Commun. Optim. Theory 2026 (2026) 11 https://doi.org/10.23952/cot.2026.11

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

ON THE APPROXIMATION OF SOLUTIONS OF QUASI-EQUILIBRIUM PROBLEMS WITH ERRORS

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Dedicated to the memory of Professor Hedy Attouch

Abstract. In this paper, we study an inexact version of the proximal point algorithm for solving quasi-equilibrium problems in Banach spaces. We show that the sequence generated by the algorithm converges strongly to a solution of the problem. Our algorithm is new and requires fewer iterative steps than our previous algorithms in [D. Rouhani, V. Mohebbi, Extragradient methods for quasi-equilibrium problems in Banach spaces, J. Aust. Math. Soc. 112 (2022) 90–114] and [D. Rouhani, V. Mohebbi, Proximal point method for quasi-equilibrium problems in Banach spaces, Numer. Funct. Anal. Optim. 41 (2020) 1007–1026]. Finally, we present some applications and numerical experiments of our results.

Keywords. Demiclosed; Proximal point method; Quasi-equilibrium problem; quasi ϕ-nonexpansive; θ-undermonotone.

2020 Mathematics Subject Classification. 90C25, 90C30, 90C33, 65K15, 58E35.

1. INTRODUCTION

Let E be a real Banach space and C be a nonempty subset of E. Suppose that $f : C \times C \to \mathbb{R}$ is a bifunction, and $K(\cdot) : C \to 2^C$ is a multivalued mapping with nonempty values. The quasi-equilibrium problem QEP(f,K) consists of finding $x^* \in K(x^*)$ such that

$$f(x^*,y) \geq 0, \quad \forall y \in K(x^*).$$

The associated Minty quasi-equilibrium problem MQEP(f,K) can be expressed as finding $x^* \in K(x^*)$ such that

$$f(y,x^*) \leq 0, \quad \forall y \in K(x^*).$$

When $K(x) = C$ for all $x \in C$, the quasi-equilibrium problem QEP(f,K) and the associated Minty quasi-equilibrium problem MQEP(f,K) become a classical equilibrium problem EP(f,C) and a classical Minty equilibrium problem MEP(f,C) respectively.

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Received October 6, 2024; Accepted February 17, 2025.

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The equilibrium problems unify many problems in nonlinear analysis such as variational inequalities, convex optimization problems, Nash equilibrium problems, fixed point problems etc. The first existence result for solutions to $\mathrm{EP}(f,C)$ was obtained by Ky Fan [9]. Then Brézis, Nirenberg and Stampacchia [6] studied the existence of solutions to $\mathrm{EP}(f,C)$ with a coercivity assumption on the bifunction f. Next, Blum and Oettli [5] showed the existence of solutions to $\mathrm{EP}(f,C)$ by assuming the monotonicity of f. Recently, equilibrium problems were studied extensively for existence of solutions (see, e.g., [10] and the references therein).

The main motivation for studying quasi-equilibrium problems arises from the relation between quasi-equilibrium problems and quasi-variational inequalities. Note that quasi-variational inequalities encompass certain problems of interest in applications, which do not fall within the scope of variational inequalities. See for example the papers by Zaslavski [26, 27], Reich et al. [23] and the references therein. An instance of such applications is the generalized Nash equilibrium problem, which models a large number of real life problems in Economics and other areas (see [19, 20, 21]). Recently, quasi-equilibrium problems were studied in [4, 7, 8]. To see some results and applications of quasi variational inequalities, see [20] and the references therein.

We focus on a popular method for solving quasi-equilibrium problems, namely the proximal point algorithm. The proximal point algorithm is one of the famous methods for solving equilibrium problems which has been applied originally in finding a zero of a maximal monotone operator. Many authors have studied this problem by investigating various types of algorithms. See for example, Attouch [2, 3] and the references therein. The following proximal point algorithm was applied by Moudafi in [18] for solving monotone equilibrium problems in Hilbert spaces:

$$f(x^k,y)+\lambda_{k-1}\langle y-x^k, x^k-x^{k-1}\rangle \geq 0, \quad \forall y\in C. \tag{1.1}$$

Hybrid proximal methods for equilibrium problems were also studied by Mordukhovich et al. in [17]. Iusem and Sosa [12] extended the existence result in [18] to pseudo-monotone bifunctions. They showed the existence of the sequence $\{x^k\}$ in (1.1) where $\lambda_k > \theta$ and $\theta \geq 0$ is the undermonotonicity constant of the bifunction f (see Proposition 3 of [12]). Then they proved the weak convergence of the sequence $\{x^k\}$ to a solution of the problem (see also [14, 15] for some extensions and more results). In [11], this result was extended to Banach spaces, i.e. the existence and uniqueness of the sequence $\{x^k\}$ given by

$$f(x^k,y)+\lambda_{k-1}\langle y-x^k, Jx^k-Jx^{k-1}\rangle \geq 0, \qquad \forall y\in C, \tag{1.2}$$

was proved with the same assumptions on the parameters, and the weak convergence of the sequence $\{x^k\}$ to a solution of $EP(f,C)$ was also established.

In [7], the authors applied the extragradient method for solving quasi-equilibrium problems in Banach spaces. In fact, they showed that the sequence generated by the extragradient method converges strongly to a solution of the quasi-equilibrium problem. Also, the quasi-equilibrium problem with different assumptions on the problem was studied in [8]. The authors showed the strong convergence of the generated sequence to a solution of the quasi-equilibrium problem.

In this paper, we study an inexact proximal point algorithm for solving quasi-equilibrium problems in Banach spaces. Our algorithm is new and requires fewer iterative steps than our previous algorithms in [7, 8]. In fact in [7, 8], in order to generate the iterative sequence $\{x^k\}$, we used three projections on half-spaces, whereas in this paper, we use only two projections.

Moreover, the first projection in each algorithm is the same. However, in [7, 8], in the k-th step of iterations, the second projection is on the intersection of $(k+1)$ half-spaces, and the third projection is the projection of the initial point x^0 on the intersection of three half-spaces, whereas in this paper, the second projection is just on one half-space. Therefore, from a computational point of view, our algorithm in this paper presents a big advantage.

This paper is organized as follows. In Section 2, we introduce some preliminary material related to the geometry of Banach spaces. In Section 3, we study an inexact version of the proximal point method for solving quasi-equilibrium problems in Banach spaces, and we prove the strong convergence of the generated sequence to a solution of the quasi-equilibrium problem. We give also a sufficient condition for the solution set of the problem to be nonempty. In Section 4, we present some applications and numerical experiments of our results.

2. Preliminaries

Let E be a real Banach space, we denote the topological dual of E by E^*. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ v \in E^* : \langle x, v \rangle = \|x\|^2 = \|v\|^2 \right\}.$$

A Banach space E is called strictly convex, if $\|\frac{1}{2}(x+y)\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Also, E is called uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, it holds that $\|\frac{1}{2}(x+y)\| < 1 - \delta$. It is known that uniformly convex Banach spaces are reflexive and strictly convex.

A Banach space E is called smooth, whenever

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U = \{z \in E : \|z\| = 1\}$. It is called uniformly smooth, whenever the limit in (2.1) exists uniformly for $x, y \in U$. Note that ℓ^p spaces $(1 < p < \infty)$, L^p spaces $(1 < p < \infty)$ and the Sobolev spaces $W^{k,p}$ $(1 < p < \infty)$ are uniformly smooth and uniformly convex.

The duality mapping J has the following properties (see [13, 25]):

(i) if E is smooth, then J is single-valued;

(ii) if E is strictly convex, then J is one-to-one; that is, $Jx \cap Jy \neq \emptyset$ implies that $x = y$;

(iii) if E is reflexive, then J is onto;

(iv) if E is strictly convex, then J is strictly monotone, that is, if $x^* \in J(x)$, $y^* \in J(y)$ and $\langle x - y, x^* - y^* \rangle = 0$, then $x = y$;

(v) if E is smooth, then J is norm-to-weak* continuous;

(vi) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

(vii) if E is smooth, reflexive and strictly convex, then J is single-valued and bijective, therefore J^{-1} is well defined.

Let E be a smooth Banach space. We define $\phi : E \times E \to \mathbb{R}$ by $\phi(x,y) = \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2$. This function can be seen as a "distance-like" function, better conditioned than the square of the metric distance, namely $\|x - y\|^2$; see, e.g., [1, 13, 22].

It is clear that

$$0 \leq (\|x\| - \|y\|)^2 \leq \phi(x,y) \tag{2.2}$$

for all $x,y \in E$. In Hilbert spaces, where the duality mapping J is the identity operator, it holds that $\phi(x,y) = \|x-y\|^2$. In the sequel, we will need the following three properties of ϕ, proved in [13].

Proposition 2.1. *Let E be a smooth and uniformly convex Banach space, and $\{x^k\}$ and $\{y^k\}$ be two sequences in E. If $\lim_{k\to\infty}\phi(x^k,y^k) = 0$ and either $\{x^k\}$ or $\{y^k\}$ is bounded, then $\lim_{k\to\infty}\|x^k - y^k\| = 0$.*

Proposition 2.2. *Let E be a reflexive, strictly convex and smooth Banach space, and C be a nonempty, closed and convex subset of E. Then for all $x \in E$ there exists a unique $\bar{x} \in C$ such that $\phi(\bar{x},x) = \inf\{\phi(z,x) : z \in C\}$.*

We define $P_C : E \to C$ by taking $P_C(x)$ as the unique $\bar{x} \in C$ given by Proposition 2.2. P_C is called the generalized projection of E onto C. When E is a Hilbert space, P_C is just the metric projection of E onto C.

The third result taken from [13] is the following.

Proposition 2.3. *Consider a smooth Banach space E, and a nonempty, closed and convex subset $C \subset E$. Let $x \in E, \bar{x} \in C$. Then $\bar{x} = P_C(x)$ if and only if $\langle z-\bar{x}, J(x) - J(\bar{x})\rangle \leq 0$ for all $z \in C$.*

Now we introduce some notations and definitions that will be used in the sequel. For a sequence $\{x^k\}$ in E, we denote strong convergence of $\{x^k\}$ to $x \in E$ by $x^k \to x$, and weak convergence by $x^k \rightharpoonup x$. We also denote the set of all solutions to QEP(f,K) by $S(f,K)$. The set of all fixed points of the multivalued mapping $K(\cdot)$ will be denoted by Fix(K).

In the following definitions, we assume that $C \subset E$ is a nonempty, closed and convex set.

Definition 2.4. $T : C \to C$ is called a quasi ϕ-nonexpansive mapping whenever Fix$(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $(p,x) \in \text{Fix}(T) \times C$.

Definition 2.5. Suppose that $K(\cdot) : C \to 2^C$ is a multivalued mapping such that for all $x \in C$, $K(x)$ is nonempty, closed and convex. $K(\cdot)$ is called quasi ϕ-nonexpansive whenever the mapping $T(\cdot) = P_{K(\cdot)}(\cdot)$ is quasi ϕ-nonexpansive where P is the generalized projection.

Definition 2.6. The multivalued mapping $K(\cdot) : C \to 2^C$ is called demiclosed, if whenever $x^k \rightharpoonup \bar{x}$ and $\lim_{k\to\infty} d(x^k, K(x^k)) = 0$, then $\bar{x} \in \text{Fix}(K)$.

Note that when T is a quasi ϕ-nonexpansive mapping, it is well known that Fix(T) is convex. Also, if T is demiclosed then Fix(T) is closed (see Remark 2.8).

Definition 2.7. The bifunction $f : C \times C \to \mathbb{R}$ is called monotone, if $f(x,y) + f(y,x) \leq 0$ for all $x,y \in C$, and pseudo-monotone if for any pair $x,y \in C$, $f(x,y) \geq 0$ implies $f(y,x) \leq 0$.
Also, f is called θ-undermonotone, if there exists $\theta \geq 0$ such that $f(x,y) + f(y,x) \leq \frac{\theta}{2}(\phi(x,y) + \phi(y,x))$, for all $x,y \in C$, where θ is the undermonotonicity constant of f.

Now, we introduce some conditions on the bifunction f and the multivalued mapping K, that we will need for the convergence analysis.

B1: $f(x,x) = 0$ for all $x \in C$.
B2: $f(\cdot,x) : C \to \mathbb{R}$ is weakly upper semicontinuous for all $x \in C$.
$\overline{\text{B}}$2: $f(\cdot,x) : C \to \mathbb{R}$ is upper semicontinuous for all $x \in C$.
B3: $f(x,\cdot) : C \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in C$.

B4: f is θ-undermonotone.
B5: f is pseudo-monotone.
B6: $K(\cdot): C \to 2^C$ is a quasi ϕ-nonexpansive and demiclosed multivalued mapping with nonempty, closed and convex values.

Remark 2.8. If $K(\cdot): C \to 2^C$ is a multivalued mapping satisfying B6, then $\mathrm{Fix}(K)$ is closed and convex.

Proof. See Remark 3.3 in [7] □

3. Strong convergence

In this section, we study an inexact proximal point algorithm for solving quasi-equilibrium problems in Banach spaces. We assume that E is uniformly smooth and uniformly convex, $C \subset E$ is nonempty, closed and convex, and $f: C \times C \to \mathbb{R}$ is a bifunction, $K(\cdot): C \to 2^C$ is a multivalued mapping and the assumptions B1–B6 are satisfied and $S^* := \mathrm{Fix}(K) \cap S(f,C) \neq \emptyset$.

Algorithm 1.
1. Initialization: Let $x^0 \in C$, $\alpha_k \in (0,1)$, $\beta_k \in [0,1]$, $\gamma_k \in [\varepsilon, 1]$ for some $\varepsilon > 0$, and $\{u^k\} \subset E$ be an arbitrary sequence such that $u^k \to u$, and $\{e^k\}$ is a bounded sequence of computational errors. Take $\beta > 0$ satisfying $\theta < \beta$ where θ is the undermonotonicity constant of f, and a sequence $\{\lambda_k\} \subset (\theta, \beta]$.
2. Iterative step: Given x^k, define

$$y^k = P_{K(x^k)}(x^k).$$

$$H_k := \left\{ y \in E : \langle y - x^k, Jx^k - Jy^k \rangle \leq \frac{-\gamma_k}{2} \phi(x^k, y^k) \right\}.$$

$$z^k = P_{H_k}(x^k). \tag{3.1}$$

$$w^k = J^{-1}(\alpha_k Ju^k + (1-\alpha_k)(\beta_k Jz^k + (1-\beta_k)Je^k), \tag{3.2}$$

Then we compute $x^{k+1} \in C$ form (3.3) below:

$$f(x^{k+1}, y) + \lambda_k \langle y - x^{k+1}, Jx^{k+1} - Jw^k \rangle \geq 0, \qquad \forall y \in C. \tag{3.3}$$

The following proposition shows the existence and uniqueness of the iterative sequence $\{x^k\}$.

Proposition 3.1. *Assume that f satisfies B1,$\overline{\mathrm{B}}$2, B3, B4 and $\lambda > \theta$. Let the bifunction $\tilde{f}$ be defined by $\tilde{f}(x,y) = f(x,y) + \lambda \langle y - x, Jx - J\bar{x} \rangle$, where $\bar{x} \in E$. Then $EP(\tilde{f}, C)$ has a unique solution.*

Proof. See [11]. □

In order to prove the strong convergence of the sequence $\{x^k\}$, we recall the following two lemmas from [16, 24] that will be needed in the sequel.

Lemma 3.2. *Consider sequences $\{s_k\} \subset [0,\infty), \{t_k\} \subset \mathbb{R}$ and $\{\gamma_k\} \subset (0,1)$ satisfying $\sum_{k=0}^{\infty} \gamma_k = \infty$. Suppose that $s_{k+1} \leq (1-\gamma_k)s_k + \gamma_k t_k$ for all $k \geq 1$. If $\limsup_{n\to\infty} t_{k_n} \leq 0$ for every subsequence $\{s_{k_n}\}$ of $\{s_k\}$ satisfying $\liminf_{n\to\infty}(s_{k_n+1} - s_{k_n}) \geq 0$, then $\lim_{k\to\infty} s_k = 0$.*

Proof. See [24]. □

We will use the map $V : E\times E^* \to \mathbb{R}$ defined by

$$V(x,v) = \|x\|^2 - 2\langle x,v\rangle + \|v\|^2. \tag{3.4}$$

Note that $V(x,v) = \phi(x,J^{-1}v)$ for all $(x,v)\in E\times E^*$ and V is convex with respect to both variables (see [1] for more details). We also need the following result in reflexive Banach spaces.

Lemma 3.3. *Let E be a strictly convex, smooth, and reflexive Banach space, and let V be as in (3.4). Then $V(x,v)\leq V(x,v+w) - 2\langle J^{-1}(v)-x,w\rangle$ for all $x\in E$ and $v,w\in E^*$.*

Proof. See [16]. □

We now proceed to the convergence analysis of Algorithm 1. We divide the proof of the main theorem into several lemmas, and prove the theorem at the end of this section.

Theorem 3.4. *Let $f : C\times C\to\mathbb{R}$ be a bifunction, and $K(\cdot) : C\to 2^C$ be a multivalued mapping such that the assumptions B1–B6 are satisfied. Let $\{\alpha_k\}\subset(0,1)$, $\{\beta_k\}\subset[0,1]$ and $\{\lambda_k\}\subset(0,\infty)$ be sequences such that $\alpha_k\to 0$, $\lambda_k>\theta$ and $\{\lambda_k\}$ is bounded above, and $\frac{1-\beta_k}{\alpha_k}\to 0$. If $S^*\neq\emptyset$, then the sequence $\{x^k\}$ generated by (3.3) converges strongly to $P_{S^*}u$.*

Lemma 3.5. *The sequences $\{x^k\}$, $\{y^k\}$, $\{z^k\}$ and $\{w^k\}$, generated by Algorithm 1, are bounded.*

Proof. First we show that the sequence $\{x^k\}$ is bounded. Note that by (3.3), we have

$$f(x^{k+1},y) + \lambda_k\langle y - x^{k+1}, Jx^{k+1} - Jw^k\rangle \geq 0, \qquad \forall y\in C. \tag{3.5}$$

Take $x^*\in S^*$, and replace y by x^*. Since f is pseudo-monotone, we have $f(x^{k+1},x^*)\leq 0$. Therefore, we get $\langle x^* - x^{k+1}, Jx^{k+1} - Jw^k\rangle \geq 0$. Equivalently, we have

$$\varphi(x^{k+1},w^k) + \varphi(x^*,x^{k+1}) - \varphi(x^*,w^k)\leq 0.$$

which implies that

$$\varphi(x^*,x^{k+1})\leq\varphi(x^*,w^k). \tag{3.6}$$

On the other hand, it is easy to see that the set H_k is a nonempty, closed and convex subset of E containing all $x^*\in S^*$.Since $z^k = P_{H_k}(x^k)$ by (3.1), Proposition 2.3 implies $\langle x^* - z^k, Jx^k - Jz^k\rangle\leq 0$, or equivalently, $\phi(z^k,x^k)+\phi(x^*,z^k)\leq\phi(x^*,x^k)$. Therefore, we have

$$\phi(x^*,z^k)\leq\phi(x^*,x^k). \tag{3.7}$$

Using (3.2), (3.6), and (3.7), we obtain

$$\begin{aligned}
\phi(x^*,x^{k+1}) &\leq \phi(x^*,w^k) = \phi(x^*,J^{-1}(\alpha_k Ju^k + (1-\alpha_k)(\beta_k Jz^k + (1-\beta_k)Je^k)))\\
&= V(x^*,\alpha_k Ju^k + (1-\alpha_k)(\beta_k Jz^k + (1-\beta_k)Je^k))\\
&\leq \alpha_k V(x^*,Ju^k) + (1-\alpha_k)V(x^*,(\beta_k Jz^k + (1-\beta_k)Je^k))\\
&\leq \alpha_k V(x^*,Ju^k) + (1-\alpha_k)\Big(\beta_k V(x^*,Jz^k) + (1-\beta_k)V(x^*,Je^k)\Big)\\
&= \alpha_k\phi(x^*,u^k) + (1-\alpha_k)\Big(\beta_k\phi(x^*,z^k) + (1-\beta_k)\phi(x^*,e^k)\Big)\\
&\leq \alpha_k\phi(x^*,u^k) + (1-\alpha_k)\Big(\beta_k\phi(x^*,x^k) + (1-\beta_k)\phi(x^*,e^k)\Big)\\
&\leq \max\Big\{\phi(x^*,u^k),\phi(x^*,x^k),\phi(x^*,e^k)\Big\}.
\end{aligned} \tag{3.8}$$

Let $L := \sup_k \left\{\phi(x^*,u^k),\phi(x^*,e^k)\right\}$. Continuing the process in (3.8), we conclude that

$$\phi(x^*,x^{k+1}) \leq \max\left\{L,\phi(x^*,x^0)\right\},$$

and hence $\{x^k\}$ is bounded by virtue of (2.2). Therefore (3.2) and (3.7) show that the sequences $\{z^k\}$ and $\{w^k\}$ are bounded. Also, since $y^k = P_{K(x^k)}(x^k)$, and $P_{K(\cdot)}(\cdot)$ is a quasi ϕ-nonexpansive mapping, we have $\phi(x^*,y^k) \leq \phi(x^*,x^k)$. Hence $\{y^k\}$ is bounded too. □

Lemma 3.6. *Let $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\gamma_k\}$ be as in Theorem 3.4, and $\{e^k\}$, $\{u^k\}$, $\{w^k\}$ and $\{z^k\}$ be the sequences given by Algorithm 1, and $x^* \in S^*$. Then we have*

$$\begin{aligned}&\alpha_k\langle x^*-u^k,Ju^k-Jw^k\rangle+(1-\alpha_k)\beta_k\langle x^*-z^k,Jz^k-Jw^k\rangle+(1-\alpha_k)(1-\beta_k)\langle x^*-e^k,Je^k-Jw^k\rangle\\&\leq\frac{-\gamma_k}{2}\Big(\alpha_k\phi(u^k,w^k)+(1-\alpha_k)\beta_k\phi(z^k,w^k)+(1-\alpha_k)(1-\beta_k)\phi(e^k,w^k)\Big).\end{aligned}\tag{3.9}$$

Proof. Note that, as shown in (3.8), we have

$$\phi(x^*,w^k)\leq\alpha_k\phi(x^*,u^k)+(1-\alpha_k)\Big(\beta_k\phi(x^*,z^k)+(1-\beta_k)\phi(x^*,e^k)\Big).\tag{3.10}$$

Using (3.10), we have

$$\begin{aligned}&\alpha_k\phi(x^*,w^k)+(1-\alpha_k)\beta_k\phi(x^*,w^k)+(1-\alpha_k)(1-\beta_k)\phi(x^*,w^k)\\&=\phi(x^*,w^k)\\&\leq\alpha_k\phi(x^*,u^k)+(1-\alpha_k)\beta_k\phi(x^*,z^k)+(1-\alpha_k)(1-\beta_k)\phi(x^*,e^k)\end{aligned}\tag{3.11}$$

Now, since $\gamma_k\in[\varepsilon,1]$, we have the following three statements:

$$\text{If } \phi(x^*,w^k)\leq\phi(x^*,u^k), \text{then } \langle x^*-u^k,Ju^k-Jw^k\rangle\leq\frac{-\gamma_k}{2}\phi(u^k,w^k),\tag{3.12}$$

$$\text{If } \phi(x^*,w^k)\leq\phi(x^*,z^k), \text{then } \langle x^*-z^k,Jz^k-Jw^k\rangle\leq\frac{-\gamma_k}{2}\phi(z^k,w^k),\tag{3.13}$$

$$\text{If } \phi(x^*,w^k)\leq\phi(x^*,e^k), \text{then } \langle x^*-e^k,Je^k-Jw^k\rangle\leq\frac{-\gamma_k}{2}\phi(e^k,w^k),\tag{3.14}$$

Multiplying (3.12) by α_k, (3.13) by $(1-\alpha_k)\beta_k$ and (3.14) by $(1-\alpha_k)(1-\beta_k)$, and then by adding them up and using (3.11), we have

$$\begin{aligned}&\alpha_k\langle x^*-u^k,Ju^k-Jw^k\rangle+(1-\alpha_k)\beta_k\langle x^*-z^k,Jz^k-Jw^k\rangle+(1-\alpha_k)(1-\beta_k)\langle x^*-e^k,Je^k-Jw^k\rangle\\&\leq\frac{-\gamma_k}{2}\Big(\alpha_k\phi(u^k,w^k)+(1-\alpha_k)\beta_k\phi(z^k,w^k)+(1-\alpha_k)(1-\beta_k)\phi(e^k,w^k)\Big).\end{aligned}$$

□

Lemma 3.7. *Let $\{x^{k_n}\}$ be a subsequence of $\{x^k\}$ such that $\liminf_{n\to\infty}\left[\phi(x^*,x^{k_n+1})-\phi(x^*,x^{k_n})\right]\geq 0$, where $x^*\in S^*$, then*

$$\lim_{n\to\infty}\|x^{k_n+1}-w^{k_n}\|=\lim_{n\to\infty}\|z^{k_n}-w^{k_n}\|=\lim_{n\to\infty}\|z^{k_n}-x^{k_n}\|=\lim_{n\to\infty}\|x^{k_n}-y^{k_n}\|=0.$$

Proof. We divide the proof of this lemma into three steps.
Step 1: We show that $\lim_{n\to\infty}\|x^{k_n+1}-w^{k_n}\| = \lim_{n\to\infty}\|z^{k_n}-x^{k_n}\| = 0$.
Note that

$$\begin{aligned}
0 &\le \liminf_{n\to\infty}\Big[\phi(x^*,x^{k_n+1})-\phi(x^*,x^{k_n})\Big]\\
&\le \liminf_{n\to\infty}\Big[V(x^*,\alpha_{k_n}Ju^{k_n}+(1-\alpha_{k_n})(\beta_{k_n}Jz^{k_n}+(1-\beta_{k_n})Je^{k_n}))-\phi(x^*,x^{k_n})\Big]\\
&\le \liminf_{n\to\infty}\Big[\alpha_{k_n}V(x^*,Ju^{k_n})+(1-\alpha_{k_n})V(x^*,\beta_{k_n}Jz^{k_n}+(1-\beta_{k_n})Je^{k_n})-\phi(x^*,x^{k_n})\Big]\\
&\le \liminf_{n\to\infty}\Big[\alpha_{k_n}\phi(x^*,u^{k_n})+(1-\alpha_{k_n})\Big(\beta_{k_n}\phi(x^*,z^{k_n})+(1-\beta_{k_n})\phi(x^*,e^{k_n})\Big)-\phi(x^*,x^{k_n})\Big]\\
&\le \liminf_{n\to\infty}\Big[\alpha_{k_n}\phi(x^*,u^{k_n})+(1-\alpha_{k_n})\Big(\phi(x^*,x^{k_n})+(1-\beta_{k_n})\phi(x^*,e^{k_n})\Big)-\phi(x^*,x^{k_n})\Big]\\
&= \liminf_{n\to\infty}\Big[\alpha_{k_n}\phi(x^*,u^{k_n})+(1-\beta_{k_n})\phi(x^*,e^{k_n})+(1-\alpha_{k_n})\phi(x^*,x^{k_n})-\phi(x^*,x^{k_n})\Big]\\
&\le \liminf_{n\to\infty}\Big[\alpha_{k_n}\phi(x^*,u^{k_n})+(1-\beta_{k_n})\phi(x^*,e^{k_n})\Big]\\
&\le \limsup_{n\to\infty}\Big[\alpha_{k_n}\phi(x^*,u^{k_n})+(1-\beta_{k_n})\phi(x^*,e^{k_n})\Big]=0
\end{aligned}$$

by using the convexity of $V(x^*,\cdot)$ in the third inequality, and $\lim_{k\to\infty}\alpha_k = 0$ in the last equality.
We conclude that

$$\lim_{n\to\infty}\Big[\phi(x^*,x^{k_n+1})-\phi(x^*,x^{k_n})\Big]=0. \tag{3.15}$$

Moreover, we have

$$\begin{aligned}
\phi(x^*,x^{k_n+1}) &\le \varphi(x^{k_n+1},w^{k_n})+\varphi(x^*,x^{k_n+1})\\
&\le \phi(x^*,w^{k_n})\\
&= \phi(x^*,J^{-1}(\alpha_{k_n}Ju^{k_n}+(1-\alpha_{k_n})(\beta_{k_n}Jz^{k_n}+(1-\beta_{k_n})Je^{k_n})))\\
&= V(x^*,\alpha_{k_n}Ju^{k_n}+(1-\alpha_{k_n})(\beta_{k_n}Jz^{k_n}+(1-\beta_{k_n})Je^{k_n}))\\
&\le \alpha_{k_n}V(x^*,Ju^{k_n})+(1-\alpha_{k_n})V(x^*,(\beta_{k_n}Jz^{k_n}+(1-\beta_{k_n})Je^{k_n}))\\
&\le \alpha_{k_n}V(x^*,Ju^{k_n})+(1-\alpha_{k_n})\Big(\beta_{k_n}V(x^*,Jz^{k_n})+(1-\beta_{k_n})V(x^*,Je^{k_n})\Big)\\
&= \alpha_{k_n}\phi(x^*,u^{k_n})+(1-\alpha_{k_n})\Big(\beta_{k_n}\phi(x^*,z^{k_n})+(1-\beta_{k_n})\phi(x^*,e^{k_n})\Big)\\
&\le \alpha_{k_n}\phi(x^*,u^{k_n})+(1-\alpha_{k_n})\phi(x^*,z^{k_n})+(1-\beta_{k_n})\phi(x^*,e^{k_n})\\
&\le \alpha_{k_n}\phi(x^*,u^{k_n})+(1-\alpha_{k_n})\Big(\phi(z^{k_n},x^{k_n})+\phi(x^*,z^{k_n})\Big)+(1-\beta_{k_n})\phi(x^*,e^{k_n})\\
&\le \alpha_{k_n}\phi(x^*,u^{k_n})+(1-\alpha_{k_n})\phi(x^*,x^{k_n})+(1-\beta_{k_n})\phi(x^*,e^{k_n})
\end{aligned} \tag{3.16}$$

Therefore, we have

$$\begin{aligned}
0 &\le \varphi(x^{k_n+1}, w^{k_n}) \\
&\le \alpha_{k_n}\phi(x^*, u^{k_n}) + (1-\alpha_{k_n})\phi(x^*, z^{k_n}) + (1-\beta_{k_n})\phi(x^*, e^{k_n}) - \varphi(x^*, x^{k_n+1}) \\
&\le \alpha_{k_n}\phi(x^*, u^{k_n}) + (1-\alpha_{k_n})\Big(\phi(z^{k_n}, x^{k_n}) + \phi(x^*, z^{k_n})\Big) + (1-\beta_{k_n})\phi(x^*, e^{k_n}) - \varphi(x^*, x^{k_n+1}) \\
&\le \alpha_{k_n}\phi(x^*, u^{k_n}) + (1-\alpha_{k_n})\phi(x^*, x^{k_n}) + (1-\beta_{k_n})\phi(x^*, e^{k_n}) - \varphi(x^*, x^{k_n+1}) \to 0.
\end{aligned}$$

Using (3.15), the first and the third inequalities imply that

$$\lim_{n\to\infty} \phi(x^{k_n+1}, w^{k_n}) = \lim_{n\to\infty} \phi(z^{k_n}, x^{k_n}) = 0. \tag{3.17}$$

Now we apply Proposition 2.1 to (3.17) to conclude that

$$\lim_{n\to\infty} \|x^{k_n+1} - w^{k_n}\| = \lim_{n\to\infty} \|z^{k_n} - x^{k_n}\| = 0.$$

Step 2: We show that $\lim_{n\to\infty} \|x^{k_n} - y^{k_n}\| = 0$.

Since $z^{k_n} \in H_{k_n}$, from the definition of H_{k_n}, we have $\gamma_{k_n}\phi(x^{k_n}, y^{k_n}) \le 2\langle x^{k_n} - z^{k_n}, Jx^{k_n} - Jy^{k_n}\rangle$. By the Cauchy-Schwarz inequality, we have

$$\gamma_n\phi(x^{k_n}, y^{k_n}) \le 2\|x^{k_n} - z^{k_n}\|\|Jx^{k_n} - Jy^{k_n}\|.$$

Since $\{x^k\}$ and $\{y^k\}$ are bounded, $\lim_{n\to\infty} \|x^{k_n} - z^{k_n}\| = 0$, and $\gamma_{k_n} \ge \varepsilon > 0$, we obtain that $\lim_{n\to\infty} \phi(x^{k_n}, y^{k_n}) = 0$. Hence Proposition 2.1 shows that $\lim_{n\to\infty} \|x^{k_n} - y^{k_n}\| = 0$.

Step 3: We show that $\lim_{n\to\infty} \|z^{k_n} - w^{k_n}\| = 0$.

Replacing k by k_n in (3.9), we deduce that

$$\limsup_{n\to\infty} \gamma_{k_n}\phi(z^{k_n}, w^{k_n}) \le \limsup_{n\to\infty} \langle z^{k_n} - x^*, Jz^{k_n} - Jw^{k_n}\rangle.$$

Then by the Cauchy-Schwarz inequality, we have

$$\limsup_{n\to\infty} \gamma_{k_n}\phi(z^{k_n}, w^{k_n}) \le \limsup_{n\to\infty} \|z^{k_n} - x^*\|\|Jz^{k_n} - Jw^{k_n}\|.$$

Since $\{z^k\}$ is bounded, $\lim_{n\to\infty} \|Jz^{k_n} - Jw^{k_n}\| = 0$ and $\gamma_{k_n} \ge \varepsilon > 0$, we have $\lim_{n\to\infty} \phi(z^{k_n}, w^{k_n}) = 0$. Hence Proposition 2.1 shows that $\lim_{n\to\infty} \|z^{k_n} - w^{k_n}\| = 0$. □

Proof of the main theorem (Theorem 3.4):

Let $x^* = P_{S^*}u$. In view of Lemma 3.3, we have

$$\begin{aligned}
\phi(x^*, x^{k+1}) &\le \phi(x^*, w^k) \\
&= \phi(x^*, J^{-1}(\alpha_k Ju^k + (1-\alpha_k)(\beta_k Jz^k + (1-\beta_k)Je^k))) \\
&= V(x^*, \alpha_k Ju^k + (1-\alpha_k)(\beta_k Jz^k + (1-\beta_k)Je^k)) \\
&\le V(x^*, \alpha_k Ju^k + (1-\alpha_k)(\beta_k Jz^k + (1-\beta_k)Je^k) - \alpha_k(Ju^k - Jx^*)) \\
&- 2\langle J^{-1}(\alpha_k Ju^k + (1-\alpha_k)(\beta_k Jz^k + (1-\beta_k)Je^k)) - x^*, -\alpha_k(Ju^k - Jx^*)\rangle \\
&= V(x^*, (1-\alpha_k)(\beta_k Jz^k + (1-\beta_k)Je^k) + \alpha_k Jx^*) + 2\alpha_k\langle w^k - x^*, Ju^k - Jx^*\rangle \\
&\le (1-\alpha_k)V(x^*, (\beta_k Jz^k + (1-\beta_k)Je^k)) + 2\alpha_k\langle w^k - x^*, Ju^k - Jx^*\rangle \\
&\le (1-\alpha_k)\beta_k V(x^*, Jz^k) + (1-\alpha_k)(1-\beta_k)V(x^*, Je^k) + 2\alpha_k\langle w^k - x^*, Ju^k - Jx^*\rangle \\
&= (1-\alpha_k)\beta_k\phi(x^*, z^k) + (1-\alpha_k)(1-\beta_k)\phi(x^*, e^k) + 2\alpha_k\langle w^k - x^*, Ju^k - Jx^*\rangle \\
&\le (1-\alpha_k)\phi(x^*, x^k) + (1-\alpha_k)(1-\beta_k)\phi(x^*, e^k) + 2\alpha_k\langle w^k - x^*, Ju^k - Jx^*\rangle,
\end{aligned}$$

by using Lemma 3.3 in the second inequality, and the convexity of $V(x^*,\cdot)$ in the third and fourth inequalities.

Next, we show that $\lim_{k\to\infty}\phi(x^*, x^k) = 0$. In view of Lemma 3.2, since $\lim_{k\to\infty}\frac{(1-\beta_k)}{\alpha_k} = 0$, it suffices to show that

$$\limsup_{n\to\infty}\langle w^{k_n} - x^*, Ju^{k_n} - Jx^*\rangle \le 0 \tag{3.18}$$

for every subsequence $\{\phi(x^*, x^{k_n})\}$ of $\{\phi(x^*, x^k)\}$ satisfying

$$\liminf_{n\to\infty}\Big[\phi(x^*, x^{k_n+1}) - \phi(x^*, x^{k_n})\Big] \ge 0.$$

Consider such a subsequence. By Lemma 3.7, we have

$$\lim_{n\to\infty}\|x^{k_n+1} - w^{k_n}\| = \lim_{n\to\infty}\|z^{k_n} - w^{k_n}\| = \lim_{n\to\infty}\|z^{k_n} - x^{k_n}\| = \lim_{n\to\infty}\|x^{k_n} - y^{k_n}\| = 0.$$

Also, Lemma 3.5 shows that the sequences $\{x^k\}$, $\{y^k\}$, $\{z^k\}$ and $\{w^k\}$ are bounded. By considering a subsequence of $\{x^{k_n+1}\}$ if needed, we may assume without loss of generality that $x^{k_n+1} \rightharpoonup p \in E$. On the other hand, by (3.5) we have

$$f(x^{k+1}, y) + \lambda_k\langle y - x^{k+1}, Jx^{k+1} - Jw^k\rangle \ge 0, \qquad \forall y \in C \tag{3.19}$$

which implies that

$$f(x^{k+1}, y) + \lambda_k\|y - x^{k+1}\|\|Jx^{k+1} - Jw^k\| \ge 0, \qquad \forall y \in C \tag{3.20}$$

Note that the sequences $\{\lambda_k\}$, $\{x^k\}$ and $\{w^k\}$ are bounded, and E is uniformly smooth, therefore the duality mapping J is uniformly norm to norm continuous on bounded subsets of E. Replacing k by k_n in (3.20) and taking liminf, we get

$$\liminf_{n\to\infty} f(x^{k_n+1}, y) \ge 0, \qquad \forall y \in C. \tag{3.21}$$

Since y is arbitrary, (3.21) shows that $p \in S(f, C)$. Now we prove that $p \in \mathrm{Fix}(K)$. Note that we have $\lim_{n\to\infty}\|x^{k_n} - y^{k_n}\| = 0$. Therefore we have $\lim_{n\to\infty} d(x^{k_n}, K(x^{k_n})) = 0$. Since K is

demiclosed, then $p \in \mathrm{Fix}(K)$. Therefore $p \in S^* = \mathrm{Fix}(K) \cap S(f,C)$.
Note that $w^{k_n} \rightharpoonup p$, therefore we get

$$\lim_{n\to\infty} \langle w^{k_n} - x^*, Ju^{k_n} - Jx^*\rangle = \langle p - x^*, Ju - Jx^*\rangle. \tag{3.22}$$

Since $x^* = P_{S^*}u$, we conclude from Proposition 2.3 that $\langle p - x^*, Ju - Jx^*\rangle \le 0$. By applying (3.22), we get

$$\limsup_{n\to\infty} \langle w^{k_n} - x^*, Ju^{k_n} - Jx^*\rangle \le 0.$$

Therefore (3.18) holds and thus by Lemma 3.2, $\lim_{k\to\infty} \phi(x^*, x^k) = 0$. Finally, Proposition 2.1 implies that $x^k \to x^* = P_{S^*}u$.

Remark 3.8. In Theorem 3.4, if we assume that $u^k \equiv u$ and $\lambda_k \to 0$, which requires the bifunction f to be monotone, then the uniform smoothness assumption on E which was used to deduce (3.21) from (3.20), as well as in (3.22) is not needed anymore. In this case, we may assume E to be only smooth and uniformly convex.

The following theorem gives a sufficient condition for the solution set S^* to be nonempty.

Theorem 3.9. *With the same assumptions as in Theorem 3.4, assume that the sequence $\{x^k\}$ generated by (3.3) has a bounded subsequence $\{x^{k_n}\}$ such that*

$$\lim_{n\to\infty} \|x^{k_n+1} - x^{k_n}\| = \lim_{n\to\infty} \|z^{k_n} - x^{k_n}\| = 0.$$

Then $S^* \neq \emptyset$ and the sequence $\{x^k\}$ converges strongly to $P_{S^*}u$.

Proof. Without loss of generality, we may assume that $x^{k_n} \rightharpoonup q \in E$, therefore $x^{k_n+1} \rightharpoonup q$. Similar to Step 2 in Lemma 3.7, we can show that $\lim_{n\to\infty} \|x^{k_n} - y^{k_n}\| = 0$. Therefore we have $\lim_{n\to\infty} d(x^{k_n}, K(x^{k_n})) = 0$. Since K is demiclosed, we get $q \in \mathrm{Fix}(K)$.
We have:

$$\begin{aligned}\|Jx^{k_n+1} - Jw^{k_n}\| &= \|Jx^{k_n+1} - (\alpha_{k_n} Ju^{k_n} + (1-\alpha_{k_n})(\beta_{k_n} Jz^{k_n} + (1-\beta_{k_n})Je^{k_n})\| \\ &\le \alpha_{k_n}\|Jx^{k_n+1} - Ju^{k_n}\| + (1-\alpha_{k_n})\beta_{k_n}\|Jx^{k_n+1} - Jz^{k_n}\| \\ &\quad + (1-\alpha_{k_n})(1-\beta_{k_n})\|Jx^{k_n+1} - Je^{k_n})\|.\end{aligned}$$

Since $\alpha_k \to 0$, $\beta_k \to 1$, and $\lim_{n\to\infty} \|x^{k_n+1} - z^{k_n}\| = 0$, and the duality mapping J is uniformly norm to norm continuous on bounded subsets of E, we get $\lim_{n\to\infty} \|Jx^{k_n+1} - Jw^{k_n}\| = 0$. Now it follows from (3.19)–(3.21) that $q \in S(f,C)$ which implies that $S^* \neq \emptyset$. The strong convergence of the sequence $\{x^k\}$ follows from Theorem 3.4. □

4. Applications

In this section, we give some examples of applications of our main result.

Example 4.1. Let $E = \ell^p = \left\{\xi = (\xi_1, \xi_2, \xi_3, \cdots) : \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{\frac{1}{p}} < \infty\right\}$ for $1 < p < \infty$, and consider a monotone operator $T : E \to E^*$, and define $f(x,y) = \langle T(x), y - x\rangle$. Let $C = \Big\{\xi = (\xi_1, \xi_2, \xi_3, \cdots) \in \ell^p : \xi_i \ge 0, \forall i \in \mathbb{N}\Big\}$, and $K(\cdot) : C \to 2^C$ be defined by $K(x) = C \cap B(x, \frac{1}{2}\|x\|)$ for each $x \in C$, where $B(x, \frac{1}{2}\|x\|)$ denotes the closed ball of radius $\frac{1}{2}\|x\|$ centered at x. It is easy to see that $K(\cdot) : C \to 2^C$ is a multivalued mapping with nonempty, closed and convex values, which is quasi ϕ-nonexpansive and demiclosed. Also it is clear that f satisfies B1, B3, B4 and B5. Now if f is weakly upper semicontinuous with respect to the first argument and $S^* \neq \emptyset$, and $\{x^k\}$ is the sequence generated by Algorithm 1, then $\{x^k\}$ converges strongly to an element $q \in S(f,K)$ by Theorem 3.4, and q is a solution to the following quasi-variational inequality problem:

$$\text{Find } x^* \in K(x^*) \text{ such that } \langle T(x^*), y - x^* \rangle \geq 0, \qquad \forall\, y \in K(x^*).$$

Example 4.2. Let $E = \mathbb{R}^n$ and C be a nonempty, closed and convex subset of E, and the bifunction $f : C \times C \to \mathbb{R}$ be defined by $f(x,y) = y^tAy + y^tb - x^tAx - x^tb$ where A is a square matrix of order n such that A is positive semidefinite, b is a vector in $\mathbb{R}^n$ and x^t denotes the transpose of the vector x in $\mathbb{R}^n$. It is easy to see that f satisfies B1-B5. We also define $K : C \to 2^C$ by

$$K(x) = \Big\{ z \in C : z_1 + z_2 + \cdots + z_n \geq \max\{x_1 + x_2 + \cdots + x_n, l\} \Big\}, \tag{4.1}$$

where l is a real number. It can be seen that K satisfies B6 for an appropriate l. If $S^* \neq \emptyset$, then the sequence $\{x^k\}$ generated by Algorithm 1 converges strongly to an element $x^* \in S(f,K)$ by Theorem 3.4. By considering $f(x,y) = \varphi(y) - \varphi(x)$ where $\varphi(x) = x^tAx + x^tb$, we see that x^* is a minimizer of the function φ and $x^* \in K(x^*)$.

As an illustration of the above example, we take $n = 2$, $l = 0$, $C = [-10,10] \times [-10,10]$, $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ -12 \end{bmatrix}$. Suppose that $\alpha_k = \frac{1}{k+1}$, $\beta_k = 1 - \frac{1}{(k+1)^2}$, $\gamma_k = \frac{1}{2}$, $\lambda_k = \frac{1}{3k+1}$, $x^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $u^k = \begin{bmatrix} \frac{1}{k+1} - 2 \\ \frac{-2}{3k+1} + 5 \end{bmatrix}$ and $e^k = \begin{bmatrix} \frac{3}{k+1} + 7 \\ \frac{5}{2k} - 4 \end{bmatrix}$. It is clear that $u^k \to u = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $P_{S^*}(u) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Indeed $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is the unique minimizer of the function $\varphi(x) = x_1^2 + 3x_2^2 + 2x_1 - 12x_2$ and a fixed point of the mapping K. Note that the conditions of Theorem 3.4 are satisfied. Hence if the sequence $\{x^k\}$ is generated by Algorithm 1, then it converges strongly to the unique element $x^* = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \in S(f,K)$.

We performed the numerical experiment for this example and the numerical results are displayed in Table 4.1. Table 4.1 shows that the sequence $\{x^k\}$ converges to $(-1,2)$, which is the solution of QEP(f,K). This problem was solved by the Optimization Toolbox in Matlab R2020a on a Laptop Intel(R) Core(TM) i7- 8665U CPU @ 1.90GHz RAM 8.00 GB.

k	x^{k+1}	$\\|x^{k+1} - x^k\\|$	$\\|x^{k+1} - x^*\\|$
1	(-0.240054512653145, 1.947130080589275)	1.961871993729646	0.761782365323158
2	(-0.496231647465926, 1.947490869107325)	0.256177388871931	0.506497543766995
3	(-0.659426939901883, 1.942973963426576)	0.163257789738150	0.345314317849505
10	(-0.900560368698428, 1.971395673844252)	0.011369716632352	0.103471966001507
100	(-0.990010951588912, 1.996714652291586)	$1.060151216208788 \times 10^{-4}$	0.010515445674066
1000	(-0.999000221116849, 1.999667125252295)	$1.036698383583439 \times 10^{-6}$	0.001053737829279
2000	(-0.999500102994629, 1.999833448327051)	$2.729744802440216 \times 10^{-7}$	$5.269122087606682 \times 10^{-4}$

Table 4.1

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