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COMPACT SETS IN THE SPACE OF BOUNDED LINEAR OPERATORS AND APPLICATIONS TO OPTIMAL FEEDBACK CONTROL

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In memory of Professor S.K. Mitter

Abstract. In this paper, we present some results characterizing compact sets in the space of bounded linear operators from one Banach space to another and their applications to feedback control theory. We present necessary and sufficient conditions characterizing compact sets in the weak operator topology followed by similar results with respect to strong operator topology. These results are then used to develop feedback control theory for evolution equations on Banach spaces.
Keywords. Spaces of Bounded Linear Operators, Weak and Strong Operator topologies, Characterization of Compact sets, Optimal feedback controls.

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1. Introduction

Let X,Y be any pair of real Banach spaces and $\mathscr{L}(Y,X)$ the space of bounded linear operators from Y to X. Furnished with the uniform operator topology, this is a Banach space. In fact the space $\mathscr{L}(Y,X)$ can be given many different topologies. Some of the most popular topologies are (1) the uniform operator topology, (2) the topology of convergence on compact sets, (3) the topology of point-wise convergence, also called strong operator topology and (4) the weak operator topologies. Here we are interested in the strong and weak operator topologies. For convenience of the reader we present the method of construction of strong and weak operator topologies. In the last section 5, we develop optimal feedback control theory and present several applications.

Strong Operator Topology: Let $\mathscr{F}$ denote the class of all finite subsets of the set Y, $B \in \mathscr{L}(Y,X)$, $\varepsilon > 0$, $F \in \mathscr{F}$ and define

$$\mathscr{N}_\varepsilon(B,F) \equiv \{T \in \mathscr{L}(Y,X) : \| (T-B)y \|_X < \varepsilon, \ \forall \ y \in F\}.$$

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The family of sets $\{\mathscr{N}_\varepsilon(B,F) : B \in \mathscr{L}(Y,X), F \in \mathscr{F}, \varepsilon > 0\}$ forms a local base for the strong operator topology. Let $D \equiv (D, \geq)$ denote a directed set. A sequence or a net (= generalized sequence) $\{T_\alpha\}$ converges to T in this topology if and only if, for every $\varepsilon > 0$, there exists a $\beta \in D$ such that $T_\alpha \in \mathscr{N}_\varepsilon(T,F)$ for all $\alpha > \beta$ and for every $F \in \mathscr{F}$. This is a locally convex topology and with respect to this topology, $\mathscr{L}(Y,X)$ is a locally convex Hausdorff topological vector space. We denote this topological space by $(\mathscr{L}(Y,X), \tau_{so}) \equiv \mathscr{L}_{so}(Y,X)$. This topology can be generated also by the family of seminorms $\{\rho_y(T) \equiv \| Ty \|_X, y \in Y, T \in \mathscr{L}(Y,X)\}$.

Weak Operator Topology: Let $\mathscr{G}$ denote the class of all finite subsets of the set X^*, the topological dual of the Banach space X. Let $B \in \mathscr{L}(Y,X)$, $F \in \mathscr{F}$ and $G \in \mathscr{G}$ and $\varepsilon > 0$ arbitrary, and define the set

$$\mathscr{N}_\varepsilon(B,F,G) \equiv \{T \in \mathscr{L}(Y,X) : |(T-B)y, x^*)| < \varepsilon, \ \forall\, y \in F, x^* \in G\}.$$

The family of sets $\{\mathscr{N}_\varepsilon(B,F,G), B \in \mathscr{L}(Y,X), F \in \mathscr{F}, G \in \mathscr{G}, \varepsilon > 0\}$ forms a local base for the weak operator topology. Again, a generalized sequence (or a net) T_α converges to an element $T \in \mathscr{L}(Y,X)$ in this topology if, and only if, there exists a $\beta \in D$ such that for all $\alpha > \beta$,

$$T_\alpha \in \mathscr{N}_\varepsilon(T,F,G)\ \forall\, F \in \mathscr{F} \text{ and } \forall\, G \in \mathscr{G}.$$

We denote the corresponding topological vector space by $(\mathscr{L}(Y,X), \tau_{wo}) \equiv \mathscr{L}_{wo}(Y,X)$. This is also a locally convex Hausdorff topological vector space. This topology can be generated also by the family of seminorms $\{\rho_{y,x^*}(T) \equiv |(Ty, x^*)|, y \in Y, x^* \in X^*, T \in \mathscr{L}(Y,X)\}$.

The author believes that the following result is known to the specialists in the field. But the author is not aware of any paper where the proof is given. For completeness we present a proof.

Proposition 1 Let X,Y be a pair of real Banach spaces. Then, the space $\mathscr{L}_{so}(Y,X)$ is a locally convex sequentially complete Hausdorff topological vector space. If X is also weakly sequentially complete [8], then the space $\mathscr{L}_{wo}(Y,X)$ is also a sequentially complete locally convex Hausdorff topological vector space.

Proof It is clear from the definition of their bases for the topology that the spaces $\mathscr{L}_{so}(Y,X)$ and $\mathscr{L}_{wo}(Y,X)$ are locally convex Hausdorff spaces. So it remains to verify that they are sequentially complete. First, we consider the space $\mathscr{L}_{so}(Y,X)$. Let $\{T_n\}$ be a Cauchy sequence in the strong operator topology. Clearly, for each $y \in Y$, $\{T_n y\} \equiv \{x_y^n\}$ is a Cauchy sequence in X and since X is a Banach space there exists an element $x_y \in X$ such that $x_y^n \xrightarrow{s} x_y$. Then it follows from Banach-Steinhaus theorem [Dunford-Schwartz, 10] that there exists a unique $T \in \mathscr{L}(Y,X)$ so that $x_y = Ty$. Thus $\mathscr{L}_{so}(Y,X)$ is a locally convex sequentially complete topological Hausdorff space. Next, we consider the space $\mathscr{L}_{wo}(Y,X)$. Let $\{T_n\} \in \mathscr{L}(Y,X)$ be a Cauchy sequence in the weak operator topology. Then, for each $y \in Y$, $\{T_n y\} \equiv \{x_y^n\}$ is a weak Cauchy sequence in X. Since, by hypothesis, X is weakly sequentially complete, there exists a unique element $x_y \in X$ so that, along a subsequence if necessary, $x_y^n \xrightarrow{w} x_y$ in X. Since for each $y \in Y$, $\{T_n y\}$ is a weak Cauchy sequence, it is a bounded sequence. By the Banach-Steinhaus theorem, $\{T_n\}$ is a bounded sequence in the uniform operator topology. Clearly, the operator T defined by $x_y = Ty$ is linear and T_n converges to T in the weak operator topology. The limit T is unique because $\mathscr{L}_{wo}(Y,X)$ is Hausdorff. This shows that $\mathscr{L}_{wo}(Y,X)$ is a locally convex sequentially complete topological Hausdorff space. This completes the proof. •

2. Characterization of Compact Sets in $\mathscr{L}(Y,X)$

Continuity and compactness are two of the major concepts in topology. Compact sets in topological spaces play a crucial role in the study of calculus of variations, optimal control theory and many other fields. Feedback control theory is based on the compactness of admissible sets of operators, more generally, sets of operator valued functions with a compatible topology. In this paper, we are interested in feedback control theory using suitable operators and operator valued functions. This raises the question of characterization of compact sets in the space of bounded linear operators endowed with topologies such as weak (and strong) operator topologies. With this goal in mind, in this section, we present some necessary and sufficient conditions for a set $\mathscr{A} \subset \mathscr{L}(Y,X)$ to be compact in the weak operator topology followed by a similar result with respect to the strong operator topology.

For characterization of compact sets, we use the celebrated result of Rosenthal [9] known as Rosenthal's ℓ_1 theorem. For detailed proof; see [11]. This theorem states that in a Banach space a bounded sequence has a weakly convergent subsequence if, and only if, it does not contain an isomorphic copy of ℓ_1. For an excellent account of this result; see [11]. For convenience of the reader we state this result below.

Lemma 2.1 (Rosenthal ℓ_1 theorem) Let $\{x_n\}$ be a bounded sequence in a Banach space X. Then, (i): either $\{x_n\}$ has a weakly Cauchy subsequence or (ii)): $\{x_n\}$ has a ℓ_1 subsequence.

We use the above result, among others, to prove the following theorem giving the necessary and sufficient conditions for a set in $\mathscr{L}(Y,X)$ to be compact in the weak operator topology.

Theorem 2.2 Let $\{X,Y\}$ be a pair of real Banach spaces with X being weakly sequentially complete. Then a set $\mathscr{A} \subset \mathscr{L}_{wo}(Y,X)$ is sequentially compact (in the weak operator topology) if, and only if,(i): $\mathscr{A}$ is closed and bounded and (ii): X contains no isomorphic copy of ℓ_1.

Proof First we prove that the given conditions are sufficient. Consider any sequence $\{T_n\} \in \mathscr{A}$. Then by assumption (i), for any $y \in Y$, $\{x_n\} \equiv \{T_n y\}$ is a bounded sequence in X. Since, by assumption (ii), X contains no copy of ℓ_1, it follows from Rosenthal's dichotomy (Lemma 2.1) that the sequence $\{x_n\}$ has a weakly Cauchy subsequence. By relabeling, if necessary, we may consider that $\{x_n\}$ itself is the weakly Cauchy subsequence of the original sequence in X. The subsequence may depend on y and hence differ from point to point. In an arbitrary Banach space a weak Cauchy sequence need not converge. However, by our assumption, X is weakly sequentially complete. Sufficient conditions for a Banach space to be weakly sequential complete see [8]. Thus there exists a unique $x_o \in X$, dependent on the point $y \in Y$, such that $x_n \longrightarrow x_o = x_o(y)$ weakly. In other words, $T_n(y) \xrightarrow{w} x_o(y)$. This is true for every $y \in Y$. Hence it follows from a corollary of uniform boundedness principle and Banach-Steinhaus theorem that there exists a $T_o \in \mathscr{L}(Y,X)$ such that $x_o(y) = T_o y$. Since, by assumption (i) $\mathscr{A}$ is closed, we have $T_o \in \mathscr{A}$ proving that $\mathscr{A}$ is sequentially compact in the weak operator topology. This proves the sufficiency of the conditions. Next we prove that these conditions are also necessary. Suppose that $\mathscr{A}$ is sequentially compact in the weak operator topology. Clearly, this implies that $\mathscr{A}$ is a (sequentially) closed subset of $\mathscr{L}_{wo}(Y,X)$ and norm bounded. Consider any sequence $\{T_n\} \in \mathscr{A}$. Since $\mathscr{A}$ is sequentially compact in the weak operator topology, there exists a $T_o \in \mathscr{A}$ such that, along a subsequence if necessary, $T_n \xrightarrow{wo} T_o$ in $\mathscr{A}$. The weak operator topology is Hausdorff and hence the limit is unique. Clearly, for every $y \in Y$, the sequence $\{x_n \equiv T_n y\}$ is a bounded sequence in X and converges weakly to $x_o = T_o y$. Thus it follows from Rosenthal's dichotomy that X cannot contain a copy of ℓ_1 proving the necessity of the conditions. This completes the proof. •

It is well known that reflexive Banach spaces are weakly sequentially complete. An example of a nonreflexive Banach space which is weakly sequentially complete is the Lebesgue space $L_1(S,\Sigma,\nu)$ where ν is a sigma finite positive measure. However, this space contains a copy of ℓ_1 [11] and thus Theorem 2.2 does not hold in case $X = L_1(S,\Sigma,\nu)$. There are many examples of nonreflexive Banach spaces which are weakly sequentially complete. It follows from Rosenthal's dichotomy that a weakly sequentially complete Banach space is either reflexive or has an isomorphic copy of ℓ_1. Thus the condition (ii) of Theorem 2.2 is equivalent to the assumption that X is reflexive. Next, we present a result on the necessary and sufficient conditions for a set in $\mathscr{L}(Y,X)$ to be compact in the strong operator topology τ_{so}.

Theorem 2.3 Let $\{Y,X\}$ be any pair of real Banach spaces and $\mathscr{L}_{so}(Y,X)$ denote the space of bounded linear operators from Y to X endowed with the strong operator topology τ_{so}. A bounded set $\mathscr{A} \subset \mathscr{L}(Y,X)$ is compact in the strong operator topology if, and only if,

(i): for each $y \in Y$, the set $\mathscr{A}(y) \equiv \{Ty, T \in \mathscr{A}\}$ is a relatively compact subset of X, and

(ii): the set $\mathscr{A}$ is point-wise closed as a subset of $\mathscr{L}_{so}(Y,X)$.

Proof For proof we use the celebrated theorem of Tychonoff which states that an arbitrary product of compact sets is compact in the product topology [Willard, 14]. We present a brief outline of our proof using this fundamental result. Since, by assumption (i), for each $y \in Y$, $\mathscr{A}(y) \equiv \{Ty, T \in \mathscr{A}\}$ is relatively compact in X, it follows from Tychonoff's theorem that $\mathscr{A}$, as a subspace of X^Y, is relatively compact in the Tychonoff product topology on X^Y. By assumption (ii) $\mathscr{A}$ is point-wise closed. Thus it follows from these facts that $\mathscr{A}$ is compact in the topology of point-wise convergence which is equivalent to convergence in the strong operator topology. This proves that the conditions (i) and (ii) are sufficient. The necessity of the conditions is obvious. This completes the outline of our proof. •

Remark 2.4 It is known that the closed unit ball $\mathscr{B}_1 \subset \mathscr{L}(Y,X)$ is compact in the weak operator topology if, and only if, X is reflexive [Dunford-Schwartz, 10, p512]. Thus, for nonreflexive Banach space X, it may be interesting to find additional conditions for a bounded set $\mathscr{A} \subset \mathscr{L}(Y,X)$ to be compact in the weak operator topology.

In the following remark we present a simple example.

Remark 2.5 Let (Ω,Σ,μ) be a finite positive measure space and take $X = L_1(\Omega,\Sigma,\mu)$. It is well known that X is a weakly sequentially complete Banach space. However, it contains an isomorphic copy of ℓ_1 as demonstrated eloquently in [Diestel,11, p201]. So the closed unit ball $\mathscr{B}_1 \subset \mathscr{L}(Y,X)$ is not compact in the weak operator topology. However, the following is true.

A set $\mathscr{A} \subset \mathscr{L}(Y,X)$ is sequentially compact in the weak operator topology iff (i) it is bounded and (ii) it is uniformly integrable in the sense that for each $y \in Y$, the set $\mathscr{A}(y) \equiv \{Ty, T \in \mathscr{A}\}$ is uniformly μ integrable.

3. COMPACT SETS IN $\mathscr{K}(Y,X)$

Let $\mathscr{K}(Y,X)$ denote the class of linear compact operators from Y to X, that is, for each $T \in \mathscr{K}(Y,X)$, the set $T(B_1(Y))$ is a relatively compact subset of X where $B_1(Y)$ stands for the closed unit ball in Y. Note that $\mathscr{K}(Y,X)$ is a closed subspace of the Banach space $\mathscr{L}(Y,X)$ in the uniform operator topology. Hence $\mathscr{K}(Y,X)$ is also a Banach space. Let Z be another real Banach space. The space $\mathscr{K}(Y,X)$ has the following properties. For any $T\in\mathscr{K}(Y,X)$ and $L\in\mathscr{L}(X,Z)$, $LT\in\mathscr{K}(Y,Z)$ and similarly for any $S\in\mathscr{L}(Z,Y)$, $TS\in\mathscr{K}(Z,X)$. In particular, $\mathscr{K}(X)$ is a two sided ideal in the Banach algebra of bounded linear operators $\mathscr{L}(X)$. In recent years, several interesting results on characterization of compact subsets of $\mathscr{K}(Y,X)$ have appeared in the literature [12,13]. We present here a result due to Mayoral [12]. This result has found interesting applications in control theory [1,2]. For some degree of completeness we state this result here.

Theorem 3.1 Let $\{X,Y\}$ be a pair of real Banach spaces with Y not containing a copy of ℓ_1. Then, a set $M\subset\mathscr{K}(Y,X)$ is relatively compact in the uniform operator topology if, and only if, (i):The set M is uniformly completely continuous and (ii): For each $y\in Y$, $M(y)\equiv\{Ty,T\in M\}$ is a relatively compact subset of X.

For detailed proof of the above result see Mayoral [12, Theorem 1]. The proof makes elegant use of the abstract Ascoli's theorem [Willard,14, Theorem 43.15, p287] characterizing precontract sets of $C(Y,X)$ where Y and X are uniform (topological) spaces.

4. Application to Feedback Control and Optimization

Results presented in the preceding sections have interesting applications in control theory for systems governed differential equations on Banach spaces [1, 2]. Here we are interested in feedback control and its optimization. Let X,Y be any pair of real Banach spaces with X representing the state space and Y the output space. Consider the dynamic system on the Banach space X with an observer G that maps the state space X to the observable output space Y,

$$\dot{x}(t)=Ax(t)+F(x(t))+B(t)y(t),x(0)=x_0, \tag{4.1}$$

$$y(t)=G(x(t)),t\in I, \tag{4.2}$$

where A is the infinitesimal generator of a C_0 semigroup $S(t),t\geq 0$, on X, F is a Lipschitz map in X and G is a Lipschitz map from the state space X to the output space Y and B is an operator valued function (considered as an output feedback operator) with values in $\mathscr{L}(Y,X)$. The observation space is the Banach space Y considered as the output space. In applications, Y may be a closed subspace of the state space X or even a finite dimensional space determined by practical constraints or limitations on the acquisition of data on the state of the system. Using the semigroup $S(t),t\in I$, the system (4.1)-(4.2) can be written as the following integral equation

$$x(t)=S(t)x_0+\int_0^t S(t-s)F(x(s))ds+\int_0^t S(t-s)B(s)G(x(s))ds,t\in I. \tag{4.3}$$

For each given $x_0\in X$, and the operator valued function $B(t),t\in I$, with values in $\mathscr{L}(Y,X)$, and the nonlinear maps $\{F,G\}$ satisfying Lipschitz conditions, one can easily verify that this integral equation has a unique solution $x\in C(I,X)$. In other words, the system (4.1)-(4.2) has a unique mild solution $x\in C(I,X)$. Let $\mathscr{L}_{so}(Y,X)$ denote the space $\mathscr{L}(Y,X)$ endowed with the strong operator topology. The space $\mathscr{B}_\infty(I,\mathscr{L}_{so}(Y,X))$ is the class of bounded measurable functions on I, taking values in $\mathscr{L}_{so}(Y,X)$. The admissible set of output feedback control operators is given by the set $\mathscr{B}_{ad}\equiv B_\infty(I,\mathscr{A})$ where $\mathscr{A}$ is a compact subset of the topological space

$\mathscr{L}_{so}(Y,X)$. It is known that the set $\mathscr{B}_{ad}$, endowed with the Tychonoff product topology τ_p [14], is a compact Hausdorff space. Recently, the author of this paper introduced a more general class of admissible feedback control laws containing Lipschitz maps; see [3]. The problem is to find an element $B \in \mathscr{B}_{ad}$ that minimizes the following cost functional

$$J(B) \equiv \int_0^T \ell(t,x(t))dt + \Phi(x(T)), \tag{4.4}$$

where ℓ and Φ are suitable Borel measurable functions defined on $I \times X$ and X respectively. We present a result on existence of optimal feedback law in the following theorem.

Theorem 4.1 Consider the system given by equations (4.1),(4.2) with the admissible set of feedback control laws $\mathscr{B}_{ad} = B_\infty(I,\mathscr{A})$ as introduced above. Suppose the operator A is the infinitesimal generator of a C_0-semigroup $\{S(t), t \in I\}$ of operators in the Banach space X satisfying $\sup\{\| S(t) \|_{\mathscr{L}(X)}, t \in I\} \leq M$ with $0 < M < \infty$ and the pair $\{F,G\}$ are nonlinear Borel measurable maps satisfying the following Lipschitz conditions

$$\| F(x_1) - F(x_2) \|_X \leq K \| x_1 - x_2 \|_X, x_1, x_2 \in X$$
$$\| G(x_1) - G(x_2) \|_{\mathscr{L}(X,Y)} \leq K \| x_1 - x_2 \|_X, x_1, x_2 \in X$$

for some constant $K > 0$. The functions $\{\ell, \Phi\}$ are real valued Borel measurable and lower semicontinuous in the state variable on X satisfying

$$|\ell(t,x)| \leq \alpha + \beta \| x \|_X^p \qquad |\Phi(x)| \leq \alpha + \beta \| x \|_X^p$$

for any pair of positive numbers $\{\alpha, \beta\}$ and $1 \leq p < \infty$. Then there exists an optimal control law $B_o \in \mathscr{B}_{ad}$ minimizing the cost functional (4.4).

Proof First we verify that the control to solution map $B \longrightarrow x = x(B)$ is continuous from $\mathscr{B}_{ad}$ to $C(I,X)$ with respect to Tychonoff product topology τ_p on $\mathscr{B}_{ad}$ and the uniform norm topology on $C(I,X)$. Let $B_n \xrightarrow{\tau_p} B_o$ and let $x_n \in C(I,X)$ and $x_o \in C(I,X)$ denote the corresponding mild solutions of equations (4.1),(4.2) in the sense that

$$x_n(t) = S(t)x_0 + \int_0^t S(t-s)F(x_n(s))ds + \int_0^t S(t-s)B_n(s)G(x_n(s))ds, t \in I, \tag{4.5}$$

$$x_o(t) = S(t)x_0 + \int_0^t S(t-s)F(x_o(s))ds + \int_0^t S(t-s)B_o(s)G(x_o(s))ds, t \in I. \tag{4.6}$$

By assumption the set $\mathscr{A}$ is a compact subset of $\mathscr{L}_{so}(Y,X)$ and hence there exists finite positive number b such that

$$\sup\{\| \Gamma \|_{\mathscr{L}(Y,X)}, \Gamma \in \mathscr{A}\} \leq b.$$

Since the maps F, G satisfy Lipschitz conditions and $\{S(t), t \in I\}$ is a C_0 semigroup in $\mathscr{L}(X)$, and $\mathscr{A}$ is a compact set in $\mathscr{L}_{so}(Y,X)$ each of the above integral equations has unique solution $x_n \in C(I,X)$ and $x_o \in C(I,X)$ respectively. Subtracting equation (4.5) from equation (4.6) term by term and using triangle inequality one can easily verify that

$$\| x_o(t) - x_n(t) \|_X \leq e_n(t) + MK(1+b) \int_0^t \| x_o(t) - x_n(t) \|_X, t \in I, \tag{4.7}$$

where

$$e_n(t) = M\int_0^t \parallel [B_o(s) - B_n(s)]G(x_o(s)) \parallel_X ds, t \in I. \tag{4.8}$$

Using Gronwall inequality applied to the expression (inequality) (4.7) we obtain the following inequality

$$\parallel x_o(t) - x_n(t) \parallel_X \leq e_n(t) + \int_0^t e^{MK(1+b)(t-s)} e_n(s) ds, t \in I. \tag{4.9}$$

It follows from the same expression (4.8) and the fact that the set $\mathscr{A}$ is compact in the strong operator topology (and hence bounded) that $\{e_n(t), t \in I\}$ is uniformly bounded. Since $G(x_o(s)) \in Y$ for $s \in I$ and B_n converges to B_o in the Tychonoff product topology τ_p, it follows from the expression (4.8) that $e_n(t) \longrightarrow 0$ for each $t \in I$. Thus letting $n \to \infty$ in the expression (4.9) we find that x_n converges to x_o in the norm topology of $C(I,X)$. Since both ℓ and Φ are lower semicontinuous in the state variable we have

$$\ell(t, x_o(t)) \leq \underline{\lim}\, \ell(t, x_n(t)) \ a.e\ t \in I\ , \Phi(x_o(T)) \leq \underline{\lim}\, \Phi(x_n(T)). \tag{4.10}$$

Hence

$$\begin{aligned} J(B_o) = \int_I \ell(t, x_o(t))dt + \Phi(x_o(T)) &\leq \int_I \underline{\lim}\, \ell(t, x_n(t))dt + \underline{\lim}\, \Phi(x_n(T)) \\ &\leq \underline{\lim} \int_I \ell(t, x_n(t))dt + \underline{\lim}\, \Phi(x_n(T)) = \underline{\lim}\, J(B_n). \end{aligned} \tag{4.11}$$

Note that by virtue of the growth properties of ℓ and Φ as stated in the theorem, and the fact that the solutions $\{x_o, x_n\}_{n\geq 1}$ are contained in a bounded subset of the Banach space $C(I,X)$, all the integrals in the expression (4.11) are well defined (finite) and $J(B_o) > -\infty$. Hence

$$J(B_o) \leq \underline{\lim}\, J(B_n)$$

proving that $B \longrightarrow J(B)$ is lower semicontinuous in the Tychonoff product topology τ_p on $\mathscr{B}_{ad}$. Since $\mathscr{B}_{ad}$ is compact in this topology, J attains its minimum on it. This proves the existence of an optimal feedback control law. •

Next, we consider an interesting confinement problem. Let $B_r \subset X$ denote the closed ball of radius $r > 0$ centered at the origin containing the initial state x_0 in its interior. The objective is to find a feedback law that forces the solution trajectory to stay inside the ball as long as possible. Let $I_{B_r}(\cdot)$ denote the indicator function of the set B_r given by $I_{B_r}(x) = 1$ for $x \in B_r$ and 0 for $x \notin B_r$. The problem as stated is equivalent to finding a feedback control law in $\mathscr{B}_{ad}$ that maximizes the following functional

$$J_s(B) = \int_0^T I_{B_r}(x(B)(t))dt.$$

Corollary 4.2 Consider the system (4.1), (4.2) and suppose the assumptions of Theorem 4.1 hold. Then there exists an optimal (feedback control) policy $B_o \in \mathscr{B}_{ad}$ that maximizes the functional J_s.

Proof Let $\{B_n, B_o\} \in \mathscr{B}_{ad}$ and $\{x_n, x_o\} \in C(I,X)$ denote the corresponding mild solutions of the system (4.1), (4.2). It follows from Theorem 4.1 that as $B_n \xrightarrow{\tau_p} B_o$, $x_n \xrightarrow{s} x_o$ in $C(I,X)$. It is known that indicator functions of closed sets are upper semicontinuous. Hence

$$\overline{\lim} I_{B_r}(x_n(t)) \leq I_{B_r}(x_o(t)) \text{ for each } t \in I.$$

Integrating the above expression on either side, one can easily conclude that

$$\overline{\lim} \int_0^T I_{B_r}(x_n(t))dt \leq \int_0^T \overline{\lim} I_{B_r}(x_n(t))dt \leq \int_0^T I_{B_r}(x_o(t))dt.$$

Hence

$$\overline{\lim} J_s(B_n) \leq J_s(B_o)$$

proving upper semicontinuity of J_s on $\mathscr{B}_{ad}$ in the τ_p topology. It follows from compactness of the set $\mathscr{B}_{ad}$ in the same topology that J_s attains its maximum on it. This proves the existence of an optimal policy. •

One can interpret the above feedback law as an stabilizing control forcing the state trajectory to stay inside the closed ball B_r for the longest period of time. For example, if the nonlinear terms F, G are only locally Lipschitz, having possibly polynomial growth, the system (4.1),(4.2) has solution only locally in the sense that there exists an $r > 0$ such that the solution trajectory exits the ball $B_r \subset X$ in finite time. In this situation one is interested to find a stabilizing feedback control law using the output provided by the observer G.

Another closely related problem is to find a feedback control law that maximizes the first exit time. Let B_r' denote the complement of the closed ball B_r. Define

$$\tau(B) = \inf\{t \geq 0 : x(B)(t) \in B_r'\}.$$

If the underlying set is empty, take $\tau(B) = T+$. The problem is to find a feedback law $B \in \mathscr{B}_{ad}$ that maximizes the exit time $\tau(B)$. Interested reader may like to prove the existence of an optimal policy.

5. Necessary Conditions of Optimality

For simplicity, here we assume that X is a reflexive Banach space. Let $DF(\xi) \in \mathscr{L}(X)$ and $DG(\xi) \in \mathscr{L}(X,Y)$ denote the Gâteaux differentials of F and G evaluated at any point $\xi \in Y$. Given that the optimal policy exists, one may consider the question of construction of such policies. We present a result which can be used to determine the optimal policy.

Theorem 5.1 Consider the system given by equations (4.1),(4.2) with the admissible feedback control laws $\mathscr{B}_{ad} = B_\infty(I, \mathscr{A})$ where $\mathscr{A}$ is a compact convex subset of the space $\mathscr{L}_{so}(Y,X)$ and the cost functional given by (4.4) with ℓ and Φ continuously Gâteaux differentiable in the state variable on X. Suppose the operators $\{A, F, G\}$ satisfy the assumptions as stated above with F and G being continuously Gâteaux differentiable in the state variable on X. Let $B_o \in \mathscr{B}_{ad}$ with $x_o \in C(I,X)$ being the corresponding mild solution of the system (4.1),(4.2). Then, in order for B_o to be the optimal feedback control law, it is necessary that there exists a $\psi \in C(I, X^*)$ such that the triple $\{B_o, x_o, \psi\}$ satisfy the following inequality and the evolution equations:

$$dJ(B_o, B - B_o) = \int_I < (B(t) - B_o(t))G(x_o(t)), \psi(t) >_{X,X^*} dt \geq 0, \ \forall\, B \in \mathscr{B}_{ad} \tag{5.1}$$

$$-\dot{\psi}(t) = A^*\psi(t) + DF(x_o(t))^*\psi(t) + (B_o(t)DG(x_o(t)))^*\psi(t) + \ell_x(t, x_o(t)), \quad (5.2)$$
$$\psi(T) = \Phi_x(x_o(T)) = D\Phi(x_o(T)),\ t \in [0,T)$$

and

$$\dot{x}_o(t) = Ax_o(t) + F(x_o(t)) + B_o(t)G(x_o(t)), \quad (5.3)$$
$$x_o(0) = x_0,\ t \in (0,T],$$

where $dJ(B_o, B-B_o)$ denotes the Gâteaux differential (or the directional derivative) of J evaluated at B_o in the direction $B-B_o$.

Proof Let $B_o \in \mathscr{B}_{ad}$ denote the optimal feedback (control) strategy and $B \in \mathscr{B}_{ad}$ any other feedback control. It follows from convexity of the set $\mathscr{B}_{ad}$ that for any $\varepsilon \in [0,1]$, $B_\varepsilon \equiv B_o + \varepsilon(B-B_o) \in \mathscr{B}_{ad}$. Clearly, it follows from optimality of B_o that

$$(1/\varepsilon)[J(B_\varepsilon) - J(B_o)] \geq 0 \ \forall\ \varepsilon \in (0,1].$$

Letting $\varepsilon \downarrow 0$ in the above expression we obtain the Gâteaux differential of J at B_o in the direction $B-B_o$ giving the following inequality

$$dJ(B_o, B-B_o) \geq 0 \ \forall\ B \in \mathscr{B}_{ad}. \quad (5.4)$$

Let $\{x_o, x_\varepsilon\} \in C(I,X)$ denote the mild solutions of the system (4.1),(4.2) corresponding to the elements $\{B_o, B_\varepsilon\}$. Using the cost functional J given by (4.4) and computing the above expression and letting $\varepsilon \downarrow 0$, we obtain the Gâteaux differential of J at B_o in the direction $B-B_o$ as follows

$$dJ(B_o, B-B_o) =< \Phi_x(x_o(T)), z(T) >_{X^*,X} + \int_0^T < \ell_x(t, x_o(t)), z(t) >_{X^*,X} dt, \quad (5.5)$$

where $z \in C(I,X)$, given by $z(t) = \lim_{\varepsilon\to 0}(1/\varepsilon)\big(x_\varepsilon(t) - x_o(t)\big), t \in (0,T]$, is the unique mild solution of the variational equation

$$\dot{z} = Az + DF(x_o(t))z + B_o(t)DG(x_o(t))z + (B-B_o)G(x_o(t)), t \in I,$$
$$z(0) = 0. \quad (5.6)$$

This follows readily from the facts that A is the generator of C_0 semigroup on X and $DF(x_o(t))$ and $B_o(t)DG(x_o(t))$ are bounded operator valued functions with values in $\mathscr{L}(X)$. For $B, B_o \in \mathscr{B}_{ad}$ and $G(x_o(\cdot)) \in C(I,Y)$, we have $(B-B_o)G(x_o) \in B_\infty(I,X)$. Thus it follows from the existence of mild solution of the variational equation (5.6) that the map $(B-B_o)G(x_o) \longrightarrow z$ is a continuous linear map from $B_\infty(I,X)$ to $z \in C(I,X)$. Since, by hypothesis, both ℓ and Φ are continuously Gâteaux differentiable, and $x_o \in C(I,X)$, we conclude that $\Phi_x(x_o(T)) \in X^*$ and $\ell_x(\cdot, x_o(\cdot) \in L_1(I,X^*)$. Hence the expression on the righthand side of equation (5.5), denoted by

$$L(z) \equiv< \Phi_x(x_o(T)), z(T) >_{X^*,X} + \int_0^T < \ell_x(t, x_o(t)), z(t) >_{X^*,X} dt, \quad (5.7)$$

is a continuous linear functional of $z \in C(I,X)$. Thus the composition map $\tilde{L}$ given by

$$(B-B_o)G(x_o) \longrightarrow z \longrightarrow L(z) \equiv \tilde{L}((B-B_o)G(x_o)) \quad (5.8)$$

is a continuous linear functional on $B_\infty(I,X)$. In general, continuous linear functionals on $B_\infty(I,X) \subset L_\infty(I,X)$ are given by finitely additive measures with values in the dual X^* of X.

Thus there exists a $\mu \in \mathscr{M}_{bfa}(\Sigma_I, X^*)$ (space of finitely additive measures having bounded variations) such that

$$\tilde{L}((B-B_o)G(x_o)) = \int_I < (B(t)-B_o(t))G(x_o(t)), \mu(dt) >_{X,X^*},$$

where Σ_I is the algebra of subsets of the set I. Since X is reflexive, X^* is reflexive, and so it has Radon-Nikodym property. If μ were continuous with respect to Lebesgue measure on Σ_I, we would have a $\psi \in L_1(I, X^*)$ such that $\mu(E) = \int_E \psi(t)dt$ for every $E \in \Sigma_I$. We do not know if the vector measure μ is continuous with respect to Lebesgue measure. Thus the existence of such a Radon-Nikodym derivative ψ cannot be guaranteed. So we follow a different route. We note that equation (5.6) has a unique mild solution for each given $f \in L_1(I,X)$ in place of $(B-B_o)G(x_o) \in B_\infty(I,X)$. Since I is a finite interval it is evident that $B_\infty(I,X) \subset L_1(I,X)$ and so $(B-B_o)G(x_o) \in L_1(I,X)$ for all $B \in \mathscr{B}_{ad}$. Thus $\tilde{L}$ is also a continuous linear functional on $L_1(I,X)$ which is the predual of $L_\infty(I,X^*)$. Hence there exists a $\psi \in L_\infty(I,X^*)$ such that

$$\tilde{L}((B-B_o)G(x_o)) = \int_I < (B(t)-B_o(t))G(x_o(t)), \psi(t) >_{X,X^*} dt. \tag{5.9}$$

In fact ψ has a better regularity property as seen below. Using the non homogeneous component $(B-B_o)G(x_o)$ from equation (5.6) and substituting in equation (5.9), we obtain

$$\begin{aligned}&\tilde{L}((B-B_o)G(x_o))\\ &= \int_I < \dot{z} - Az - DF(x_o(t))z - B_o(t)DG(x_o(t))z, \psi(t) >_{X,X^*} dt.\end{aligned} \tag{5.10}$$

Integrating by parts one can easily verify that

$$\begin{aligned}&\tilde{L}((B-B_o)G(x_o))\\ &=< z(T), \psi(T) >_{X,X^*} + \int_I < z(t), -\dot{\psi}(t) >_{X,X^*}\\ &+ \int_I < z(t), -A^*\psi(t) - (DF(x_o(t)))^*\psi(t) - (B_o(t)DG(x_o(t)))^*\psi(t) >_{X,X^*} dt.\end{aligned} \tag{5.11}$$

Setting

$$\begin{aligned}&-\dot{\psi} = A^*\psi + (DF(x_o(t)))^*\psi + (B_o(t)DG(x_o(t)))^*\psi + \ell_x(t, x_o(t)),\\ &\psi(T) = D\Phi(x_o(T)), t \in I,\end{aligned} \tag{5.12}$$

and using these identities in the expression (5.11) we arrive at the following expression

$$\begin{aligned}&\tilde{L}((B-B_o)G(x_o))\\ &\quad =< z(T), D\Phi(x_o(T) >_{X,X^*} + \int_I < z(t), \ell_x(t, x_o(t)) >_{X,X^*} dt.\end{aligned} \tag{5.13}$$

Note that the expression on the right hand side of the above identity coincides with the functional (5.7) as required by the implications stated in the expression (5.8). Thus the necessary condition given by the inequality (5.1) follows from the expressions (5.4),(5.5) (5.9) and (5.13). The necessary condition(5.2) follows from (5.12). Thus the adjoint variable ψ, whose existence was guaranteed by the continuity of the linear functional $\tilde{L}$ on $L_1(I,X)$, can be determined by solving the backward evolution equation (5.2) (same as equation (5.12)). Since by assumption X is reflexive, the adjoint semi group $\{S^*(t), t \geq 0\}$ is also strongly continuous and hence this equation has a unique mild solution $\psi \in C(I,X^*) \subset L_\infty(I,X^*)$. The necessary condition (5.3) is simply the system equation (4.1) and (4.2) corresponding to the optimal policy B_o; so nothing to prove. This completes the proof. •

Algorithm and its Convergence: Using Theorem 5.1 we can construct the optimal feedback control law. Here we present the necessary steps in the following theorem. For simplicity, we assume that both the state space X and the output space Y are reflexive Banach spaces.

Theorem 5.2 Consider the necessary conditions of optimality (5.1),(5.2),(5.3) given by theorem 5.1 and suppose both X and Y are reflexive Banach spaces. Then there exists a sequence of feedback control laws $\{B_n\}$ in $\mathscr{B}_{ad}$ along which the cost functional J decreases monotonically as indicated below

$$J(B_1) \geq J(B_2) \geq J(B_3) \geq \cdots J(B_{n-1}) \geq J(B_n) \geq \cdots.$$

Proof The proof is based on several steps. Choose an arbitrary $B_1 \in \mathscr{B}_{ad}$ and solve the state equation (5.3) with B_1 replacing B_o and denote the solution by $x_1 \in C(I,X)$. Use the pair $\{B_1,x_1\}$ in place of $\{B_o,x_o\}$ and solve the adjoint equation (5.2) giving $\psi_1 \in C(I,X^*)$. Define the operator valued function C_1 given by the tensor product $C_1(t) \equiv G(x_1(t)) \otimes \psi_1(t) \in Y \otimes X^*, t \in I$. Let $\mathscr{S}_1(Z)$ denote the unit sphere of any real Banach space Z. Choose $v_1^* \in \mathscr{S}_1(Y^*)$ and $w_1 \in \mathscr{S}_1(X)$ so that

$$< v_1^*, G(x_1(t)) >_{Y^*,Y} = \| G(x_1(t)) \|_Y, \text{ and } < w_1, \psi_1(t) >_{X,X^*} = \| \psi_(t) \|_{X^*}.$$

Existence of such a pair follows from [Dunford & Schwartz, [10], p65] and the fact that X is a reflexive Banach space. It is clear from the above expressions that the pair $\{v_1^*, w_1\}$ also depends on $t \in I$. Introduce the operator valued function $D_1(t) \equiv w_1(t) \otimes v_1^*(t) \in \mathscr{L}(Y,X)$ for $t \in I$, and note that $\| D_1(t) \|_{\mathscr{L}(Y,X)} = 1$. Then for any $\varepsilon > 0$, define the operator valued function B_2 as

$$B_2(t) = B_1(t) - \varepsilon D_1(t), t \in I.$$

Choose $\varepsilon > 0$ sufficiently small, so that we have $B_2 \in \mathscr{B}_{ad}$. If the inequality (5.1) holds for $\{B_1, G(x_1), \psi_1\}$, we have B_1 optimal (a rare event). Ignoring this, we continue the algorithm with B_2 and we find that the Gâteaux differential of J at B_1 in the direction $B_2 - B_1$, denoted by $dJ(B_1, B_2 - B_1)$, is given by

$$\begin{aligned} dJ(B_1,B_2-B_1) &= \int_I < (B_2(t)-B_1(t))G(x_1(t)), \psi_1(t) >_{X.X^*} dt, \\ &= -\varepsilon \int < D_1(t)G(x_1(t)), \psi_1(t) >_{X,X^*} dt + o(\varepsilon) \\ &= -\varepsilon \int < v_1^*(t), G(x_1(t)) >_{Y^*,Y} < w_1(t), \psi_1(t) >_{X,X^*} dt + o(\varepsilon) \\ &= -\varepsilon \int \| G(x_1(t)) \|_Y \| \psi_1(t) \|_{X^*} dt + o(\varepsilon). \end{aligned}$$

Computing the cost functional J at B_2 and using the above expression we obtain

$$\begin{aligned} J(B_2) &= J(B_1) + dJ(B_1, B_2 - B_1) + o(\varepsilon) \\ &= J(B_1) - \varepsilon \int \| G(x_1(t)) \|_Y \| \psi_1(t) \|_{X^*} dt + o(\varepsilon). \end{aligned}$$

This shows that for $\varepsilon > 0$ sufficiently small, $J(B_2) < J(B_1)$. To continue the process, use the operator B_2 (in place of B_o) and solve the state equation (5.3) giving x_2, and the adjoint equation

(5.2) corresponding to the pair $\{B_2, x_2\}$ giving ψ_2. Then take $v_2^* \in S_1(Y^*)$ and $w_2 \in S_1(X)$ satisfying

$$< v_2^*, G(x_2(t)) >_{Y^*,Y} = \| G(x_2(t)) \|_Y, \text{ and } < w_2, \psi_2(t) >_{X,X^*} = \| \psi_2(t) \|_{X^*}$$

for $t \in I$. Again,recalling that the pair $\{v_2^*, w_2\}$ may very well depend on $t \in I$, we construct the operator valued function $D_2(t) \equiv w_2(t) \otimes v_2^*(t) \in \mathscr{L}(Y,X), t \in I$. Then for any $\varepsilon > 0$, define the operator valued function B_3 as

$$B_3(t) = B_2(t) - \varepsilon D_2(t), t \in I.$$

Using the above operators in the expression (5.1) and replacing the triple $\{B_o, x_o, \psi\}$ by the triple $\{B_2, x_2, \psi_2\}$, again we find that the directional derivative of the functional J evaluated at B_2 in the direction $B_3 - B_2$ is given by

$$\begin{aligned} dJ(B_2, B_3 - B_2) &= \int_I < (B_3(t) - B_2(t))G(x_2(t)), \psi_2(t) >_{X.X^*} dt, \\ &= -\varepsilon \int << D_2, G(x_2) \otimes \psi_2 >> dt + o(\varepsilon) \\ &= -\varepsilon \int \| G(x_2) \|_Y \| \psi_2 \|_{X^*} dt + o(\varepsilon). \end{aligned}$$

Computing the cost functional corresponding to B_3, we find that

$$\begin{aligned} J(B_3) &= J(B_2) + dJ(B_2, B_3 - B_2) + o(\varepsilon) \\ &= J(B_2) - \varepsilon \int \| G(x_2) \|_Y \| \psi_2 \|_{X^*} dt + o(\varepsilon). \end{aligned}$$

This shows that for $\varepsilon > 0$ small, $J(B_3) < J(B_2)$. Repeating this step-by-step process ad infinitum, we arrive at a sequence $\{B_n\}$ in $\mathscr{B}_{ad}$ satisfying

$$J(B_1) \geq J(B_2) \geq J(B_3) \geq \cdots J(B_{n-1}) \geq J(B_n) \geq \cdots.$$

This completes the proof. •

Some Recent Developments: Here we present references to some notable recent developments on systems governed by differential equations on infinite dimensional Banach spaces and their optimal control. In a recent book [6], we developed optimal control theory for systems driven by vector measures. In [5], we considered optimal feedback control problems for stochastic systems on Hilbert spaces presenting solutions to a broad range of standard and nonstandard optimal control problems. We have also studied a very large class of nonlinear deterministic as well as stochastic systems admitting unbounded and measurable vector fields [7]. These systems do not admit classical, strong, or even mild solutions. It is shown that they have only generalized (or measure valued) solutions which can be controlled to optimize both standard and nonstandard objective functionals. For optimal control of stochastic systems on Hilbert spaces subject to Brownian motion and Lévy process admitting only generalized solutions (measure-valued), interested reader is referred to the recent paper of the author [4].

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