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VARIATIONAL CONVERGENCE OF NONLOCAL INTEGRODIFFERENTIAL DIFFUSION PROBLEMS OF GRADIENT FLOW TYPE

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Dedicated to the memory of Professor Hedy Attouch

Abstract. In this paper, we continue our study of nonlocal problems of gradient flow type that we developed in our previous papers [O. Anza Hafsa, J.-P. Mandallena, G. Michaille, Nonlocal time delays reaction-diffusion problems of gradient flow type: Existence, stochastic homogenization, Evol. Equ. Control Theory 17 (2026) 23-61] and [O. Anza Hafsa, J.-P. Mandallena, G. Michaille, Stochastic homogenization of nonlocal reaction-diffusion problems of gradient flow type, J. Elliptic Parabol. Equ. 10 (2024) 415-474]. We consider here nonlocal integrodifferential diffusion problems. We present existence, uniqueness and compactness results and investigate stochastic homogenization.
Keywords. Convergence of integrodifferential nonlocal diffusion equations; Integrodifferential nonlocal diffusion equations; Mosco-convergence; Stochastic homogenization; Γ-convergence.
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1. INTRODUCTION

Consider a compactly supported radial function $J:\mathbb{R}^d\to[0,\infty[$ and the corresponding integrals $\int_O J(x-y)(u(t,y)-u(t,x))dy$ at time t and $\int_O J(x-y)(u(t-\tau,y)-u(t-\tau,x))dy$ at some past time $t-\tau$ with $\tau>0$. In many examples of population dynamics, $u(t,x)$ represents the density of some population at time t located at x in a bounded domain $O\subset\mathbb{R}^d$, and the term

$$\mathcal{J}(t,x)=\int_O J(x-y)(u(t,y)-u(t,x))dy$$

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accounts for the population flux of individuals at time t in O which jump from y to x in O. The second population flux

$$\mathcal{I}_\tau(t,x) = \int_O J(x-y)(u(t-\tau,y)-u(t-\tau,x))dy$$

is superimposed at each t on the first flux $\mathcal{I}(t,x)$ with a delay time τ. It may represent a maturation period, a resource regeneration time or an incubation period. Assuming that τ is small, the first order approximation $\mathcal{I}(t,x) = \mathcal{I}_\tau(t+\tau,x) \approx \mathcal{I}_\tau(t,x) + \tau\frac{\partial \mathcal{I}_\tau}{\partial t}(t,x)$ together with $\mathcal{I}_\tau(0,\cdot) = 0$ yields to

$$\mathcal{I}_\tau(t,x) = -\frac{1}{\tau}\int_0^t \exp\left(\frac{s-t}{\tau}\right)\mathcal{I}(s,x)ds.$$

Consider a term $F(t)$ which accounts for the source of the population growth at (t,x). Then, the differential form of the balance law leads to the following nonlocal equation:

$$\frac{\partial u}{\partial t}(t,x) - \mathcal{I}(t,x) - \frac{1}{\tau}\int_0^t \exp\left(\frac{s-t}{\tau}\right)\mathcal{I}(s,x)ds = F(t).$$

Noticing that $\mathcal{I}(t,x) = -\nabla\mathcal{J}(u(t,x))$, where

$$\mathcal{J}(u(t,x)) = \frac{1}{4}\int_O\int_O J(x-y)(u(t,x)-u(t,y))^2 dxdy,$$

and adding a suitable initial condition $u_0(x)$, we see that $u(t,x)$ satisfies the Neumann-Cauchy homogeneous nonlocal problem of the type:

$$\begin{cases} \dfrac{\partial u}{dt}(t,x)+\nabla\mathcal{J}(u(t,x))+\dfrac{1}{\tau}\displaystyle\int_0^t \exp\left(\frac{s-t}{\tau}\right)\nabla\mathcal{J}(u(s,x))ds=F(t) \text{ for } \mathcal{L}^1\text{-a.a.} t\in[0,T] \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Thus, motivated by the dynamic of the nonlocal flux of populations flowing in random medium, and presenting some delay time, we continue our study of nonlocal problems of gradient flow type that we developed in our previous papers [6, 7] by considering stochastic homogenization of nonlocal integrodifferential diffusion problems of type (1.1) with a suitable scaling depending on $\varepsilon > 0$. More precisely, given $(\Omega,\mathcal{F},\mathbb{P})$ a complete probability space, $T > 0$ and $O \subset \mathbb{R}^d$ a bounded open domain with Lipschitz boundary, we are interested in the stochastic homogenization as $\varepsilon \to 0$ of the problems

$$(\mathcal{P}_\varepsilon^\omega)\begin{cases} \dfrac{du_\varepsilon^\omega}{dt}(t)+\nabla\mathcal{J}_\varepsilon(\omega,u_\varepsilon^\omega(t))+\displaystyle\int_0^t \mathcal{K}(t-s)\nabla\mathcal{J}_\varepsilon(\omega,u_\varepsilon^\omega(s))ds=F_\varepsilon(\omega,t) \quad \text{for } \mathcal{L}^1\text{-a.a. } t\in[0,T] \\ u_\varepsilon^\omega(0) = u_{0,\varepsilon}^\omega \in L^2(O), \end{cases}$$

where $\mathcal{K}$ is a general smooth kernel. For each $\varepsilon > 0$, $F_\varepsilon : \Omega\times[0,T] \to L^2(O)$ is a random source and $\mathcal{J}_\varepsilon : \Omega\times L^2(O) \to [0,\infty[$ is given by

$$\mathcal{J}_\varepsilon(\omega,u) := \frac{1}{4\varepsilon^d}\int_O\int_O J\left(\omega,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{x-y}{\varepsilon}\right)\left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 dxdy$$

with a random density $J : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty[$. For the meaning of the scale parameter ε we refer to [7, Section 5].

In a deterministic framework, the variational convergence as $\varepsilon \to 0$ of problems of type $(\mathscr{P}_\varepsilon^\omega)$ without a convolution term and with a density J depending solely on the third variable was first addressed by Andreu, Mazón, Rossi, and Toledo in [2, 3] (see also the book [4]) using semigroup theory and the convergence of their resolvents. They proved convergence to a local Cauchy problem. In the context of homogenization, the variational convergence of nonlocal energies of type $\mathscr{J}_\varepsilon$ was recently studied by Braides and Piatnitski in [13] for the periodic case and in [12] for the stochastic case (see also the book [1]).

In this paper, we prove (see Corollary 3.9) that as $\varepsilon \to 0$, $(\mathscr{P}_\varepsilon^\omega)$ converges almost surely, in a variational sense, to a standard (local) homogenized problem

$$(\mathscr{P}_\omega^{\rm hom})\begin{cases} \dfrac{du^\omega}{dt}(t)+\nabla\mathscr{J}_{\rm hom}(\omega,u^\omega(t))+\displaystyle\int_0^t\mathcal{K}(t-s)\nabla\mathscr{J}_{\rm hom}(\omega,u_\varepsilon^\omega(s))ds=F(\omega,t) \text{ for} \mathscr{L}^1\text{-a.a.} t\in[0,T] \\ u^\omega(0)=u_0^\omega\in \operatorname{dom}(\mathscr{J}_{\rm hom}(\omega,\cdot)). \end{cases}$$

The density of the quadratic inegral functional $\mathscr{J}_{\rm hom} : \Omega \times L^2(O) \to [0,\infty]$ is defined as the limit of a suitable subadditive process already considered in [7, Propositions 3.14 and 3.17]. The proof is based upon a general compactness result (see Theorem 2.2) with respect to the Mosco$\times$weak Γ-convergence for problems of the type:

$$(\mathscr{P}_\varepsilon)\begin{cases} \dfrac{du_\varepsilon}{dt}(t)+\nabla\mathscr{E}_\varepsilon(u_\varepsilon(t))+\displaystyle\int_0^t\mathcal{K}(t-s)\nabla\mathscr{G}_\varepsilon(u_\varepsilon(s))ds=F_\varepsilon(t) \quad \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,T] \\ u_\varepsilon(0)=u_{0,\varepsilon}\in L^2(O) \end{cases}$$

where, for each $\varepsilon > 0$, $\mathscr{E}_\varepsilon, \mathscr{G}_\varepsilon : L^2(O) \to [0,\infty[$ are convex and Fréchet-differentiable and satisfy the following condition: $\langle \nabla\mathscr{E}_\varepsilon(u), \nabla\mathscr{G}_\varepsilon(u)\rangle \geqslant \alpha\|\nabla\mathscr{G}_\varepsilon(u)\|^2_{L^2(O)}$ for all $u \in L^2(O)$ with $\alpha > 0$ independent of ε.

The plan of the paper is as follows. Section 2 is devoted to existence, uniqueness (see Theorem 2.1) and compactness (see Theorem 2.2) for problems of the type $(\mathscr{P}_\varepsilon)$. After specifying in §2.1 the framework of our study, Theorem 2.1 and Theorem 2.2) are stated and proved in §2.2 and §2.3 respectively. Section 3 is devoted to stochastic homogenization. In §3.1 we specify the probability setting, and by applying Theorem 2.1 we obtain existence and uniqueness of solutions for problems of the type $(\mathscr{P}_\varepsilon^\omega)$, see Corollary 3.5. Finally, the almost surely variational convergence as $\varepsilon \to 0$ of $(\mathscr{P}_\varepsilon^\omega)$ to $(\mathscr{P}_\omega^{\rm hom})$ is stated (see Corollary 3.9) and proved, by using Theorem 2.2, in §3.2.

For convenience of the reader, in the appendix we recall some classical definitions and results that we use in the paper.

Notation. Throughout the paper we will use the following notation.

- The Lebesgue measure on $\mathbb{R}^d$ with $d \in \mathbb{N}^*$ is denoted by $\mathscr{L}^d$ and for each Borel set $A \subset \mathbb{R}^d$, the measure of A with respect to $\mathscr{L}^d$ is denoted by $\mathscr{L}^d(A)$.

- The class of bounded Borel subsets of $\mathbb{R}^d$ is denoted by $\mathscr{B}_{\rm b}(\mathbb{R}^d)$.
- The space of continuous functions from $[0,T]$ to $L^2(O)$ is denoted by $C([0,T];L^2(O))$.
- The space of absolutely continuous functions from $[0,T]$ to $L^2(O)$ is denoted by
$$AC([0,T];L^2(O)).$$
- By $u_n \to u$ in $C([0,T];L^2(O))$ we mean that $\lim\limits_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t)-u(t)\|_{L^2(O)} = 0$.
- By $u_n \rightharpoonup u$ in $L^2([0,T];L^2(O))$ we mean that for every $v \in L^2([0,T];L^2(O))$,
$$\int_0^T \langle u_n(t), v(t)\rangle dt \to \int_0^T \langle u(t), v(t)\rangle dt$$
as $n \to \infty$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(O)$.

2. Existence, Uniqueness and Compactness for Integrodifferential Diffusion Problems of Gradient Flow Type

2.1. Preliminaries. Given $T > 0$ and $\mathscr{E}, \mathscr{G} : L^2(O) \to [0,\infty[$ be two convex and Fréchet-differentiable functionals, we consider the following integrodifferential diffusion problem of gradient flow type:

$$(\mathscr{P})\begin{cases} \dfrac{du}{dt}(t) + \nabla\mathscr{E}(u(t)) + \mathcal{K} * (\nabla\mathscr{G}\circ u)(t) = F(t) & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,T] \\ u(0) = u_0 \in L^2(O), \end{cases}$$

where
$$\mathcal{K} * (\nabla\mathscr{G}\circ u)(t) := \int_0^t \mathcal{K}(t-s)(\nabla\mathscr{G}\circ u)(s)ds = \int_0^t \mathcal{K}(t-s)\nabla\mathscr{G}(u(s))ds$$
with $\mathcal{K} \in C^1([0,T];[0,\infty[)$.

The map $F : [0,T] \to L^2(O)$ is Borel measurable and satisfies the following condition:

(R) $F \in L^2([0,T];L^2(O)) \cap AC([0,T];L^2(O))$.

The functionals $\mathscr{E}, \mathscr{G} : L^2(O) \to [0,\infty[$ are convex and Fréchet-differentiable and satisfy the following conditions:

(D_1) $\nabla\mathscr{G}(0) = 0$;

(D_2) there exists $C > 0$ such that, for every $u, v \in L^2(O)$,
$$\|\nabla\mathscr{G}(u) - \nabla\mathscr{G}(v)\|_{L^2(O)} \leqslant C\|u-v\|_{L^2(O)}.$$

(D_3^α) there exists $\alpha > 0$ such that for every $u \in L^2(O)$,
$$\langle \nabla\mathscr{E}(u), \nabla\mathscr{G}(u)\rangle \geqslant \alpha\|\nabla\mathscr{G}(u)\|^2_{L^2(O)}.$$

From now on, the class of Borel maps $F : [0,T] \to L^2(O)$ (resp. the class of convex and Fréchet-différentiable pair of functionals $(\mathscr{E},\mathscr{G})$ with $\mathscr{E},\mathscr{G} : L^2(O) \to [0,\infty[)$ verifying (R) (resp. (D_1)–(D_2) and (D_3^α)) is denoted by $\mathscr{F}_{(\mathrm{R})}$ (resp. $\mathscr{F}_{(\mathrm{D}_1)\text{-}(\mathrm{D}_2)\text{-}(\mathrm{D}_3^\alpha)}$).

2.2. **Existence and uniqueness.** The problem $(\mathscr{P})$ can be equivalently rewritten as follows:

$$(\mathscr{P})\begin{cases}\dfrac{du}{dt}(t)+\nabla\mathscr{E}(u(t))=G(t,u) & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,T]\\[2ex] u(0)=u_0\in L^2(O)\end{cases}$$

with $G:[0,T]\times L^2([0,T];L^2(O))\to L^2(O)$ given by

$$G(t,u):=F(t)-\mathcal{K}*(\nabla\mathscr{G}\circ u)(t). \tag{2.1}$$

The following theorem states the existence and uniqueness of a solution to problem $(\mathscr{P})$.

Theorem 2.1. *If* (R) *and* (D_1)–(D_2) *hold, then there exists* $\overline{u}\in C([0,T];L^2(O))$ *such that*:

- $\overline{u}$ *is the unique solution of* $(\mathscr{P})$;
- $\nabla\mathscr{E}\circ\overline{u}\in L^2([0,T];L^2(O))$;
- $\frac{d\overline{u}}{dt}\in L^2([0,T];L^2(O))$;

($\mathcal{R}$) $\overline{u}$ *admits a right derivative* $\frac{d^+\overline{u}}{dt}(t)$ *at every* $t\in[0,T[$ *which satisfies* $\frac{d^+\overline{u}}{dt}(t)+\nabla\mathscr{E}(\overline{u}(t))=G(t,\overline{u})$.

Proof of Theorem 2.1. First of all, by (D_2), for every $u,v\in C([0,T];L^2(O))$ and every $\tau\in[0,T]$,

$$\|\mathcal{K}*(\nabla\mathscr{G}\circ u)-\mathcal{K}*(\nabla\mathscr{G}\circ v)\|_{C([0,\tau];L^2(O))}\leqslant\widehat{C}\|u-v\|_{C([0,\tau];L^2(O))} \tag{2.2}$$

with $\widehat{C}:=C\int_0^T K(t)dt$. Fix any $u\in C([0,T];L^2(O))$ and consider the following problem:

$$(\mathscr{P}_u)\begin{cases}\dfrac{dv}{dt}(t)+\nabla\mathscr{E}(v(t))=G(t,u) & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,T]\\[2ex] v(0)=u_0\in L^2(O).\end{cases}$$

By (R), (D_1) and (2.2) it follows that

$$\int_0^T\|G(t,u)\|^2_{L^2(O)}dt\leqslant 2T\widehat{C}^2\|u\|^2_{C([0,T];L^2(O))}+2\int_0^T\|F(t)\|^2_{L^2(O)}dt<\infty,$$

and so $G(\cdot,u)\in L^2([0,T];L^2(O))$. Hence, by [9, Theorem 17.2.5, p. 701], the problem $(\mathscr{P}_u)$ admits a unique solution $\Lambda u\in C([0,T];L^2(O))$. This establishes the existence of a map $\Lambda: C([0,T];L^2(O))\to C([0,T];L^2(O))$ which, to each $u\in C([0,T];L^2(O))$, associates the unique solution Λu of $(\mathscr{P}_u)$.

Step 1: Existence and Uniqueness. To prove that $(\mathscr{P})$ has a unique solution $u\in C([0,T];L^2(O))$ it is sufficient to establish the following:

($\mathcal{C}$) there exists $n\in\mathbb{N}^*$ such that the iterated map Λ^n is a strict contraction.

Indeed, ($\mathcal{C}$) implies that Λ has a unique fixed point $\overline{u}\in C([0,T];L^2(O))$ which is a solution of $(\mathscr{P})$. On the other hand, if $\widetilde{u}\in C([0,T];L^2(O))$ is another solution of $(\mathscr{P})$, then $\widetilde{u}$ and $\Lambda\widetilde{u}$ are two solutions of $(\mathscr{P}_{\widetilde{u}})$. Thus $\Lambda\widetilde{u}=\widetilde{u}$ by uniqueness of the solution of $(\mathscr{P}_{\widetilde{u}})$. This means that $\widetilde{u}$ is a fixed point of Λ. Consequently $\widetilde{u}=\overline{u}$ because $\overline{u}$ is the unique fixed point of Λ.

Let us prove $(\mathcal{C})$. Fix any $u\in C([0,T];L^2(O))$ and any $v\in C([0,T];L^2(O))$. Then, by definition of Λ, we have:

$$\frac{d\Lambda u}{dt}(t)+\nabla\mathscr{E}(\Lambda u(t))=G(t,u) \text{ for } \mathscr{L}^1\text{-a.a. } t\in[0,T]; \tag{2.3}$$

$$\frac{d\Lambda v}{dt}(t)+\nabla\mathscr{E}(\Lambda v(t))=G(t,u) \text{ for } \mathscr{L}^1\text{-a.a. } t\in[0,T]; \tag{2.4}$$

$$\Lambda u(0)=\Lambda v(0)=u_0. \tag{2.5}$$

From (2.3) and (2.4), as $\nabla\mathscr{E}$ is monotone[1] we see that

$$\left\langle\frac{d\Lambda u}{d\tau}(\tau)-\frac{d\Lambda v}{d\tau}(\tau),\Lambda u(\tau)-\Lambda v(\tau)\right\rangle\leqslant\langle G(\tau,u)-G(\tau,v),\Lambda u(\tau)-\Lambda v(\tau)\rangle$$

for $\mathscr{L}^1$-a.a. $\tau\in[0,T]$. Hence

$$\frac{1}{2}\frac{d}{d\tau}\|\Lambda u(\tau)-\Lambda v(\tau)\|^2_{L^2(O)}\leqslant\langle G(\tau,u)-G(\tau,v),\Lambda u(\tau)-\Lambda v(\tau)\rangle.$$

Fix any $t\in[0,T]$. For each $s\in[0,t]$, by integration over $[0,s]$ and by using (2.4) and Cauchy-Schwarz's inequality, it follows that

$$\begin{aligned}\|\Lambda u(s)-\Lambda v(s)\|^2_{L^2(O)} &\leqslant \int_0^s\langle G(\tau,u)-G(\tau,v),\Lambda u(\tau)-\Lambda v(\tau)\rangle d\tau\\ &\leqslant \int_0^s\|G(\tau,u)-G(\tau,v)\|_{L^2(O)}\|\Lambda u(\tau)-\Lambda v(\tau)\|_{L^2(O)}d\tau.\end{aligned}$$

From Lemma D.1 (that we apply with $p=2$, $a=0$, $\phi(\cdot)=\|\Lambda u(\cdot)-\Lambda v(\cdot)\|_{L^2(O)}$ and $\mathrm{m}(\cdot)=\|G(\cdot,u)-G(\cdot,v)\|_{L^2(O)}$) we deduce that for every $s\in[0,t]$,

$$\|\Lambda u(s)-\Lambda v(s)\|_{L^2(O)}\leqslant\int_0^s\|G(\tau,u)-G(\tau,v)\|_{L^2(O)}d\tau. \tag{2.6}$$

On the other hand, taking (2.1) and (2.2) into account, we see that for every $\tau\in[0,s]$,

$$\begin{aligned}\|G(\tau,u)-G(\tau,v)\|_{L^2(O)} &\leqslant \|\mathcal{K}*(\nabla\mathscr{G}\circ u)(\tau)-\mathcal{K}*(\nabla\mathscr{G}\circ v)(\tau)\|_{L^2(O)}\\ &\leqslant \|\mathcal{K}*(\nabla\mathscr{G}\circ u)-\mathcal{K}*(\nabla\mathscr{G}\circ v)\|_{C([0,\tau];L^2(O))}\\ &\leqslant \widehat{C}\|u-v\|_{C([0,\tau];L^2(O))}.\end{aligned}$$

Consequently, for every $s\in[0,t]$,

$$\begin{aligned}\int_0^s\|G(\tau,u)-G(\tau,v)\|_{L^2(O)}d\tau &\leqslant \widehat{C}\int_0^s\|u-v\|_{C([0,\tau];L^2(O))}d\tau\\ &\leqslant \widehat{C}\int_0^t\|u-v\|_{C([0,\tau];L^2(O))}d\tau \end{aligned} \tag{2.7}$$

[1] As $\mathscr{E}:L^2(O)\to[0,\infty[$ is assumed to be convex and Fréchet-differentiable, $\nabla\mathscr{E}$ is monotone, i.e. $\langle\nabla\mathscr{E}(u)-\nabla\mathscr{E}(v),u-v\rangle\geqslant 0$ for all $u,v\in L^2(O)$.

From (2.6) and (2.7) it follows that for every $u,v\in C([0,T];L^2(O))$ and every $t\in[0,T]$,

$$\|\Lambda u-\Lambda v\|_{C([0,t];L^2(O))}\leqslant \widehat{C}\int_0^t\|u-v\|_{C([0,\tau];L^2(O))}d\tau. \tag{2.8}$$

By iterating (2.8) we deduce that for every $n\in\mathbb{N}^*$, every $u,v\in C([0,T];L^2(O))$ and every $t\in[0,T]$,

$$\begin{aligned}\|\Lambda^n u-\Lambda^n v\|_{C([0,t];L^2(O))} &\leqslant \widehat{C}^n\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}}\|u-v\|_{C([0,\tau_n];L^2(O))}d\tau_n\cdots d\tau_1\\ &\leqslant \widehat{C}^n\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}}d\tau_n\cdots d\tau_1\|u-v\|_{C([0,T];L^2(O))}\\ &=\frac{(\widehat{C}t)^n}{n!}\|u-v\|_{C([0,T];L^2(O))}.\end{aligned}$$

Consequently, for every $n\in\mathbb{N}^*$ and every $u,v\in C([0,T];L^2(O))$,

$$\|\Lambda^n u-\Lambda^n v\|_{C([0,T];L^2(O))}\leqslant\frac{(\widehat{C}T)^n}{n!}\|u-v\|_{C([0,T];L^2(O))},$$

and $(\mathcal{C})$ follows because $\frac{(\widehat{C}T)^n}{n!}\to 0$ as $n\to\infty$.

Step 2: Regularity. From Step 1, let $\overline{u}\in C([0,T];L^2(O))$ be the unique solution of $(\mathscr{P})$. Then $\nabla\mathscr{E}\circ\overline{u}\in L^2([0,T];L^2(O))$ and so $\frac{d\overline{u}}{dt}\in L^2([0,T];L^2(O))$ because $G(\cdot,\overline{u})\in L^2([0,T];L^2(O))$. According to [9, Theorem 17.2.6], to establish $(\mathcal{R})$ it suffices to show that $G(\cdot,\overline{u})\in AC([0,T];L^2(O))$.

First of all, for $\mathscr{L}^1$-a.e. $t\in[0,T]$, we have

$$\frac{d\mathcal{K}*(\nabla\mathscr{G}\circ\overline{u})}{dt}(t)=\mathcal{K}(0)\nabla\mathscr{G}(\overline{u}(t))+\frac{d\mathcal{K}}{dt}*(\nabla\mathscr{G}\circ\overline{u})(t),$$

so that

$$\left\|\frac{d\mathcal{K}*(\nabla\mathscr{G}\circ\overline{u})}{dt}\right\|_{L^2([0,T];L^2(O))}\leqslant\sqrt{2}\left(\mathcal{K}(0)+\sqrt{T}\left\|\frac{d\mathcal{K}}{dt}\right\|_{L^2([0,T])}\right)\|\nabla\mathscr{G}\circ\overline{u})\|_{L^2([0,T];L^2(O))} < \infty. \tag{2.9}$$

Consequently

$$\mathcal{K}*(\nabla\mathscr{G}\circ\overline{u})\in W^{1,2}([0,T];L^2(O)). \tag{2.10}$$

From (2.10) and (R) we conclude that $G(\cdot,\overline{u})\in AC([0,T];L^2(O))$, which completes the proof. ■

2.3. **Compactness.** For each $\varepsilon>0$, let $F_\varepsilon:[0,T]\to L^2(O)$ be a Borel measurable map such that $F_\varepsilon\in\mathscr{F}_{(\mathrm{R})}$, let $\mathscr{E}_\varepsilon,\mathscr{G}_\varepsilon:L^2(O)\to[0,\infty[$ be convex and Fréchet-differentiable functionals such that $(\mathscr{E}_\varepsilon,\mathscr{G}_\varepsilon)\in\mathscr{F}_{(\mathrm{D}_1)\text{-}(\mathrm{D}_2)\text{-}(\mathrm{D}_3^\alpha)}$ and consider the following integrodifferential diffusion problem of gradient flow type:

$$(\mathscr{P}_\varepsilon)\begin{cases}\dfrac{du_\varepsilon}{dt}(t)+\nabla\mathscr{E}_\varepsilon(u_\varepsilon(t))+\mathcal{K}*(\nabla\mathscr{G}_\varepsilon\circ u_\varepsilon)(t)=F_\varepsilon(t)\quad\text{for }\mathscr{L}^1\text{-a.a. }t\in[0,T]\\ u_\varepsilon(0)=u_{0,\varepsilon}\in L^2(O).\end{cases}$$

Let $F : [0,T] \to L^2(O)$ be a Borel measurable map and let $\mathscr{E}, \mathscr{G} : L^2(O) \to [0,\infty]$ be proper, convex and lower semicontinuous functionals that are Fréchet-differentiable on $\mathrm{dom}(\partial\mathscr{E})$ and $\mathrm{dom}(\partial\mathscr{G})$ respectively. The following result gives sufficient conditions for the compactness of $(\mathscr{P}_\varepsilon)$ as $\varepsilon \to 0$.

Theorem 2.2. *For every $\varepsilon > 0$, let $\overline{u}_\varepsilon \in AC([0,T];L^2(O))$ be the unique solution of $(\mathscr{P}_\varepsilon)$ obtained by Theorem 2.1, and assume that*:

- (C$_1$) $\sup\limits_{\varepsilon>0} \mathscr{G}_\varepsilon(u_{0,\varepsilon}) < \infty$;
- (C$_2$) $u_{0,\varepsilon} \to u_0$ *in* $L^2(O)$;
- (C$_3$) $\sup\limits_{\varepsilon>0} \|F_\varepsilon(0)\|_{L^2(O)} < \infty$;
- (C$_4$) $F_\varepsilon \rightharpoonup F$ *in* $L^2([0,T];L^2(O))$;
- (C$_5$) $\sup\limits_{\varepsilon>0} \left\| \frac{dF_\varepsilon}{dt} \right\|_{L^1([0,T];L^2(O))} < \infty$;
- (C$_6$) $\mathscr{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathscr{E}$;
- (C$_7$) $\mathscr{G}_\varepsilon \xrightarrow{\Gamma_{\mathrm{w}}} \mathscr{G}$;
- (C$_8$) *for every* $\{v_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$, *if* $\sup\limits_{\varepsilon>0} \mathscr{G}_\varepsilon(v_\varepsilon) < \infty$ *then* $\{v_\varepsilon\}_{\varepsilon>0}$ *is relatively compact in* $L^2(O)$.[2]

Then, there exists $\overline{u} \in C([0,T];L^2(O))$ *such that (up to a subsequence)*

$$\overline{u}_\varepsilon \to \overline{u} \text{ in } C([0,T];L^2(O))$$

and $\overline{u}$ *is a solution of the following integrodifferential diffusion problem of gradient flow type*:

$$(\mathscr{P}) \begin{cases} \dfrac{du}{dt}(t) + \nabla\mathscr{E}(u(t)) + \mathcal{K} * (\nabla\mathscr{G} \circ u)(t) = F(t) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u(0) = u_0 \in \mathrm{dom}(\mathscr{G}). \end{cases}$$

Remark 2.3. As $F_\varepsilon \in \mathscr{F}_{(\mathrm{R})}$ for all $\varepsilon > 0$ we have

$$\sup_{\varepsilon>0} \|F_\varepsilon(t)\|_{L^2(O)} \leqslant \sup_{\varepsilon>0} \|F_\varepsilon(0)\|_{L^2(O)} + \sup_{\varepsilon>0} \left\| \frac{dF_\varepsilon}{d\tau} \right\|_{L^1([0,T];L^2(O))} \quad \text{for all } t \in [0,T].$$

In particular, if (C$_3$)–(C$_5$) hold then $\sup_{t\in[0,T]} \sup_{\varepsilon>0} \|F_\varepsilon(t)\|_{L^2(O)} < \infty$.

Proof of Theorem 2.2. Fix any $\varepsilon > 0$ and let $\overline{u}_\varepsilon \in AC([0,T];L^2(O))$ be the unique solution of $(\mathscr{P}_\varepsilon)$.
Let $\widetilde{T} > 0$ be such that $\sqrt{\widetilde{T}}\|\mathcal{K}\|_{L^2([0,T])} < \min\{\alpha, T\}$, so that

$$\alpha - \sqrt{\widetilde{T}}\|\mathcal{K}\|_{L^2([0,T])} > 0, \tag{2.11}$$

and set $\ell := \max\{k \in \mathbb{N} : k\widetilde{T} \leqslant T\}$ and let $\{T_i\}_{i\in\{0,\cdots,\ell+1\}}$ be defined by

$$T_i := \begin{cases} i\widetilde{T} & \text{if } i \in \{0,\cdots,\ell\} \\ T & \text{if } i = \ell+1. \end{cases}$$

[2]The condition (C$_8$) implies that $\{\mathscr{G}_\varepsilon\}_{\varepsilon>0}$ is equicoercive, i.e. for all $\{v_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$, if $\sup_{\varepsilon>0} \mathscr{G}_\varepsilon(v_\varepsilon) < \infty$ then $\sup_{\varepsilon>0} \|v_\varepsilon\|_{L^2(O)} < \infty$.

For each $i\in\{0,\cdots,\ell\}$, let $\overline{u}^i_\varepsilon\in AC([0,\widetilde{T}];L^2(O))$ be defined by

$$\overline{u}^i_\varepsilon(t):=\overline{u}_\varepsilon(t+T_i). \tag{2.12}$$

Let $F^i_\varepsilon:[0,\widetilde{T}]\to L^2(O)$ be defined by

$$\begin{aligned}F^i_\varepsilon(t)&:=F_\varepsilon(t+T_i)-\int_0^{T_i}\mathcal{K}(t+T_i-s)\nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon(s))ds\\&=F_\varepsilon(t+T_i)-\sum_{k=0}^{i-1}\int_{T_k}^{T_{k+1}}\mathcal{K}(t+T_i-s)\nabla\mathcal{G}_\varepsilon(\overline{u}^k_\varepsilon(s-T_k))ds\end{aligned} \tag{2.13}$$

with the convention $\sum_{k=0}^{i-1}=0$ if $i=0$, and let $(\mathscr{P}^i_\varepsilon)$ be given by

$$(\mathscr{P}^i_\varepsilon)\begin{cases}\dfrac{du^i_\varepsilon}{dt}(t)+\nabla\mathscr{E}_\varepsilon(u^i_\varepsilon(t))=F^i_\varepsilon(t)-\mathcal{K}*(\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon)(t)\quad\text{for }\mathscr{L}^1\text{-a.a. }t\in[0,\widetilde{T}]\\ u^i_\varepsilon(0)=\overline{u}^{i-1}_\varepsilon(\widetilde{T})\end{cases}$$

with the convention $\overline{u}^{i-1}_\varepsilon(\widetilde{T})=u_{0,\varepsilon}$ if $i=0$. Note that the problem $(\mathscr{P}^i_\varepsilon)$ can be equivalently rewritten as follows:

$$(\mathscr{P}^i_\varepsilon)\begin{cases}\dfrac{du^i_\varepsilon}{dt}(t)+\nabla\mathscr{E}_\varepsilon(u^i_\varepsilon(t))=G^i_\varepsilon(t)\quad\text{for }\mathscr{L}^1\text{-a.a. }t\in[0,\widetilde{T}]\\ u^i_\varepsilon(0)=\overline{u}^{i-1}_\varepsilon(\widetilde{T})\end{cases}$$

with $G^i_\varepsilon:[0,\widetilde{T}]\to L^2(O)$ given by

$$G^i_\varepsilon(t):=F^i_\varepsilon(t)-\mathcal{K}*(\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon)(t). \tag{2.14}$$

According to Theorem 2.1, it is easy to see that for every $i\in\{0,\cdots,\ell\}$, $\overline{u}^i_\varepsilon$ given by (2.12) is a solution of $(\mathscr{P}^i_\varepsilon)$ and satisfies the following regularity condition:

- $(\mathcal{R}^i_\varepsilon)$ $\overline{u}^i_\varepsilon$ admits a right derivative $\frac{d^+\overline{u}^i_\varepsilon}{dt}(t)$ at every $t\in[0,\widetilde{T}[$ which satisfies $\frac{d^+\overline{u}^i_\varepsilon}{dt}(t)+\nabla\mathscr{E}_\varepsilon(\overline{u}^i_\varepsilon(t))=G^i_\varepsilon(t)$.

Moreover, from [5, Lemma 2.5, p. 45] we can assert that

- $(\widehat{\mathcal{R}}^i_\varepsilon)$ for every $t\in]0,\widetilde{T}[$,

$$\left\|\frac{d^+\overline{u}^i_\varepsilon}{dt}(t)\right\|_{L^2(O)}\leqslant\frac{1}{t}\int_0^t\left\|\frac{d\overline{u}^i_\varepsilon}{ds}(s)\right\|_{L^2(O)}ds+\int_0^t\left\|\frac{dG^i_\varepsilon}{ds}(s)\right\|_{L^2(O)}ds.$$

Roughly, our strategy is as follows. We first show that for every $i\in\{0,\cdots,\ell\}$, there exists $\overline{u}^i\in C([0,\widetilde{T}];L^2(O))$ such that (up to a subsequence) $\overline{u}^i_\varepsilon\to\overline{u}^i$ in $C([0,\widetilde{T}];L^2(O))$. Then, we define $\overline{u}\in C([0,T];L^2(O))$ by $\overline{u}(t)=\overline{u}^i(t-T_i)$ if $t\in[T_i,T_{i+1}]$ and we prove that $\overline{u}_\varepsilon\to\overline{u}$ in $C([0,T];L^2(O))$ and that $\overline{u}$ is a solution of $(\mathscr{P})$. To implement this stategy we proceed into three steps.

Step 1: Convergence for each *i*. For each $i\in\{0,\cdots,\ell\}$ we consider the following five assertions:

- (C^i_1) $\sup_{\varepsilon>0}\mathcal{G}_\varepsilon(\overline{u}^i_\varepsilon(0))<\infty$;

(C_2^i) $\{\overline{u}_\varepsilon^i(0)\}_{\varepsilon>0}$ is relatively compact in $L^2(O)$;
(C_3^i) $\sup\limits_{\varepsilon>0}\|F_\varepsilon^i(t)\|_{L^2(O)}<\infty$ for all $t\in[0,\widetilde{T}]$;
(C_4^i) $\sup\limits_{\varepsilon>0}\|F_\varepsilon^i\|_{L^2([0,\widetilde{T}];L^2(O))}<\infty$;
(C_5^i) $\sup\limits_{\varepsilon>0}\left\|\frac{dF_\varepsilon^i}{dt}\right\|_{L^1([0,\widetilde{T}];L^2(O))}<\infty$,

and the following five properties:

(P_1^i) $\sup\limits_{\varepsilon>0}\|\nabla\mathscr{G}_\varepsilon\circ\overline{u}_\varepsilon^i\|_{L^2([0,\widetilde{T}];L^2(O))}<\infty$;
(P_2^i) $\sup\limits_{\varepsilon>0}\left\|\frac{d\overline{u}_\varepsilon^i}{dt}\right\|_{L^2([0,\widetilde{T}];L^2(O))}<\infty$;
(P_3^i) there exists $C_i>0$ such that $\sup\limits_{\varepsilon>0}\|\nabla\mathscr{G}_\varepsilon(\overline{u}_\varepsilon^i(t))\|_{L^2(O)}<C_i\left(\frac{1}{\sqrt{t}}+1\right)$ for all $t\in]0,\widetilde{T}[$;
(P_4^i) $\sup\limits_{\varepsilon>0}\mathscr{G}_\varepsilon(\overline{u}_\varepsilon^i(t))<\infty$ for all $t\in]0,\widetilde{T}]$;
(P_5^i) there exists $\overline{u}^i\in C([0,\widetilde{T}];L^2(O))$ such that (up to a subsequence):

$$\overline{u}_\varepsilon^i\to\overline{u}^i \text{ in } C([0,\widetilde{T}];L^2(O)); \tag{2.15}$$
$$\nabla\mathscr{G}_\varepsilon\circ\overline{u}_\varepsilon^i\rightharpoonup\nabla\mathscr{G}\circ\overline{u}^i \text{ in } L^2([0,\widetilde{T}];L^2(O)). \tag{2.16}$$

The goal of this step is to prove that (P_1^i)–(P_5^i) hold for all $i\in\{0,\cdots,\ell\}$. For this, it suffices to prove the following three assertions:

(I_0) for $i=0$, (C_1^i)–(C_5^i) are satisfied;

(I_1) for every $i\in\{0,\cdots,\ell\}$, if (C_1^i)–(C_5^i) hold then (P_1^i)–(P_5^i) are satisfied;

(I_2) for every $i\in\{1,\cdots,\ell\}$, if (P_4^{i-1}) holds and if (P_1^k), (P_3^k) and (P_5^k) hold for all $k\in\{0,\cdots,i-1\}$, then (C_1^i)–(C_5^i) are satisfied.

Taking (C_1)–(C_5) into account, it is clear that (I_0) is verified. So, we only need to establish (I_1) and (I_2).

Step 1-1: Proving (I_1). Fix $i\in\{0,\cdots,\ell\}$ and assume that (C_1^i)–(C_5^i) hold.

Proof of (P_1^i). As $\overline{u}_\varepsilon^i$ is a solution of $(\mathscr{P}_\varepsilon^i)$, we have

$$\int_0^{\widetilde{T}}\left\langle\frac{d\overline{u}_\varepsilon^i}{dt}(t),\nabla\mathscr{G}_\varepsilon(\overline{u}_\varepsilon^i(t))\right\rangle dt+\int_0^{\widetilde{T}}\left\langle\nabla\mathscr{E}_\varepsilon(\overline{u}_\varepsilon^i(t)),\nabla\mathscr{G}_\varepsilon(\overline{u}_\varepsilon^i(t))\right\rangle dt$$
$$+\int_0^{\widetilde{T}}\left\langle\mathscr{K}*(\nabla\mathscr{G}_\varepsilon\circ\overline{u}_\varepsilon^i)(t),\nabla\mathscr{G}_\varepsilon(\overline{u}_\varepsilon^i(t))\right\rangle dt=\int_0^{\widetilde{T}}\left\langle F_\varepsilon^i(t),\nabla\mathscr{G}_\varepsilon(\overline{u}_\varepsilon^i(t))\right\rangle dt.$$

Hence

$$\begin{aligned}\int_0^{\widetilde{T}} \left\langle \nabla\mathscr{E}_\varepsilon(\overline{u}^i_\varepsilon(t)), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt \\ + \int_0^{\widetilde{T}} \left\langle \mathcal{K} * (\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon)(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt &\leqslant -\int_0^{\widetilde{T}} \left\langle \frac{d\overline{u}^i_\varepsilon}{dt}(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt \\ &\quad + \int_0^{\widetilde{T}} \left\langle F^i_\varepsilon(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt. \end{aligned} \tag{2.17}$$

First of all, as $\langle \frac{d\overline{u}^i_\varepsilon}{dt}(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\rangle = \frac{d(\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon)}{dt}(t)$, we have

$$-\int_0^{\widetilde{T}} \langle \frac{d\overline{u}^i_\varepsilon}{dt}(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\rangle dt = \mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0)) - \mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(\widetilde{T})) \leqslant \mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0)).$$

But $\sup_{\varepsilon>0} \mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0)) < \infty$ by (C^i_1), hence

$$M^i_0 := \sup_{\varepsilon>0} \left(-\int_0^{\widetilde{T}} \left\langle \frac{d\overline{u}^i_\varepsilon}{dt}(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt \right) < \infty. \tag{2.18}$$

Secondly, by (D^α_3), we have

$$\int_0^{\widetilde{T}} \left\langle \nabla\mathscr{E}_\varepsilon(\overline{u}^i_\varepsilon(t)), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt \geqslant \alpha \|\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon\|^2_{L^2([0,\widetilde{T}];L^2(O))}. \tag{2.19}$$

Thirdly, from the Cauchy-Schwarz inequality, we have

$$\int_0^{\widetilde{T}} \left\langle \mathcal{K} * (\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon)(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt \leqslant \left\|\mathcal{K} * (\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon)\right\|_{L^2([0,\widetilde{T}];L^2(O))} \left\|\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon\right\|_{L^2([0,\widetilde{T}];L^2(O))}.$$

But, by an easy calculation,

$$\left\|\mathcal{K} * (\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon)\right\|_{L^2([0,\widetilde{T}];L^2(O))} \leqslant \sqrt{\widetilde{T}} \|\mathcal{K}\|_{L^2([0,T])} \left\|\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon\right\|_{L^2([0,\widetilde{T}];L^2(O))}, \tag{2.20}$$

hence

$$\int_0^{\widetilde{T}} \left\langle \mathcal{K} * (\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon)(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt \leqslant \sqrt{\widetilde{T}} \|\mathcal{K}\|_{L^2([0,T])} \left\|\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon\right\|^2_{L^2([0,\widetilde{T}];L^2(O))}. \tag{2.21}$$

Fourthly, we have

$$\int_0^{\widetilde{T}} \left\langle F^i_\varepsilon(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt \leqslant \|F^i_\varepsilon\|_{L^2([0,\widetilde{T}];L^2(O))} \left\|\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon\right\|_{L^2([0,\widetilde{T}];L^2(O))}.$$

But, by (C^i_4),

$$M^i_1 := \sup_{\varepsilon>0} \|F^i_\varepsilon\|_{L^2([0,\widetilde{T}];L^2(O))} < \infty, \tag{2.22}$$

hence

$$\int_0^{\widetilde{T}} \left\langle F^i_\varepsilon(t), \nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt \leqslant M^i_1 \left\|\nabla\mathscr{G}_\varepsilon \circ \overline{u}^i_\varepsilon\right\|_{L^2([0,\widetilde{T}];L^2(O))}. \tag{2.23}$$

Combining (2.17) with (2.18), (2.19), (2.21) and (2.23), we have

$$\left(\alpha-\sqrt{\widetilde{T}}\|\mathcal{K}\|_{L^2([0,T])}\right)\|\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon\|^2_{L^2([0,\widetilde{T}];L^2(O))}\leqslant M^i_0+M^i_1\|\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon\|_{L^2([0,\widetilde{T}];L^2(O))}$$

Thus, for every $\varepsilon>0$,

$$A\|\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon\|^2_{L^2([0,\widetilde{T}];L^2(O))}-M^i_1\|\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon\|_{L^2([0,\widetilde{T}];L^2(O))}-M^i_0\leqslant 0,$$

with $A=\alpha-\sqrt{\widetilde{T}}\|\mathcal{K}\|_{L^2([0,T])}>0$ from (2.11), which yields (P^i_1).

Proof of (P^i_2). As $\overline{u}^i_\varepsilon$ is a solution of $(\mathscr{P}^i_\varepsilon)$ we have

$$\int_0^{\widetilde{T}}\left\langle\frac{d\overline{u}^i_\varepsilon}{dt}(t),\frac{d\overline{u}^i_\varepsilon}{dt}(t)\right\rangle dt+\int_0^{\widetilde{T}}\left\langle\nabla\mathscr{E}_\varepsilon(\overline{u}^i_\varepsilon(t)),\frac{d\overline{u}^i_\varepsilon}{dt}(t)\right\rangle dt$$
$$+\int_0^{\widetilde{T}}\left\langle\mathcal{K}*(\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon)(t),\frac{d\overline{u}^i_\varepsilon}{dt}(t)\right\rangle dt=\int_0^{\widetilde{T}}\left\langle F^i_\varepsilon(t),\frac{d\overline{u}^i_\varepsilon}{dt}(t)\right\rangle dt.$$

Hence

$$\begin{aligned}\left\|\frac{d\overline{u}^i_\varepsilon}{dt}\right\|^2_{L^2([0,\widetilde{T}];L^2(O))}&\leqslant\int_0^{\widetilde{T}}\left\langle F^i_\varepsilon(t),\frac{d\overline{u}^i_\varepsilon}{dt}(t)\right\rangle dt+\left|\int_0^{\widetilde{T}}\left\langle\mathcal{K}*(\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon)(t),\frac{d\overline{u}^i_\varepsilon}{dt}(t)\right\rangle dt\right|\\&\quad-\int_0^{\widetilde{T}}\left\langle\nabla\mathscr{E}_\varepsilon(\overline{u}^i_\varepsilon(t)),\frac{d\overline{u}^i_\varepsilon}{dt}(t)\right\rangle dt\\&\leqslant\left(\|F^i_\varepsilon\|_{L^2([0,\widetilde{T}];L^2(O))}+\|\mathcal{K}*(\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon)\|_{L^2([0,\widetilde{T}];L^2(O))}\right)\left\|\frac{d\overline{u}^i_\varepsilon}{dt}\right\|_{L^2([0,\widetilde{T}];L^2(O))}\\&\quad+M^i_0\\&\leqslant\left(M^i_1+\|\mathcal{K}*(\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon)\|_{L^2([0,\widetilde{T}];L^2(O))}\right)\left\|\frac{d\overline{u}^i_\varepsilon}{dt}\right\|_{L^2([0,\widetilde{T}];L^2(O))}+M^i_0\end{aligned}\tag{2.24}$$

with $M^i_0<\infty$ and $M^i_1<\infty$ given by (2.18) and (2.22) respectively. On the other hand, setting

$$M^i_2:=\sup_{\varepsilon>0}\|\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon\|_{L^2([0,\widetilde{T}];L^2(O))}\tag{2.25}$$

we have $M^i_2<\infty$ by (P^i_1), and by (2.20) we see that

$$\|\mathcal{K}*(\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon)\|_{L^2([0,\widetilde{T}];L^2(O))}\leqslant M^i_2\sqrt{\widetilde{T}}\|\mathcal{K}\|_{L^2([0,T])}.$$

From (2.24) it follows that for every $\varepsilon>0$,

$$\left\|\frac{d\overline{u}^i_\varepsilon}{dt}\right\|^2_{L^2([0,\widetilde{T}];L^2(O))}-\left(M^i_1+M^i_2\sqrt{\widetilde{T}}\|\mathcal{K}\|_{L^2([0,T])}\right)\left\|\frac{d\overline{u}^i_\varepsilon}{dt}\right\|_{L^2([0,\widetilde{T}];L^2(O))}-M^i_0\leqslant 0,$$

which implies (P^i_2).

Proof of (P^i_3). First of all, from (2.9), (2.20) and (P^i_1), we can assert that

$$\sup_{\varepsilon>0}\|\mathcal{K}*(\nabla\mathcal{G}_\varepsilon\circ\overline{u}^i_\varepsilon)\|_{W^{1,2}([0,\widetilde{T}];L^2(O))}<\infty.\tag{2.26}$$

According to (2.14), from (C_5^i) and (2.26) we deduce that

$$M_3^i := \sup_{\varepsilon>0} \left\| \frac{dG_\varepsilon^i}{dt} \right\|_{L^1([0,\widetilde{T}];L^2(O))} < \infty. \tag{2.27}$$

Setting

$$M_4^i := \sup_{\varepsilon>0} \left\| \frac{d\overline{u}_\varepsilon^i}{dt} \right\|_{L^2([0,\widetilde{T}];L^2(O))}$$

where $M_4^i < \infty$ by (P_2^i), from ($\widehat{\mathcal{R}}_\varepsilon^i$) we see that for every $t \in]0,\widetilde{T}[$,

$$\begin{aligned} \left\| \frac{d^+\overline{u}_\varepsilon^i}{dt}(t) \right\|_{L^2(O)} &\leqslant \frac{1}{\sqrt{t}} \left\| \frac{d\overline{u}_\varepsilon^i}{dt} \right\|_{L^2([0,\widetilde{T}];L^2(O))} + M_3^i \\ &\leqslant \frac{1}{\sqrt{t}} M_4^i + M_3^i \end{aligned} \tag{2.28}$$

with $M_3^i < \infty$ given by (2.27). Fix any $t \in]0,\widetilde{T}[$. By ($\mathcal{R}_\varepsilon^i$) we have

$$\left\langle \frac{d^+\overline{u}_\varepsilon^i}{dt}(t), \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\rangle + \left\langle \nabla\mathcal{E}_\varepsilon(\overline{u}_\varepsilon^i(t)), \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\rangle = \left\langle G_\varepsilon^i(t), \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\rangle, \tag{2.29}$$

and by using (D_3^α) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \alpha \left\| \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\|_{L^2(O)}^2 &\leqslant \left| \left\langle \frac{d^+\overline{u}_\varepsilon^i}{dt}(t), \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\rangle \right| + \left\langle G_\varepsilon^i(t), \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\rangle \\ &\leqslant \left(\left\| \frac{d^+\overline{u}_\varepsilon^i}{dt}(t) \right\|_{L^2(O)} + \left\| G_\varepsilon^i(t) \right\|_{L^2(O)} \right) \left\| \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\|_{L^2(O)}. \end{aligned} \tag{2.30}$$

From (2.28) and (2.30), we deduce that

$$\alpha \left\| \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\|_{L^2(O)} \leqslant \frac{1}{\sqrt{t}} M_4^i + M_3^i + \left\| G_\varepsilon^i(t) \right\|_{L^2(O)}. \tag{2.31}$$

Set

$$M_5^i := \sup_{t\in[0,\widetilde{T}]} \sup_{\varepsilon>0} \left\| F_\varepsilon^i(t) \right\|_{L^2(O)},$$

where $M_5^i < \infty$ by ($\widehat{C}_3^i$). From the definition of $G_\varepsilon^i(t)$ in (2.14) it is easy to see that

$$\begin{aligned} \left\| G_\varepsilon^i(t) \right\|_{L^2(O)} &\leqslant \|F_\varepsilon^i(t)\|_{L^2(O)} + \|\mathcal{K}\|_{L^2([0,T])} \left\| \nabla\mathcal{G}_\varepsilon \circ \overline{u}_\varepsilon^i \right\|_{L^2([0,\widehat{T}];L^2(O))} \\ &\leqslant M_5^i + M_2^i \|\mathcal{K}\|_{L^2([0,T])}, \end{aligned} \tag{2.32}$$

where M_2^i is given by (2.25). Combining (2.31) and (2.32) we concude that for every $\varepsilon > 0$ and every $t \in]0,\widetilde{T}[$,

$$\left\| \nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^i(t)) \right\|_{L^2(O)} \leqslant C_i \left(\frac{1}{\sqrt{t}} + 1 \right)$$

with $C_i := \frac{1}{\alpha} \max \left\{ M_4^i, M_3^i + M_5^i + M_2^i \|\mathcal{K}\|_{L^2([0,T])} \right\}$, which implies ($P_3^i$).

Proof of (P_4^i). By using (2.29), (D_3^α), the Cauchy-Schwarz inequality, (2.32) and (P_3^i), it is easily seen that for every $t\in]0,\widetilde{T}[$,

$$\begin{aligned}\left\langle\frac{d^+\overline{u}^i_\varepsilon}{dt}(t),\nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle &\leqslant \left\langle G^i_\varepsilon(t),\nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle\\ &\leqslant \left\|G^i_\varepsilon(t)\right\|_{L^2(O)}\left\|\nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\|_{L^2(O)}\\ &\leqslant \widehat{C}_i\left(\frac{1}{\sqrt{t}}+1\right)\end{aligned}\tag{2.33}$$

with $\widehat{C}_i:=\left(M^i_5+M^i_2\|\mathcal{K}\|_{L^2([0,T])}\right)C_i$. Fix any $s\in]0,\widetilde{T}]$. By integrating (2.33) over $[0,s]$, we get

$$\int_0^s\left\langle\frac{d^+\overline{u}^i_\varepsilon}{dt}(t),\nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt\leqslant\int_0^{\widetilde{T}}\widehat{C}_i\left(\frac{1}{\sqrt{t}}+1\right)dt.\tag{2.34}$$

As $\langle\frac{d^+\overline{u}^i_\varepsilon}{dt}(t),\nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\rangle=\frac{d(\mathscr{G}_\varepsilon\circ\overline{u}^i_\varepsilon)}{dt}(t)$ for all $t\in]0,s[$, we have

$$\int_0^s\left\langle\frac{d^+\overline{u}^i_\varepsilon}{dt}(t),\nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))\right\rangle dt=\int_0^s\frac{d(\mathscr{G}_\varepsilon\circ\overline{u}^i_\varepsilon)}{dt}(t)dt=\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(s))-\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0)).\tag{2.35}$$

From (2.34) and (2.35), we deduce that, for every $\varepsilon>0$ and every $s\in]0,\widetilde{T}]$,

$$\begin{aligned}\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(s)) &\leqslant \mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0))+\int_0^{\widetilde{T}}\widehat{C}_i\left(\frac{1}{\sqrt{t}}+1\right)dt\\ &\leqslant M^i_6+\int_0^{\widetilde{T}}\widehat{C}_i\left(\frac{1}{\sqrt{t}}+1\right)dt\end{aligned}$$

with $M^i_6:=\sup_{\varepsilon>0}\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0))<\infty$ by (C_1^i), which implies (P_4^i).

Proof of (P_5^i). As $\overline{u}^i_\varepsilon\in AC([0,\widetilde{T}];L^2(O))$ and $\sup_{\varepsilon>0}\|\frac{d\overline{u}^i_\varepsilon}{dt}\|_{L^2([0,\widetilde{T}];L^2(O))}<\infty$ by (P_2^i), it is clear that $\{\overline{u}^i_\varepsilon\}_{\varepsilon>0}$ is equicontinuous. On the other hand, by (C_2^i), $\{\overline{u}^i_\varepsilon(0)\}_{\varepsilon>0}$ is relatively compact in $L^2(O)$. For $t\in]0,\widetilde{T}]$ we have $\sup_{\varepsilon>0}\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t))<\infty$ by (P_4^i), so that by (C_8), $\{\overline{u}^i_\varepsilon(t)\}_{\varepsilon>0}$ is relatively compact in $L^2(O)$. From Ascoli's theorem it follows that there exists $\overline{u}^i\in C([0,\widetilde{T}];L^2(O))$ such that (up to a subsequence)

$$\overline{u}^i_\varepsilon\to\overline{u}^i\text{ in }C([0,\widetilde{T}];L^2(O)),\tag{2.36}$$

which proves (2.15).

Remark 2.4. Since $\overline{u}^0_\varepsilon(0)=\overline{u}_\varepsilon(0)=u_{0,\varepsilon}$, by ($C_2$) we have $\overline{u}^0_\varepsilon(0)\to u_0$ in $L^2(O)$, and so $\overline{u}^0(0)=u_0$.

Let us now prove (2.16). Fix any $t\in]0,\widetilde{T}[$ and any subsequence $\{\overline{u}^i_{\sigma(\varepsilon)}\}_{\varepsilon>0}$ of $\{\overline{u}^i_\varepsilon\}_{\varepsilon>0}$ given by (2.36). By (P_3^i) there exists $\xi^i(t)\in L^2(O)$ and a subsequence $\{\overline{u}^i_{\sigma(\theta(\varepsilon))}\}_{\varepsilon>0}$ of $\{\overline{u}^i_{\sigma(\varepsilon)}\}_{\varepsilon>0}$ such that

$$\nabla\mathscr{G}_{\sigma(\theta(\varepsilon))}\left(\overline{u}^i_{\sigma(\theta(\varepsilon))}(t)\right)\rightharpoonup\xi^i(t)\text{ in }L^2(O).\tag{2.37}$$

Moreover, by (2.36), we have

$$\overline{u}^i_{\sigma(\theta(\varepsilon))}(t)\to\overline{u}^i(t)\text{ in }L^2(O).\tag{2.38}$$

Taking (C_7) and (C_8) into account, from Theorem C.4(c), we have

$$\nabla\mathscr{G}_\varepsilon \xrightarrow{\text{(s,w)-graph}} \partial\mathscr{G},$$

and therefore, from (2.37) and (2.38), we deduce that $\xi^i(t) = \nabla\mathscr{G}(\overline{u}^i(t))$. Hence

$$\nabla\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(t)) \rightharpoonup \nabla\mathscr{G}(\overline{u}^i(t)) \text{ in } L^2(O) \text{ for all } t \in]0,\widetilde{T}[. \tag{2.39}$$

Remark 2.5. We have also proved that for $\mathscr{L}^1$-a.e. $t\in[0,\widetilde{T}]$, $\overline{u}^i(t)\in\mathrm{dom}(\partial\mathscr{G})$.

On the other hand, consider any subsequence $\{\overline{u}^i_{\sigma(\varepsilon)}\}_{\varepsilon>0}$ of $\{\overline{u}^i_\varepsilon\}_{\varepsilon>0}$ given by (2.36). By (P^i_1) there exists $\Psi^i \in L([0,\widetilde{T}];L^2(O))$ and a subsequence $\{\nabla\mathscr{G}_{\sigma(\theta(\varepsilon))}\circ\overline{u}^i_{\sigma(\theta(\varepsilon))}\}_{\varepsilon>0}$ of $\{\nabla\mathscr{G}_{\sigma(\varepsilon)}\circ\overline{u}^i_{\sigma(\varepsilon)}\}_{\varepsilon>0}$ such that

$$\nabla\mathscr{G}_{\sigma(\theta(\varepsilon))}\circ\overline{u}^i_{\sigma(\theta(\varepsilon))} \rightharpoonup \Psi^i \text{ in } L^2([0,\widetilde{T}];L^2(O)). \tag{2.40}$$

Taking (2.39) and (P^i_3) into account, from Lebesgue's dominated convergence theorem, we can assert that

$$\int_0^{\widetilde{T}}\left\langle\nabla\mathscr{G}_{\sigma(\theta(\varepsilon))}(\overline{u}^i_{\sigma(\theta(\varepsilon))}(t)),\varphi(t)\right\rangle dt \to \int_0^{\widetilde{T}}\left\langle\nabla\mathscr{G}(\overline{u}^i(t)),\varphi(t)\right\rangle dt \text{ for all } \varphi\in C([0,\widetilde{T}];L^2(O)),$$

so that we infer from (2.40) that $\Psi^i = \nabla\mathscr{G}\circ\overline{u}^i$. Thus (2.16) holds.

Step 1-2: Proving (I_2). Fix $i\in\{1,\cdots,\ell\}$ and assume that (C^k_1)–(C^k_5) and (P^k_1)–(P^k_5) hold for all $k\in\{0,\cdots,i-1\}$.

Proof of (C^i_1). As $\overline{u}^i_\varepsilon$ is a solution of ($\mathscr{P}^i_\varepsilon$), we have $\overline{u}^i_\varepsilon(0) = \overline{u}^{i-1}_\varepsilon(\widetilde{T})$. Hence

$$\sup_{\varepsilon>0}\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0)) = \sup_{\varepsilon>0}\mathscr{G}_\varepsilon(\overline{u}^{i-1}_\varepsilon(\widetilde{T})).$$

From (P^{i-1}_4), we deduce that $\sup_{\varepsilon>0}\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0)) < \infty$, i.e., ($C^i_1$) is verified.

Proof of (C^i_2). As $\overline{u}^i_\varepsilon$ is a solution of ($\mathscr{P}^i_\varepsilon$), we have $\overline{u}^i_\varepsilon(0) = \overline{u}^{i-1}_\varepsilon(\widetilde{T})$. But by ($P^{i-1}_5$)-(2.15) we see that $\overline{u}^{i-1}_\varepsilon(\widetilde{T}) \to \overline{u}^{i-1}(\widetilde{T})$ in $L^2(O)$, so $\overline{u}^i_\varepsilon(0)\to\overline{u}^i(0)$ in $L^2(O)$, which implies (C^i_2).

Remark 2.6. In fact, we have proved a stronger condition than (C^i_2), i.e.,

($\widehat{C}^i_2$) $\overline{u}^i_\varepsilon(0)\to\overline{u}^i(0)$ in $L^2(O)$.

Proof of (C^i_3). As $\mathcal{K}\in C^1([0,T];[0,\infty[)$, we infer from (2.13) that, for every $t\in[0,\widetilde{T}]$,

$$\begin{aligned}\left\|F^i_\varepsilon(t)\right\|_{L^2(O)} &\leqslant \|F_\varepsilon(t+T_i)\|_{L^2(O)} + \|\mathcal{K}\|_{C([0,T])}\sum_{k=0}^{i-1}\int_{T_k}^{T_{k+1}}\left\|\nabla\mathscr{G}(\overline{u}^k(s-T_k))\right\|_{L^2(O)} ds\\ &\leqslant \sup_{\varepsilon>0}\|F_\varepsilon(t+T_i)\|_{L^2(O)} + \|\mathcal{K}\|_{C([0,T])}\sum_{k=0}^{i-1}\int_0^{\widetilde{T}}\sup_{\varepsilon>0}\left\|\nabla\mathscr{G}_\varepsilon(\overline{u}^k_\varepsilon(s))\right\|_{L^2(O)} ds.\end{aligned}$$

By (C_3) and Remark 2.3, we have $M := \sup_{s\in[0,T]}\sup_{\varepsilon>0}\|F_\varepsilon(s)\|_{L^2(O)} < \infty$ and by using (P^k_3) for all $k\in\{0,\cdots,i-1\}$, we deduce that, for every $t\in[0,\widetilde{T}]$,

$$\left\|F^i_\varepsilon(t)\right\|_{L^2(O)} \leqslant M + \|\mathcal{K}\|_{C([0,T])}\sum_{k=0}^{i-1}\int_0^{\widetilde{T}} C_k\left(\frac{1}{\sqrt{s}}+1\right)ds < \infty,$$

and (C_3^i) follows.

Remark 2.7. In fact, we have proved a stronger condition than (C_3^i), i.e.,

$(\widehat{\mathrm{C}}_3^i)$ $\sup\limits_{t\in[0,\widetilde{T}]}\sup\limits_{\varepsilon>0}\left\|F_\varepsilon^i(t)\right\|_{L^2(O)}<\infty.$

Proof of (C_4^i). For every $k\in\{0,\cdots,i-1\}$, by using (P_5^k)-(2.16), we see that

$$\int_{T_k}^{T_{k+1}}\mathcal{K}(\cdot+T_i-s)\nabla\mathcal{G}_\varepsilon(\overline{u}_\varepsilon^k(s-T_k))ds\rightharpoonup\int_{T_k}^{T_{k+1}}\mathcal{K}(\cdot+T_i-s)\nabla\mathcal{G}(\overline{u}^k(s-T_k))ds \text{ in } L^2([0,\widetilde{T}];L^2(O)).$$

Hence, taking (2.13) into account and by using (C_4),

$$F_\varepsilon^i\rightharpoonup F^i \text{ in } L^2([0,\widetilde{T}];L^2(O)),$$

where $F^i:[0,\widetilde{T}]\to L^2(O)$ if $i\in\{1,\cdots,\ell-1\}$ and $F^i:[0,T-T_k]\to L^2(O)$ if $i=\ell$, is given by

$$F^i(t):=F(t+T_i)-\sum_{k=0}^{i-1}\int_{T_k}^{T_{k+1}}\mathcal{K}(t+T_i-s)\nabla\mathcal{G}(\overline{u}^k(s-T_k))ds, \tag{2.41}$$

which implies (C_4^i).

Remark 2.8. In fact, we have proved a stronger condition than (C_4^i), i.e.,

$(\widehat{\mathrm{C}}_4^i)$ $F_\varepsilon^i\rightharpoonup F^i$ in $L^2([0,\widetilde{T}];L^2(O))$

with F^i given by (2.41).

Proof of (C_5^i). For $\mathscr{L}^1$-a.e. $t\in[0,\widetilde{T}]$, by deriving (2.13), we have

$$\frac{dF_\varepsilon^i}{dt}(t)=\frac{dF_\varepsilon}{dt}(t+T_i)-\sum_{k=0}^{i-1}\int_{T_k}^{T_{k+1}}\frac{d\mathcal{K}}{dt}(t+T_i-s)\nabla\mathcal{G}(\overline{u}^k(s-T_k))ds,$$

hence, since $\mathcal{K}\in C^1([0,T];[0,\infty[)$,

$$\left\|\frac{dF_\varepsilon^i}{dt}(t)\right\|_{L^2(O)}\leqslant\left\|\frac{dF_\varepsilon}{dt}(t+T_i)\right\|_{L^2(O)}+\left\|\frac{d\mathcal{K}}{dt}\right\|_{C([0,T])}\sum_{k=0}^{i-1}\sup_{\varepsilon>0}\left\|\nabla\mathcal{G}_\varepsilon\circ\overline{u}_\varepsilon^k\right\|_{L^2([0,\widetilde{T}];L^2(O))}.$$

Consequently, we get

$$\begin{aligned}\int_0^{\widetilde{T}}\left\|\frac{dF_\varepsilon^i}{dt}(t)\right\|_{L^2(O)}dt&\leqslant\int_0^{\widetilde{T}}\left\|\frac{dF_\varepsilon}{dt}(t+T_i)\right\|_{L^2(O)}dt+\widetilde{T}\left\|\frac{d\mathcal{K}}{dt}\right\|_{C([0,T])}\sum_{k=0}^{i-1}\sup_{\varepsilon>0}\left\|\nabla\mathcal{G}_\varepsilon\circ\overline{u}_\varepsilon^k\right\|_{L^2([0,\widetilde{T}];L^2(O))}\\&\leqslant\sup_{\varepsilon>0}\left\|\frac{dF_\varepsilon}{dt}\right\|_{L^1([0,T];L^2(O))}+\widetilde{T}\left\|\frac{d\mathcal{K}}{dt}\right\|_{C([0,T])}\sum_{k=0}^{i-1}\sup_{\varepsilon>0}\left\|\nabla\mathcal{G}_\varepsilon\circ\overline{u}_\varepsilon^k\right\|_{L^2([0,\widetilde{T}];L^2(O))}.\end{aligned}$$

By using (C_5) and (P_1^k) for all $k\in\{0,\cdots,i-1\}$, we deduce that $\sup_{\varepsilon>0}\left\|\frac{dF_\varepsilon^i}{dt}\right\|_{L^1([0,\widetilde{T}];L^2(O))}<\infty$ for all $t\in[0,\widetilde{T}]$, i.e. (C_5^i) is verified.

Step 2: Existence of a solution of $(\mathscr{P}^i)$ for each i. Fix any $i \in \{0,\cdots,\ell\}$ and consider the following problem:

$$(\mathscr{P}^i)\begin{cases} \dfrac{du^i}{dt}(t)+\nabla\mathscr{E}(u^i(t)) = F^i(t)-\mathcal{K}*(\nabla\mathscr{G}\circ\overline{u}^i)(t) & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,\widetilde{T}] \\ u^i(0)=\overline{u}^{i-1}(\widetilde{T}) \end{cases}$$

with F^i given by (2.41), where, for each $k\in\{0,\cdots,\ell\}$, $\overline{u}^k\in C([0,\widetilde{T}];L^2(O))$ is given by (P_5^k)-(2.15). Recall that by convention, $\overline{u}^{i-1}(\widetilde{T})=u_0$ if $i=0$ (see Remark 2.4). Problem $(\mathscr{P}^i)$ can be equivalently rewritten as follows:

$$(\mathscr{P}^i)\begin{cases} \dfrac{du^i}{dt}(t)+\nabla\mathscr{E}(u^i(t)) = G^i(t) & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,\widetilde{T}] \\ u^i(0)=\overline{u}^{i-1}(\widetilde{T}) \end{cases}$$

with $G^i:[0,\widetilde{T}]\to L^2(O)$ given by

$$G^i(t) := F^i(t)-\mathcal{K}*(\nabla\mathscr{G}\circ\overline{u}^i)(t). \tag{2.42}$$

We are going to prove that $\overline{u}^i$ is a solution of $(\mathscr{P}^i)$.

Step 2-1: Legendre-Fenchel transform of $(\mathscr{P}^i_\varepsilon)$. Fix any $\varepsilon>0$ and denote the Legendre-Fenchel conjugates of $\mathscr{E}_\varepsilon$ and $\mathscr{E}$ by $\mathscr{E}^*_\varepsilon$ and $\mathscr{E}^*$ respectively. Recalling that $\overline{u}^i_\varepsilon$ is a solution of $(\mathscr{P}^i_\varepsilon)$, from Fenchel's extremality relation (see Proposition A.4(b)), we see that $(\mathscr{P}^i_\varepsilon)$ is equivalent to

$$\begin{cases} \mathscr{E}_\varepsilon(\overline{u}^i_\varepsilon(t))+\mathscr{E}^*_\varepsilon(G^i_\varepsilon(t)-\dfrac{d\overline{u}^i_\varepsilon}{dt}(t))+\left\langle\dfrac{d\overline{u}^i_\varepsilon}{dt}(t)-G^i_\varepsilon(t),\overline{u}^i_\varepsilon(t)\right\rangle=0 & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,\widetilde{T}] \\ \overline{u}^i_\varepsilon(0)=\overline{u}^{i-1}_\varepsilon(\widetilde{T}) \end{cases}$$

with G^i_ε given by (2.14). Using Legendre-Fenchel's inequality (see Theorem A.2(b)), we obtain

$$(\mathscr{P}^i_\varepsilon)\Longleftrightarrow\begin{cases} \displaystyle\int_0^{\widetilde{T}}\left[\mathscr{E}_\varepsilon(\overline{u}^i_\varepsilon(t))+\mathscr{E}^*_\varepsilon(G^i_\varepsilon(t)-\frac{d\overline{u}^i_\varepsilon}{dt}(t))+\left\langle\frac{d\overline{u}^i_\varepsilon}{dt}(t)-G^i_\varepsilon(t),\overline{u}^i_\varepsilon(t)\right\rangle\right]dt=0 \\ \overline{u}^i_\varepsilon(0)=\overline{u}^{i-1}_\varepsilon(\widetilde{T}). \end{cases}$$

On the other hand, we have

$$\begin{aligned} \int_0^{\widetilde{T}}\left\langle\frac{d\overline{u}^i_\varepsilon}{dt}(t)-G^i_\varepsilon(t),\overline{u}^i_\varepsilon(t)\right\rangle dt &= \int_0^{\widetilde{T}}\left[\frac{d}{dt}(\frac{1}{2}\|\overline{u}^i_\varepsilon\|^2)(t)-\left\langle G^i_\varepsilon(t),\overline{u}^i_\varepsilon(t)\right\rangle\right]dt \\ &= \frac{1}{2}(\|\overline{u}^i_\varepsilon(\widetilde{T})\|^2-\|\overline{u}^i_\varepsilon(0)\|^2)-\int_0^{\widetilde{T}}\left\langle G^i_\varepsilon(t),\overline{u}^i_\varepsilon(t)\right\rangle dt. \end{aligned}$$

Hence, for every $\varepsilon > 0$,

$$(\mathscr{P}^i_\varepsilon)\Longleftrightarrow\begin{cases}\displaystyle\int_0^{\widetilde{T}}\left[\mathscr{E}_\varepsilon(\overline{u}^i_\varepsilon(t))+\mathscr{E}^*_\varepsilon(G^i_\varepsilon(t)-\frac{d\overline{u}^i_\varepsilon}{dt}(t))\right]dt+\frac{1}{2}(\|\overline{u}^i_\varepsilon(\widetilde{T})\|^2-\|\overline{u}^i_\varepsilon(0)\|^2)\\ \displaystyle-\int_0^{\widetilde{T}}\langle G^i_\varepsilon(t),\overline{u}^i_\varepsilon(t)\rangle dt=0\\ \overline{u}^i_\varepsilon(0)=\overline{u}^{i-1}_\varepsilon(\widetilde{T}).\end{cases}\tag{2.43}$$

Step 2-2: Passing to the limit. First of all, by $(\widehat{\mathrm{C}}^i_2)$, we have

$$\overline{u}^i_\varepsilon(0)\to\overline{u}^i(0)\text{ in }L^2(O).\tag{2.44}$$

On the other hand, $\overline{u}^i_\varepsilon(0)=\overline{u}^{i-1}_\varepsilon(\widetilde{T})$ and, by using (P^{i-1}_5)-(2.15) if $i\in\{1,\cdots,\ell\}$ and (C_2) if $i=0$,

$$\overline{u}^{i-1}_\varepsilon(\widetilde{T})\to\overline{u}^{i-1}(\widetilde{T})\text{ in }L^2(O).$$

Hence,

$$\overline{u}^i(0)=\overline{u}^{i-1}(\widetilde{T});\tag{2.45}$$

$$\lim_{\varepsilon\to0}\|\overline{u}^i_\varepsilon(0)\|^2_{L^2(O)}=\|\overline{u}^i(0)\|^2_{L^2(O)}.\tag{2.46}$$

In the same way, by (P^i_5)-(2.15), we have

$$\lim_{\varepsilon\to0}\|\overline{u}^i_\varepsilon(\widetilde{T})\|^2_{L^2(O)}=\|\overline{u}^i(\widetilde{T})\|^2_{L^2(O)}.\tag{2.47}$$

By (P^i_2), we can assert that (up to a subsequence)

$$\frac{d\overline{u}^i_\varepsilon}{dt}\rightharpoonup\frac{d\overline{u}^i}{dt}\text{ in }L^2([0,\widetilde{T}];L^2(O)).\tag{2.48}$$

Taking (2.14) and (2.42) into account, from $(\widehat{\mathrm{C}}^i_4)$ and (P^i_5)-(2.16), we infer that

$$G^i_\varepsilon\rightharpoonup G^i\text{ in }L^2([0,\widetilde{T}];L^2(O)).\tag{2.49}$$

From (C^i_1), (2.44) and (C_7), we have $\mathscr{G}(\overline{u}^i(0))\leqslant\underline{\lim}_{\varepsilon\to0}\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0))\leqslant\sup_{\varepsilon>0}\mathscr{G}_\varepsilon(\overline{u}^i_\varepsilon(0))<\infty$. Hence,

$$\overline{u}^i(0)\in\mathrm{dom}(\mathscr{G}).\tag{2.50}$$

Let $E,E^*:L^2([0,\widetilde{T}];L^2(O))\to[0,\infty]$ be defined by

$$\begin{cases}\displaystyle E(u):=\int_0^{\widetilde{T}}\mathscr{E}(u(t))dt\\ \displaystyle E^*(u):=\int_0^{\widetilde{T}}\mathscr{E}^*(u(t))dt\end{cases}$$

and, for each $\varepsilon > 0$, let $E_\varepsilon, E_\varepsilon^* : L^2([0,\widetilde{T}];L^2(O)) \to [0,\infty]$ be defined by

$$\begin{cases} E_\varepsilon(u) := \displaystyle\int_0^{\widetilde{T}} \mathscr{E}_\varepsilon(u(t))dt \\ E_\varepsilon^*(u) := \displaystyle\int_0^{\widetilde{T}} \mathscr{E}_\varepsilon^*(u(t))dt. \end{cases}$$

From (C$_6$) and Theorem B.4, we have $\mathscr{E}_\varepsilon^* \xrightarrow{M} \mathscr{E}^*$. Hence $E_\varepsilon \xrightarrow{M} E$ and $E_\varepsilon^* \xrightarrow{M} E^*$ by Theorem B.5. From (P$_5^i$)-(2.15), (2.48), and (2.49), it follows that

$$\begin{aligned} &\varliminf_{\varepsilon\to 0} E_\varepsilon(\overline{u}_\varepsilon^i) \geqslant E(\overline{u}^i), \text{ i.e.} \\ &\varliminf_{\varepsilon\to 0} \int_0^{\widetilde{T}} \mathscr{E}_\varepsilon(\overline{u}_\varepsilon^i(t))dt \geqslant \int_0^{\widetilde{T}} \mathscr{E}(\overline{u}^i(t))dt; \qquad (2.51) \\ &\varliminf_{\varepsilon\to 0} E_\varepsilon^*\left(G_\varepsilon^i - \frac{d\overline{u}_\varepsilon^i}{dt}\right) \geqslant E^*\left(G^i - \frac{d\overline{u}^i}{dt}\right), \text{ i.e.} \\ &\varliminf_{\varepsilon\to 0} \int_0^{\widetilde{T}} \mathscr{E}_\varepsilon^*\left(G_\varepsilon^i(t) - \frac{d\overline{u}_\varepsilon^i}{dt}(t)\right)dt \geqslant \int_0^{\widetilde{T}} \mathscr{E}^*\left(G^i(t) - \frac{d\overline{u}^i}{dt}(t)\right)dt. \qquad (2.52) \end{aligned}$$

Taking (2.44), (2.45), (2.46), (2.47), (2.50), (2.51), and (2.52) into account and letting $\varepsilon \to 0$ in (2.43), we obtain

$$\begin{cases} \displaystyle\int_0^{\widetilde{T}} \left[\mathscr{E}(\overline{u}^i(t)) + \mathscr{E}^*\left(G^i(t) - \frac{d\overline{u}^i}{dt}(t)\right)\right]dt + \frac{1}{2}(\|\overline{u}^i(\widetilde{T})\|^2 - \|\overline{u}^i(0)\|^2) \\ \quad - \displaystyle\int_0^{\widetilde{T}} \langle G^i(t), \overline{u}^i(t)\rangle dt \leqslant 0 \\ \overline{u}^i(0) = \overline{u}^{i-1}(\widetilde{T}) \in \mathrm{dom}(\mathscr{G}), \end{cases}$$

i.e.,

$$\begin{cases} \displaystyle\int_0^{\widetilde{T}} \left[\mathscr{E}(\overline{u}^i(t)) + \mathscr{E}^*\left(G^i(t) - \frac{d\overline{u}^i}{dt}(t)\right) + \left\langle \frac{d\overline{u}^i}{dt}(t) - G^i(t), \overline{u}^i(t)\right\rangle\right]dt \leqslant 0 \\ \overline{u}^i(0) = \overline{u}^{i-1}(\widetilde{T}) \in \mathrm{dom}(\mathscr{G}). \end{cases}$$

But, by using again Legendre-Fenchel's inequality (see Theorem A.2(b)), we have

$$\mathscr{E}(\overline{u}^i(t)) + \mathscr{E}^*\left(G^i(t) - \frac{d\overline{u}^i}{dt}(t)\right) + \left\langle \frac{d\overline{u}^i}{dt}(t) - G^i(t), \overline{u}^i(t)\right\rangle \geqslant 0 \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,\widetilde{T}],$$

hence

$$\begin{cases} \displaystyle\int_0^{\widetilde{T}} \left[\mathscr{E}(\overline{u}^i(t)) + \mathscr{E}^*\left(G^i(t) - \frac{d\overline{u}^i}{dt}(t) \right) + \left\langle \frac{d\overline{u}^i}{dt}(t) - G^i(t), \overline{u}^i(t) \right\rangle \right] dt = 0 \\ \overline{u}^i(0) = \overline{u}^{i-1}(\widetilde{T}) \in \mathrm{dom}(\mathscr{G}). \end{cases} \tag{2.53}$$

Using again Fenchel's extremality relation (see Proposition A.4(b)), we see that (2.53) is equivalent to

$$\begin{cases} \displaystyle\frac{d\overline{u}^i}{dt}(t) + \nabla\mathscr{E}(\overline{u}^i(t)) = G^i(t) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, \widetilde{T}] \\ \overline{u}^i(0) = \overline{u}^{i-1}(\widetilde{T}) \in \mathrm{dom}(\mathscr{G}), \end{cases}$$

which shows that $\overline{u}^i$ is a solution of $(\mathscr{P}^i)$.

Step 3: Existence of a solution of $(\mathscr{P})$**.** Let $\overline{u} : [0,T] \to L^2(O)$ be defined by

$$\overline{u}(t) := \overline{u}^i(t - T_i) \text{ if } t \in [T_i, T_{i+1}] \text{ with } i \in \{0, \cdots, \ell\}.$$

For each $i \in \{1, \cdots, \ell\}$, as $\overline{u}^i$ is a solution of $(\mathscr{P}^i)$ we have $\overline{u}^i(0) = \overline{u}^{i-1}(\widetilde{T})$, i.e. $\overline{u}^i(T_i - T_i) = \overline{u}^{i-1}(T_i - T_{i-1})$, which means that $\overline{u}(T_i^-) = \overline{u}(T_i^+)$. Since $\overline{u}^i \in C([0,\widetilde{T}]; L^2(O))$ for all $i \in \{1, \cdots, \ell\}$, it follows that $\overline{u} \in C([0,T]; L^2(O))$.

On the other hand, we have $\overline{u}(0) = \overline{u}^0(0)$, and $\overline{u}^0(0) = u_0$ because $\overline{u}^0$ is solution of $(\mathscr{P}^0)$. Moreover, for every $i \in \{0, \cdots, \ell\}$, taking (2.42) and (2.41) into account, as $\overline{u}^i$ is a solution of $(\mathscr{P}^i)$, for $\mathscr{L}^1$-a.e. $t \in [T_i, T_{i+1}]$, we have

$$\begin{aligned} \frac{d\overline{u}}{dt}(t) + \nabla\mathscr{E}(\overline{u}(t)) &= \frac{d\overline{u}^i}{dt}(t - T_i) + \nabla\mathscr{E}(\overline{u}^i(t - T_i)) \\ &= G^i(t - T_i) \\ &= F^i(t - T_i) - \mathcal{K} * (\nabla\mathscr{G} \circ \overline{u}^i)(t - T_i) \\ &= F(t) - \sum_{k=0}^{i-1} \int_{T_k}^{T_{k+1}} \mathcal{K}(t-s)\nabla\mathscr{G}(\overline{u}^k(s - T_k))ds - \int_0^{t-T_i} \mathcal{K}(t - T_i - s)\nabla\mathscr{G}(\overline{u}^i(s))ds \\ &= F(t) - \sum_{k=0}^{i-1} \int_{T_k}^{T_{k+1}} \mathcal{K}(t-s)\nabla\mathscr{G}(\overline{u}(s))ds - \int_{T_i}^{t} \mathcal{K}(t-s)\nabla\mathscr{G}(\overline{u}(s))ds \\ &= F(t) - \int_0^t \mathcal{K}(t-s)\nabla\mathscr{G}(\overline{u}(s))ds \\ &= F(t) - \mathcal{K} * (\nabla\mathscr{G} \circ \overline{u})(t). \end{aligned}$$

Consequently $\overline{u}$ is a solution of $(\mathscr{P})$, and the proof is complete. ■

3. Stochastic Homogenization for Integrodifferential Nonlocal Diffusion Problems of Gradient Flow Type

3.1. **Random nonlocal integrodifferential diffusion problems of gradient flow type.** From now on we consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a family $\{T_z\}_{z\in\mathbb{Z}^d}$ satisfying the following three properties:

- (mesurability) $T_z : \Omega \to \Omega$ is $\mathscr{F}$-measurable for all $z \in \mathbb{Z}^d$;
- (group property) $T_z \circ T_{z'} = T_{z+z'}$ and $T_{-z} = T_z^{-1}$ for all $z, z' \in \mathbb{Z}^d$;
- (mass invariance) $\mathbb{P}(T_z A) = \mathbb{P}(A)$ for all $A \in \mathscr{F}$ and all $z \in \mathbb{Z}^d$.

Definition 3.1. The family $\{T_z\}_{z\in\mathbb{Z}^d}$ is said to be a (discrete) group of $\mathbb{P}$-preserving transformation on $(\Omega, \mathscr{F}, \mathbb{P})$ and the quadruplet $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is called a (discrete) dynamical system.

Let $\mathscr{I} := \{A \in \mathscr{F} : \mathbb{P}(T_z A \Delta A) = 0 \text{ for all } z \in \mathbb{Z}^d\}$ be the σ-algebra of invariant sets with respect to $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$.

Definition 3.2. When $\mathbb{P}(A) \in \{0,1\}$ for all $A \in \mathscr{I}$, the measurable dynamical system

$$(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$$

is said to be ergodic.

Remark 3.3. A sufficient condition to ensure the ergodicity of $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is the so-called mixing condition, i.e. for every $(E,F) \in \mathscr{F} \times \mathscr{F}$,

$$\lim_{|z|\to\infty} \mathbb{P}(T_z E \cap F) = \mathbb{P}(E)\mathbb{P}(F).$$

For each $X \in L^1_{\mathbb{P}}(\Omega)$, $\mathbb{E}^{\mathscr{I}}(X)$ denotes the conditional mathematical expectation of X with respect to $\mathscr{I}$, i.e. the unique $(\mathscr{I}, \mathscr{B}(\overline{\mathbb{R}}))$-measurable function in $L^1_{\mathbb{P}}(\Omega)$ such that for every $E \in \mathscr{I}$,

$$\int_E \mathbb{E}^{\mathscr{I}}(X)(\omega)d\mathbb{P}(\omega) = \int_E X(\omega)d\mathbb{P}(\omega).$$

Remark 3.4. If $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is ergodic then $\mathbb{E}^{\mathscr{I}}(X)$ is constant and equal to the mathematical expectation $\mathbb{E}(X)$ of X, i.e. $\mathbb{E}^{\mathscr{I}}(X) = \mathbb{E}(X) := \int_\Omega X(\omega)d\mathbb{P}(\omega)$.

Let $J : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty[$ be a $(\mathscr{F} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$-measurable satisfying the following conditions:

(NL_1) J is symmetric, i.e. for every $(\omega, x, y, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$J(\omega, x, y, \xi) = J(\omega, y, x, \xi),$$

and J is bi-stationary with respect to $(T_z)_{z\in\mathbb{Z}^d}$, i.e. for every $z \in \mathbb{Z}^d$ and every $(\omega, x, y, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$J(\omega, x+z, y+z, \xi) = J(T_z\omega, x, y, \xi);$$

(NL_2) there exist $\underline{J} : \mathbb{R}^d \to [0,\infty[$ and $\overline{J} \in L^\infty(\mathbb{R}^d; [0,\infty[)$ with

$$\begin{cases} \underline{J} \not\equiv 0 \\ \text{for every } (\xi,\zeta) \in \mathbb{R}^d \times \mathbb{R}^d, \text{ if } |\xi| \leqslant |\zeta| \text{ then } \underline{J}(\xi) \geqslant \underline{J}(\zeta) \\ \operatorname{supp}(\overline{J}) = \overline{B}_{R_J}(0) \text{ is compact with } R_J > 0, \end{cases} \tag{3.1}$$

such that, for every $(\omega,x,y,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\underline{J}(\xi)\leqslant J(\omega,x,y,\xi)\leqslant\overline{J}(\xi).$$

Let $O\subset\mathbb{R}^d$ be an open set and. For each $\varepsilon>0$, define $\mathscr{J}_\varepsilon:\Omega\times L^2(O)\to[0,\infty[$ by

$$\mathscr{J}_\varepsilon(\omega,u):=\frac{1}{4\varepsilon^d}\int_O\int_O J\left(\omega,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{x-y}{\varepsilon}\right)\left(\frac{u(x)-u(y)}{\varepsilon}\right)^2dxdy \tag{3.2}$$

and let $F_\varepsilon:\Omega\times L^2(O)\to L^2(O)$ be such that, for every $\omega\in\Omega$, $F_\varepsilon(\omega,\cdot)\in\mathscr{F}_{(\mathrm{R})}$. For each $\omega\in\Omega$ and each $\varepsilon>0$, let us consider the following nonlocal integrodifferential diffusion problem:

$$(\mathscr{P}_\varepsilon^\omega)\begin{cases}\dfrac{du_\varepsilon^\omega}{dt}(t)+\nabla\mathscr{J}_\varepsilon(\omega,u_\varepsilon^\omega(t))+\mathcal{K}*(\nabla\mathscr{J}_\varepsilon(\omega,\cdot)\circ u_\varepsilon^\omega)(t)=F_\varepsilon(\omega,t) & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,T]\\ u_\varepsilon^\omega(0)=u_{0,\varepsilon}^\omega\in L^2(O)\end{cases}$$

with $\mathcal{K}\in C^1([0,T];[0,\infty[)$. The following result is a consequence of Theorem 2.1.

Corollary 3.5. *For each $\omega\in\Omega$ and each $\varepsilon>0$, there exists $\overline{u}_\varepsilon^\omega\in C([0,T];L^2(O))$ such that*:

- *$\overline{u}_\varepsilon^\omega$ is the unique solution of $(\mathscr{P}_\varepsilon^\omega)$*;
- $\nabla\mathscr{J}_\varepsilon(\omega,\cdot)\circ\overline{u}_\varepsilon^\omega\in L^2([0,T];L^2(O))$;
- $\frac{d\overline{u}_\varepsilon^\omega}{dt}\in L^2([0,T];L^2(O))$;
- *$\overline{u}_\varepsilon^\omega$ admits a right derivative $\frac{d^+\overline{u}_\varepsilon^\omega}{dt}(t)$ at every $t\in[0,T[$ which satisfies*

$$\frac{d^+\overline{u}_\varepsilon^\omega}{dt}(t)+\nabla\mathscr{J}_\varepsilon(\overline{u}_\varepsilon^\omega(t))+\mathcal{K}*(\nabla\mathscr{J}_\varepsilon(\omega,\cdot)\circ\overline{u}_\varepsilon^\omega)(t)=F_\varepsilon(\omega,t).$$

Proof of Corollary 3.5. It suffices to apply Theorem 2.1 with $F=F_\varepsilon(\omega,\cdot)$ and $\mathscr{E}=\mathscr{G}=\mathscr{J}_\varepsilon(\omega,\cdot)$, where $\omega\in\Omega$ and $\varepsilon>0$. As $F_\varepsilon(\omega,\cdot)\in\mathscr{F}_{(\mathrm{R})}$ we only need to prove that $\mathscr{J}_\varepsilon(\omega,\cdot)$ satisfies (D_1)–(D_2) with $\mathscr{G}=\mathscr{J}_\varepsilon(\omega,\cdot)$. From (3.2) we see that for every $u\in L^2(O)$,

$$\nabla\mathscr{J}_\varepsilon(\omega,u)(x)=-\frac{1}{\varepsilon^{d+2}}\int_O J\left(\omega,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{x-y}{\varepsilon}\right)(u(y)-u(x))dy. \tag{3.3}$$

Hence $\nabla\mathscr{J}_\varepsilon(\omega,0)=0$, i.e. (D_1) holds with $\mathscr{G}=\mathscr{J}_\varepsilon(\omega,\cdot)$. On the other hand, taking (3.3) and (NL_2) into account, for every $u,v\in L^2(O)$, we have

$$\begin{aligned}\|\nabla\mathscr{J}_\varepsilon(\omega,u)-\nabla\mathscr{J}_\varepsilon(\omega,v)\|_{L^2(O)}^2&=\int_O|\nabla\mathscr{J}_\varepsilon(\omega,u)(x)-\nabla\mathscr{J}_\varepsilon(\omega,v)(x)|^2dx\\&\leqslant C_\varepsilon\int_O\left[\int_O\left(|v(y)-u(y)|^2+|u(x)-v(x)|^2\right)dy\right]dx\\&=C_\varepsilon\mathscr{L}^d(O)\left(\int_O|v(y)-u(y)|^2dy+\int_O|u(x)-v(x)|^2dx\right)\\&=2C_\varepsilon\mathscr{L}^d(O)\|u-v\|_{L^2(O)}^2\end{aligned}$$

with $C_\varepsilon:=\frac{2}{\varepsilon^{d+2}}\mathscr{L}^d(O)\|\overline{J}\|_{L^\infty(\mathbb{R}^d)}^2$, which proves (D_2) with $\mathscr{G}=\mathscr{J}_\varepsilon(\omega,\cdot)$, and the proof is complete. ■

3.2. **Stochastic homogenization theorem.** For each $\theta \in \mathbb{R}^d$, each $R > 0$ and each $A \in \mathscr{B}_{\rm b}(\mathbb{R}^d)$, set

$$L^2_{{\rm loc},\theta,R,A}(\mathbb{R}^d) := \Big\{u \in L^2_{\rm loc}(\mathbb{R}^d) : u = \ell_\theta \text{ in } \partial_R(A)\Big\}, \tag{3.4}$$

where $\ell_\theta : \mathbb{R}^d \to \mathbb{R}$ is the linear map defined by $\ell_\theta(x) = \theta x$ and $\partial_R(A)$ denotes the R-neighborhood of the boundary ∂A of A, i.e.

$$\partial_R(A) := \Big\{x \in \mathbb{R}^d : {\rm dist}(x, \partial A) < R\Big\}. \tag{3.5}$$

Let $\mathscr{S} : \mathscr{B}_{\rm b}(\mathbb{R}^d) \times \Omega \times \mathbb{R}^d \to [0,\infty[$ be defined by

$$\mathscr{S}_A(\omega,\theta) := \inf\Big\{\mathscr{J}(\omega,u,\mathbb{R}^d,A) : u \in L^2_{{\rm loc},\theta,R_J,A}(\mathbb{R}^d)\Big\},$$

where $R_J > 0$ is given by (NL$_2$) and $\mathscr{J} : \Omega \times L^2_{\rm loc}(\mathbb{R}^d) \times \mathscr{B}_{\rm b}(\mathbb{R}^d) \times \mathscr{B}_{\rm b}(\mathbb{R}^d) \to [0,\infty[$ is defined by

$$\mathscr{J}(\omega,u,A,B) := \frac{1}{4}\int_A\int_B J(\omega,x,y,x-y)(u(x)-u(y))^2 dxdy.$$

Let $f_{\rm hom} : \Omega \times \mathbb{R}^d \to [0,\infty[$ be defined by

$$f_{\rm hom}(\omega,\theta) := \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathscr{I}}\left(\frac{\mathscr{S}_{[0,k[^d}(\cdot,\theta)}{k^d}\right)(\omega).$$

Remark 3.6. It is easy to see that $f_{\rm hom}(\omega,\cdot)$ is quadratic, i.e. there exists a symmetric $d \times d$ matrix $A^\omega_{\rm hom}$ such that for every $\theta \in \mathbb{R}^d$,

$$f_{\rm hom}(\omega,\theta) = \frac{1}{2}\langle A^\omega_{\rm hom}\theta, \theta\rangle, \tag{3.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$ (see [7, Propositions 3.14 and 3.17] for more details on the definition of $f_{\rm hom}$).

Let $\mathscr{J}_{\rm hom} : \Omega \times L^2(O) \to [0,\infty]$ be defined by

$$\mathscr{J}_{\rm hom}(\omega,u) := \begin{cases} \displaystyle\int_O f_{\rm hom}(\omega,\nabla u(x))dx & \text{if } u \in H^1(O) \\ \infty & \text{if } u \in L^2(O)\backslash H^1(O). \end{cases}$$

In [7, Lemma 4.2 and Theorem 4.8], we proved the following Mosco-convergence result.

Theorem 3.7. *There exists $\Omega' \in \mathscr{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$, we have:*

(a) *for every $\{v_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$, if $\sup_{\varepsilon>0}\mathscr{J}_\varepsilon(\omega,v_\varepsilon) < \infty$ then $\{v_\varepsilon\}_{\varepsilon>0}$ is relatively compact in $L^2(O)$;*

(b) *$\{\mathscr{J}_\varepsilon(\omega,\cdot)\}_{\varepsilon>0}$ Mosco-converges to $\mathscr{J}_{\rm hom}(\omega,\cdot)$.*

Remark 3.8. By Remark 3.6, $\mathscr{J}_{\rm hom}(\omega,\cdot)$ is proper, convex and lower semicontinuous, and Fréchet-differentiable on ${\rm dom}(\partial\mathscr{J}_{\rm hom}(\omega,\cdot))$.

Let $F : [0,T] \times \Omega \to L^2(O)$ be a Borel measurable map. As a consequence of Theorem 2.2, we obtain the following stochastic homogenization result.

Corollary 3.9. *For $\mathbb{P}$-a.e. $\omega\in\Omega$ and every $\varepsilon>0$, let $\overline{u}_\varepsilon^\omega\in AC([0,T];L^2(O))$ be the unique solution of $(\mathscr{P}_\varepsilon^\omega)$, see Corollary 3.5, and assume that*:

- (H_1^ω) $\sup\limits_{\varepsilon>0}\mathscr{J}_\varepsilon(\omega,u_{0,\varepsilon}^\omega)<\infty$;
- (H_2^ω) $u_{0,\varepsilon}^\omega\to u_0^\omega$ *in* $L^2(O)$;
- (H_3^ω) $\sup\limits_{\varepsilon>0}\|F_\varepsilon(\omega,0)\|_{L^2(O)}<\infty$;
- (H_4^ω) $F_\varepsilon(\omega,\cdot)\rightharpoonup F(\omega,\cdot)$ *in* $L^2([0,T];L^2(O))$;
- (H_5^ω) $\sup\limits_{\varepsilon>0}\left\|\frac{dF_\varepsilon(\omega,\cdot)}{dt}\right\|_{L^1([0,T];L^2(O))}<\infty$.

Then, there exists $\widehat{\Omega}\in\mathscr{F}$ with $\mathbb{P}(\widehat{\Omega})=1$ such that for every $\omega\in\widehat{\Omega}$ there exists $\overline{u}^\omega\in C([0,T];L^2(O))$ such that (up to a subsequence)

$$\overline{u}_\varepsilon^\omega\to\overline{u}^\omega \text{ in } C([0,T];L^2(O))$$

and $\overline{u}^\omega$ is a solution of the following local integrodifferential diffusion problem of gradient flow type:

$$(\mathscr{P}_\omega^{\mathrm{hom}})\begin{cases}\dfrac{du^\omega}{dt}(t)+\nabla\mathscr{J}_{\mathrm{hom}}(\omega,u^\omega(t))+\mathcal{K}*(\nabla\mathscr{J}_{\mathrm{hom}}(\omega,\cdot)\circ u)(t)=F(\omega,t) \text{ for } \mathscr{L}^1\text{-a.a. } t\in[0,T]\\ u^\omega(0)=u_0^\omega\in\mathrm{dom}(\mathscr{J}_{\mathrm{hom}}(\omega,\cdot)).\end{cases}$$

Proof of Corollary 3.9. Let $\Omega''\in\mathscr{F}$ be such that $\mathbb{P}(\Omega'')=1$ and (H_1^ω)–(H_5^ω) (in Corollary 3.9) hold. Set $\widehat{\Omega}=\Omega'\cap\Omega''$ where $\Omega'\in\mathscr{F}$, with $\mathbb{P}(\Omega')=1$, is given by Theorem 3.7. Then $\widehat{\Omega}\in\mathscr{F}$ and $\mathbb{P}(\widehat{\Omega})=1$. Fix $\omega\in\widehat{\Omega}$. We are going to apply Theorem 2.2.
Firstly, it is easy to see that (C_1)–(C_5) hold with $\mathscr{G}_\varepsilon=\mathscr{J}_\varepsilon(\omega,\cdot)$, $u_{0,\varepsilon}=u_{0,\varepsilon}^\omega$, $u_0=u_0^\omega$, $\overline{u}_\varepsilon=\overline{u}_\varepsilon^\omega$, $F_\varepsilon=F_\varepsilon(\omega,\cdot)$ and $F=F(\omega,\cdot)$. Secondly, by Theorem 3.7(b), (C_6)–(C_7) are satisfied with $\mathscr{E}_\varepsilon=\mathscr{G}_\varepsilon=\mathscr{J}_\varepsilon(\omega,\cdot)$ and $\mathscr{E}=\mathscr{G}=\mathscr{J}_{\mathrm{hom}}(\omega,\cdot)$. Thirdly, by Theorem 3.7(a), (C_8) is verified with $\mathscr{G}_\varepsilon=\mathscr{J}_\varepsilon(\omega,\cdot)$, and the proof is complete. ■

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APPENDIX A. ELEMENTS OF LEGENDRE-FENCHEL CALCULUS

Let X be a normed space and let X^* be its topological dual. In what follows, for any $u \in X$ and any $u^* \in X^*$, we write $u^*(u) = \langle u^*, u\rangle$. We begin with the following definition.

Definition A.1. Let $\Phi : X \to]-\infty, \infty]$ be a proper[3] function. The Legendre-Fenchel conjugate (or the conjugate) of Φ is the function $\Phi^* : X^* \to]-\infty, \infty]$ defined by

$$\Phi^*(u^*) := \sup\left\{\langle u^*, u\rangle - \Phi(u) : u \in X\right\}.$$

(As Φ is proper and $\Phi > -\infty$ we have $\Phi^* > -\infty$.) The Legendre-Fenchel biconjugate (or the biconjugate) of Φ is the function $\Phi^{**} : X \to [-\infty, \infty]$ defined by

$$\Phi^{**}(u) := \sup\left\{\langle u^*, u\rangle - \Phi^*(u^*) : u^* \in X^*\right\}.$$

(Since $\Phi^* > -\infty$, $u^* \in \mathrm{dom}(\Phi^*)$ if and only if there exists $\alpha \in \mathbb{R}$ such that $\Phi^*(u^*) \leqslant \alpha$, i.e. $\Phi(u) \geqslant \langle u^*, u\rangle - \alpha$ for all $u \in X$. Hence, if Φ admits a continuous affine minorant function[4] then Φ^* is proper and $\Phi^{**} > -\infty$.) The following theorem gives the main properties of the Legendre-Fenchel conjugate and biconjugate (see [9, §9.3, p. 343] for more details).

Theorem A.2. *Let $\Phi : X \to]-\infty, \infty]$ be a proper function.*

(a) *If Φ is convex and lower semicontinuous then Φ^* is proper, convex and lower semicontinuous.*

(b) *(Legendre-Fenchel's inequality.) For every $u \in X$ and every $u^* \in X^*$,*

$$\Phi(u) + \Phi^*(u^*) - \langle u^*, u\rangle \geqslant 0.$$

(c) *(Fenchel-Moreau-Rockafellar's theorem.) If Φ is convex and lower semicontinuous then*

$$\Phi^{**} = \Phi.$$

(d) *If Φ is convex and admits a continuous affine minorant function then*

$$\Phi^{**} = \overline{\Phi},$$

where $\overline{\Phi}$ denotes the lower semicontinuous envelope of Φ.

Here is the definition of the subdifferential of a function.

Definition A.3. Let $\Phi : X \to]-\infty, \infty]$ be a proper function. The subdifferential of Φ is the multi-valued operator $\partial\Phi : X \rightrightarrows X^*$ defined by

$$\partial\Phi(u) := \left\{u^* \in X^* : \Phi(v) \geqslant \Phi(u) + \langle u^*, v - u\rangle \text{ for all } v \in X\right\}.$$

(Note that $\mathrm{dom}(\Phi) \supset \mathrm{dom}(\partial\Phi) := \left\{u \in X : \partial\Phi(u) \neq \varnothing\right\}$.)

For the subdifferentials of convex functions we have the following result (see [9, §9.5, pp. 355 and Lemma 17.4.1, pp. 737] for more details).

Proposition A.4. *Let $\Phi : X \to]-\infty, \infty]$ be a proper and convex function.*

[3] We say that $\Phi : X \to]-\infty, \infty]$ is proper if (its effective domain) $\mathrm{dom}(\Phi) := \{u \in X : \Phi(u) < \infty\} \neq \varnothing$.

[4] This is true if $\Phi : X \to]-\infty, \infty]$ is a proper, convex and lower semicontinuous function, because Φ is then equal to the supremum of all its continuous affine minorant functions.

(a) *If* Φ *is Fréchet-differentiable at* $u \in X$, *then*

$$\partial\Phi(u) = \{\nabla\Phi(u)\}.$$

(b) *(Fenchel's extremality relation.) If* Φ *is lower semicontinuous, then*

$$u^* \in \partial\Phi(u) \iff \Phi(u) + \Phi^*(u^*) - \langle u^*, u\rangle = 0.$$

(c) *(Brønsted-Rockafellar's lemma) If* Φ *is lower semicontinuous, then*

$$\overline{\mathrm{dom}(\partial\Phi)} = \overline{\mathrm{dom}(\Phi)}.$$

Appendix B. Mosco-convergence

Let X be a Banach space and let X^* be its topological dual. In what follows, "$\to$" (resp. "$\rightharpoonup$") denotes the strong (resp. the weak) convergence. We begin with the definition of De Giorgi Γ-convergence (see [10, 11, 14] for more details).

Definition B.1. Let $\Phi : X \to]-\infty,\infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty,\infty]$. We say that $\{\Phi_\varepsilon\}_{\varepsilon>0}$ strongly Γ-converges (resp. weakly Γ-converges) to Φ, and we write

$$\Phi = \Gamma_{\mathrm{s}}\text{-}\lim_{\varepsilon\to 0}\Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\Gamma_{\mathrm{s}}} \Phi \quad (\text{resp. } \Phi = \Gamma_{\mathrm{w}}\text{-}\lim_{\varepsilon\to 0}\Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\Gamma_{\mathrm{w}}} \Phi),$$

if the following two assertions hold:

- for every $u \in X$, $\Gamma_{\mathrm{s}}\text{-}\varliminf_{\varepsilon\to 0}\Phi_\varepsilon(u) \geqslant \Phi(u)$(resp. $\Gamma_{\mathrm{w}}\text{-}\varliminf_{\varepsilon\to 0}\Phi_\varepsilon(u) \geqslant \Phi(u)$) with

$$\Gamma_{\mathrm{s}}\text{-}\varliminf_{\varepsilon\to 0}\Phi_\varepsilon(u) := \inf\left\{\varliminf_{\varepsilon\to 0}\Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\right\}$$

$$(\text{resp. } \Gamma_{\mathrm{w}}\text{-}\varliminf_{\varepsilon\to 0}\Phi_\varepsilon(u) := \inf\left\{\varliminf_{\varepsilon\to 0}\Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \rightharpoonup u\right\})$$

  or equivalently, for every $u \in X$ and every $\{u_\varepsilon\}_{\varepsilon>0} \subset X$, if $u_\varepsilon \to u$ (resp. $u_\varepsilon \rightharpoonup u$) then

$$\varliminf_{\varepsilon\to 0}\Phi_\varepsilon(u_\varepsilon) \geqslant \Phi(u);$$

- for every $u \in X$, $\Gamma_{\mathrm{s}}\text{-}\varlimsup_{\varepsilon\to 0}\Phi_\varepsilon(u) \leqslant \Phi(u)$(resp. $\Gamma_{\mathrm{w}}\text{-}\varlimsup_{\varepsilon\to 0}\Phi_\varepsilon(u) \leqslant \Phi(u)$) with

$$\Gamma_{\mathrm{s}}\text{-}\varlimsup_{\varepsilon\to 0}\Phi_\varepsilon(u) := \inf\left\{\varlimsup_{\varepsilon\to 0}\Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\right\}$$

$$(\text{resp. } \Gamma_{\mathrm{w}}\text{-}\varlimsup_{\varepsilon\to 0}\Phi_\varepsilon(u) := \inf\left\{\varlimsup_{\varepsilon\to 0}\Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \rightharpoonup u\right\})$$

  or equivalently, for every $u \in X$ there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset X$ such that $u_\varepsilon \to u$ (resp. $u_\varepsilon \rightharpoonup u$) and

$$\varlimsup_{\varepsilon\to 0}\Phi_\varepsilon(u_\varepsilon) \leqslant \Phi(u).$$

From Γ-convergence we can define Mosco-convergence (which was introduced by Mosco, see [15]).

Definition B.2. Let $\Phi : X \to]-\infty,\infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty,\infty]$. We say that $\{\Phi_\varepsilon\}_{\varepsilon>0}$ Mosco-converges to Φ, and we write

$$\Phi = \mathrm{M}\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\mathrm{M}} \Phi,$$

if $\Phi = \Gamma_{\mathrm{s}}\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon = \Gamma_{\mathrm{w}}\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon$ or equivalently $\Gamma_{\mathrm{s}}\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon \leqslant \Phi \leqslant \Gamma_{\mathrm{w}}\text{-}\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon$.

From Definition B.2, it is easy to see that under a suitable compactness condition strong Γ-convergence is equivalent to Mosco-convergence.

Proposition B.3. *Let $\Phi : X \to]-\infty,\infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty,\infty]$. Assume that the following compactness condition hold*:

- *for all $\{u_\varepsilon\}_{\varepsilon>0} \subset X$, if $\sup_{\varepsilon>0} \Phi_\varepsilon(u_\varepsilon) < \infty$ then $\{u_\varepsilon\}_{\varepsilon>0}$ is strongly relatively compact in X.*

Then, $\Phi_\varepsilon \xrightarrow{\Gamma_{\mathrm{s}}} \Phi$ if and only if $\Phi_\varepsilon \xrightarrow{\mathrm{M}} \Phi$.

As stated in the following theorem due to Mosco (see [15, Theorem 1]), in the reflexive case and for lower semicontinuous, convex and proper functions, the Legendre-Fenchel transform is continuous with respect to Mosco-convergence.

Theorem B.4. *Let $\Phi : X \to]-\infty,\infty]$ be a proper, convex and lower semicontinuous function and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty,\infty]$ be a proper, convex and lower semicontinuous function. If X is reflexive then $\Phi_\varepsilon \xrightarrow{\mathrm{M}} \Phi$ if and only if $\Phi^*_\varepsilon \xrightarrow{\mathrm{M}} \Phi^*$.*

The following result allows to pass from Mosco-convergence in X to Mosco-convergence in $L^2([0,T]; X)$ (see [5, Lemma 2.6, p. 50] for a proof).

Theorem B.5. *Fix $T > 0$ and assume that X is a Hilbert space. Let $\Phi : X \to [0,\infty]$ be a proper, convex and lower semicontinuous function, let $\Theta : L^2([0,T];X) \to [0,\infty]$ be defined by*

$$\Theta(u) := \int_0^T \Phi(u(t))dt$$

and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to [0,\infty]$ be a lower semicontinuous, proper and convex function and let $\Theta_\varepsilon : L^2([0,T];X) \to [0,\infty]$ be defined by

$$\Theta_\varepsilon(u) := \int_0^T \Phi_\varepsilon(u(t))dt.$$

If $\Phi_\varepsilon \xrightarrow{\mathrm{M}} \Phi$ then $\Theta_\varepsilon \xrightarrow{\mathrm{M}} \Theta$.

Appendix C. Graph-convergence

Let $(X,\mathscr{T})$ and $(X^*,\mathscr{T}^*)$ be two topological spaces. We begin with the definition of Kuratowski-Painlevé convergence.

Definition C.1. We say that $\{E_\varepsilon\}_{\varepsilon>0} \subset \mathscr{P}(X \times X^*)$ converges to $E \in \mathscr{P}(X \times X^*)$ in the sense of Kuratowski-Painlevé with respect to the product topology $\mathscr{T} \times \mathscr{T}^*$, and we write

$$E = \text{KP-}\lim_{\varepsilon\to 0} E_\varepsilon \text{ or } E_\varepsilon \xrightarrow{\text{KP}} E,$$

if the following two assertions hold:

- $E \subset \text{KP-}\varliminf_{\varepsilon\to 0} E_\varepsilon$ with

$$\text{KP-}\varliminf_{\varepsilon\to 0} E_\varepsilon := \left\{(u,u^*) \in X \times X^* : \exists (u_\varepsilon, u_\varepsilon^*)_{\varepsilon>0} \in \prod_{\varepsilon>0} E_\varepsilon \text{ s.t. } u_\varepsilon \xrightarrow{\mathscr{T}} u \text{ and } u_\varepsilon^* \xrightarrow{\mathscr{T}^*} u^*\right\}$$

  or equivalently, for all $(u,u^*) \in E$ there exists $(u_\varepsilon, u_\varepsilon^*)_{\varepsilon>0} \in \prod_{\varepsilon>0} E_\varepsilon$ such that $u_\varepsilon \xrightarrow{\mathscr{T}} u$ and $u_\varepsilon^* \xrightarrow{\mathscr{T}^*} u^*$.
- $\text{KP-}\varlimsup_{\varepsilon\to 0} E_\varepsilon \subset E$ with

$$\text{KP-}\varlimsup_{\varepsilon\to 0} E_\varepsilon := \left\{(u,u^*) \in X \times X^* : \exists (u_{\sigma(\varepsilon)}, u_{\sigma(\varepsilon)}^*)_{\varepsilon>0} \in \prod_{\varepsilon>0} E_{\sigma(\varepsilon)} \text{ s.t. } u_{\sigma(\varepsilon)} \xrightarrow{\mathscr{T}} u \text{ and } u_{\sigma(\varepsilon)}^* \xrightarrow{\mathscr{T}^*} u^*\right\}$$

  or equivalently, for all $(u,u^*) \in X \times X^*$ and all subsequence $(u_{\sigma(\varepsilon)}, u_{\sigma(\varepsilon)}^*)_{\varepsilon>0} \in \prod_{\varepsilon>0} E_{\sigma(\varepsilon)}$, if $u_{\sigma(\varepsilon)} \xrightarrow{\mathscr{T}} u$ and $u_{\sigma(\varepsilon)}^* \xrightarrow{\mathscr{T}^*} u^*$ then $(u,u^*) \in E$.

Remark C.2. It is clear that the sets $\text{KP-}\varliminf_{\varepsilon\to 0} E_\varepsilon$ and $\text{KP-}\varlimsup_{\varepsilon\to 0} E_\varepsilon$ are closed subsets of $X \times X^*$ and $\text{KP-}\varliminf_{\varepsilon\to 0} E_\varepsilon \subset \text{KP-}\varlimsup_{\varepsilon\to 0} E_\varepsilon$. Hence $E_\varepsilon \xrightarrow{\text{KP}} E$ if and only if $\text{KP-}\varliminf_{\varepsilon\to 0} E_\varepsilon = \text{KP-}\varlimsup_{\varepsilon\to 0} E_\varepsilon = E$. In particular, the KP-limit of a set is its closure.

From now on, X is a Banach space and X^* is its topological dual. Here is the definition of graph-convergence for sequences of subdifferentials.

Definition C.3. Let $\partial\Phi : X \rightrightarrows X^*$ and, for each $\varepsilon > 0$, let $\partial\Phi_\varepsilon : X \rightrightarrows X^*$, where $\Phi, \Phi_\varepsilon : X \to]-\infty, \infty]$ are proper functions. We say that $\{\partial\Phi_\varepsilon\}_{\varepsilon>0}$ graph-converges to $\partial\Phi$ with respect to the product topology $\mathscr{T} \times \mathscr{T}^*$, and we write

$$\partial\Phi_\varepsilon \xrightarrow{\text{graph}} \partial\Phi,$$

if $\{\mathrm{G}(\partial\Phi_\varepsilon)\}_{\varepsilon>0}$ KP-converges to $\mathrm{G}(\partial\Phi)$ with respect to the product topology $\mathscr{T} \times \mathscr{T}^*$, i.e.

$$\mathrm{G}(\partial\Phi_\varepsilon) \xrightarrow{\text{KP}} \mathrm{G}(\partial\Phi),$$

where $\mathrm{G}(\partial\Phi_\varepsilon)$ and $\mathrm{G}(\partial\Phi)$ denote the graph of $\partial\Phi_\varepsilon$ and $\partial\Phi$ respectively, i.e.

$$\mathrm{G}(\partial\Phi_\varepsilon) := \left\{(u,u^*) \in \mathrm{dom}(\Phi_\varepsilon) \times X^* : u^* \in \partial\Phi_\varepsilon(u)\right\};$$
$$\mathrm{G}(\partial\Phi) := \left\{(u,u^*) \in \mathrm{dom}(\Phi) \times X^* : u^* \in \partial\Phi(u)\right\}.$$

In other words, $\partial\Phi_\varepsilon \xrightarrow{\text{graph}} \partial\Phi$ if and only if the following two assertions hold:

- for all $(u,u^*) \in X \times X^*$ with $u^* \in \partial\Phi(u)$ there exists $\{(u_\varepsilon, u^*_\varepsilon)\}_{\varepsilon>0} \subset X \times X^*$ with $u^*_\varepsilon \in \partial\Phi(u_\varepsilon)$ for all $\varepsilon > 0$ such that $u_\varepsilon \xrightarrow{\mathscr{T}} u$ and $u^*_\varepsilon \xrightarrow{\mathscr{T}^*} u^*$;
- for all $(u,u^*) \in X \times X^*$ and all subsequence $\{(u_{\sigma(\varepsilon)}, u^*_{\sigma(\varepsilon)})\}_{\varepsilon>0} \subset X \times X^*$ with $u^*_{\sigma(\varepsilon)} \in \partial\Phi(u_{\sigma(\varepsilon)})$ for all $\varepsilon > 0$, if $u_{\sigma(\varepsilon)} \xrightarrow{\mathscr{T}} u$ and $u^*_{\sigma(\varepsilon)} \xrightarrow{\mathscr{T}^*} u^*$ then $u^* \in \partial\Phi(u)$.

In what follows, we use the notation "$\xrightarrow{\text{(s,s)-graph}}$", "$\xrightarrow{\text{(w,s)-graph}}$" and "$\xrightarrow{\text{(s,w)-graph}}$" to denote the graph-convergence with respect to the product topology $\mathscr{T} \times \mathscr{T}^*$ with $(\mathscr{T}, \mathscr{T}^*) = (\mathscr{T}_{\rm s}, \mathscr{T}_{\rm s}^*)$, $(\mathscr{T}, \mathscr{T}^*) = (\mathscr{T}_{\rm w}, \mathscr{T}_{\rm s}^*)$ and $(\mathscr{T}, \mathscr{T}^*) = (\mathscr{T}_{\rm s}, \mathscr{T}_{\rm w}^*)$ respectively, where $\mathscr{T}_{\rm s}$ and $\mathscr{T}_{\rm s}^*$ (resp. $\mathscr{T}_{\rm w}$ and $\mathscr{T}_{\rm w}^*$) the strong topology (resp. the weak topology) on X and X^* respectively.

For convex, lower semincontinuous and proper functions, links between graph-convergence and Γ-convergence can be established (for a proof we refer to [8, Proposition 3.68, p. 378]).

Theorem C.4. *Let $\Phi : X \to]-\infty, \infty]$ be a proper, convex and lower semicontinuous function and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty, \infty]$ be a proper, convex and lower semicontinuous function. Then*:

(a) $\Phi_\varepsilon \xrightarrow{\rm M} \Phi \Longrightarrow \partial\Phi_\varepsilon \xrightarrow{\text{(s,s)-graph}} \partial\Phi$;
(b) $\Phi_\varepsilon \xrightarrow{\Gamma_{\rm w}} \Phi \Longrightarrow \partial\Phi_\varepsilon \xrightarrow{\text{(w,s)-graph}} \partial\Phi$;
(c) $\Big(X$ *is reflexive,* $\Phi_\varepsilon \xrightarrow{\Gamma_{\rm w}} \Phi$ *and* $\{\Phi_\varepsilon\}_{\varepsilon>0}$ *is equicoercive*[5]$\Big) \Longrightarrow \partial\Phi_\varepsilon \xrightarrow{\text{(s,w)-graph}} \partial\Phi$.

Appendix D. Grönwall's Lemma

In the paper we use the following version of so-called Grönwall's lemmas (for the proof we refer to [5, Lemma A.2 pp. 277]).

Lemma D.1. *Let $p \in [1, \infty[$, let $T > 0$, let $a \in [0, \infty[$, let $\mathrm{m} \in L^1([0,T])$ be such that $\mathrm{m}(s) \geqslant 0$ for $\mathscr{L}^1$-a.a. $s \in [0,T]$ and let $\phi \in C([0,T];\mathbb{R})$ be such that $\frac{1}{p}\phi^p(s) \leqslant \frac{1}{p}a^p + \int_0^s \phi^{p-1}(t)\mathrm{m}(t)dt$ for all $s \in [0,T]$. Then $\phi(s) \leqslant a + \int_0^s \mathrm{m}(t)dt$ for all $s \in [0,T]$.*

[5] We say that $\{\Phi_\varepsilon\}_{\varepsilon>0}$ is equicoercive if the following assertion holds: for every $\{v_\varepsilon\}_{\varepsilon>0} \subset X$, if $\sup_{\varepsilon>0} \Phi_\varepsilon(v_\varepsilon) < \infty$ then $\sup_{\varepsilon>0} \|v_\varepsilon\|_X < \infty$.