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SUFFICIENT CONDITIONS OF OPTIMALITY IN AN INFINITE-HORIZON OPTIMAL CONTROL PROBLEM WITH VANISHING DISCOUNTING

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Dedicated to R. Vinter on the occasion of his 75th birthday

Abstract. We derive sufficient optimality conditions for an infinite horizon optimal control problem with a vanishing discounting factor and demonstrate the obtained results by examples.

Keywords. Infinite-horizon optimal control problem; Sufficient conditions of optimality; Vanishing discounting.

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1. Introduction

In this paper, we consider the controlled system

$$\begin{aligned} &\dot{x}(t) = f(x(t),u(t)),\ t \geq 0,\\ &x(0) = x_0,\\ &x(t) \in X,\\ &u(t) \in U, \end{aligned} \tag{1.1}$$

where $f(\cdot,\cdot) : \mathbb{R}^n \times U \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$, and $X \subset \mathbb{R}^n$ plays a role of a state constraint.

A control $u(\cdot)$ and the pair $(x(\cdot),u(\cdot))$ are called admissible control and an admissible process, respectively, if $u(\cdot)$ is measurable, $x(\cdot)$ is absolutely continuous, and the relationships (1.1) are satisfied. The set of admissible controls is denoted by $\mathscr{U}(x_0)$, which we assume to be not empty.

The optimal control problem often considered on the trajectories of (1.1) is that of finding the infimum over admissible processes of

$$\int_0^\infty e^{-\lambda t} g(x(t),u(t))\,dt, \tag{1.2}$$

where $\lambda > 0$ is a parameter and $g(x,u) : \mathbb{R}^n \times U \to \mathbb{R}$ is a given function. Here $e^{-\lambda t}$ is the discounting factor, which is introduced to ensure convergence of the integral when appropriate assumptions on g are in place. λ in (1.2) is often chosen arbitrarily. If it is desired to take λ

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close to zero, then the following optimization problem with a *vanishing discounting factor* can be considered:

$$\inf_{u(\cdot)\in\mathscr{U}(x_0)} \limsup_{\lambda\to 0^+} \lambda \int_0^\infty e^{-\lambda t} g(x(t),u(t))\,dt. \tag{1.3}$$

The multiplicative factor λ is introduced here to ensure that the limit as $\lambda \to 0^+$ is bounded. We could as well have taken $\liminf_{\lambda\to 0^+}$ rather than $\limsup_{\lambda\to 0^+}$ in (1.3), but we chose the latter because taking $\inf_{u(\cdot)\in\mathscr{U}(x_0)} \limsup_{\lambda\to 0^+}$ can be interpreted as "minimization in the worst case scenario". In this paper we establish sufficient optimality conditions for problem (1.3).

It should also be mentioned that there are optimality criteria on infinite horizon that do not involve a discounting factor; see, e.g., [3, 19] and references therein.

We assume that $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are Borel measurable and that g is bounded below. We do *not* assume that f and g are continuous, and neither do we assume any structure of U and X such as closedness or compactness.

If we interchange the infimum and the limit in (1.3), if the latter exists, we obtain the so-called *Abel limit*

$$\lim_{\lambda\to 0^+} \inf_{u(\cdot)\in\mathscr{U}(x_0)} \lambda \int_0^\infty e^{-\lambda t} g(x(t),u(t))\,dt. \tag{1.4}$$

The so-called *Cesàro limit* of the long-run averages

$$\lim_{T\to\infty} \inf_{u(\cdot)\in\mathscr{U}_T(x_0)} \frac{1}{T}\int_0^T g(x(t),u(t))\,dt \tag{1.5}$$

is closely related to it. (Here $\mathscr{U}_T(x_0)$ is the set of admissible controls on the interval $[0,T]$.)

A lot of literature is devoted to estabishing conditions of existence and equality of Cesàro and Abel limits in problems of dynamic programming and optimal control in discrete and continuous time, see, e.g., [1, 6, 7, 10, 15, 16]. Relatively weak conditions ensuring equality of the limits (1.4) and (1.5) are established in the recent paper [7].

Optimality conditions in the problem

$$\inf_{u(\cdot)\in\mathscr{U}_T(x_0)} \liminf_{T\to\infty} \frac{1}{T}\int_0^T g(x(t),u(t))\,dt,$$

as well as in its discrete counterpart, were studied in [2, 7, 8, 9, 12] using the approach of representing the original nonlinear problem of optimal control as a problem of linear programming in a certain infinite-dimensional space and applying duality results. Important contributions to this methodology were made by Vinter [17]. In [2, 7, 8, 9] it is assumed that the functions defining the dynamics and the cost of the dynamical system are continuous and that the constraint sets X and U are compact. Since these assumptions are not imposed in this paper, we are considering a more general class of systems.

Optimality conditions for the discrete-time version of problem (1.3) recently appeared in [13]. Although the ideas in treatment of discrete-time and continuous-time problems are similar, there are also significant differences; e.g., nonsmooth analysis is used in the present paper, but not in discrete time. Results of this paper were announced without proofs in [14].

The rest of the paper is organized as follows. In Section 2 we formulate and prove sufficient optimality conditions in problem (1.3) which are illustrated by examples in Section 3.

2. SUFFICIENT OPTIMALITY CONDITIONS

Denote

$$d(x_0) := \sup_{(\psi,\eta)} \inf_{(x,u,p)} \{g(x,u) + (\psi(x_0) - \psi(x)) + pf(x,u)\}, \tag{2.1}$$

where supremum is taken over functions $\psi : \mathbb{R}^n \to \mathbb{R}$ non-decreasing along admissible trajectories, that is, such that

$$t \mapsto \psi(x(t)) \text{ is non-decreasing for any admissible } x(\cdot), \tag{2.2}$$

and bounded locally Lipschitz $\eta : \mathbb{R}^n \to \mathbb{R}$ (we write $\eta \in Lip$). Infimum in (2.1) is taken over $(x,u) \in X \times U$ and $p \in \partial\eta(x)$, where $\partial\eta$ stands for Clarke's generalized gradient ([4]), which for Lipschitz η is equal to the convex hull of the limits of its gradients, that is, has representation

$$\partial\eta(x) = \operatorname{conv}\{p \,|\, p = \lim_{i\to\infty} \nabla\eta(x_i) \text{ for some } x_i \to x\}.$$

Since g is assumed to be bounded below, by taking constant ψ and η in (2.1), we see that $d(x_0) > -\infty$.

For a fixed $\lambda > 0$ denote

$$h_\lambda(x_0) := \lambda \inf_{u(\cdot)\in\mathscr{U}(x_0)} \int_0^\infty e^{-\lambda t} g(x(t),u(t))\,dt. \tag{2.3}$$

Since g is bounded below, so is $h_\lambda(x_0)$. Assume in addition that there exists an admissible process $(x(\cdot),u(\cdot))$ such that $g(x(t),u(t)) \le M$ for all t. Then $h_\lambda(x_0) \le M$ for all λ.

We note that functions non-decreasing along the state trajectories were first used to characterize the limit behavior in optimal control problems with vanishing discounting in [11].

Proposition 2.1. *It holds that*

$$\liminf_{\lambda\to 0^+} h_\lambda(x_0) \ge d(x_0). \tag{2.4}$$

Proof. Assume that (2.4) is not true. Then there exist $\beta > 0$, $\psi(\cdot)$ satisfying (2.2) and $\eta \in Lip$, such that for all $(x,u) \in X \times U$ and $p \in \partial\eta(x)$

$$g(x,u) + (\psi(x_0) - \psi(x)) + pf(x,u) \ge \liminf_{\lambda'\to 0^+} h_{\lambda'}(x_0) + \beta. \tag{2.5}$$

Take an arbitrary admissible admissible process $(x(\cdot),u(\cdot))$. For all t and $p(t) \in \partial\eta(x)\big|_{x=x(t)}$, we get from (2.5) that

$$g(x(t),u(t)) + (\psi(x_0) - \psi(x(t))) + p(t)f(x(t),u(t)) \ge \liminf_{\lambda'\to 0^+} h_{\lambda'}(x_0) + \beta. \tag{2.6}$$

Since η is Lipschitz, the function $t \mapsto \eta(x(t))$ is absolutely continuous, and for any t where it is differentiable we have due to the chain rule ([4], Theorem 2.3.10 or [18], p.46) that $\frac{d}{dt}\eta(x(t)) \in \partial\eta(x)\big|_{x=x(t)} f(x(t),u(t))$, that is, there exists $p(t) \in \partial\eta(x)\big|_{x=x(t)}$ such that

$$\frac{d}{dt}\eta(x(t)) = p(t)f(x(t),u(t)). \tag{2.7}$$

For $p(t)$ satisfying (2.7), we obtain from (2.6) that

$$g(x(t),u(t)) + (\psi(x_0) - \psi(x(t))) + \frac{d}{dt}\eta(x(t)) \ge \liminf_{\lambda'\to 0^+} h_{\lambda'}(x_0) + \beta.$$

Multiplying both sides by $e^{-\lambda t}$, integrating, multiplying by λ, and taking into account that $\psi(\cdot)$ is non-decreasing, we obtain

$$\lambda \int_0^\infty e^{-\lambda t} g(x(t),u(t))\,dt + \lambda \int_0^\infty e^{-\lambda t} \frac{d}{dt}\eta(x(t))\,dt \ge \liminf_{\lambda'\to 0^+} h_{\lambda'}(x_0) + \beta. \tag{2.8}$$

Since

$$\lambda \int_0^\infty e^{-\lambda t} \frac{d}{dt}\eta(x(t))\,dt = -\lambda\eta(x_0) + \lambda^2 \int_0^\infty e^{-\lambda t}\eta(x(t))\,dt \to 0 \text{ as } \lambda \to 0^+ \tag{2.9}$$

due to η being bounded, from (2.8) and (2.9) for sufficiently small λ, we have

$$\lambda \int_0^\infty e^{-\lambda t} g(x(t),u(t))\,dt \ge \liminf_{\lambda'\to 0^+} h_{\lambda'}(x_0) + \beta/2$$

which implies that $h_\lambda(x_0) \ge \liminf_{\lambda'\to 0^+} h_{\lambda'}(x_0) + \beta/2$. Thus

$$\liminf_{\lambda\to 0^+} h_\lambda(x_0) \ge \liminf_{\lambda'\to 0^+} h_{\lambda'}(x_0) + \beta/2,$$

which is a contradiction. The proposition is proved. □

The following theorem, which is the main result of the paper, provides conditions ensuring the existence of the limit $\lim\limits_{\lambda\to 0^+} h_\lambda(x_0)$ and of optimality of a given process in problem (1.3).

Theorem 2.2. *Assume that a pair* $(\bar\psi,\bar\eta)$ *of maximizers in problem* (2.1) *exists and for some admissible process* $(x^*(\cdot),u^*(\cdot))$ *and all* $t \ge 0$,

$$(x^*(t),u^*(t)) = \operatorname{argmin}_{(x,u)}\{g(x,u) - \bar\psi(x) + pf(x,u)\} \quad \textit{for all } p \in \partial\bar\eta(x) \tag{2.10}$$

and

$$\psi(x^*(t)) = \text{const}. \tag{2.11}$$

Then

(a) *there exists the limit* $h(x_0) := \lim\limits_{\lambda\to 0^+} h_\lambda(x_0)$*;*

(b) *there is equality*

$$V(x_0) = h(x_0) = d(x_0), \tag{2.12}$$

where

$$V(x_0) := \inf_{u(\cdot)\in\mathscr{U}(x_0)} \limsup_{\lambda\to 0^+} \lambda \int_0^\infty e^{-\lambda t} g(x(t),u(t))\,dt$$

is the value function in (1.3)*;*

(c) *the process* $(x^*(\cdot),u^*(\cdot))$ *is optimal in (1.3).*

Remark 2.3. If function $\bar\eta$ is smooth, then (2.10) becomes

$$(x^*(t),u^*(t)) = \operatorname{argmin}_{(x,u)}\{g(x,u) - \bar\psi(x) + \nabla\bar\eta(x)f(x,u)\}. \tag{2.13}$$

However, as we will see in examples below, $\bar\eta$ may be nonsmooth even in simple situations.

Remark 2.4. The limiting function h may be discontinuous and functions $\bar\psi,\bar\eta$ may depend on x_0, as shown in Example 3.2 below.

Remark 2.5. For (2.10) to hold it is necessary that for all t

$$u^*(t) = \text{argmin}_{u \in U}\{g(x^*(t),u) + pf(x^*(t),u)\} \text{ for all } p \in \partial\bar{\eta}(x)\big|_{x=x^*(t)}, \tag{2.14}$$

which implies the optimal feedback control law

$$u^f[x] = \text{argmin}_{u \in U}\{g(x,u) + pf(x,u)\} \text{ for all } p \in \partial\bar{\eta}(x).$$

In the case when $\bar{\eta}$ is smooth, the latter becomes

$$u^f[x] = \text{argmin}_{u \in U}\{g(x,u) + \nabla\bar{\eta}(x)f(x,u)\}. \tag{2.15}$$

If $\bar{\eta}$ is known, this law can be used to construct an optimal control in (1.3). If the maximizing η is not known, it may be possible to approximate it and to construct a control close to the optimal. This approach is demonstrated in [5] in a problem with a fixed dicounting factor. In the case of the vanishing dicounting factor, developing a method for constructing an approximately optimal control may be a subject of further research.

Proof of Theorem 2.2. From (2.10) and (2.1), we have that for all t and any $p(t) \in \partial\eta(x)\big|_{x=x^*(t)}$

$$g(x^*(t),u^*(t)) + (\bar{\psi}(x_0) - \bar{\psi}(x^*(t))) + p(t)f(x^*(t),u^*(t)) = d(x_0). \tag{2.16}$$

Similarly to how the left-hand side of (2.6) was transformed into the left-hand side of (2.8), we multiply both sides of (2.16) by $e^{-\lambda t}$, integrate, multiply by λ, and take into account that $\bar{\psi}(x_0) - \bar{\psi}(x^*(t)) = 0$ due to (2.11). We obtain

$$\lambda\int_0^\infty e^{-\lambda t}g(x^*(t),u^*(t))\,dt + \lambda\int_0^\infty e^{-\lambda t}\frac{d}{dt}\eta(x^*(t))\,dt = d(x_0). \tag{2.17}$$

Since $h_\lambda(x_0) \le \lambda\int_0^\infty e^{-\lambda t}g(x^*(t),u^*(t))\,dt$ and taking into account that the second integtal in (2.17) vanishes as $\lambda \to 0^+$ (see (2.9)), we conclude that

$$\limsup_{\lambda\to 0^+} h_\lambda(x_0) \le d(x_0).$$

Along with (2.4), the last inequality implies that the limit $h(x_0) = \lim_{\lambda\to 0^+} h_\lambda(x_0)$ exists and is equal to $d(x_0)$. Part (a) of the theorem and the second equality in (2.12) are proved. From (2.17) and the equality $d(x_0) = h(x_0)$ we conclude that

$$\lim_{\lambda\to 0^+}\lambda\int_0^\infty e^{-\lambda t}g(x^*(t),u^*(t))\,dt = h(x_0). \tag{2.18}$$

For any admissible process $(x(\cdot),u(\cdot))$

$$\lambda\int_0^\infty e^{-\lambda t}g(x(t),u(t))\,dt \ge h_\lambda(x_0).$$

Therefore,

$$\limsup_{\lambda\to 0^+}\lambda\int_0^\infty e^{-\lambda t}g(x(t),u(t))\,dt \ge \limsup_{\lambda\to 0^+} h_\lambda(x_0) = h(x_0),$$

hence,

$$V(x_0) \ge h(x_0). \tag{2.19}$$

From (2.18) and (2.19), it follows that

$$\lim_{\lambda\to 0^+}\lambda\int_0^\infty e^{-\lambda t}g(x^*(t),u^*(t))\,dt \le V(x_0). \tag{2.20}$$

Since the opposite inequality is implied by the definition of $V(x_0)$, (2.20) holds as equality, that is, the process $(x^*(\cdot), u^*(\cdot))$ is optimal. The theorem is proved. □

Theorem 2.2 establishes sufficient conditions of optimality in terms of the maximizing functions ψ, η in (2.1). Theorem 2.7 below demonstrates a possible way for finding one such pair of functions. (It may be not unique.)

Prior to this theorem we prove the monotonicity property of the function h.

Proposition 2.6. *Function $h(\cdot)$ is non-decreasing over admissible trajectories, that is, it satisfies* (2.2).

Proof. This assertion follows from the the dynamic programming principle. For an arbitrary admissible process $(x(\cdot), u(\cdot))$ and any $\tau > 0$ we have

$$\begin{aligned} h_\lambda(x_0) &\le \lambda \int_0^\tau e^{-\lambda t} g(x(t), u(t))\, dt + \lambda \int_\tau^\infty e^{-\lambda t} g(x(t), u(t))\, dt \\ &= \lambda \int_0^\tau e^{-\lambda t} g(x(t), u(t))\, dt + \lambda e^{-\lambda \tau} \int_0^\infty e^{-\lambda s} g(x(\tau + s), u(\tau + s))\, dt. \end{aligned}$$

Taking infimum with respect to admissible processes in the second integral we conclude that

$$h_\lambda(x_0) \le \lambda \int_0^\tau e^{-\lambda t} g(x(t), u(t))\, dt + e^{-\lambda \tau} h_\lambda(x(\tau)).$$

Passing to the limit as $\lambda \to 0^+$ we obtain $h(x_0) \le h(x(\tau))$. The proposition is proved. □

Theorem 2.7. *Let the pointwise limit $h(x_0) = \lim_{\lambda \to 0^+} h_\lambda(x_0)$ exist for all $x_0 \in X$ and $\bar{\eta}(\cdot) \in Lip$ be such that*

$$\inf_{(x,u,p)} \{g(x,u) - h(x) + pf(x,u)\} = 0, \tag{2.21}$$

where infimum is taken over $(x,u) \in X \times U$ and $p \in \partial \bar{\eta}(x)$. Then the supremum in (2.1) *is reached at the functions $\psi = h$ and $\eta = \bar{\eta}$.*

Remark. If $\bar{\eta}$ is smooth, then (2.21) becomes

$$\inf_{(x,u)} \{g(x,u) - h(x) + \nabla \bar{\eta}(x) f(x,u)\} = 0. \tag{2.22}$$

Proof of Theorem 2.7. It is proved in Proposition 2.6 that h satisfies (2.2). It follows from (2.4) that

$$h(x_0) \ge d(x_0), \tag{2.23}$$

therefore, from (2.1) and (2.23),

$$\sup_{\eta(\cdot) \in Lip} \inf_{(x,u,p)} \{g(x,u) + (h(x_0) - h(x)) + pf(x,u)\} \le d(x_0) \le h(x_0), \tag{2.24}$$

where infimum is taken over $(x,u) \in X \times U$ and $p \in \partial \eta(x)$. Hence,

$$\sup_{\eta(\cdot) \in Lip} \inf_{(x,u,p)} \{g(x,u) - h(x) + pf(x,u)\} \le 0. \tag{2.25}$$

From (2.21), it follows that the supremum on the left side is reached at $\eta = \bar{\eta}$, in which case the last inequality holds as equality:

$$\max_{\eta(\cdot) \in Lip} \inf_{(x,u,p)} \{g(x,u) - h(x) + pf(x,u)\} = 0.$$

The latter implies that $\max_{\eta(\cdot)\in Lip}\inf_{(x,u,p)}\{g(x,u)+(h(x_0)-h(x))+pf(x,u)\}=h(x_0)$, which implies via (2.24) that

$$\max_{\eta(\cdot)\in Lip}\inf_{(x,u,p)}\{g(x,u)+(h(x_0)-h(x))+pf(x,u)\}=d(x_0),$$

which means that supremum in (2.1) is reached at $\psi=h$ and $\eta=\bar{\eta}$. The proposition is proved. □

3. EXAMPLES

In this section, applications of Theorems 2.2 and 2.7 are demonstrated.

Example 3.1. Consider the problem

$$\inf_{u(\cdot)\in\mathscr{U}(x_0)}\limsup_{\lambda\to 0^+}\lambda\int_0^\infty e^{-\lambda t}(1-x(t))^2\,dt \tag{3.1}$$

on the trajectories of the system

$$\begin{aligned}&\dot{x}(t)=(1-x(t))^2u(t),\ t>0,\\&x(0)=x_0,\\&x(t)\in(0,2),\\&u\in[-1,1].\end{aligned}\tag{3.2}$$

In this example, $g(x)=(1-x)^2$ and $X=(0,2)$.

Clearly, the control that makes the system approach $x=1$ as quicky as possible, is optimal. The corresponding optimal feedback control law is

$$u^f[x]=\begin{cases}1, & x\in(0,1),\\ -1, & x\in(1,2),\\ \text{any}, & x=1,\end{cases}\tag{3.3}$$

and the corresponding optimal trajectory is

$$x^*(t)=\begin{cases}1-\frac{1}{t+1/(1-x_0)}, & x_0\in(0,1)\\ 1+\frac{1}{t+1/(x_0-1)}, & x_0\in(1,2),\\ 1, & x_0=1.\end{cases}\tag{3.4}$$

(In fact, the optimal control is not unique; we will consider other optimal feedback laws after we investigate the one given by (3.3).)

Let us show that $h(x)=0$ for all $x\in X$. We see from (3.4) that the integral

$$\int_0^T(1-x^*(t))^2\,dt$$

is uniformly bounded with respect to T, hence

$$h(x_0)=\lim_{\lambda\to 0^+}\lambda\int_0^\infty e^{-\lambda t}(1-x^*(t))^2=0.\tag{3.5}$$

Next we will show that (2.21) holds with $\bar{\eta}(x) = |1-x|$. We have

$$\partial\bar{\eta}(x) = \begin{cases} -1, & x < 1, \\ 1, & x > 1, \\ [-1,1], & x = 1 \end{cases}$$

and for $p \in \partial\bar{\eta}(x)$ we have

$$\begin{aligned} g(x) - h(x) + pf(x,u) &= (1-x)^2 + p(1-x)^2 u \\ &= (1-x)^2(1+pu) \\ &= \begin{cases} (1-x)^2(1-u), & x \in (0,1), \\ (1-x)^2(1+u), & x \in (1,2), \\ 0, & x = 1. \end{cases} \end{aligned} \tag{3.6}$$

Therefore,

$$\min_{(x,u,p)} \{g(x) - h(x) + pf(x,u)\} = 0, \tag{3.7}$$

that is, (2.21) holds. Due to Theorem 2.7, maximizing functions in (2.1) are $\bar{\psi} = 0$ and $\bar{\eta} = |1-x|$. From (3.3) and (3.6), one can see that, for $u^*(t) := u^f[x^*(t)]$ and for all t and $p(t) \in \partial\bar{\eta}(x)\big|_{x=x^*(t)}$,

$$g(x^*(t)) - h(x^*(t)) + p(t)f(x^*(t), u^*(t)) = 0.$$

This implies via (3.7) that

$$(x^*(t), u^*(t)) = \text{argmin}_{(x,u,p)}\{g(x) - \bar{\psi}(x) + pf(x,u)\} \text{ for all } p \in \partial\bar{\eta}(x).$$

Since $h(x) = 0$, we have $h(x^*(t)) = \text{const}$. Thus (2.10) and (2.11) hold, hence, the process $(x^*(\cdot), u^*(\cdot))$ is optimal due to Theorem 2.2, which agrees with our earlier observation.

If the parameter $\lambda > 0$ was fixed, the optimal feedback control law (3.3) would be unique. But in the situation with the vanishing discounting, the feedback control law

$$u^f[x] = \begin{cases} a_1, & x \in (0,1), \\ -a_2, & x \in (1,2), \\ \text{any}, & x = 1, \end{cases} \tag{3.8}$$

with any $a_1, a_2 \in (0,1]$ is also optimal. Indeed, it can be verified that, similarly to (3.5), $h(x_0) = 0$ for the corresponding trajectory. It can be similarly shown that for the feedback law (3.8), relations (2.10) and (2.11) hold with $\bar{\psi} = 0$ and

$$\bar{\eta}(x) = \begin{cases} \frac{1}{a_1}(1-x), & x \le 1 \\ \frac{1}{a_2}(x-1), & x > 1. \end{cases}$$

Example 3.2. Consider the problem

$$\inf_{u(\cdot)\in\mathscr{U}(x_0)} \limsup_{\lambda\to 0^+} \lambda \int_0^\infty e^{-\lambda t}(-x(t))\,dt \tag{3.9}$$

on the trajectories of the system

$$\begin{aligned}&\dot{x}(t) = x(t)u(t),\ t \geq 0,\\&x(0) = x_0,\\&x(t) \in [0,1],\\&u \in [-1,1].\end{aligned}$$

In this example, $g(x) = -x$ and $X = [0,1]$.

It is clear that the feedback control law below is optimal in problem (3.9):

$$u^f[x] = \begin{cases} \text{any } u, & x = 0,\\ 1, & x \in (0,1),\\ 0, & x = 1.\end{cases} \tag{3.10}$$

If $x_0 = 0$, then $x(t) \equiv 0$ and $h_\lambda(0) = 0$ for all $\lambda > 0$. Otherwise, if $x_0 \in (0,1]$, the optimal trajectory reaches $x = 1$ at some time τ independent of λ and stays there for $t \geq \tau$. Therefore,

$$\begin{aligned}h_\lambda(x_0) &= \lambda \int_0^\tau e^{-\lambda t}(-x^*(t))\,dt + \lambda \int_\tau^\infty e^{-\lambda t}(-1)\,dt\\&= \lambda \int_0^\tau e^{-\lambda t}(-x^*(t))\,dt - e^{-\lambda\tau},\end{aligned}$$

from which we conclude that $h(x_0) = \lim_{\lambda\to 0^+} h_\lambda(x_0) = -1$. Thus,

$$h(x) = \lim_{\lambda\to 0^+} h_\lambda(x) = \begin{cases} 0, & x = 0,\\ -1, & x \in (0,1].\end{cases} \tag{3.11}$$

Notice that h is discontinuous.

Let us construct $\bar{\eta}_{x_0}$ such that (2.21) holds. (In this example $\bar{\eta}_{x_0}$ depends on x_0, for this reason we keep it in the subscript.)

For $x_0 = 0$ set $\bar{\eta}_{x_0}(x) \equiv 0$. In this case

$$g(x) - h(x) + \nabla\bar{\eta}_{x_0}(x) f(x,u) = \begin{cases} 0, & x = 0,\\ -x+1, & x \in (0,1],\end{cases}$$

and (2.22) holds.

If $x_0 \in (0,1]$, set

$$\bar{\eta}_{x_0}(x) = \begin{cases} x_0 - \ln x_0, & x \in [0,x_0)\\ x - \ln x, & x \in [x_0,1].\end{cases}$$

In this case, at the points where $\bar{\eta}_{x_0}$ is differentiable, we have

$$\begin{aligned}&g(x) - h(x) + \nabla\bar{\eta}_{x_0}(x) f(x,u) =\\&\begin{cases} 0, & x = 0,\\ -x+1, & x \in (0,x_0),\\ -x+1+(1-1/x)xu = (1-x)(1-u), & x \in (x_0,1].\end{cases}\end{aligned} \tag{3.12}$$

At $x = x_0$ function $\bar{\eta}_{x_0}$ is not differentiable and from the properties of the generalized gradient it follows that $g(x_0) - h(x_0) + \partial\bar{\eta}_{x_0}(x_0) f(x_0,u)$ is equal to the interval between the points $-x_0+1$ and $(1-x_0)(1-u)$. From these formulas we see that (2.21)-(2.22) also hold.

Due to Theorem 2.7, maximizing functions in (2.1) are $\bar{\psi} = h$ given by (3.11) and $\bar{\eta}_{x_0}$. As seen from (3.10) and the bottom line of (3.12), for $u^*(t) := u^f[x^*(t)]$, we have for all $t > 0$

$$g(x^*(t)) - h(x^*(t)) + \nabla\bar{\eta}_{x_0}(x^*(t))f(x^*(t),u^*(t)) = 0.$$

This implies via (2.22) and (3.12) that

$$(x^*(t),u^*(t)) = \text{argmin}_{(x,u)}\{g(x) - \bar{\psi}(x) + \nabla\bar{\eta}_{x_0}(x)f(x,u)\}.$$

From (3.11), we see that $h(x^*(t)) = \text{const}$ (if $x_0 = 0$ then $h(x^*(t)) \equiv 0$, if $x_0 \in (0,1]$ then $h(x^*(t)) \equiv -1$.) Thus, (2.10) and (2.11) hold. Hence, $(x^*(\cdot),u^*(\cdot))$ is optimal due to Theorem 2.2, which agrees with the observation made above.

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