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OPTIMAL IMPULSIVE CONTROL PROBLEMS WITH MEASURABLE TIME DEPENDENCE

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In Memory of Sanjoy Mitter

Abstract. A standard framework for studying 'impulse' control systems, in which the control input is a measure, is to interpret state trajectories as limits of classical state trajectories for a conventional control system, with affine control dependence and for which the control is unbounded. Discontinuities in the state trajectory can occur at the atoms of the measure control. If the measure control is scalar valued, then all limiting state trajectories are the same and discontinuities in the state trajectories at any particular time are determined by the solution to a differential equation, which depends on the magnitude of the atom and the state-dependent input gain function. By means of reparameterization techniques, which accord time the role of a state variable and which replace the original impulse control system by a classical control system, we can establish important properties of impulse control systems, such as the representation of the reachable set as the closure of the reachable set for classical control inputs and existence of minimizers for optimal impulse control problems, by simply demonstrating analogous properties of the reparameterized control system. This approach breaks down when the time dependence of the drift term in the impulse control system is not Lipschitz continuous because the right side of the reparameterized dynamic equations are not Lipschitz continuous w.r.t. the state variables and therefore is not amenable to standard analysis. In this paper, it is shown that density and closure properties of impulse control systems are retained, for impulse control systems in which the drift term is merely measurable regarding its time dependence and when the input gain function is state dependent. Difficulties arising from the irregular state dependence of the reparameterized control system can be simply overcome by switching between the original and reparameterized control system descriptions in the convergence analysis.

Keywords. Existence of optimal controls, Impulse control systems; Optimal control.

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1. INTRODUCTION

Consider the impulsive control system

$$
\text{(I)} \begin{cases} dx(t) = f(t,x(t))dt + g(x(t))\mu(dt),\ t \in [0,T] \\ \text{in which } \mu \text{ is a non-negative Borel measure on } [0,T], \text{ and} \\ x(0) = x_0\,, \end{cases}
$$

involving data: a constant $T > 0$, functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ and a vector $x_0 \in \mathbb{R}^n$.

The interpretation of 'state trajectory' $x : [0,T] \to \mathbb{R}^n$ for (I) corresponding to a measure control μ requires careful consideration. A first approach is to regard x as a (possibly discontinuous) bounded variation function that is a solution to the integral equation:

$$
\begin{cases} x(t) = \int_0^t f(t',x(t'))dt' + \int_{[0,t]} g(x(t'))\mu(dt'), \quad t \in (0,T] \\ x(0) = x_0\,. \end{cases} \tag{1.1}
$$

But the right side is ambiguous at any time $\bar{t}$ that is an atom for μ. This is because x may jump precisely at the time $\bar{t}$ and the meaning of the value of the integrand $g(x(t))$, at time $t = \bar{t}$, is not clear.

An important area of application of impulsive optimal control is space vehicle control, in which a measure control is interpreted as an idealization of a high intensity 'fuel burn' of short duration [9]. For such applications, we require the state trajectory to be the limit of solutions x_i to the ordinary differential equation

$$
\begin{cases} \dot{x}_i(t) = f(t,x_i(t)) + g(x_i(t))u_i(t),\ t \in [0,T] \\ x(0) = x_0\,, \end{cases}
$$

in which $\{u_i\}$ is some sequence of 'classical' controls (non-negative, integrable functions) such that the absolutely continuous measures $u_i(t)dt$ converge to μ in distribution. As is well known, the definition of state trajectory x, consistent with this interpretation, is that x is a function of bounded variation, right continuous on $(0,T)$, such that $x(0) = x_0$ and for each $t \in (0,T]$,

$$
x(t) = x_0 + \int_0^t f(t',x(t'))dt' + \int_{[0,t]} g(x(t'))\mu^{\text{cont}}(dt') + \sum_{r \in [0,t]} (\xi^r(1) - \xi^r(0)), \tag{1.2}
$$

in which, for each $r \in [0,T]$, $\xi^r : [0,1] \to \mathbb{R}^n$ is the solution of the differential equation

$$
\frac{d}{ds}\xi^r(s) = \mu(\{r\})g(\xi^r(s)),\ \text{a.e. } s \in [0,1], \tag{1.3}
$$

with initial condition $\xi^r(0) = x(r^-)$. In these relations, μ^{cont} denotes the continuous component of the measure μ. Observe that, if r is not an atom of μ, $\xi^r(1) - \xi^r(0) = 0$ and the term $\xi^r(1) - \xi^r(0)$ therefore does not contribute to the summation in (1.2). Since μ has at most a countable number of atoms, the summation in (1.2) is over a countable number of index values.

We shall call a solution to (1.2) an 'extended state trajectory', since its definition extends the classical concept of state trajectory to allow for measure controls. A couple (x,μ) comprising a measure control μ and an extended state trajectory corresponding to μ is called an extended process.

The key idea behind the definition of extended state trajectory is that the afore-mentioned difficulty of evaluating the jump in the state at an atom $\bar{t}$ of μ, according to some interpretation of the integral $\int_{\{\bar{t}\}} \mu(dt)$, is circumvented by the introduction of a μ-dependent differential equation (1.3) to describe the 'instantaneous' evolution in the state at the atom.

If u is a classical control, then the extended state trajectory corresponding to the absolutely continuous measure $u(t)dt$ coincides with the state trajectory of the classical control system for the given control u.

We can associate with the control system (I) an optimal control problem:

$$(\mathrm{P}) \begin{cases} \text{Minimize } a(x(T)) \\ \text{over extended processes } (x,\mu) \text{ s.t.} \\ x(T) \in C, \end{cases}$$

in which $a : \mathbb{R}^n \to \mathbb{R}$ is a given lower semicontinuous function and $C \subset \mathbb{R}^n$ is a closed set. Solutions to this optimal control problem are called minimizers for (P).

Many interesting questions relating to impulse control systems and associated optimal control problems involve reachable sets. In this connection, the following definition is useful: given $K > 0$, the K-extended reachable set for (I) is the set of terminal values of extended state trajectories associated with measure controls whose total variation is bounded by K:

$$\mathscr{R}^{\mathrm{e}}_K(T) := \left\{x(T) \,:\, (x,\mu) \text{ is an extended process for (I) s.t. } ||\mu||_{\mathrm{TV}} \le K\right\}$$

Consider now the following hypothesis

(H): f,g are continuously differentiable and there exist $k_f, k_g > 0$ and $c_f, c_g > 0$ such that

$$|f(t',x') - f(t,x)| \le k_f(|t'-t| + |x'-x|) \text{ for all } (t',x'),(t,x) \in \mathbb{R}\times\mathbb{R}^n,$$

$$|g(x') - g(x)| \le k_g|x'-x|) \text{ for all } x',x \in \mathbb{R}\times\mathbb{R}^n,$$

$$|f(t,x)| \le c_f \text{ for all } (t,x) \in \mathbb{R}\times\mathbb{R}^n \text{ and } |g(x)| \le c_g \text{ for all } x \in \mathbb{R}^n.$$

The following facts about impulse control systems are well-known, in the case that (H) is satisfied:

(a): *(Well-Posedness)* Given a measure control μ, there is a unique extended state trajectory in the sense of a solution to (1.2).

(b): *(Closure Property)* Given an extended state trajectory x, corresponding to a measure control μ, there exists a sequence of classical controls $\{u_i\}$ such that

$$x_i(t) \to x(t), \text{ for all } t \in \mathscr{S} \cup \{0\} \cup \{T\}\ ,$$

in which, for each i, x_i is the state trajectory corresponding to u_i. Here, $\mathscr{S} \subset [0,T]$ is the complement of some countable set.

(c): *(Compactness)* For any $K \ge 0$, $\mathscr{R}^{\mathrm{e}}_K(T)$ is a non-empty compact set.

The significance of (c) is that, under the hypotheses

(B): $\exists K$ s.t. $||\mu||_{TV} \le K$, for all extended processes (x,μ) s.t. $x(T) \in C$,

and that there exists an extended process that satisfies the state endpoint constraint, the optimal control problem (P) has a solution.

A basic technique used in earlier work on impulse control systems is to consider a reparameterization of extended state trajectories, by means of a singular change of independent variable $t=\sigma(s)$ related to the control measure. The reparameterized control system is

$$
\text{(R)}\quad \begin{cases} \dot{y}(s)=(1-\alpha(s))f(\tau(s),y(s))+\alpha(s)g(y(s)), \text{ a.e. } s\in[0,S],\\ \dot{\tau}(s)=(1-\alpha(s)), \text{ a.e. } s\in[0,S],\\ \text{for some meas. } \alpha:[0,S]\to\mathbb{R},\\ \text{subject to}\\ \alpha(s)\in[0,1], \text{ a.e. } s\in[0,S],\\ y(0)=x_0,\ \tau(0)=0,\tau(S)=T\,. \end{cases}
$$

Here, $S>0$ is a given number. (R) is a classical control system, involving the scalar control variable α, a two-component state variable (τ,y) and a new time variable s. The link between state trajectories for (R) and extended state trajectories (I) is that, if (τ,y) is state trajectory for (R) such that $\tau(S)=T$, then $x(t)=y(\sigma(t))$ satisfies precisely the equation (1.2), qualifying it to be an extended state trajectory (for the right inverse transformation $s=\sigma(t)$ of $t=\tau(s)$ appropriately defined).

The properties (a)-(c) above can all be proved, via reparameterization, by appealing to standard existence, uniqueness, closure and continuous dependence properties of solutions to controlled first order differential equations, relating to control system (R). Likewise, necessary conditions, in the form of a maximum principle, can be proved by reformulating the original optimal control problem as a classical optimal control problem involving the reparameterized control system, to which the classical maximum principle is directly applicable.

There is an extensive research literature on the interpretation of state trajectories for impulse control system, which elaborates on the ideas summarised above. Some representative papers, based on the reparameterization approach are [1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 14, 16].

The question arises whether the above concepts and results concerning extended state trajectories can be generalized to allow for a 'drift' term $f(t,x)$ in the dynamic relation (1.2) that is merely *measurably time dependent*. A reason for seeking an answer to this question is that, when the input gain $g(x)$ is independent of x, the input term $\int_0^t g d\mu$ in the state equation (1.1) is unambiguous, there is no need for reparameterization and the desired properties of impulse control systems can be simply proved for measurably time-dependent drifts [15]; consequently a positive answer would unify existing theories for impulse control systems, with state-dependent and state-independent control gains. A more practical reason to consider drift terms discontinuous w.r.t. the time variable is that, in space vehicle control, a primary control $\bar{u}$ is calculated off-line to guide the vehicle approximately to its destination then a secondary impulse control is applied, on-line, to achieve course correction. If the primary control is bang-bang then the control system associated with the impulse control system has discontinuous drift.

Earlier theory, outlined above, does not immediately adapt to allow for drift terms with measurable time dependence. This is because, in this more general context, the controlled differential equation governing reparameterized state trajectories is no longer Lipschitz continuous w.r.t. the time-related state variable component τ and classical analysis based on existence, uniqueness and regular dependence of solutions to Lipschitz differential equations is no longer directly applicable. On the other hand, the defining equation for an extended state trajectory

$$x(t) = x_0 + \int_0^t f(t',x(t'))dt' + \int_{[0,t]} g(x(t'))\mu^{\text{cont}}(dt') \quad (+\text{'discrete' terms})$$

makes sense when $f(t,x)$ is merely assumed to be measurably time dependent and, indeed, is unaffected by adjustment of $f(\cdot,x)$ on a null-set, pointing to the possibility that the theory can be extended, albeit by modifying earlier methods.

In this paper we interpret state trajectories for impulse control systems with measurably time-dependent drifts and establish some of their properties. The purpose is to show that the obstacles outlined above in achieving this goal are simply dealt with by exploiting the special structure of the state-discontinuities appearing in the reparameterized equations and by switching back and forth between state trajectories representations involving the original and the transformed time variable.

We mention that measurably time dependent drifts have been treated in earlier literature. For example, in the paper [2] by Arutyunov et al., necessary conditions are proved (by an interplay of time domain and transformed time delay analysis) for an optimal control problem associated with a generalization of (I) in which the impulse control is vector valued, f and g are not only measurably time dependent but depend also an an independent, 'ordinary' control, but which are not accompanied (as in this paper) by conditions for closedness of reachable sets, existence of mininimizers. It is also true that, in early work, Bressan [4] provided a consistent definition of extended state trajectories for versions of impulse systems, as a limits of strict sense state trajectories, which did not require reparameterization and which therefore may be adapted to allow for measurably time dependent drift terms.

Notation. Take an interval $[a,b]$ and an integer k. The spaces $W^{1,1}([a,b];\mathbb{R}^k)$, $L^1([a,b];\mathbb{R}^k)$, $L^\infty([a,b];\mathbb{R}^k)$, have their usual meaning as spaces of absolutely continuous, Lebesgue integrable and essentially bounded functions on $[a,b]$ with values in $\mathbb{R}^k$. $NBV([a,b];\mathbb{R}^k)$ denotes the space of bounded variation functions $x:[a,b]\to\mathbb{R}^k$ that are right continuous on (a,b) ('normalized' functions of bounded variation). We sometimes write, for example L^∞ in place of $L^\infty([a,b];\mathbb{R}^k)$, etc., if the domain and codomain of function involved are clear from context. Given a function $x:[a,b]\to\mathbb{R}^k$ and points $t_1\in[a,b)$ and $t_2\in(a,b]$, we denote by $x(t_1^+)$ and $x(t_2^-)$ the right and left limits of x at t_1 and t_2, respectively, when these limits exist. $C^\oplus(a,b)$ denotes the set of Borel non-negative scalar valued measures on $[0,T]$. (We will refer to such elements simply as measures). For any $\mu\in C^\oplus(a,b)$, we use both the notations $\|\mu\|_{TV}$ and $\int_{[0,T]}\mu(dt)$ for the total variation of μ.

2. Impulse Controls and Extended Processes

Consider the impulsive control system of the introduction

$$\text{(I)}\quad \begin{cases} dx(t) = f(t,x(t))dt + g(x(t))d\mu(t),\ t\in[0,T] \\ \text{for some } \mu\in C^\oplus(0,T) \text{ and } x\in NBV([0,T];\mathbb{R}^n) \text{ satisfying} \\ x(0)=x_0\,. \end{cases}$$

in which $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}^n$ are given functions, $T>0$ and $x_0\in\mathbb{R}^n$.

We shall invoke the following hypothesis:

(H^*): g is continuously differentiable. $f(\cdot,x)$ is Borel measurable for each $x\in\mathbb{R}^n$ and $f(t,\cdot)$ is continuously differentiable for each $t\in[0,T]$. Furthermore, there exist $k_f,k_g>0$ and $c_f,c_g>0$ such that

$$|f(t,x')-f(t,x)|\le k_f|x'-x| \text{ for all } x',x\in\mathbb{R}^n, \text{ a.e. } t\in[0,T],$$

$$|g(x')-g(x)|\le k_g|x'-x|) \text{ for all } x',x\in\mathbb{R}^n,$$

$$|f(t,x)|\le c_f \text{ for all } (t,x)\in\mathbb{R}\times\mathbb{R}^n \text{ and } |g(x)|\le c_g \text{ for all } x\in\mathbb{R}^n.$$

The key difference between this hypothesis and the hypothesis (H) of the introduction is that, now, we require the function $f(\cdot,x)$ to be merely measurably time dependent (not Lipschitz continuous), for each $x\in\mathbb{R}^n$. The following definitions of impulsive control, extended state trajectory etc., are the same as those of the introduction, but are now interpreted in a broader context, in which the less restrictive hypothesis (H^*) replaces (H).

A *measure control* μ is a non-negative Borel measure on $[0,T]$. A *strict sense control* is a measure control $\mu=u(t)dt$ for some nonnegative function $u\in L^1(0,T)$. We denote the strict sense control u.

An element (x,μ) is said to be an *extended process* (with extended state trajectory x) if $x\in NBV([0,T];\mathbb{R}^n)$ such that $x(0)=x_0$, μ is a measure control and, for each $t\in(0,T]$,

$$x(t)=x(0)+\int_0^t f(t',x(t'))dt'+\int_{[0,t]} g(x(t'))\mu^{\text{cont}}(dt')+\sum_{r\in[0,t]}(\xi^r(1)-\xi^r(0)), \qquad (2.1)$$

in which, for each $r\in[0,T]$, $\xi^r:[0,1]\to\mathbb{R}^n$ is the solution of the differential equation

$$\frac{d}{ds}\xi^r(s)\,=\,\mu(\{r\})g(\xi^r(s)), \text{ a.e. } s\in[0,1],$$

with initial condition $\xi^r(0)=x(r^-)$. μ^{cont} denotes the continuous component of μ. As we have earlier remarked, the summation in (2.1) is over a countable number of index values.

An extended state process (x,μ) corresponding to a strict sense control u, written simply (x,u), is called a *strict sense process*. In these circumstances x, referred to as a strict sense state trajectory, is an absolutely continuous function that satisfies the differential equation

$$\dot{x}(t)=f(t,x(t))+g(x(t))u(t), \text{ a.e. } t\in[0,T].$$

with initial condition $x(0)=x_0$.

3. REPARAMETERIZATION

Extended processes will be analysed with the help of a change of independent variable that replaces possibly discontinuous state trajectories by continuous functions. The processes that arise in this way are called reparameterized processes for (I).

To be precise, a *reparameterized process* for (I) is a collection (S,τ,y,α) of elements, comprising a number $S>0$ and functions $(\tau,y)\in W^{1,1}([0,S];\mathbb{R}\times\mathbb{R}^n)$ and $\alpha\in L^\infty(0,S)$ that satisfy

$$\begin{cases} \dot{y}(s)=(1-\alpha(s))f(\tau(s),y(s))+\alpha(s)g(y(s)), \text{ a.e. } s\in[0,S],\\ \dot{\tau}(s)=(1-\alpha(s)), \text{ a.e. } s\in[0,S],\\ \alpha(s)\in[0,1] \text{ a.e. } s\in[0,S],\\ y(0)=x_0, \tau(0)=0 \text{ and } \tau(S)=T\,. \end{cases}$$

The (τ,y) and α components of some reparameterized process (on $[0,S]$) are called reparameterized state trajectories and reparameterized controls respectively.

Proposition 3.1.

(A): *Let (S,τ,y,α) be a reparameterized process for* (I). *Then there exists an impulse process (x,μ) for* (I) *such that $x(T)=y(S)$ and*

$$||\mu||_{TV}=\int_0^S\alpha(s)ds.$$

Furthermore, the impulse process (x,μ) is a strict sense process (x,u) if there exists $c>0$ such that $1-\alpha(s)\geq c$ for all $s\in[0,S]$.

(B): *Let (x,μ) be an impulse process for* (I). *Then there exists a reparameterized process (S,τ,y,α) for* (I) *such that $y(S)=x(T)$.*

Proof of Prop. 3.1. A proof is provided, for example, in [1], in the special case when (H^*) is replaced by the stronger hypothesis (H). However essentially the same analysis confirms the validity of this more general version of the proposition.

Proposition 3.2. *Fix $S>0$.*

(A): *For every measurable $\alpha:[0,S]\to\mathbb{R}$ such that $\alpha(s)\in[0,1]$ a.e., there exist unique absolutely continuous functions $y:[0,S]\to\mathbb{R}^n$ and $\tau:[0,S]\to\mathbb{R}$ such that*

$$\begin{cases} \dot{y}(s)=(1-\alpha(s))f(\tau(s),y(s))+\alpha(s)g(y(s)), \ a.e. \ s\in[0,S],\\ \dot{\tau}(s)=(1-\alpha(s)), \ a.e. \ s\in[0,S],\\ y(0)=x_0 \ \text{and} \ \tau(0)=0\,. \end{cases}$$

(B): *Take a sequence of reparameterized processes $\{(S_i,\tau_i,y_i,\alpha_i)\}$ such that $\sup_i S_i<+\infty$. Then there exists a process (S,τ,y,α) such that, along a subsequence, $S_i\to S$ and $||(\tau_i^e,y_i^e)-(\tau^e,y^e)||_{L^\infty(-\infty,+\infty)}\to 0$ uniformly.*

($z^e:\mathbb{R}\to\mathbb{R}^k$ denotes the extension of a given function $z:[a,b]\to\mathbb{R}^k$ to all of $\mathbb{R}$, by constant extrapolation from the left and the right.)

Proof of Prop. 3.2.

(A): Fix a measurable function $\alpha:[0,S]\to[0,1]$. Let τ be the solution to the differential equation $\dot{\tau}=(1-\alpha)$ on $[0,S]$ with initial value $\tau(0)=0$. Insert this function into the differential equation for y. Then there exists a unique solution to $\dot{y}(s)=(1-\alpha(s))f(\tau(s),y(s))+\alpha(s)g(y(s))$ on $[0,S]$ with initial value $y(0)=x_0$, because the latter equation (with τ a fixed function) satisfies the standard ordinary differential equation hypotheses for existence and uniqueness of solutions.

(B): Assume, initially, that $S_i = S$ for all i , i.e. (τ_i, y_i) are defined on a fixed interval. By definition of reparameterized process, $\tau_i(S) = T$ for each i. Define the right inverse function $\sigma_i : [0,T] \to [0,S]$ of τ_i to be the function defined by the conditions $\sigma_i(0) = 0$, $\sigma_i(T) = S$ and

$$\sigma_i(t) := \inf\left\{s \in [0,S] : t < \tau_i(s)\right\} \text{ for } t \in (0, \tau_i(S)), i = 1,2\ldots \tag{3.1}$$

The functions y_i and τ_i are Lipschitz continuous and uniformly bounded. We deduce from Ascoli's Theorem that, following a sequence extraction, $(y_i, \tau_i) \to (y, \tau)$, uniformly, and $(\dot{y}_i, \dot{\tau}_i) \to (\dot{y}, \dot{\tau})$ in the weak* topology, for Lipschitz continuous functions y and τ. Then $T = \tau(S)$. For each $s \in [0,S]$,

$$\int_{[0,s]} \alpha_i(s')g(y_i(s'))ds' \to \int_{[0,s]} \alpha(s')g(y(s'))ds', \text{ uniformly.}$$

Let $\delta_i \downarrow 0$ be a sequence such that $||y_i - y||_{L^\infty} \le \delta_i$, for each i.

Since $\alpha_i(s) \in [0,1]$ a.e., we deduce from the latter property and the convexity of $[0,1]$ that $\dot{\tau}(s) = (1-\alpha(s))$ a.e., for some measurable function such that $\alpha(s) \in [0,1]$ for a.e. $s \in [0,S]$. Let σ be the right inverse of τ, defined by the right side of (3.1) when τ replaces τ_i.

We can choose $\varepsilon_i \downarrow 0$ such that $|\tau_i(s) - \tau(s)| \le \varepsilon_i$ for all $s \in [0,S]$. From the definition of the inverse functions σ_i , $i = 1,2,\ldots$ and σ, we can show that, for any $t \in [\varepsilon_i, T - \varepsilon_i]$,

$$\sigma(t+\varepsilon_i) \ge \sigma_i(t) \ge \sigma(t-\varepsilon_i), \text{ for all } i \text{ sufficiently large.} \tag{3.2}$$

Fix i. Take any $s' \in [0,S]$. Then

$$y_i(s') = x(0) + \int_0^{s'} (1-\alpha_i(s))f(\tau_i(s), y_i(s))ds + \int_{[0,s']} \alpha_i(s)g(y_i(s))ds\,.$$

Applying a singular change of variables $t = \tau_i(s)$ (see [13, Lemma 4.8, Ch. 0]) and making use of simple estimates, we arrive at

$$\int_0^{s'} (1-\alpha_i(s))f(\tau_i(s), y_i(s))ds = \int_0^{\tau_i(s')} f(t, y_i(\sigma_i(t)))dt = \int_0^{\tau(s')\wedge\tau_i(s')} f(t, y(\sigma_i(t)))dt + r_i$$

in which r_i is some number such that

$$|r_i| \le k_f \times \delta_i + c_f \times \varepsilon_i.$$

Since $\sigma : [0,S] \to \mathbb{R}$ is a monotone function, there exists a subset $\mathscr{S} \subset (0,T)$ of full Lebesgue measure on which σ is continuous. Take any point $\bar{t} \in \mathscr{S}$. Then $t \to y(\sigma(t))$ is continuous at $\bar{t}$. It follows then from (3.2) and the continuity of $f(t,\cdot)$ that

$$f(\bar{t}, y(\sigma_i(\bar{t})) \to f(\bar{t}, y(\sigma(\bar{t})) \text{ as } i \to \infty.$$

But then, making use of the dominated convergence theorem, noting that $\tau_i(s') \to \tau(s')$ and, once again, applying a change of variable $s = \sigma(t)$, we can deduce that

$$\int_0^{\tau(s')\wedge\tau_i(s')} f(t, y(\sigma_i(t)))dt \to \int_0^{\tau(s')} f(t, y(\sigma(t)))dt = \int_0^{s'} (1-\alpha(s))f(\tau(s), y(s))ds, \text{ as } i \to \infty.$$

We deduce from the above relations that y satisfies:

$$y(s') = x(0) + \int_0^{s'} (1-\alpha(s))f(\tau(s), y(s))ds + \int_{[0,s']} \alpha(s)g(y(s))ds, \text{ for all } s' \in [0,S]\,.$$

We have assumed that all the S_i's are the same. If this is not the case, we arrange, by subsequence extraction that $S_i \to S$, for some S. Now choose $S' > S$, extend each y_i to $[0,S']$

by concatenating, on the right, the solution to $\dot{y} = g(y)$ on $[S_i, S']$ with boundary condition $y(S_i) = y_i(S_i)$, and extend τ_i taking $\tau_i(s) = T$, for any $s \geq S_i$. Now apply the preceding analysis to obtain a limiting reparameterized process (S', τ', y', α') in which τ' is constant and $\alpha'(s) = 1$ on $[S, S']$. The desired limiting reparameterized process is now obtained, replacing S' by S and restricting y', τ and α' to $[0, S]$.

4. Density and Compactness Properties of Extended Trajectories

For any $K > 0$, we consider the *K-extended reachable set for* (I), defined in the introduction

$$\mathscr{R}_K^{\mathrm{e}}(T) := \left\{ x(T) \,:\, (x, \mu) \text{ is an extended process for (I) s.t. } ||\mu||_{\mathrm{TV}} \leq K \right\}$$

and we define the *K-strict sense reachable set for* (I), as

$$\mathscr{R}_K^{\mathrm{s}}(T) := \left\{ x(T) \,:\, (x, u) \text{ is a strict sense process for (I) s.t. } \int_0^T u(t)\, dt \leq K \right\}.$$

Since strict sense processes are special cases of extended processes, $\mathscr{R}_K^{\mathrm{s}}(T) \subseteq \mathscr{R}_K^{\mathrm{e}}(T)$. The following theorem tells us that the class of extended processes for (I) is a 'relaxation' of the class of strict sense processes for (I).

Theorem 4.1. *For any $K > 0$, the set $\mathscr{R}_K^{\mathrm{e}}(T)$ is compact and $\mathscr{R}_K^{\mathrm{e}}(T) = \overline{\mathscr{R}_K^{\mathrm{s}}(T)}$.*

Proof.
(i) (We show that $\mathscr{R}_K^{\mathrm{e}}(T) = \overline{\mathscr{R}_K^{\mathrm{s}}(T)}$.) Take $z \in \mathscr{R}_K^{\mathrm{e}}(T)$ and an extended process (x, μ) with $||\mu||_{\mathrm{TV}} \leq K$ and such that $x(T) = z$. Then, by Prop. 3.1, there exists a reparameterized process (S, τ, y, α) for (I) such that $x(T) = y(S)$ and $\int_0^S \alpha(s)\, ds = ||\mu||_{\mathrm{TV}}$. Set

$$\alpha_0(s) := 1 - \alpha(s), \quad \text{a.e. } s \in [0, S], \text{ and} \quad \nu := \int_0^S \alpha(s)\, ds.$$

For $j = 1, 2, \ldots$ and a.e. $s \in [0, S]$, define

$$\alpha_0^j(s) := \frac{T}{T + \frac{\nu}{j}} \left[\alpha_0(s) + \frac{1}{j} \alpha(s) \right] \text{ and } \alpha_j(s) := (1 - \alpha_0^j(s)).$$

Notice that $\alpha_0^j(s) + \alpha_j(s) = 1$ and, for some constant $c_j \in (0, 1)$, $0 \leq \alpha_j(s) \leq 1 - c_j$ a.e. Moreover, since $S = T + \nu$,

$$\int_0^S \alpha_0^j(s)\, ds = T \text{ and } \int_0^S \alpha_j(s)\, ds = \int_0^S \left(1 - \alpha_0^j(s)\right) ds = S - T = \nu \leq K.$$

Let y_j, τ_j be the unique solution to the reparameterized system associated with S and α_j. Then, for any $j \geq 1$, the quadruple $(S, \tau_j, y_j, \alpha_j)$ is a reparameterized process for (I) which, according to Prop. 3.1, corresponds a strict sense process (x_j, u_j) satisfying the conditions $x_j(T) = y_j(S)$ and $\int_0^T u_j(s)\, ds = \nu \leq K$. We see that the (reparameterized) controls α_j converge to α in the L^∞-norm. Consequently $\tau_j \to \tau$ in $L^\infty(0, S)$. Arguing as in the proof of Prop. 3.2, we show that there exists a Lipschitz continuous function $\bar{y} : [0, S] \to \mathbb{R}^n$ such that $y(0) = x$ and $||y^j - \bar{y}||_{L^\infty} \to 0$ as $j \to \infty$. For each j

$$y_j(s') = x(0) + \int_0^{s'} (1 - \alpha_j(s)) f(\tau_j(s), y_j(s)) ds + \int_{[0, s']} \alpha_j(s) g(y_j(s)) ds \text{ for all } s' \in [0, S].$$

Essentially the same analysis as that employed in the proof of Prop. 3.2, making use of the singular change of variables $t = \tau_i(s)$ to evaluate the limit of the first integral term on the right side, permits us to conclude that

$$\bar{y}(s') = x(0) + \int_0^{s'} (1-\alpha(s)) f(\tau(s), \bar{y}(s)) ds + \int_{[0,s']} \alpha(s) g(\bar{y}(s)) ds \text{ for all } s' \in [0,S].$$

But then $(S, \tau, \bar{y}, \alpha)$ is a reparameterized process for (I). Since, for a given reparameterized control α, there is a unique reparameterized state trajectory, we have $y = \bar{y}$. It follows that

$$|x_j(T) - x(T)| = |y_j(S) - y(S)| \to 0.$$

We may conclude that $\mathscr{R}^{\mathrm{e}}_K(T) \subseteq \overline{\mathscr{R}^{\mathrm{s}}_K(T)}$. Since the reverse inequality is automatically satisfied, we have shown that the two sets coincide.

(ii) *(We show that $\mathscr{R}^{\mathrm{e}}_K(T)$ is compact.)* Take a sequence $\{z_j\}$ in $\mathscr{R}^{\mathrm{e}}_K(T)$. There exists, for each j, an extended process (x_j, μ_j) such that $x_j(T) = z_j$ and $||\mu_j||_{TV} \le K$. By Prop 3.1 there exists a reparameterized process $(S_j, \tau_j, y_j, \alpha_j)$ for (I) such that $y_j(S_j) = z_j$. Moreover, $||\mu_j||_{TV} = \int_0^{S_j} \alpha_j(s) ds$ and $T = \int_{[0,S_j]} (1-\alpha_j(s)) ds = S_j - ||\mu_j||_{TV}$. It follows that $\{S_j\}$ is a bounded sequence. We can deduce from Gronwall's Lemma, applied to the differential equation of the reparameterized control system, that $y_j(S_j) = z_j$ is a bounded sequence. We may therefore arrange, by subsequence extraction, that $z_j \to z$, for some $z \in \mathbb{R}^n$.

We may now conclude from Prop. 3.2 that, following extraction of a subsequence, $S_j \to S$, for some number S, there exists a reparameterized process (S, τ, y, α) such that $y_j(S_j) \to x(S)$ as $j \to \infty$. But according to Prop 3.1 there exists an extended process (x, μ) such that $||\mu||_{TV} = \int_0^S \alpha(s) ds \le K$ and $x(T) = y(S) = z$. We have confirmed that $\mathscr{R}^{\mathrm{e}}_K(T)$ is compact.

5. Existence of Optimal Controls

Consider now the impulse optimal control problem of the introduction:

$$(\mathrm{P}) \begin{cases} \text{Minimize } a(x(T)) \\ \text{subject to} \\ dx(t) = f(t, x(t))dt + g(x(t))d\mu \\ x(0) = x_0 \text{ and } x(T) \in C \end{cases}$$

in which f and g are as before, $a : \mathbb{R}^n \to \mathbb{R}$ is a given function, and $C \subset \mathbb{R}^n$ is a given subset.

A precise formulation, in terms of our earlier definitions, is

$$\begin{cases} \text{Minimize } a(x(T)) \\ \text{over extended processes } (x, \mu) \text{ for (I) s.t.} \\ x(0) = x_0 \text{ and } x(T) \in C. \end{cases}$$

We say that an extended process (x, μ) for (I) is *admissible* if $x(T) \in C$. An admissible extended process $(\bar{x}, \bar{\mu})$ for (I) satisfying the condition

$$a(\bar{x}(T)) \le a(x(T))$$

for all admissible extended processes (x, μ) is said to be a minimizer for (P).

Theorem 5.1 (Existence of Minimizers)**.** *Consider the optimal control problem* (P)*. Assume that $a : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous and that C is closed. Assume furthermore that there exists an admissible extended process and there exists a constant $K > 0$ such that*

$$||\mu||_{TV} \leq K$$

for all admissible extended processes. Then, there exists a minimizer for (P)*.*

Proof. Since we have assumed that an admissible extended process exists, it can be deduced from Thm. 4.1 that the set $\mathscr{R}^{\mathrm{e}}_K(T) \cap C$ is non-empty and compact. Since the function a is assumed to be lower semicontinuous, there exists $\bar{z} \in \mathscr{R}^{\mathrm{e}}_K(T) \cap C$ that minimizes a over $\mathscr{R}^{\mathrm{e}}_K(T) \cap C$. Let $(\bar{x}, \bar{\mu})$ be an admissible extended process such that $\bar{x}(T) = \bar{z}$. For any admissible extended process (x, μ) write $z = x(T)$. By assumption $||\mu||_{TV} \leq K$ and so $z \in \mathscr{R}^{\mathrm{e}}_K(T) \cap C$. Then, from the minimizing property of $\bar{z}$,

$$a(\bar{x}(T)) = a(\bar{z}) \leq a(z) = a(x(T)).$$

In view of the fact that $(\bar{x}, \bar{\mu})$ is an arbitrarily chosen admissible extended process, we have shown that $(\bar{x}, \bar{\mu})$ is a minimizer for (P).

Summary and Future Directions

Important properties of impulse control systems, concerning density and closure properties of reachable sets, have previously been proved by techniques involving reparameterization of state trajectories and interpretation of time as a state variable. In ealier literature, it is typically assumed that the drift term in the impulse control system dynamics is Lipschitz continuous w.r.t. time. This assumption ensures that the reparameterized control system is Lipschitz continuous w.r.t. all state variables and, in consequence, classical continuous dependence and convergence properties of solutions to regular first order differential equations can be brought into play to analyse the impulse control system via the reparameterized control system. The approach fails when the time dependence of the drift term in the impulse control system is not Lipschitz continuous. We have shown, nonetheless, that density and closure properties of impulse control systems can still be established, for a more general class of impulse control systems in which the drift term is merely measurably time dependent.

Future work will be directed at providing generalizations, which provide density and closure properties for measurably time dependent impulse control systems, with multiple measure controls, and exploring their system-theoretic implications.

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