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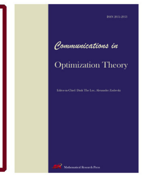

A MULTIDIMENSIONAL PALEY-WIENER THEOREM

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Dedicated to V.M. Tikhomirov on the occasion of his 90th birthday

Abstract. The Paley-Wiener theorem states that the Hilbert transform of an integrable odd function, which is monotone on $\mathbb{R}_+$, is integrable. There exists an extension of this result for functions with generalized monotonicity. In this paper, we extend the latter result to the multivariate case. What is proved under a multidimensional condition of generalized monotonicity, is the integrability of the Hilbert transforms with respect to separate variables and their combinations for the groups of the variables. In other words, the main result ensures the belonging of an integrable function odd in each variable to the product Hardy space.

Keywords. Hilbert transform; General monotone functions, Product Hardy space.

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1. Introduction

Being somewhat apart of the mainstream, the Paley-Wiener theorem (see [8]) asserts that for an odd function $g \in L^1(\mathbb{R})$ monotone decreasing on $\mathbb{R}_+ = (0,\infty)$, its Hilbert transform is also integrable, i.e., g is in the (real) Hardy space $H^1(\mathbb{R})$ (for alternative proofs, see [10] and [9, Ch.IV, 6.2]). The oddness of g is essential, in particular, because of Kober's result [3] which asserts that if $g \in H^1(\mathbb{R})$, then the cancelation property holds

$$\int_{\mathbb{R}} g(t)\,dt = 0,$$

which odd functions satisfy automatically. Monotonicity or something like that is also necessary: there is an example of an integrable odd function with non-integrable Hilbert transform in [6]. On the other hand, for even monotone functions with cancelation property such an assertion fails to hold in general. More reasons can be seen in comparison with recent results in [5].

In [7], the monotonicity assumption has been relaxed in the Paley-Wiener theorem, and in [6] weighted versions have been obtained for both odd and even functions more general than monotone ones. The periodic case has also been covered in [6] in the same manner. One can find historical background relevant to these problems in [6].

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Our goal is to figure out what may happen in the multivariate case. The choice of functions odd in each variable seems natural. We are not going to prove just an extension of the Paley-Wiener theorem by involving a certain version of multidimensional monotonicity in this study; instead a special condition is taken to substitute for monotonicity, more general and much less restrictive. The details and relations to the one-dimensional results will be given in the following section. Further problem is what singular operator should be taken in place of the Hilbert transform. Equivalently, what Hardy type space is to be involved? Roughly speaking, various superpositions of the Hilbert transform will be taken (in [2] this is called the multidimensional Hilbert transform), which in turn, assigns the so-called product Hardy space $H^1(\mathbb{R}\times\ldots\times\mathbb{R})$; of course, more details will be given below. In words, the main result reads as

If a function is odd in each of the n Euclidean variables and belongs to a special class of general monotone functions, then it belongs to the product Hardy space $H^1(\mathbb{R}\times\ldots\times\mathbb{R})$.

In the rest of the paper, each of the notions in the above non-strict formulation will be detailed and the proof of the main result will be given. The former issue will be given in the following section, along with the precise formulation of the main result. Then the proof will be given. We will start with the reformulation and proof of the one-dimensional result. This is done not only for completeness but also since each of the steps of that proof will be used in several dimensions. After that we present a two-dimensional proof. Only then, we give a general proof, using the introduced notions and notation in full.

Let C denote a positive absolute constant, maybe different in different occurrences. We shall use the notation $\lesssim$ as abbreviation for $\le C$.

2. Preliminaries and main result

In order to be able to formulate the main result and then proceed to the proof, we must recall one-dimensional notions and introduce their multivariate analogs, with appropriate notation.

2.1. **One-dimensional notions.**

The Hilbert transform of a function $g\in L^1(\mathbb{R})$ is

$$\mathscr{H}g(x)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{g(t)}{x-t}\,dt,$$

where the integral is understood in the improper (Principal Value) sense, as

$$\mathscr{H}g(x)=\lim_{\delta\to0+}\mathscr{H}_\delta g(x),$$

with

$$\mathscr{H}_\delta g(x)=\frac{1}{\pi}\int_{|t-x|>\delta}\frac{g(t)}{x-t}\,dt$$

being the *truncated Hilbert transform*.

If g is integrable, its Hilbert transform exists almost everywhere but is not necessarily integrable. Moreover, it can be even not locally integrable. When the Hilbert transform is integrable, we say that g is in *the (real) Hardy space* $H^1:=H^1(\mathbb{R})$.

Among various generalizations of monotonicity we choose (see [7])

$$GM := \left\{ h : \|dh\|_{L^1(x,2x)} \le C \int_{\frac{x}{c}}^{cx} |h(t)|\,\frac{dt}{t} \right\}. \tag{2.1}$$

Here we integrate h over $(x,2x)$, $x>0$, in the Stieltjes sense. In other words, we define a class of functions h locally of bounded variation and such that there exist $C>0$ and $c>1$ so that each h in this class satisfies (2.1). Of course, the point is that C and c are independent of x. It is plain that every monotone function is GM. Obviously, any $c'>c$ can be taken in the definition of GM. Condition (2.1) is given for $x\in(0,\infty)$. Because of oddness of the considered functions, there is no need for special conventions on the negative half-axis.

2.2. **Multivariate notation.**

Let $\eta=(\eta_1,...,\eta_n)$ be an n-dimensional Boolean vector, that is, its entries are either 0 or 1 only, with $\mathbf{0}=(0,0,...,0)$ and $\mathbf{1}=(1,1,...,1)$. Its main task is to indicate the variables in which a certain action be fulfilled. Correspondingly, $|\eta|=\eta_1+...+\eta_n$. The inequality of vectors is meant coordinate wise. If the only 1 entry is on the j-th place, while the rest are zeros, such a (basis) vector will be denoted by e_j. By x_η we denote the $|\eta|$-tuple consisting only of x_j such that $\eta_j=1$ and

$$dx_\eta := \prod_{j:\eta_j=1} dx_j.$$

We shall freely use $\prod$ for both usual multiplication and repeated operator action. It will be clear each time what is meant and hopefully this will cause no confusion.

When we apply the Hilbert transform to the j-th variable it will be defined by $\mathscr{H}_j$ and, consequently, $\mathscr{H}_j\mathscr{H}_k...\mathscr{H}_l := \mathscr{H}_{jk...l}$. For the latter case, the introduced indicator notation is more convenient. Naturally,

$$\mathscr{H}_\eta := \prod_{j:\eta_j=1} \mathscr{H}_j.$$

Similarly, $d^\eta g$ is used in the Stieltjes integration of the function g and other issues with respect to the indicated variables. This needs certain explanation. Like the integration over an interval is defined by partitions of this interval, the Stieltjes integration with respect to the η-variables is defined by partition by means of the parallelepipeds $\prod\limits_{j:\eta_j=1}[a_j,b_j]$ and taking the corresponding difference (see, e.g., [4, Chapter V])

$$\prod_{j:\eta_j=1} \Big(g(u_1,...,u_{j-1},b_j,u_{j+1},...,u_n) - g(u_1,...,u_{j-1},a_j,u_{j+1},...,u_n) \Big).$$

The Hardy space which can be defined by means of the Hilbert transform only is the product Hardy space $H^1(\mathbb{R}\times...\times\mathbb{R})$. One of the ways the norm in this space can be defined is

$$\|g\|_{H^1(\mathbb{R}\times...\times\mathbb{R})} = \sum_{\mathbf{0}\le\eta\le\mathbf{1}} \left\| \mathscr{H}_\eta g \right\|_{L^1(\mathbb{R}^n)}.$$

Naturally, $\mathscr{H}_{\mathbf{0}}g$ is just g itself. Again, [4, Chapter V] is the source where all these are given in a concentrated form.

It is worth mentioning that $H^1(\mathbb{R}\times...\times\mathbb{R}) \subsetneq H^1(\mathbb{R}^n)$. There is Uchiyama's example that this inclusion is proper (see, e.g., [1]).

In dimension one, GM functions are locally of bounded variation. A similar restriction is applied in the multivariate case. Functions considered are locally of Hardy bounded variation. We do not give the details, they can be found in various sources, see, e.g., [4], where definitions are given by means of the above indicator notation. In what follows, we shall assume the functions analyzed be locally of bounded variation without mentioning this each time. Thus, in the multiple case, GM_n functions (with $GM := GM_1$) are defined as those locally of bounded variation for which there exist $C > 0$ and $c > 1$ such that

$$\Big(\prod_{j:\eta_j=1}\int_{x_j}^{2x_j}\Big)|d^\eta g(t_\eta, x_{\mathbf{1}-\eta})| \le C\Big(\prod_{j:\eta_j=1}\int_{\frac{x_j}{c}}^{cx_j}\Big)|g(t_\eta, x_{\mathbf{1}-\eta})|\prod_{j:\eta_j=1}\frac{dt_j}{t_j} \tag{2.2}$$

for every $\eta \ne \mathbf{0}$ and $x_{\mathbf{1}-\eta} \in (0,\infty)^{|\mathbf{1}-\eta|}$.

We mention that also χ will be used as an indicator vector. It will possess all the same properties as η does. It is needed because one has to apply certain additional actions to the groups of variables within those already indicated by η.

By a weaker notion of monotonicity, preserving the signs of the differences underlying the Stieltjes integration is understood, like in dimension one monotonicity means preserving the sign of the first difference.

2.3. **Formulation of the main result.**

We reformulate the main result given in the introduction in a non-strict form. In fact, the formulation will be in the same way as in dimension one but every notion used in assumptions and assertions is replaced by a certain multidimensional generalization.

Theorem 1. *Let g be a Lebesgue integrable function on $\mathbb{R}^n$, odd in each variable. Let g also be GM_n. Then $\mathscr{H}_\eta g \in L^1(\mathbb{R}^n)$ for every $\eta \ne \mathbf{0}$. In other words, such a g belongs to $H^1(\mathbb{R}\times\ldots\times\mathbb{R})$.*

The rest of the paper will be the proof of this result, where the relations between the one-dimensional setting and multivariate one will be revealed in full.

3. PROOFS

As promised, we will present the proof in three settings.

3.1. **One-dimensional proof.**

There are many advantages in reproducing the one-dimensional proof given in [7]. Of course, this will make the presentation self-contained. This will also assist in better understanding of multivariate operations. Last but not least, almost every step in the one-dimensional proof is a special calculation that will be referred to in higher dimensions by applying separately to a corresponding variable.

Theorem 1 can be reformulated in dimension one as follows:

Let g be an odd function integrable on $\mathbb{R}$. If g is general monotone in the sense given by (2.1), then its Hilbert transform is also integrable, that is, the function is in the real Hardy space H^1.

Proof. Let $u > 0$, calculations for $u < 0$ are the same. Since g is odd and integrable, we obtain

$$\int_0^\infty \left| \left(\int_{\frac{3u}{2}}^\infty + \int_{-\infty}^{-\frac{3u}{2}} \right) \frac{g(t)}{u-t}\, dt \right| du$$
$$\le \int_0^\infty \int_{3u/2}^\infty |g(t)| \frac{2t}{t^2-u^2}\, dt\, du \le C \int_0^\infty |g(t)|\, dt, \tag{3.1}$$

and, similarly,

$$\int_0^\infty \left| \left(\int_0^{\frac{u}{2}} + \int_{-\frac{u}{2}}^0 \right) \frac{g(t)}{u-t}\, dt \right| du \le C \int_0^\infty |g(t)|\, dt. \tag{3.2}$$

We also have

$$\int_0^\infty \left| \int_{-\frac{3u}{2}}^{-\frac{u}{2}} \frac{g(t)}{u-t}\, dt \right| du \le \int_0^\infty \int_{\frac{u}{2}}^{\frac{3u}{2}} \frac{|g(t)|}{u+t}\, dt\, du \le C \int_0^\infty |g(t)|\, dt. \tag{3.3}$$

Therefore, by simple substitutions,

$$\int_0^\infty \left| \int_{-\infty}^\infty \frac{g(t)}{u-t}\, dt \right| du = \int_0^\infty \left| \int_{\frac{u}{2}}^{\frac{3u}{2}} \frac{g(t)}{u-t}\, dt + \int_{\mathbb{R}\setminus[\frac{u}{2},\frac{3u}{2}]} \frac{g(t)}{u-t}\, dt \right| du$$
$$+ \int_0^\infty \left| \int_{\frac{u}{2}}^{\frac{3u}{2}} \frac{g(t)}{u-t}\, dt \right| du + O\Big(\int_0^\infty |g(t)|\, dt \Big)$$
$$\le C \Big(I + \int_0^\infty |g(t)|\, dt \Big), \tag{3.4}$$

where

$$I = \int_0^\infty \left| \int_0^{\frac{u}{2}} [g(u+t) - g(u-t)] \frac{dt}{t} \right| du. \tag{3.5}$$

Changing the order of integration and substituting then $(u-t) \to u$, we obtain

$$I \le \int_0^\infty \int_0^{\frac{u}{2}} \Big(\int_{u-t}^{u+t} |dg(s)| \Big) \frac{dt}{t}\, du \le \int_0^\infty \int_t^\infty \int_u^{u+2t} |dg(s)|\, du \frac{dt}{t}. \tag{3.6}$$

Changing then the order of the two inner integrals, we get

$$I \le \int_0^\infty \left[\int_t^{3t} |dg(s)| \int_t^s du + \int_{3t}^\infty |dg(s)| \int_{s-2t}^s du \right] \frac{dt}{t}$$
$$\le C \int_0^\infty \int_t^\infty |dg(s)|\, dt. \tag{3.7}$$

Taking into account the simple relation

$$\int_{\frac{t}{c}}^\infty \frac{1}{s} \int_s^{2s} |dg(z)|\, ds = \int_{\frac{t}{c}}^{\frac{2t}{c}} \ln \frac{cz}{t} |dg(z)| + \ln 2 \int_{\frac{2t}{c}}^\infty |dg(z)|, \tag{3.8}$$

we see that

$$\int_t^\infty |dg(s)| \le \int_{\frac{2t}{c}}^\infty |dg(z)| \lesssim \int_{\frac{t}{c}}^\infty \frac{1}{s}\int_s^{2s} |dg(z)|\, ds, \tag{3.9}$$

provided $c \ge 2$ (cf. a remark after (2.1)). We now have

$$\begin{aligned} I &\le C\int_0^\infty \int_{\frac{t}{c}}^\infty s^{-1}\left(\int_{\frac{s}{c}}^{ds} z^{-1}|g(z)|\,dz\right) ds\,dt \\ &\le C\int_0^\infty \left(\int_{\frac{s}{c}}^{cs} z^{-1}|g(z)|\,dz\right) ds \le C\int_0^\infty |g(s)|\,ds, \end{aligned} \tag{3.10}$$

which completes the proof. □

3.2. **Two-dimensional proof.**

In this subsection, we present a two-dimensional proof. Of course, the reader can skip it. On the other hand, for many just two-dimensional proof will be convincing. In any case, it allows one to better understand the ideas being possibly somewhat hidden in the general notation. First of all, it is worth reformulating the theorem and its main ingredients in terms of direct notation, without subscripts. The functions $g(s,t)$ we consider are odd in s and t and belong to the class GM_2, that is, satisfy

$$\int_x^{2x} |d^{(1,0)}g(s,y)| \le C\int_{\frac{x}{c}}^{cx} |g(s,y)|\,\frac{ds}{s}, \quad \int_y^{2y} |d^{(0,1)}g(x,t)| \le C\int_{\frac{y}{c}}^{cy} |g(x,t)|\,\frac{dt}{t}, \tag{3.11}$$

and

$$\int_x^{2x}\int_y^{2y} |d^{(1,1)}g(s,t)| \le C\int_{\frac{x}{c}}^{cx}\int_{\frac{y}{c}}^{cy} |g(s,t)|\,\frac{ds\,dt}{st}. \tag{3.12}$$

The Hilbert transforms to be estimated are seen from the norm of g in $H^1(\mathbb{R}\times\mathbb{R})$:

$$\begin{aligned} \|g\|_{H^1(\mathbb{R}\times\mathbb{R})} &= \int_{\mathbb{R}^2} |g(x,y)|\,dx\,dy \\ &+ \frac{1}{\pi}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}} \frac{g(s,y)}{x-s}\,ds\right| dx\,dy + \frac{1}{\pi}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}} \frac{g(x,t)}{y-t}\,dt\right| dx\,dy \\ &+ \frac{1}{\pi^2}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{g(s,t)}{(x-s)(y-t)}\,ds\,dt\right| dx\,dy. \end{aligned} \tag{3.13}$$

Thus, what we are going to prove reads as follows.

Let g be a Lebesgue integrable function on $\mathbb{R}^2$, odd in each variable. Let g also be GM_2, that is, (3.11) *and* (3.12) *are satisfied. Then the right-hand side of* (3.13) *is finite. In other words, such a function g belongs to $H^1(\mathbb{R}\times\mathbb{R})$.*

Proof. The estimates for the second and third integrals on the right-hand side of (3.13) are mainly familiar one-dimensional ones. Therefore, we have to estimate only the last term on the right. Doing this over $\mathbb{R}^2_+$ will suffice, the rest is analogous. Splitting the inner integrals in

$$\int_0^\infty\int_0^\infty\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{g(s,t)}{(x-s)(y-t)}\,ds\,dt\right|dx\,dy$$

in the same way as in the first line of (3.4), we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\left|\int_{\frac{x}{2}}^{\frac{3x}{2}}\int_{\frac{y}{2}}^{\frac{3y}{2}}\frac{g(s,t)}{(x-s)(y-t)}\,ds\,dt\right|dx\,dy\\
+&\int_0^\infty\int_0^\infty\left|\int_{\frac{x}{2}}^{\frac{3x}{2}}\int_{\mathbb{R}\setminus[\frac{y}{2},\frac{3y}{2}]}\frac{g(s,t)}{(x-s)(y-t)}\,ds\,dt\right|dx\,dy\\
+&\int_0^\infty\int_0^\infty\left|\int_{\mathbb{R}\setminus[\frac{x}{2},\frac{3x}{2}]}\int_{\frac{y}{2}}^{\frac{3y}{2}}\frac{g(s,t)}{(x-s)(y-t)}\,ds\,dt\right|dx\,dy\\
+&\int_0^\infty\int_0^\infty\left|\int_{\mathbb{R}\setminus[\frac{x}{2},\frac{3x}{2}]}\int_{\mathbb{R}\setminus[\frac{y}{2},\frac{3y}{2}]}\frac{g(s,t)}{(x-s)(y-t)}\,ds\,dt\right|dx\,dy.
\end{aligned}$$

For the last summand, we apply (3.1)-(3.3) in each variable to get $O\Big(\int_{\mathbb{R}}|g(s,t)|\,ds\,dt\Big)$. The second and third ones are similar: in one variable we again apply (3.1)-(3.3), while in the other one estimates are like (3.4)-(3.10), with the use of the corresponding marginal GM_2 condition in (3.11). Each again leads to $O\Big(\int_{\mathbb{R}}|g(s,t)|\,ds\,dt\Big)$. What remains is the first summand, which, as above, reduces to

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\Bigg|\int_0^{\frac{x}{2}}\int_0^{\frac{y}{2}}\Big[g(x+s,y+t)-g(x+s,y-t)\\
&-g(x-s,y+t)+g(x-s,y-t\Big]\frac{ds}{s}\frac{dt}{t}\Bigg|dx\,dy.
\end{aligned}$$

Making use of (3.6) and (3.7) in each of the two variables, we arrive at

$$\int_0^\infty\int_0^\infty\int_x^\infty\int_y^\infty|d^{(1,1)}g(s,t)|.$$

Applying now (3.8) and (3.9) in each of the two inner integrals and then using (3.12) and estimates of type (3.10) in each of the two variables, we again arrive at the bound $O\Big(\int_{\mathbb{R}}|g(s,t)|\,ds\,dt\Big)$ and complete the proof. □

After proving the main result in these two setting, there is a hope that the general proof will be more transparent.

3.3. **General proof.**

In fact, the proof in the general case does not differ much from that in dimension two. The groups to which calculations in the one-dimensional proof will be applied are going to be larger, with more variables in most of them, but the mentioned calculations will still be applied to separate variables. Similarly, for simplicity, we restrict ourselves to the estimates over $\mathbb{R}^n_+$.

Proof. We have to estimate $\left\|\mathscr{H}_\eta g\right\|_{L^1(\mathbb{R}^n_+)}$ for any $\eta\ne\mathbf{0}$. Let η be fixed and let χ be a $|\eta|$-dimensional indicator vector. First, applying (3.4) to each of the η-variables, we estimate

$$\int_{\mathbb{R}^n_+}\left|\left(\prod_{j:\chi_j=1}\int_{\frac{x_j}{2}}^{\frac{3x_j}{2}}\frac{1}{x_j-s_j}\right)\left(\prod_{i:\chi_i=0}\int_{\mathbb{R}\setminus[\frac{x_i}{2},\frac{3x_i}{2}]}\frac{1}{x_i-s_i}\right)g(s_\eta,x_{\mathbf{1}-\eta})\,ds_\eta\right|dx$$

for all possible combinations of χ.

We now apply (3.1)-(3.3) with respect to every i-th variable for which $\chi_i=0$, with

$$\left(\prod_{j:\chi_j=1}\int_{\frac{x_j}{2}}^{\frac{3x_j}{2}}\frac{1}{x_j-s_j}\right)g(s_\eta,x_{\mathbf{1}-\eta})\,ds_\chi$$

as a function. It remains to estimate

$$\int_{\mathbb{R}^n_+}\left|\left(\prod_{j:\chi_j=1}\int_{\frac{x_j}{2}}^{\frac{3x_j}{2}}\frac{1}{x_j-s_j}\right)g(s_\chi,x_{\mathbf{1}-\chi})\,ds_\chi\right|dx.$$

Using (3.5) in each of the remaining s_χ, we turn to the following quantity:

$$\int_{\mathbb{R}^n_+}\left|\left(\prod_{j:\chi_j=1}\int_0^{\frac{x_j}{2}}\frac{1}{s_j}\right)G_\chi(s_\chi,x_{\mathbf{1}-\chi})\,ds_\chi\right|dx,$$

where

$$\begin{aligned}G_\chi(s_\chi,x_{\mathbf{1}-\chi}) &= g((x+s)_\chi,x_{\mathbf{1}-\chi})-\sum_{j:\chi_j=1}g((x+s)_{\chi-e_j},x_j-s_j,x_{\mathbf{1}-\chi})\\ &\quad+\sum_{\substack{j:\chi_j=1\\ i:\chi_i=1;i\ne j}}g((x+s)_{\chi-e_j-e_i},x_j-s_j,x_i-s_i,x_{\mathbf{1}-\chi})-\ldots\\ &\quad+(-1)^{|\chi|}g((x-s)_\chi,x_{\mathbf{1}-\chi})\end{aligned}$$

is the repeated $|\chi|$-th difference, where for the j-th variable next in turn the value already obtained is taken at x_j+s_j minus the same value at x_j-s_j. Following the familiar lines, we apply (3.6) and (3.7) to each of the χ-variables and arrive to the bound

$$\int_{\mathbb{R}^n_+}\prod_{j:\chi_j=1}\int_{x_j}^{\infty}|d^\chi g(s_\chi,x_{\mathbf{1}-\chi})|\,dx.$$

Using then (3.8) and (3.9) leads to the bound

$$\int_{\mathbb{R}^n_+}\left(\prod_{j:\chi_j=1}\int_{\frac{x_j}{c}}^{\infty}\frac{1}{s_j}\int_{s_j}^{2s_j}\right)|d^\chi g(s_\chi,x_{\mathbf{1}-\chi})|\,dx.$$

Making use of the corresponding condition in (2.2), with χ in place of η, and of Fubini's theorem, we end up with the bound $\int_{\mathbb{R}^n_+}|g(x)|\,dx$, times constant multiple independent of χ. Since this argument works for every η and χ, we complete the proof. □

4. Conclusions

Most of the known conditions are about belonging to the real Hardy space $H^1(\mathbb{R}^n)$. The main result of this paper gives a sufficient condition for belonging to the product Hardy space $H^1(\mathbb{R}\times ... \times \mathbb{R})$. As mentioned above, the latter is smaller than the former. What made it possible was that just this space could be characterized by means of the Hilbert transforms. It might be beneficial to use different types of generalized monotonicity in this problem. On the other hand, it could be interesting to figure out whether this is the only option or there is a way to characterize other singular operators in a similar manner, say the Riesz transforms.

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