\,dt=0,\end{aligned}\tag{2.15}$$

for any $(\rho_1,\rho_2)\in\mathscr{C}^\infty([0,t_f]\times\overline{\Omega}_1)\times\mathscr{C}^\infty([0,t_f]\times\overline{\Omega}_2)$ with $(\rho_1(t_f),\rho_2(t_f))=(0,0)$.

Since $\partial_t\rho_i\in L^2(0,T;L^2(\Omega_i))$, $\nabla\rho_i\in L^2((0,t_f)\times\Omega)$, $\rho_i(0)\in H^1(\Omega_i), i=1,2$, and

$$\int_{\Gamma_i}\theta_i^O\vec{b}_i\rho_i\,dx\in L^2(0,t_f),\ i=1,2,\qquad\int_{\Gamma_2}\theta_1^I\vec{b}_2\rho_1\,dx\in L^2(0,t_f),\qquad\int_{\Gamma_1}\theta_2^I\vec{b}_1\rho_2\,dx\in L^2(0,t_f),$$

with the help of (2.6), (2.9) and (2.12)–(2.13), it is straightforward to pass to the limit in all the terms on the left hand side of (2.14)–(2.15). Finally, by density (2.14)–(2.15) hold for any $(\rho_1,\rho_2)\in\mathscr{C}^1([0,t_f]\times\overline{\Omega}_1)\times\mathscr{C}^1([0,t_f]\times\overline{\Omega}_2)$ with $(\rho_1(t_f),\rho_2(t_f))=(0,0)$. Therefore, (θ_1^*,θ_2^*) is a pair of weak solution associated with $(\vec{u}_1^*,\vec{u}_2^*)$ to the transport system (1.1)–(1.5).

Lastly, using the lower semicontinuity of the cost functional J over $U_{\text{ad}}^1\times U_{\text{ad}}^2$, we have

$$J(\vec{u}_1^*,\vec{u}_2^*)\le\lim_{m\to\infty}\inf J(\vec{u}_{1m}^*,\vec{u}_{2m}^*),$$

and hence $(\vec{u}_1^*,\vec{u}_2^*)$ is a pair of optimal solution to (P), which completes the proof. □

3. First-order Optimality Conditions

Next we derive the first-order necessary optimality conditions for solving problem (P) using a variational inequality (e.g. [22, 26]). To this end, we further assume that $(\theta_{10},\theta_{20}) \in (L^\infty(\Omega_1)\cap H^1(\Omega_1))\times(L^\infty(\Omega_2)\cap H^1(\Omega_2))$.

To deal with the additional boundary constraint (1.9), we introduce a Lagrangian function $L\colon U^1_{ad}\times U^2_{ad}\times L^2(0,t_f)\to\mathbb{R}$ by

$$L(\vec{u}_1,\vec{u}_2,\lambda) = J(\vec{u}_1,\vec{u}_2) + \int_0^{t_f}\lambda G(\vec{u}_1,\vec{u}_2)\,dt, \tag{3.1}$$

where $G(\vec{u}_1,\vec{u}_2) = \int_{\Gamma_1}\vec{u}_1^T\vec{b}_1\,dx - \int_{\Gamma_2}\vec{u}_2^T\vec{b}_2\,dx$ and $\lambda=\lambda(t)$ is the corresponding Lagrangian multiplier. If $(\vec{u}_1,\vec{u}_2)^T$ is an optimal solution to problem (P), then there exists $\lambda\in L^2(0,t_f)$ such that

$$D_{(\vec{u}_1,\vec{u}_2)}L(\vec{u}_1,\vec{u}_2,\lambda)\cdot\big((\vec{h}_1,\vec{h}_2)-(\vec{u}_1,\vec{u}_2)\big)\geq 0, \quad \forall(\vec{h}_1,\vec{h}_2)\in U^1_{ad}\times U^2_{ad}, \tag{3.2}$$

where $D_{(\vec{u}_1,\vec{u}_2)}$ denotes the partial Gâteaux derivative with respect to $(\vec{u}_1,\vec{u}_2)$. The associated variational inequality reads, in explicit form,

$$J'(\vec{u}_1,\vec{u}_2)\cdot\big((\vec{h}_1,\vec{h}_2)-(\vec{u}_1,\vec{u}_2)\big) + \int_0^{t_f}\lambda G'(\vec{u}_1,\vec{u}_2)\cdot\big((\vec{h}_1,\vec{h}_2)-(\vec{u}_1,\vec{u}_2)\big)\,dt\geq 0, \tag{3.3}$$

$\forall(\vec{h}_1,\vec{h}_2)^T\in U^1_{ad}\times U^2_{ad}$.

To analyze the Gâteaux derivative of J with respect to $(\vec{u}_1,\vec{u}_2)$, we first derive the first variation of our control-to-state map (1.12). To this end, we let $z_i=\theta_i'(\vec{u}_1,\vec{u}_2)\cdot(\vec{h}_1,\vec{h}_2)^T$ and $w_i = v_i'(\vec{u}_1,\vec{u}_2)\cdot(\vec{h}_1,\vec{h}_2)^T$ be the Gâteaux derivatives of θ_i and v_i, respectively, with respect to $(\vec{u}_1,\vec{u}_2)$ in the direction of $(\vec{h}_1,\vec{h}_2)^T\in U^1_{ad}\times U^2_{ad}$. Then $(z_1,z_2)^T$ and $(w_1,w_2)^T$ satisfy the following transport system

$$\begin{cases} \frac{\partial z_1}{\partial t} + v_1\cdot\nabla z_1 + w_1\cdot\nabla\theta_1 = 0,\\ \frac{\partial z_2}{\partial t} + v_2\cdot\nabla z_2 + w_2\cdot\nabla\theta_2 = 0, \end{cases} \tag{3.4}$$

where $\nabla\cdot w_i = 0$ and (θ_1,θ_2) satisfies (1.1)–(1.5). The boundary conditions are given by

$$\begin{cases} w_1\cdot n_1|_{\Gamma_1} = \vec{h}_1^T\vec{b}_1\geq 0,\\ z_1|_{\Gamma_2}=0, \quad \text{where} \quad w_1\cdot n_1|_{\Gamma_2} = -w_2\cdot n_2|_{\Gamma_2} = -\vec{h}_2^T\vec{b}_2\leq 0,\\ w_1\cdot n_1|_{\partial\Omega_1\setminus(\Gamma_1\cup\Gamma_2)} = 0,\\ w_1\cdot\tau_1|_{\partial\Omega_1}=0, \end{cases} \tag{3.5}$$

and

$$\begin{cases} w_2\cdot n_2|_{\Gamma_2} = \vec{h}_2^T\vec{b}_2\geq 0,\\ z_2|_{\Gamma_1}=0, \quad \text{where} \quad w_2\cdot n_2|_{\Gamma_1} = -w_1\cdot n_1|_{\Gamma_1} = -\vec{h}_1^T\vec{b}_1\leq 0,\\ w_2\cdot n_2|_{\partial\Omega_2\setminus(\Gamma_1\cup\Gamma_2)} = 0,\\ w_2\cdot\tau_2|_{\partial\Omega_2}=0. \end{cases} \tag{3.6}$$

The initial conditions become

$$z_1(x,0)=0, \quad z_2(x,0)=0. \tag{3.7}$$

According to (1.16)–(1.17), we have

$$\begin{aligned}(w_1,w_2) &= \Big(D_1(\vec{h}_1^T\vec{b}_1,-\vec{h}_2^T\vec{b}_2),\ D_2(-\vec{h}_1^T\vec{b}_1,\vec{h}_2^T\vec{b}_2,\Big)\\ &= \left(\sum_{j=1}^{M}\Big(h_{1j}D_1(b_{1j},0)-h_{2j}D_1(0,b_{2j})\Big),\ \sum_{j=1}^{M}\Big(-h_{1j}D_2(b_{1j},0)+h_{2j}D_2(0,b_{2j}\Big)\right)\\ &\in L^2(0,t_f;H^{5/2}(\Omega_i)). \qquad (3.8)\end{aligned}$$

Recall that $\sup_{0\le t\le t_f}\|\nabla\theta_i\|_{L^2}<\infty$ as discussed in Section 1.1. We have $w_i\cdot\nabla\theta_i\in L^1(0,t_f;L^2(\Omega_i)), i=1,2$. By [7, Theorem 6.1], there exists a unique pair of solutions

$$(z_1,z_2)\in L^\infty(0,t_f;L^2(\Omega_1))\times L^\infty(0,t_f;L^2(\Omega_2))$$

satisfying (3.4)–(3.7).

Next, let (ρ_1,ρ_2) be the adjoint state of (θ_1,θ_2) satisfying

$$\begin{cases} -\frac{\partial\rho_1}{\partial t}-v_1\cdot\nabla\rho_1=\beta_1(\theta_1-\Theta_1^d),\\ -\frac{\partial\rho_2}{\partial t}-v_2\cdot\nabla\rho_2=\beta_2(\theta_2-\Theta_2^d),\end{cases} \qquad (3.9)$$

with the boundary conditions

$$\rho_i|_{\Gamma_1}=0,\quad \rho_i|_{\Gamma_2}=0,\quad i=1,2, \qquad (3.10)$$

and the final time conditions

$$\rho_1(x,t_f)=\alpha_1(\theta_1(x,t_f)-\theta_1^d),\quad \rho_2(x,t_f)=\alpha_2(\theta_2(x,t_f)-\theta_2^d), \qquad (3.11)$$

where (v_1,v_2) satisfies (1.2)–(1.4). Since $\theta_i-\Theta_i^d\in L^2((0,t_f)\times\Omega_i)$ and $\theta_i(t_f)-\theta_i^d\in L^2(\Omega_i), i=1,2$, following the same argument as in [7, Theorem 6.1], we know that there exists a unique pair of weak solutions to (3.9)–(3.11) satisfying

$$(\rho_1,\rho_2)\in L^\infty(0,t_f;L^2(\Omega_1))\times L^\infty(0,t_f;L^2(\Omega_2)).$$

The following theorem states the first-order necessary conditions for solving the optimal solution to problem (P).

Theorem 3.1. *Let $(\theta_{10},\theta_{20})\in(L^\infty(\Omega_1)\cap H^1(\Omega_1))\times(L^\infty(\Omega_2)\cap H^1(\Omega_2))$, $(\vec{b}_1,\vec{b}_2)\in(H_{00}^2(\Gamma_1))^M\times(H_{00}^2(\Gamma_2))^M$, and $(\theta_1^I,\theta_2^I)\in L^\infty((0,t_f)\times\Gamma_1,d\mu_{v_1}^-)\times L^\infty((0,t_f)\times\Gamma_2,d\mu_{v_2}^-)$. If $(\vec{u}_1,\vec{u}_2)\in U_{ad}^1\times U_{ad}^2$ is an optimal solution to problem (P), (θ_1,θ_2) and (ρ_1,ρ_2) are the corresponding solutions to state equations* (1.1)–(1.5) *and the adjoint equations* (3.9)–(3.11)*, respectively, then there exists some $\lambda\in L^2(0,t_f)$ such that $(\vec{u}_1,\vec{u}_2)$ satisfies*

$$\begin{aligned}u_{1j}(t)=\mathscr{P}_{[0,\bar{u}_1]}\Big\{\frac{1}{\zeta_{1j}}\Big[&\Big(D_1(b_{1j},0),\rho_1\nabla\theta_1\Big)_{\Omega_1}-\Big(D_2(b_{1j},0),\rho_2\nabla\theta_2\Big)_{\Omega_2}\\ &-\gamma_{1j}-\lambda(t)B_{1j}\Big]\Big\}, \qquad (3.12)\end{aligned}$$

$$\begin{aligned}u_{2j}(t)=\mathscr{P}_{[0,\bar{u}_2]}\Big\{\frac{1}{\zeta_{2j}}\Big[&-\Big(D_1(b_{1j},0),\rho_1\nabla\theta_1\Big)_{\Omega_1}+\Big(D_2(b_{1j},0),\rho_2\nabla\theta_2\Big)_{\Omega_2}\\ &-\gamma_{2j}+\lambda(t)B_{2j}\Big]\Big\}, \qquad (3.13)\end{aligned}$$

for each $j=1,2,\dots M$, and the equality constraint (1.9)*, where $\mathscr{P}_{[a,b]}$ denotes the projection of $\mathbb{R}$ onto $[a,b]$ with $a\le b$, i.e., $\mathscr{P}_{[a,b]}(f):=\min\{b,\max\{a,f\}\}$, and $B_{ij}=\int_{\Gamma_i} b_{ij}\,dx$. Furthermore, if $\zeta_{ij}=0$ for all $i=1,2$, and $j=1,\dots,M$, then* (3.12)–(3.13) *are reduced to*

$$u_{1j}(t)=\begin{cases}\bar u_1, & \text{if } \Big(D_1(b_{1j},0),\rho_1\nabla\theta_1\Big)_{\Omega_1}-\Big(D_2(b_{1j},0),\rho_2\nabla\theta_2\Big)_{\Omega_2}-\lambda(t)B_{1,j}>\gamma_{1j},\\ 0, & \text{if } \Big(D_1(b_{1j},0),\rho_1\nabla\theta_1\Big)_{\Omega_1}-\Big(D_2(b_{1j},0),\rho_2\nabla\theta_2\Big)_{\Omega_2}-\lambda(t)B_{1,j}<\gamma_{1j},\end{cases} \tag{3.14}$$

and

$$u_{2j}(t)=\begin{cases}\bar u_2, & \text{if } -\Big(D_1(0,b_{2j}),\rho_1\nabla\theta_1\Big)_{\Omega_1}+\Big(D_2(0,b_{2j}),\rho_2\nabla\theta_2\Big)_{\Omega_2}+\lambda(t)B_{2j}>\gamma_{2j},\\ 0, & \text{if } -\Big(D_1(0,b_{2j}),\rho_1\nabla\theta_1\Big)_{\Omega_1}+\Big(D_2(0,b_{2j}),\rho_2\nabla\theta_2\Big)_{\Omega_2}+\lambda(t)B_{2j}<\gamma_{2j}.\end{cases} \tag{3.15}$$

If there exists a $t^*\in[0,t_f]$ such that the inequality in (3.14) *or* (3.15) *becomes equally, then $u_{ij}(t^*)$ can take any value in $[0,\bar u_i], i=1,2$, accordingly.*

Proof. First taking an inner product of (3.4) with (ρ_1,ρ_2) over space and time, we have

$$\int_0^{t_f}(\frac{\partial z_1}{\partial t},\rho_1)\,dt+\int_0^{t_f}(v_1\cdot\nabla z_1,\rho_1)\,dt+\int_0^{t_f}(w_1\cdot\nabla\theta_1,\rho_1)\,dt=0,$$
$$\int_0^{t_f}(\frac{\partial z_2}{\partial t},\rho_2)\,dt+\int_0^{t_f}(v_1\cdot\nabla z_2,\rho_2)\,dt+\int_0^{t_f}(w_2\cdot\nabla\theta_2,\rho_2)\,dt=0.$$

Integrating by parts using the boundary conditions (3.5)–(3.6) and (3.10), we obtain

$$(z_1(t_f),\rho_1(t_f))-\int_0^{t_f}(z_1,\frac{\partial\rho_1}{\partial t})\,dt-\int_0^{t_f}(z_1,v_1\cdot\nabla\rho_1)\,dt+\int_0^{t_f}(w_1\cdot\nabla\theta_1,\rho_1)\,dt=0,$$
$$(z_2(t_f),\rho_2(t_f))-\int_0^{t_f}(z_2,\frac{\partial\rho_2}{\partial t})\,dt-\int_0^{t_f}(z_2,v_2\cdot\nabla\rho_2)\,dt+\int_0^{t_f}(w_2\cdot\nabla\theta_2,\rho_2)\,dt=0.$$

With the help of the adjoint equations (3.9), we get

$$(z_1(t_f),\alpha_1(\theta_1(t_f)-\theta_1^d))+\int_0^{t_f}\beta_1(z_1,\theta_1-\Theta_1^d)\,dt+\int_0^{t_f}(w_1\cdot\nabla\theta_1,\rho_1)\,dt=0, \tag{3.16}$$
$$(z_2(t_f),\alpha_2(\theta_2(t_f)-\theta_2^d))+\int_0^{t_f}\beta_2(z_2,\theta_2-\Theta_2^d)\,dt+\int_0^{t_f}(w_2\cdot\nabla\theta_2,\rho_2)\,dt=0. \tag{3.17}$$

On the other hand, for any $(\vec h_1,\vec h_2)\in U_{ad}^1\times U_{ad}^2$, we have

$$\begin{aligned}J'(\vec u_1,\vec u_2)\cdot(\vec h_1,\vec h_2)=&\sum_{i=1}^2\alpha_i(z_i(t_f),\theta_i(t_f)-\theta_i^d)+\sum_{i=1}^2\beta_i\int_0^{t_f}(z_i,\theta_i-\Theta_i^d)\,dt\\&+\sum_{i=1}^2\sum_{j=1}^M\gamma_{ij}\int_0^{t_f}h_{ij}\,dt+\sum_{i=1}^2\sum_{j=1}^M\zeta_{ij}\int_0^{t_f}u_{ij}h_{ij}\,dt.\end{aligned} \tag{3.18}$$

Combining (3.18) with (3.16)–(3.17) and (3.8) yields

$$
\begin{aligned}
J'(\vec{u}_1,\vec{u}_2)\cdot(\vec{h}_1,\vec{h}_2) &= -\sum_{i=1}^{2}\int_0^{t_f}(w_i\cdot\nabla\theta_i,\rho_i)_{\Omega_i}\,dt \\
&\quad+\sum_{i=1}^{2}\sum_{j=1}^{M}\gamma_{ij}\int_0^{t_f}h_{ij}(t)\,dt+\sum_{i=1}^{2}\sum_{j=1}^{M}\zeta_{ij}\int_0^{t_f}u_{ij}h_{ij}(t)\,dt \\
&= -\sum_{j=1}^{M}\int_0^{t_f}h_{1j}(t)\Big(D_1(b_{1j},0),\rho_1\nabla\theta_1\Big)_{\Omega_1}+h_{2j}(t)\Big(D_1(0,b_{2j}),\rho_1\nabla\theta_1\Big)_{\Omega_1} \\
&\quad+h_{1j}(t)\Big(D_2(b_{1j},0),\rho_2\nabla\theta_2\Big)_{\Omega_2}-h_{2j}(t)\Big(D_2(0,b_{2j}),\rho_2\nabla\theta_2\Big)_{\Omega_2}\,dt \\
&\quad+\sum_{i=1}^{2}\sum_{j=1}^{M}\gamma_{ij}\int_0^{t_f}h_{ij}(t)\,dt+\sum_{i=1}^{2}\sum_{j=1}^{M}\zeta_{ij}\int_0^{t_f}u_{ij}h_{ij}(t)\,dt. \qquad (3.19)
\end{aligned}
$$

In addition,

$$
G'(\vec{u}_1,\vec{u}_2)\cdot(\vec{h}_1,\vec{h}_2)=\int_{\Gamma_1}\vec{h}_1^T(t)\vec{b}_1(x)\,dx-\int_{\Gamma_2}\vec{h}_2^T(t)\vec{b}_2(x)\,dx.
$$

We have

$$
\begin{aligned}
\int_0^{t_f}\lambda G'(\vec{u}_1,\vec{u}_2)\cdot(\vec{h}_1,\vec{h}_2)\,dt &= \int_0^{t_f}\lambda(t)\left(\int_{\Gamma_1}\vec{h}_1^T(t)\vec{b}_1(x)\,dx-\int_{\Gamma_2}\vec{h}_2^T(t)\vec{b}_2(x)\,dx\right)dt \\
&= \sum_{j=1}^{M}\int_0^{t_f}\lambda(t)\left(B_{1j}h_{1j}(t)-B_{2j}h_{2j}(t)\right)dt. \qquad (3.20)
\end{aligned}
$$

Since $(\vec{h}_1,\vec{h}_2)\in U_{ad}^1\times U_{ad}^2$ is arbitrary, we can set it with only one entry $h_{ij}\neq 0$ and the rest zeros. Then combining (3.3) with (3.19)–(3.20) gives

$$
\begin{aligned}
&\int_0^{t_f}\Big[-\Big(D_1(b_{1j},0),\rho_1\nabla\theta_1\Big)_{\Omega_1}+\Big(D_2(b_{1j},0),\rho_2\nabla\theta_2\Big)_{\Omega_2}+\gamma_{1j}+\zeta_{1j}u_{1j}(t)+\lambda(t)B_{1,j}\Big] \\
&\qquad\cdot(h_{1j}(t)-u_{1j}(t))\,dt\geq 0, \qquad (3.21)
\end{aligned}
$$

for $\forall h_{1j}\in U_{ad}^1, j=1,2,\ldots,M$, and

$$
\begin{aligned}
&\int_0^{t_f}\Big[\Big(D_1(0,b_{2j}),\rho_1\nabla\theta_1\Big)_{\Omega_1}-\Big(D_2(0,b_{2j}),\rho_2\nabla\theta_2\Big)_{\Omega_2}+\gamma_{2j}+\zeta_{2j}u_{2j}(t)-\lambda(t)B_{2j}\Big] \\
&\qquad\cdot(h_{2j}(t)-u_{2j}(t))\,dt\geq 0, \qquad (3.22)
\end{aligned}
$$

for $\forall h_{2j}\in U_{ad}^2, j=1,2,\ldots,M$.

Finally, incorporating with the box type of constraints in $U_{ad}^1\times U_{ad}^2$, we obtain the desired optimality conditions for $(u_{1j},u_{2,j})$, $j=1,\ldots,M$, given by (3.12)–(3.13) and the additional constraint (1.9). The proof follows the similar arguments as in that of [26, Theorem 2.28, p. 71]. Furthermore, if $\zeta_{ij}=0$ for all $i=1,2$ and $j=1,\ldots,M$, the optimality conditions (3.12)–(3.13) are reduced to (3.14)–(3.15). In this case, the L^1-penalization promotes the sparsity of the controls in time, and this completes our proof. □

4. Conclusions

This work focuses on optimal data transport via active control of the velocity at the intersection of two data sets. The results of this work lay a theoretical foundation for the construction of gradient decent based algorithms for implementing the control design. The model and analysis can be naturally extended for the case of N data sets or a network of large dimension. This work contributes to the understanding of boundary control in transport systems and its potential applications to various fields such as control of dynamic clustering and flows over networks.

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