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ON THE COMPARISON METHOD IN THE STUDY OF PROBLEMS OF MINIMIZATION OF FUNCTIONALS

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Dedicated with infinite gratitude to Vladimir Mikhailovich Tikhomirov, a distinguished scientist and wonderful teacher, on the occasion of his 90th anniversary

Abstract. The problem of minimization of a functional U, defined on the metric space (X,ρ) (where ρ is a metric on the non-empty set X), is considered. It is assumed that the values of the functional U are bounded from below by some number γ, i.e., $U(x) \geq \gamma$ for all $x \in X$. This functional is compared with a "model" function u, which is continuous and decreasing on the interval $[0,r]$ and such that $u(0) = U(x_0) - \gamma$ for some $x_0 \in X$ and $u(r) = 0$. Conditions which guarantee that the functional U has a minimum at some point $x \in X$, with $\rho(x_0,x) \leq r$ are obtained. It is shown that the theorems on the minimum of functional that use Caristi-type conditions can be derived from the obtained statement. Applications of the obtained statement to the study of fixed points of mappings in metric spaces are also provided.

Keywords. Comparison method; Caristi condition; Fixed point; Minimization of functionals; Metric spaces.

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1. INTRODUCTION

The authors of this article had the pleasure of listening to the presentations of Vladimir Mikhailovich Tikhomirov dedicated to issues of extremum theory at the conferences "Kolmogorov Readings" in Tambov from 2000 to 2013. Under the influence of those brilliant reports, we became interested in this topic.

The mathematical tools of extremum theory in normed spaces are rather well-established [1, 2] and allow for the investigation of many specific problems. In the case of metric spaces, the tools for studying extremal problems are certainly not that diverse and effective. This article proposes an investigation of the existence of a minimum of a functional defined on a metric

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space. This study is based on comparing the functional under consideration with a "model" function of a real argument. It should be noted that although the idea of comparison with similar but well-studied objects is often used in the research of various mathematical objects (in particular, equations, inclusions, fixed points, and coincidence points), such an approach apparently has not been applied to minimization problems before.

Statements, in which "good" properties of the "model" object imply that the object under consideration possesses the respective properties, are called comparison theorems. With a successful choice of the "model", one can obtain statements for objects to which direct analytical methods can not be applied.

2. Theorem on the Minimum of a Functional

Let X be a complete metric space, $x_0 \in X$, and $r > 0$. We denote $B_X(x_0,r) := \{x \in X : \rho_X(x,x_0) \le r\}$. Let a functional $U : X \to \mathbb{R} \cup \{\infty\}$ be bounded from below by some number γ (i.e., $U(x) \ge \gamma$ for all $x \in X$) such that $U(x_0) \ne \infty$ be given. Let also a function $u : [0,r] \to \mathbb{R}$ be given. We assume that the following conditions are satisfied:

(a) *for any convergent sequence* $\{x_i\}_{i=0}^{\infty} \subset B_X(x_0,r)$, $x_i \to x$, *if the numerical sequence* $\{U(x_i)\}_{i=0}^{\infty}$ *is decreasing, then* $\lim_{i\to\infty} U(x_i) \ge U(x)$.

(b) *the function u is a continuous, non-strictly decreasing function on* $[0,r]$, *and the equalities* $u(0) = U(x_0) - \gamma$, $u(r) = 0$ *hold true.*

Note that condition **(a)** is obviously satisfied if functional U is lower semi-continuous or closed, or *a fortiori* continuous on the ball $B_X(x_0,r)$. The problem of conditional minimum of such a functional U with the use of a Caristi-type inequality was considered in [3].

Definition 2.1. We say that the function u majorizes the functional U on the ball $B_X(x_0,r)$ if, for any $x \in B_X(x_0,r)$, $t \in [0,r)$ such that

$$\gamma < U(x) \le U(x_0), \;\; u(t) = U(x) - \gamma,$$

there exist $x' \in X$ and $t' \in (t,r]$, for which the inequalities

$$U(x) - U(x') \ge u(t) - u(t'), \;\; \rho(x,x') \le t' - t \tag{2.1}$$

hold true.

In the definition above, due to the fact that u is decreasing on the interval $[0,r]$ and the equality $u(t) = U(x) - \gamma$, the relations (2.1) are equivalent to the inequality

$$U(x') - \gamma \le u\big(t + \rho(x,x')\big). \tag{2.2}$$

Note that the second inequality in (2.1) yields that

$$\rho(x_0,x') \le \rho(x_0,x) + \rho(x,x') \le t + (t'-t) = t' \le r \;\Rightarrow\; x' \in B_X(x_0,r).$$

Theorem 2.2. *Let the functional U and the function u satisfy the conditions **(a)**,**(b)**, and let the function u majorizes the functional U on the ball* $B_X(x_0,r)$. *Then in this ball there exists a point* $\bar{x} \in B_X(x_0,r)$ *such that the minimum of the functional U is achieved at* $\bar{x}$ *and* $U(\bar{x}) = \gamma$.

Proof. We define an order on the set $X_0 := B_X(x_0,r)$ by stating for arbitrary elements $x,x' \in X_0$ that the inequality $x' \prec x$ holds true if and only if $\gamma < U(x) \le U(x_0)$ and for any $t \in [0,r)$ such that $u(t) = U(x) - \gamma$, relation (2.2) holds true. We also define the non-strict inequality

$$x' \preceq x \;\Leftrightarrow\; x' \prec x \text{ or } x' = x.$$

Obviously, this relation is reflexive and antisymmetric. We will show that it is also transitive.

Let $x' \prec x$ and $x'' \prec x'$. Then, for any $t, t' \in [0,r)$ such that $u(t) = U(x) - \gamma$ and $u(t') = U(x') - \gamma$, we have

$$U(x') - \gamma \leq u\big(t + \rho(x,x')\big), \;\; U(x'') - \gamma \leq u\big(t' + \rho(x',x'')\big).$$

The latter implies that $u(t') = U(x') - \gamma \leq u\big(t + \rho(x,x')\big)$. Since u is decreasing, we obtain $t' \geq t + \rho(x,x')$ and $t' + \rho(x',x'') \geq t + \rho(x,x') + \rho(x',x'') \geq t + \rho(x,x'')$. From this inequality and the fact that function u is decreasing, we obtain

$$U(x'') - \gamma \leq u\big(t + \rho(x,x'')\big).$$

Thus, it is proved that $x'' \prec x$, so the binary relation under consideration is transitive.

In terms of the order $\preceq$ defined here, the assumption that the function u majorizes the functional U on the ball X_0 means that for any $x \in X_0$ such that $\gamma < U(x) \leq U(x_0)$, there exists $x' \in X_0$, such that $x' \prec x$.

For arbitrary $x, x' \in X_0$, $x' \prec x$, by choosing $t \in [0,r)$ such that the equality $U(x) = u\big(t\big) + \gamma$ holds true, we obtain

$$U(x') \leq u\big(t + \rho(x,x')\big) + \gamma \leq u\big(t\big) + \gamma = U(x)$$

Thus, the restriction of the functional U on X_0 is monotone. According to the Hausdorff maximal principle, there exists a maximal chain S in the partially ordered space $(X_0, \preceq)$, which contains the point x_0. For the bounded set of numbers $U(S)$, we define $\overline{y} := \inf U(S)$. We will show that $\overline{y} = \gamma$, and there exists an element $\overline{x} \in S$ such that $U(\overline{x}) = \gamma$.

There are two possible situations: $\overline{y} \in U(S)$ and $\overline{y} \notin U(S)$.

Let us first consider the case $\overline{y} \notin U(S)$. We define a (strictly) decreasing sequence $\{y_i\}_{i=0}^{\infty} \subset U(S)$ such that $y_i \to \overline{y}$. There exists a decreasing sequence $\{x_i\}_{i=0}^{\infty} \subset S$ such that $U(x_i) = y_i$, $i = 0, 1, \ldots$. This sequence is coinitial in the chain S, which means that for any element of the chain $x \in S$ there exists an element x_i of this sequence such that $x_i \preceq x$. Indeed, otherwise there exists $x \in S$ such that $x \prec x_i$ for all i. However, then $U(x) < U(x_i)$, $i = 0, 1, \ldots$, and therefore $U(x) \leq \overline{y}$. This inequality can only be valid if $U(x) = \overline{y}$, which leads to a contradiction with the assumption $\overline{y} \notin U(S)$.

For any i, we define $t_i \in [0,r]$ for which $u(t_i) = U(x_i) - \gamma$. The sequence $\{t_i\}_{i=0}^{\infty}$ of numbers is increasing and bounded, hence it is a Cauchy sequence. Furthermore, from the relations

$$u(t_{i+1}) = U(x_{i+1}) - \gamma \leq u\big(t_i + \rho(x_i, x_{i+1})\big)$$

we obtain the inequality $t_{i+1} \geq t_i + \rho(x_i, x_{i+1})$. Thus, $\rho(x_i, x_{i+1}) \leq t_{i+1} - t_i$, and therefore the sequence $\{x_i\}_{i=0}^{\infty} \subset S$ is also a Cauchy sequence.

Due to the completeness of the metric space X and the closedness of the ball X_0 in X, the sequence $\{x_i\}_{i=0}^{\infty}$ converges to some $\overline{x} \in X_0$. By the virtue of condition **(a)**, we have $\lim_{i\to\infty} U(x_i) \geq U(\overline{x})$. We will now show that $\overline{x} \preceq x_i$, $i = 0, 1, \ldots$. For any natural number j, we have $x_{i+j} \prec x_i$. Let us choose $t_i \in [0,r)$ such that $U(x_i) - \gamma = u\big(t_i\big)$. By the definition of the order $\prec$, we have

$$U(\overline{x}) - \gamma \leq U(x_{i+j}) - \gamma \leq u\big(t_i + \rho(x_i, x_{i+j})\big).$$

Let us take the limit in this inequality. Taking into account continuity of the function u, we obtain $U(\overline{x}) - \gamma \leq u\big(t_i + \rho(x_i, \overline{x})\big)$. Thus, $\overline{x} \preceq x_i$, $i = 0, 1, \ldots$ and since the sequence $\{x_i\}_{i=0}^{\infty}$ is coinitial in the chain S, we have $\overline{x} \preceq x$ for any $x \in S$. Due to the maximality of the chain S, we obtain $\overline{x} \in S$, $\overline{y} = U(\overline{x}) \in U(S)$. Hence, the situation $\overline{y} \notin U(S)$ is impossible.

Thus, $\bar{y} \in U(S)$, i.e., for some $\bar{x} \in S$ it holds true that $\bar{y} = U(\bar{x})$. In this case, if $U(\bar{x}) \neq \gamma$, then there exists $x' \in X$ such that $x' \prec \bar{x}$, which contradicts with the maximality of the chain S. Therefore, $U(\bar{x}) = \gamma$. □

Let us now consider some consequences of Theorem 2.2, in particular, from this theorem we will derive the statements on the minimum of a functional satisfying the Caristi-type conditions obtained in [4, 5, 6].

Corollary 2.3. *Let a continuous strictly decreasing function $\chi : \Upsilon \to \mathbb{R}$ be defined on the interval $\Upsilon := [0, U(x_0) - \gamma]$ such that $\chi(0) = r$, $\chi\big(U(x_0) - \gamma\big) = 0$. Then, if for the functional U satisfying condition **(a)** the relation*

$$\forall x \in B_X(x_0, r) \;\; \gamma < U(x) \leq U(x_0) \;\; \Rightarrow$$
$$\exists x' \in X \;\; U(x') < U(x) \text{ and } \chi\big(U(x') - \gamma\big) - \chi\big(U(x) - \gamma\big) \geq \rho(x, x') \quad (2.3)$$

holds true, then there exists a point $\bar{x} \in B_X(x_0, r)$, where the minimum of the functional U is attained and $U(\bar{x}) = \gamma$.

Proof. Define the function u on the interval $[0, r]$ as the inverse of the function χ. Such a function u satisfies condition **(b)**. We will show that the validity of the relation (2.3) implies that the function u majorizes the functional U on the ball $B_X(x_0, r)$.

In (2.3), for x, x' it holds true that $U(x), U(x') \in \big(\gamma, U(x_0)\big]$, so for these values there exist unique

$$t := \chi\big(U(x) - \gamma\big), \;\; t' := \chi\big(U(x') - \gamma\big). \quad (2.4)$$

According to (2.3), we have

$$t' - t \geq \rho(x, x'). \quad (2.5)$$

The relations (2.4) can be rewritten in the following equivalent form

$$u(t) = U(x) - \gamma, \;\; u(t') = U(x') - \gamma.$$

From the latter relations, considering the inequality (2.5), due to the fact that the function u is a decreasing function, we obtain

$$U(x') - \gamma = u(t') \leq u\big(t + \rho(x, x')\big).$$

Thus, the inequality (2.2) is proved, and, hence, it is proved that the function u majorizes the functional U on the ball $B_X(x_0, r)$.

All conditions of Theorem 2.2 are satisfied, which implies that $U(\bar{x}) = \gamma$ for some $\bar{x} \in B_X(x_0, r)$. □

We note that the statements of Theorem 2.2 and Corollary 2.3 are not equivalent. Specifically, under the assumptions of Corollary 2.3, the function $u = \chi^{-1}$ on $[0, r]$ is strictly decreasing, while in Theorem 2.2, a less restrictive condition of non-strict decrease is imposed on this function.

Now we will use Theorem 2.2 in deriving the theorems on the minimum of functional, which employ Caristi-type conditions.

According to [4], *a functional U satisfies the Caristi-type condition with coefficient $k > 0$ if for any $x \in X$ such that $\gamma < U(x) \leq U(x_0)$, there exists $x' \in X$, $x' \neq x$, for which the inequality*

$$U(x') + k\rho(x, x') \leq U(x) \quad (2.6)$$

holds true.

In order to derive the statement on the minimum of a functional satisfying such a condition (see [4, Theorem 3], as well as [5, Theorem 1.2]), we put

$$r = \frac{U(x_0) - \gamma}{k}$$

and define the function $u : [0, r] \to \mathbb{R}$ by the relation

$$u(t) = U(x_0) - \gamma - kt, \ \ t \in [0, r].$$

Note that this function satisfies condition **(b)**. It is obvious that under the Caristi-type condition, the function u majorizes the functional U on the ball $B_X(x_0, r)$. Indeed, let for $x \in B_X(x_0, r)$, $t \in [0, r)$, the relations

$$\gamma < U(x) \leq U(x_0), \ \ u(t) = U(x) - \gamma$$

hold true, and let $x' \in X$ satisfy the inequality (2.6). Then we have

$$\begin{aligned} U(x') - \gamma \leq U(x) - \gamma - k\rho(x, x') = u(t) - k\rho(x, x') = U(x_0) - \gamma - kt - k\rho(x, x') \\ = U(x_0) - \gamma - k\big(t + \rho(x, x')\big) = u\big(t + \rho(x, x')\big). \end{aligned}$$

Thu inequality (2.2) is valid.

Therefore, from Theorem 2.2 we obtain the following result.

Theorem 2.4. *Let the functional U satisfy condition **(a)**, and let the Caristi-type condition be satisfied. Then there exists a point $\bar{x} \in X$ at which the minimum of the functional U is attained and $U(\bar{x}) = \gamma$, and $\rho(x_0, \bar{x}) \leq \frac{U(x_0) - \gamma}{k}$.*

Theorem 2.4 was proved in [4] and refined in [5]. In these works, a more restrictive assumption of lower semicontinuity of the functional U (compared to the condition **(a)**) was used.

In [6], the following generalization of the Caristi condition was proposed.

Define the class $\mathscr{K}$ of all functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy the following conditions: *a function κ is continuous, strictly increasing, $\kappa(\upsilon) > 0$ for all $\upsilon > 0$, and the function $\frac{1}{\kappa(\cdot)}$ is summable in some neighborhood of zero.* According to [6], *a functional U satisfies the generalized Caristi-type condition (with the function $\kappa(\cdot)$) if for any $x \in X$ such that $\gamma < U(x) \leq U(x_0)$, there exists $x' \in X$, $x' \neq x$, for which the inequality*

$$U(x') + \kappa\big(U(x) - \gamma\big)\rho(x, x') \leq U(x)$$

holds true.

Theorem 2.5. *(see [6, Theorem 2]) Let the functional U satisfy condition **(a)**, and let the generalized Caristi-type condition hold true. Then there exists a point $\bar{x} \in X$ where the minimum of the functional U is attained and $U(\bar{x}) = \gamma$, and*

$$\rho(x_0, \bar{x}) \leq \int_0^{U(x_0) - \gamma} \frac{d\upsilon}{\kappa(\upsilon)}.$$

In [6], in this theorem, instead of condition **(a)** it was assumed that the functional U is lower semi-continuous.

We will derive Theorem 2.5 from Theorem 2.2.

Define the function $\chi : \Upsilon \to \mathbb{R}$ (where $\Upsilon := [0, U(x_0) - \gamma]$) by the formula

$$\chi(u) = \int_u^{U(x_0)-\gamma} \frac{d\upsilon}{\kappa(\upsilon)}, \ \ u \in \Upsilon.$$

This function satisfies the conditions of Corollary 2.3, i.e., it is continuous, strictly decreasing, and $\chi(0) = r$, $\chi\big(U(x_0) - \gamma\big) = 0$. Moreover,

$$\frac{d\chi}{du} = -\frac{1}{\kappa(u)}, \ \ u \in \Upsilon.$$

There exists an inverse function to χ, denoted as $u := \chi^{-1} : [0, r] \to \mathbb{R}$, which is continuous, strictly decreasing on $[0, r]$, and $u(0) = U(x_0) - \gamma$, $u(r) = 0$. Thus, the function u satisfies condition **(b)**.

We will show that the function u defined here majorizes the functional U on the ball $B_X(x_0, r)$. First, note that its derivative is given by the formula

$$\frac{du}{dt} = -\kappa(u(t)), \ \ t \in [0, r].$$

Therefore, according to Lagrange's mean value theorem, we have

$$\forall t, t' \in [0, r] \ \ t < t' \ \Rightarrow \ \exists \tau \in (t, t') \ \ u(t) - u(t') = \kappa(u(\tau))(t' - t).$$

As the function u is decreasing and the function κ is increasing, we obtain

$$\forall t, t' \in [0, r] \ \ t < t' \ \Rightarrow \ \exists \tau \in (t, t') \ \ u(t) - u(t') \leq \kappa(u(t))(t' - t).$$

Considering this relation, for any $x \in X$ such that $\gamma < U(x) \leq U(x_0)$ and the corresponding $t \in [0, r)$ such that $u(t) = U(x) - \gamma$, from the generalized Caristi condition we obtain that there exists $x' \in X$, $x' \neq x$, for which the following relation holds true

$$U(x) - U(x') \geq \kappa\big(U(x) - \gamma\big)\rho(x, x') = \kappa\big(u(t)\big)\rho(x, x') \geq u(t) - u(t + \rho(x, x')).$$

This relation directly implies the relation (2.2). Thus, it has been proven that the function u majorizes the functional U on the ball $B_X(x_0, r)$.

We note that the theorems on the existence of a minimum of a functional, based on the Caristi condition and its generalizations, are used, in particular, to prove variational principles (see the monograph [7] as well as the articles [6, 8, 9, 10]). The results on the minima of functionals presented here are related not only to the theory of extremal problems, but are even more in demand in the studies of nonlinear operator equations and inclusions, as well as various specific classes of functional equations and inclusions. In the next paragraph, we will demonstrate their application to the problem of fixed points of mappings in metric spaces.

3. Applications to the Fixed Point Problem

Theorem 2.2 allows to obtain conditions for the existence of a fixed point of a continuous mapping $f : X \to X$ in a complete metric space X, i.e., an element $\overline{x} \in X$ such that

$$\overline{x} = f(\overline{x}).$$

We define the functional $U : X \to \mathbb{R}$ by the formula

$$U(x) := \rho(x, f(x)), \ \ x \in X. \tag{3.1}$$

Note that, due to the continuity of f, 1 U is continuous. Indeed, for any convergent sequence in X, $x_i \to x$, we have

$$\left|\rho(x_i, f(x_i)) - \rho(x, f(x))\right| \leq \rho(x_i, x) + \rho(f(x_i), f(x)),$$

and therefore $\rho(x_i, f(x_i)) \to \rho(x, f(x))$.

Furthermore, the functional (3.1) is bounded from below, i.e., $\rho(x, f(x)) \geq 0$ for any $x \in X$. Due to this inequality, we set $\gamma := 0$. It is obvious that if the functional U attains its minimum at some point $\bar{x} \in X$, i.e., $U(\bar{x}) = 0$, then $\bar{x}$ is a fixed point of the mapping f. This simple observation is widely used in the study of fixed points based on results on the minimum of the functional (see, for example, [4, 5, 10]). We also apply this observation in the following theorem.

Theorem 3.1. *Let $x_0 \in X$, $r > 0$ and the function $u : [0, r] \to \mathbb{R}$ satisfy condition **(b)**. Assume that the following relation holds true:*

$$\forall x \in B_X(x_0, r)\ \ \forall t \in [0, r) \begin{cases} 0 < \rho(x, f(x)) \leq \rho(x_0, f(x_0)) \\ u(t) = \rho(x, f(x)) \end{cases} \\ \Rightarrow\ \exists x' \in X\ \ \rho(x', f(x')) \leq u\big(t + \rho(x, x')\big). \tag{3.2}$$

Then the mapping f has a fixed point in $B_X(x_0, r)$.

Proof. Due to the assumption (3.2), the function u majorizes the functional U defined by the formula (3.1). Thus, all conditions of Theorem 2.2 are satisfied. According to Theorem 2.2, there exists a point $\bar{x} \in B_X(x_0, r)$ such that $U(\bar{x}) = \rho(\bar{x}, f(\bar{x})) = 0$, i.e., $\bar{x} = f(\bar{x})$. □

Applying Corollary 2.3 to the functional (3.1), we obtain

Corollary 3.2. *Let a continuous strictly decreasing function $\chi : \Upsilon \to \mathbb{R}$ be defined on the interval $\Upsilon := [0, \rho(x_0, f(x_0))]$ such that $\chi(0) = r$, $\chi\big(\rho(x_0, f(x_0))\big) = 0$. Then, if*

$$\forall x \in B_X(x_0, r)\ \ 0 < \rho(x, f(x)) \leq \rho(x_0, f(x_0))$$

$$\Rightarrow\ \exists x' \in X\ \ \rho(x', f(x')) < \rho(x, f(x)) \text{ and } \chi\big(\rho(x', f(x'))\big) - \chi\big(\rho(x, f(x))\big) \geq \rho(x, x'), \tag{3.3}$$

then the mapping f has a fixed point in the ball $B_X(x_0, r)$.

Consider the majorant functions $u : [0, r] \to \mathbb{R}$ of certain specific types and obtain the corresponding particular cases of Theorem 3.1.

First, we will derive a Caristi-type theorem (see [5, Theorem 3.2′]), by choosing the function u to be linear

$$u(t) := \rho(x_0, f(x_0)) - kt,\ \ t \in [0, r],$$

where $r := \frac{\rho(x_0, f(x_0))}{k}$.

Corollary 3.3. *(see* [5, Theorem 3.2′]*) Let the following Caristi-type condition be satisfied:*

$$\exists k > 0\ \ \forall x \in B_X(x_0, r)\ \ 0 < \rho(x, f(x)) \leq \rho(x_0, f(x_0))$$

$$\Rightarrow\ \exists x' \in X\ \ x' \neq x,\ \ \rho(x', f(x')) - k\rho(x, x') \geq \rho(x, f(x))\big). \tag{3.4}$$

Then, the mapping f has a fixed point in the ball $B_X(x_0, r)$ of the radius $r = \frac{\rho(x_0, f(x_0))}{k}$.

Now, using Theorem 2.5, we obtain a generalized Caristi-type condition for the existence of a fixed point of the mapping f.

Corollary 3.4. *Let the following generalized Caristi-type condition be satisfied:*

$$\exists \kappa \in \mathscr{K} \;\; \forall x \in B_X(x_0,r) \;\; 0 < \rho(x,f(x)) \leq \rho(x_0,f(x_0))$$
$$\Rightarrow \;\; \exists x' \in X \;\; x' \neq x, \;\; \rho(x',f(x')) - \kappa\big(\rho(x,f(x))\big)\rho(x,x') \geq \rho(x,f(x))\big).$$

Then, the mapping f has a fixed point in the ball $B_X(x_0,r)$ of the radius $r = \int_0^{\rho(x_0,f(x_0))} \frac{d\upsilon}{\kappa(\upsilon)}$.

The function u majorizing the functional (3.1) can be naturally defined as follows. Let a continuous function $\varphi : [0,r] \to \mathbb{R}$ such that

$$\forall t \in [0,r) \;\; \varphi(t) > t, \;\; \varphi(0) = \rho(x_0,f(x_0)) \text{ and } \varphi(r) = r \tag{3.5}$$

be given. Put $u(t) := \varphi(t) - t$, $t \in [0,r]$. We will also assume that the function $u : [0,r] \to \mathbb{R}$ defined by this equality is strictly decreasing. Obviously, for this function satisfies the condition **(b)**.

It is easy to notice that a reverse construction is always possible: if a function u satisfying condition **(b)** is given, then the formula $\varphi(t) := u(t) + t$, $t \in [0,r]$, defines a function $\varphi : [0,r] \to \mathbb{R}$ that satisfies the given conditions (3.5). Thus, Theorem 3.1 can be expressed in the following equivalent form.

Theorem 3.1′ *Let $x_0 \in X$, $r > 0$ be given, and let $\varphi : [0,r] \to \mathbb{R}$ be a continuous function satisfying the condition (3.5). Let the following relation hold true*

$$\forall x \in B_X(x_0,r) \;\; \forall t \in [0,r) \left\{ \begin{array}{l} 0 < \rho(x,f(x)) \leq \rho(x_0,f(x_0)) \\ u(t) = \rho(x,f(x)) \end{array} \right.$$
$$\Rightarrow \;\; \exists x' \in X \;\; \rho(x',f(x')) \leq \varphi\big(t + \rho(x,x')\big) - t - \rho(x,x').$$

Then the mapping f has a fixed point in the ball $B_X(x_0,r)$.

A similar statement was obtained in [11, Theorem 2.1] (as a generalization of Kantorovich's theorem on fixed points [12, Chapter XVIII, Section 1.2, Theorem 1] to mappings of metric spaces).

In [5], an example of a mapping f that has a fixed point but does not satisfy the Caristi-type condition (i.e., the assumptions of [5, Theorem 3.2′] are not fulfilled for this mapping, and, equivalently, the corollaries 3.3) was given. We will present here this function and demonstrate that it satisfies the assumptions of Theorem 3.1, and that the function u required in this statement can be easily found.

Example 3.5. Let $X = \mathbb{R}_+$. In [5, Example 4.4], the function

$$f(x) = \frac{x}{1+x}, \;\; x \in \mathbb{R}_+,$$

is considered. It is shown there that the assumptions of [5, Theorem 3.2′] (and, consequently, the corollaries 3.3) are not fulfilled for this function. At the same time, f has a unique fixed point $\bar{x} = 0$.

We will check the fulfillment of the conditions of Theorem 3.1.

We have

$$\rho(x,f(x)) = x - \frac{x}{1+x} = \frac{x^2}{1+x}, \;\; x \in \mathbb{R}_+.$$

Let $x_0 = 1$, $r = 2$, then $B_X(x_0,r) = [0,3]$, $\rho(x_0, f(x_0)) = \frac{1}{2}$. Define the function $u : [0,2] \to \mathbb{R}$ by the formula

$$u(t) := \frac{(t-2)^2}{8}, \ \ t \in [0,2].$$

It is obvious that the function u is continuous, decreasing on $[0,2]$, and the equalities

$$u(0) = \frac{1}{2} = \rho(x_0, f(x_0)), \ \ u(2) = 0$$

hold true. Thus, this function satisfies condition **(b)**.

Define the function $\chi : [0, \frac{1}{2}] \to \mathbb{R}$ as the inverse of the function u,

$$\chi(\upsilon) := 2 - 2\sqrt{2\upsilon}, \ \ \upsilon \in \big[0, \frac{1}{2}\big].$$

For any $x \in (0,3]$, let us choose an arbitrary $x' \in [0,3]$ satisfying the inequalities $x' \leq 1$ and $x' < x$. We have

$$\chi\big(\rho(x', f(x'))\big) - \chi\big(\rho(x, f(x))\big) = \frac{2\sqrt{2}(x-x')}{\sqrt{(x'+1)(x+1)}} \geq \frac{2\sqrt{2}(x-x')}{\sqrt{2\cdot 4}} = \rho(x,x').$$

Thus, the relation (3.3) holds true, and Corollary 3.2 guarantees the existence of a fixed point of the mapping f in the set $[0,3]$.

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