Commun. Optim. Theory 2025 (2025) 8 https://doi.org/10.23952/cot.2025.8

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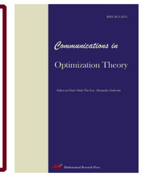

GEOMETRIC APPROACHES IN QLPV FDI PROBLEMS

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Dedicated to the memory of Professor Josef Shinar on the occasion of his 90th birthday

Abstract. This paper gives a fairly complete overview on geometric approaches to design detection filters for unknown signals representing possible changes or faults in a dynamic system. The focus is put on the qLPV system representation since this can include LTI, LTV and certain input affine nonlinear systems. Detection algorithms for filter design will be presented and the FPRG problem will be solved based on the concept of LPV invariant subspaces, inversion based approaches and also null-space based methods will be elaborated.

Keywords. Fault detection; LPV system; Geometric approach; Model based.

2020 Mathematics Subject Classification. 93B27.

1. Introduction

Detection of unknown inputs affecting dynamical systems appears in various applications. One is called fault detection (FD) and isolation (FDI) where the changes, interpreted as faults in some cases, enter the system as an additional disturbance (fault) signal. The goal is then either to decouple the effect of these signals from certain sensor measurements (outputs) of the system or to detect (reconstruct) their occurrence based on the available information.

A prototype problem for FDI was formulated originally in the linear time invariant (LTI) framework as the fundamental problem of residual generation (FPRG). This was solved using geometric approaches based on state-space modelling and construction of suitable invariant subspaces [33]. The design ended in fault detection filters that were thoroughly analysed for sensitivity, robustness and performance properties. Fault detection and isolation is also relevant for constructing safety-related control systems. Here the estimated fault signal can be used in control reconfiguration. Unlike robust control, reconfigurable control systems allow to reuse the fault related information in order to modify the control law. Related algebraic approaches include unknown input observer (UIO) design and parity space methods. Statistical approaches rely on generalized likelihood, multiple model and parameter estimation strategies.

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Received June 2, 2023; Accepted February 18, 2024.

The motivation writing the paper comes from a problem specified in [40], where the goal is detection of unknown signals representing random manoeuvring of a target in a pursuit-evading game [41, 42]. This problem was defined first in terms of an LTI state-space framework and a detection filter was designed using geometric methods. Since the problem which was set up needs nonlinear modeling originally, a next requirement was to look for finding solutions for the nonlinear case. While the geometric theory for nonlinear FPRG had existed [13], the implementation was not straightforward at all. Construction of invariant subspaces for affine linear parameter varying (qLPV) systems was already available, see [2, 5]. This is why the embedding of nonlinear state-space models into qLPV models was a promising solution.

This effort inspired the development of several solutions to the qLPV FPRG problem [7, 8, 16, 17, 45, 48, 50, 51]. These will be summarized later in this paper providing some simple examples for illustration.

After some general considerations given in Section 2 in Section 3 we provide a short introduction to LPV modeling and give the elements of an I/O interpretation of an LPV systems. Based on this interpretation in Section 4 the fundamental problems concerning LPV fault detection are formulated. Section 5 provides an overview of geometric tools and solutions such as FPRG and inversion based FD which rely on the concept of parameter-varying invariant subspaces. Finally Section 6 presents some techniques for LPV null-space computation.

2. General considerations

There are several different abstraction levels in the modeling of plants at different stages of engineering design. This ranges from simple static models to complex nonlinear dynamic models incorporating lookup tables, multibody physics or partial differential equations solved by computational fluid dynamics and finite element methods.

Control design and the related model based FDI methods require the use of high accuracy models, described by ODEs. The plant in general is described in the form

$$\dot{x}(t) = f(x(t), u(t), d(t), \nu(t)) \tag{2.1}$$

$$y(t) = h(x(t), u(t), d(t), \nu(t)) \tag{2.2}$$

where $x \in \mathscr{X} \subset \mathbb{R}^n$ is the state, $u \in \mathscr{U} \subset \mathbb{R}^{m_u}$ is the control input and $y \in \mathscr{Y} \subset \mathbb{R}^p$ is the measured output, respectively. The fault signals to be detected are represented by the vector $\nu \in \mathscr{F} \subset \mathbb{R}^{m_f}$, while $d \in \mathscr{D} \subset \mathbb{R}^{m_d}$ is the disturbance vector, which can capture sensor noise, unmodelled dynamics, etc.

The basic objective of a fault detection methodology applied to dynamic systems is to provide techniques for detection and isolation of failed components. The development of model based diagnostic systems is aimed at detecting incipient faults, and following permanently the state of the supervised process on the basis of some a priori information of the plant dynamics. This a priori information is captured in the form of a mathematical representation which is called model. Most of the model-based methods rely on the idea of analytical redundancy in which, — in contrast to physical or hardware redundancy — real physical measurements are supplemented with analytically computed redundant variables.

In order to achieve fault detection, a residual $r(t)$ is to be generated that is related to the fault ν: it should be equal to zero in the fault-free case and different from zero if a fault occurs, i.e.,

$$\begin{cases} \nu(t) = 0 \Rightarrow r(t) = 0 \\ \nu(t) \neq 0 \Rightarrow r(t) \neq 0 \end{cases} \quad \forall u(t), y(t), d(t). \tag{2.3}$$

If simultaneous faults might be present in the system, in order to achieve fault detection and isolation (FDI), the conditions

$$\begin{cases} \nu_i(t) = 0 \Rightarrow r_i(t) = 0 \\ \nu_i(t) \neq 0 \Rightarrow r_i(t) \neq 0, \, r_j(t) = 0 \end{cases} \tag{2.4}$$

are required additionally.

However, in real systems, the residual will not be exactly equal to zero for the fault-free case due to noise and modelling errors. To deal with this case a milder set of conditions is imposed, i.e., each residual $r_i(t)$ must fulfil the following conditions:

- robustness to the disturbance $d(t)$
- sensitivity to the fault(vector) ν_i
- asymptotic convergence in the fault free case, i.e., if $\nu_i = 0$, then $lim_{t\to\infty} r_i(t) = 0$.

This set of conditions is known as the FPRG.

In the solution of these problems two major directions compete: the one that uses differential geometric tools (geometric methods) while the other is based on algebraic tools (parity space methods). In the algebraic approach applying a successive derivation of the measured outputs one tries to eliminate the unknown state variable in order to obtain differential polynomial relations for the faults ν_i ($d = 0$) of the type

$$R_j(\nu_i, u, \ldots, \nu_l^{(\gamma_j)}, u^{(\gamma_j)}, y^{(\beta_j)}) = P_j(u, \ldots, u^{(\beta_j)}, y^{(\beta_j)}), \tag{2.5}$$

where R_j are candidates to be residuals. Then the subspace spanned by the differential polynomials P_j is called the parity space while the relation that corresponds to each element of this space is a parity function. For some of the systems it is possible to factorize R_j in a fault dependent and a fault independent term; as a very special case one has $R_j = \nu_i L_j(u, \ldots, u^{(\gamma_j)}, y^{(\gamma_j)})$. There is a freedom in the choice of the parity function which can be used to fulfill additional requirements concerning sensitivity of the detection and robustness against external disturbances ($d \neq 0$), see, e.g., [4, 23, 38, 43].

In the LTI case the parity relations are often interpreted in a temporal way instead of considering them merely as an algebraic relation. In this context some moving average models can be formulated, see, e.g., [11, 24]. In contrast to the linear case, where the state is eliminated by a projection on a suitable subspace, for nonlinear and qLPV systems this step might be quite involved. Moreover, with nonconstant, nonlinear coefficients the elimination will be harder and in general the relations will be much complicated, which implies limited practical applicability. Concerning the nonlinear realization problem – due to the lack of a general theory and working algorithms – the situation is even worse. Therefore, usually, a more definite problem is tackled: the set of input affine systems (with respect to the faults) is considered, and the residuals are defined to be the fault signals, i.e., $r_i(t) = \nu_i(t)$ when the corresponding problem involves the notion of the left invertibility of the systems.

Since the theory of robust control of nonlinear systems is, generally, computationally not tractable useful progress is required in an intermediate level of complexity that, for example, incorporates scheduling requirements whilst still remaining computationally feasible. One way of systematically handling gain-scheduling control analysis and design techniques is the use of linear parameter varying (LPV) plant models. The computational techniques for obtaining controllers and filters in LPV form rely on the solution of a set of linear matrix inequalities (LMIs) which, although challenging, are convex problems with reliable solvers available.

3. LPV MODELING

Rooted in the idea of gain scheduling, linear parameter varying (LPV) modelling has proven to be an efficient approach in many areas of control and filtering in treating nonlinear problems in the past decades. A broad class of nonlinear system models can be converted into a quasi-linear form, obtaining the system $\Sigma_\rho^u|_{x_0}$:

$$\dot{x}(t) = A(\rho)x(t) + B(\rho)u(t), \quad x(0) = x_0, \tag{3.1}$$

$$y(t) = C(\rho)x(t) + D(\rho)u(t), \tag{3.2}$$

where $x \in \mathscr{X} \subset \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the input and output functions, respectively, while $\rho \in \mathscr{P}$ is the vector of scheduling functions, which are determined by the measured variables. This means that their values are known in operational time by measurement. The scheduling variables in LPV models are often in fact system states (e.g. airspeed, height or angle of attack), rather than bounded external variables. These are called quasi-LPV models.

The approach is particularly appealing when a natively nonlinear problem, embedded in the LPV framework, can be solved by using traditional linear techniques. Accordingly, in the sequel we will use the following notation for a (q)LPV system:

$$\Sigma_\rho^u|_{x_0} = \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & D(\rho) \end{array}\right]_{x_0}, \tag{3.3}$$

where $x_0 = 0$ will be omitted from the notation.

Depending on the actual model, the collection of the allowed scheduling variable might vary from a subset $\mathscr{P}$ of the measurable functions (when we actually have a switched system) to a subset of constant functions (when we actually have a class of LTI systems: more precisely an LTI system with some uncertain parameters). The later class should not be confused with the class of (q)LPV systems and, as a collection of LTI systems, is not considered in the sequel.

In order to decrease the conservativeness of the design, often the elements of $\mathscr{P}$ are also supposed to be sufficiently smooth, taking values from a compact set $\mathscr{P}$. Usually smooth means that the scheduling parameter is of class C^1, i.e., it has a continuous derivative. It is a standard assumption that $\mathscr{P}$ is of box type, i.e., each parameter ρ_i ranges between its known extremal values $\rho_i(t) \in [\underline{\rho}_i, \overline{\rho}_i]$. While the derivatives of the scheduling variables usually are not measured, in control design problems they are supposed to be bounded, i.e., $\rho^{(i)}(t) \in \mathscr{P}_i \subseteq \mathbb{R}^{n_\rho}$. Typically, $i = 1$. We will denote by $\mathscr{P}_\infty$ the case when the scheduling variables are measurable functions taking values from $\mathscr{P}$, while $\mathscr{P}_1$ stands for the case when the scheduling variables are smooth, their values being constrained by the condition $(\rho, \dot{\rho}) \in \mathscr{P} \times \mathscr{P}_1$, respectively.

For LTV systems [29] sets the fundamental concept: a state transformation $\xi = T(t)x$ with nonsingular $T(t)$ on the time axis defines an equivalent system (algebraic equivalence), while if both $T(t)$ and $T^{-1}(t)$ is bounded we have a so-called topological equivalence that preserves stability. Having an LPV system it is natural to consider parameter varying state transformations, i.e., $\tilde{x} = T(\rho)x$ for $\rho \in \mathscr{P}$ that leads to

$$\tilde{\Sigma}(\rho) \sim \left[\begin{array}{c|c} \tilde{A}(\rho,\dot{\rho}) & \tilde{B}(\rho) \\ \hline \tilde{C}(\rho) & \tilde{D}(\rho) \end{array}\right] = \left[\begin{array}{c|c} T(\rho)A(\rho)T^{-1}(\rho)+\dot{T}(\rho)T^{-1}(\rho) & T(\rho)B(\rho) \\ \hline C(\rho)T^{-1}(\rho) & D(\rho) \end{array}\right], \tag{3.4}$$

provided that the scheduling variables are smooth.

If the system matrices are sufficiently smooth, [44] provides additional details. Considering the controllability and observability matrices, i.e.,

$$Q_k(t) = \begin{bmatrix} P_0(t) & \cdots & P_{k-1}(t) \end{bmatrix} \quad \text{and} \quad R_k^T(t) = \begin{bmatrix} S_0^T(t) & \cdots & S_{k-1}^T(t) \end{bmatrix},$$

respectively, where

$$P_{i+1}(t) = -A(t)P_i(t) + \frac{d}{dt}P_i(t), \quad P_0(t) = B(t),$$
$$S_{i+1}(t) = S_i(t)A(t) + \frac{d}{dt}S_i(t), \quad S_0(t) = C(t),$$

we have that the constant rank system representation is completely controllable (observable) if there exists integers α (β) such that if $\text{rank}Q_k(t) = n$ ($\text{rank}R_k(t) = n$) for all t and $k \geq \alpha$ ($k \geq \beta$), where n is the dimension of the state. Moreover, two controllable constant rank system representations (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ of order n are algebraically equivalent if and only if $\bar{P}_k(t) = T(t)P_k(t)$ and $\bar{C}(t) = C(t)T^{-1}(t)$, i.e, $T(t) = \bar{Q}_\gamma(t)Q_\gamma^\dagger(t)$ for all t, and $\gamma = \max\{\bar{\alpha},\alpha\}$. By duality we have an analogous statement for observability. Note, that while it is hard to test it in practice, in the context of LPV systems it is desirable to have a constant rank representation for every ρ. E.g., considering the restriction of $\mathscr{P}$ to constant functions and the minimality of the resulting LTI representations it is desirable to have the same state space dimension.

Constant state transformations are intimately related to the concept of invariant subspace known from the geometric theory of LTI systems and they were extended to LPV dynamics by introducing the notion of parameter-varying invariant subspace, see [2]. In introducing the various parameter-varying invariant subspaces an important goal was to set notions that lead to computationally tractable algorithms for the case when the parameter dependency of the system matrices is affine. These invariant subspaces play the same role in the solution of the fundamental problems, such as disturbance decoupling, unknown input observer design, fault detection, as their counterparts in the time invariant context, see [48, 5].

LTI systems are often represented through their transfer functions. While transfer functions are tied to frequency domain, nothing prevents us to identify them with time domain input-output (I/O) operators that stands for the same LTI system. In the time varying context we do not have a sound frequency domain description and transfer functions. Traditionally the LPV framework is formulated in terms of a state space representation. There is a possibility, however, to develop a sound input-output (I/O) description related to this model class. For a motivating example see the LPV null space generator methods, [49], and its applications to fault detection and reconfiguration [37]. Viewed at a fixed parameter trajectory ρ, the linear system that obeys to (3.1)-(3.2) can be cast as a linear time varying (LTV) system. Thus, it is convenient to consider the LPV system as a collection of time varying systems, e.g.:

$$\left\{\Sigma_\rho^u \mid \rho \in \mathscr{P}\right\}. \tag{3.5}$$

In this sense we can talk on Σ_ρ^u as an I/O map, regardless to the possible/actual state space representation. Thus, by a slight abuse of the notation, in what follows we identify and denote the LPV system as $(\Sigma_\rho^u, \mathscr{P})$. If it is clear from the context, we will drop $\mathscr{P}$ from the notation. We emphasise, however, that the parameter set – as a collection of different parameter trajectories – is an essential part in the definition of the LPV system. Note, that while this embedding of the LPV plant as a class of LTV systems bears significant advantages there are some issues that evade the LTV framework.

Thus, the well known algebraic operations as $\Gamma(\rho_1)+\Sigma(\rho_2)$ or $\Gamma(\rho_1)\Sigma(\rho_2)$, make sense among the corresponding qLPV systems provided that the input(output) dimensions are compatible.

We arrive here to some problematic points which makes a clear difference between LTV and qLPV systems. If ρ is not smooth, e.g., the LPV system is in class $\mathscr{P}_\infty$, then state transformations might send the system description to a different class, namely to the class of impulsive systems. As we have already seen, if the system is sufficiently smooth, the system equivalence is exhausted by transformations that might depend on up to the $(n-1)^{\text{th}}$ derivative of the scheduling variable. Even ρ is supposed to be smooth, such a state transformation will send our description outside the LPV framework, in general. Actually there are two problems here. The first problem is more apparent: even if we allow reparametrization (inflation of the parameter space) the type of the problem might change. Starting from an $\mathscr{P}_1$ type system we might obtain an $\tilde{\mathscr{P}}_\infty$ type of system. The second problem is more subtle: derivation is not a causal operation, thus we violate our assumption on the availability of the information. To make this point more clear, consider the same transformation in discrete time:

$$A(\rho_k) \mapsto T(\rho_{k+1})A(\rho_k)T^{-1}(\rho_k), \quad B(\rho_k) \mapsto T(\rho_{k+1})B(\rho_k)$$

Stability of an LPV system can be defined in a straightforward manner: the LPV system is stable if for each $\rho \in \mathscr{P}$ the LTV system $\Sigma(\rho)$ is stable. Observe that stability of LPV systems are tight to the parameter set. The two cases encountered in practice are $\mathscr{P}_\infty$ and $\mathscr{P}_1$, respectively. We prefer to term the first case as strong stability (as a hint for switched systems) and as parameter varying stability the second. Closely related to stability is the concept of stabilizability, i.e., the ability to obtain a stabilizing controller K. For practical reasons this concept is traditionally closely related to a state space representation of the linear system and it boils down to finding a stabilizing parameter varying state feedback gain.

Analogous to the LTI case we call a pair $(A(\rho),B(\rho))$ stabilizable if there is a (state) feedback gain $F(\rho)$ such that the LTV system $\dot{x} = (A(\rho)+B(\rho)F(\rho))x$ is exponentially stable provided that $\rho \in \mathscr{P}$. The pair $(A(\rho),C(\rho))$ is detectable if there is a (output injection) gain $L(\rho)$ such that the LTV system $\dot{x} = (A(\rho)-L(\rho)C(\rho))x$ is exponentially stable provided that $\rho \in \mathscr{P}$.

From a practical point of view it is important to know how to test the stability and how to design the corresponding stabilizing gain (output injection). An obvious choice is to consider a quadratic Lyapunov function $V(x) = x^T Qx$, $(P = Q^{-1})$, and the quadratic stability tests:

$$(A(\rho)+B(\rho)F(\rho))Q+Q(A(\rho)+B(\rho)F(\rho))^T < 0, \quad Q > 0, \tag{3.6}$$

and

$$P(A(\rho)-L(\rho)C(\rho))+(A(\rho)-L(\rho)C(\rho))^T P<0, \quad P>0, \tag{3.7}$$

for all $\rho \in \mathscr{P}$.

Another choice might be a quadratic parameter dependent Lyapunov function $V(x)=x^T Q(\rho)x$, $(P(\rho)=Q^{-1}(\rho))$, and the parameter dependent quadratic stability tests:

$$(A(\rho)+B(\rho)F(\rho))Q(\rho)+Q(\rho)(A(\rho)+B(\rho)F(\rho))^T+\frac{dQ(\rho)}{dt}<0, \quad Q(\rho)>0, \tag{3.8}$$

and

$$P(\rho)(A(\rho)-L(\rho)C(\rho))+(A(\rho)-L(\rho)C(\rho))^T P(\rho)+\frac{dP(\rho)}{dt}<0, \quad P(\rho)>0, \tag{3.9}$$

for all $\rho \in \mathscr{P}$.

In contrast to the fact that for linear systems stability is a global property, stability of qLPV systems has only a local validity due to the constraints imposed by the parameter space $\mathscr{P}$. Due to the nonlinearity of the underlying system this is not surprising. If the systems involved are stable then the measurement values also tend to zero, so the assumption that $\rho \in \mathscr{P}$ is automatically fulfilled. However, if the original system is started from an initial condition such that the corresponding ρ is outside the admissible set $\mathscr{P}$, the embedding is broken. Thus, a stabilizing gain (output injection) ensures stability of the original system only on a stability domain that is compatible with $\mathscr{P}$.

Recall that, analogous to the LTI case, having a stabilizing state feedback gain F and a stabilizing output injection gain L one has the following double coprime factorization result:

Proposition 3.1. *If the qLPV system $\Sigma(\rho) \sim \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & D(\rho)\end{array}\right]$ is stabilizable and detectable with the gain $F(\rho)$ and output injection gain $L(\rho)$ then*

$$\begin{pmatrix} \tilde{V}_\rho & -\tilde{U}_\rho \\ -\tilde{N}_\rho & \tilde{M}_\rho \end{pmatrix}\begin{pmatrix} M_\rho & U_\rho \\ N_\rho & V_\rho \end{pmatrix}=\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{3.10}$$

with

$$\begin{pmatrix} M_\rho & U_\rho \\ N_\rho & V_\rho \end{pmatrix}=\left[\begin{array}{c|cc} A(\rho)+B(\rho)F(\rho) & B(\rho) & L(\rho) \\ \hline F(\rho) & I & 0 \\ C(\rho)+D(\rho)F(\rho) & D(\rho) & I \end{array}\right] \tag{3.11}$$

and

$$\begin{pmatrix} \tilde{V}_\rho & -\tilde{U}_\rho \\ -\tilde{N}_\rho & \tilde{M}_\rho \end{pmatrix}=\left[\begin{array}{c|cc} A(\rho)-L(\rho)C(\rho) & -B(\rho)+L(\rho)D(\rho) & -L(\rho) \\ \hline F(\rho) & I & 0 \\ C(\rho) & -D(\rho) & I \end{array}\right]. \tag{3.12}$$

Moreover, the qLPV system Σ_ρ has a right and left (doubly) coprime factorization as follows:

$$\Sigma_\rho=N_\rho M_\rho^{-1}=\tilde{M}_\rho^{-1}\tilde{N}_\rho. \tag{3.13}$$

Proof. Recall the system multiplication formula

$$\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22}\end{array}\right]\left[\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22}\end{array}\right]=\left[\begin{array}{cc|c} M_{11} & M_{12}N_{21} & M_{12}N_{22} \\ 0 & N_{11} & N_{12} \\ \hline M_{21} & M_{22}N_{21} & M_{22}N_{22}\end{array}\right].$$

Then, one has

$$\left[\begin{array}{c|cc} A(\rho)-L(\rho)C(\rho) & -B(\rho)+L(\rho)D(\rho) & -L(\rho) \\ \hline F(\rho) & I & 0 \\ C(\rho) & -D(\rho) & I \end{array}\right]\left[\begin{array}{c|cc} A(\rho)+B(\rho)F(\rho) & B(\rho) & L(\rho) \\ \hline F(\rho) & I & 0 \\ C(\rho)+D(\rho)F(\rho) & D(\rho) & I \end{array}\right]=$$

$$=\left[\begin{array}{cc|cc} A(\rho)-L(\rho)C(\rho) & -B(\rho)F(\rho)-L(\rho)C(\rho) & -B(\rho) & -L(\rho) \\ 0 & A(\rho)+B(\rho)F(\rho) & B(\rho) & L(\rho) \\ \hline F(\rho) & F(\rho) & I & 0 \\ C(\rho) & C(\rho) & 0 & I \end{array}\right]$$

and

$$\left[\begin{array}{c|cc} A(\rho)+B(\rho)F(\rho) & B(\rho) & L(\rho) \\ \hline F(\rho) & I & 0 \\ C(\rho)+D(\rho)F(\rho) & D(\rho) & I \end{array}\right]\left[\begin{array}{c|cc} A(\rho)-L(\rho)C(\rho) & -B(\rho)+L(\rho)D(\rho) & -L(\rho) \\ \hline F(\rho) & I & 0 \\ C(\rho) & -D(\rho) & I \end{array}\right]=$$

$$=\left[\begin{array}{cc|cc} A(\rho)+B(\rho)F(\rho) & B(\rho)F(\rho)+L(\rho)C(\rho) & B(\rho)-L(\rho)D(\rho) & L(\rho) \\ 0 & A(\rho)-L(\rho)C(\rho) & -B(\rho)+L(\rho)D(\rho) & -L(\rho) \\ \hline F(\rho) & F(\rho) & I & 0 \\ C(\rho)+D(\rho)F(\rho) & C(\rho)+D(\rho)F(\rho) & 0 & I \end{array}\right].$$

Using a change of state variable one can observe that the zero initial state response of this system becomes the identity map. $\square$

Given a double coprime factorization the set of the stabilizing controllers is provided through the extension of the well-known Youla parametrization:

$$\mathscr{K}_{stab}=\{K=\mathfrak{M}_{\Sigma_P}(Q)\mid Q\in\mathbb{Q},(V+NQ)^{-1}\text{ exists}\},$$

where $\mathbb{Q}=\{Q\,|\,Q \text{ stable }\}$ and $\mathfrak{M}_{\Sigma_P}(Q)=(U+MQ)(V+NQ)^{-1}$. Note that $Q=\mathfrak{M}_{\tilde{\Sigma}_P}(K)$ and thus $Q=0_K$ corresponds to $K_0=UV^{-1}$, see [46, 47]. It is obvious that the entire scheme remains valid for the LPV framework, too. Then Q is interpreted as a qLPV system. The only constraint is to respect the stability concept set by the given qLPV model, i.e., by the parameter set. In practice one can use either the strong stability or the parameter varying stability, when selecting the elements of the parametrization.

Some authors prefer to qualify stability of qLPV systems, and hence the corresponding Youla parametrization, according to our ability to provide for them a (quadratic) Lyapunov function, see, e.g., [59]. Thus, if we have a stability guarantee for some $\mathscr{P}_\infty$ proved by a constant Lyapunov matrix then we have quadratic stability. For $\mathscr{P}_1$ a parameter varying Lyapunov matrix is associated, and the corresponding stability is termed as parameter varying quadratic stability. Note, that these stability tests provide only sufficient conditions.

4. Fault detection for LPV systems

Let us consider the fault affected qLPV system $\Sigma_\rho|_{x_0}$:

$$\dot{x}(t)=A(\rho)x(t)+B_u(\rho)u(t)+B_d(\rho)d(t)+B_f(\rho)\nu(t),\quad x(0)=x_0 \tag{4.1}$$

$$y(t)=C(\rho)x(t)+D_u(\rho)u(t)+D_d(\rho)d(t)+D_f(\rho)\nu(t), \tag{4.2}$$

where $x\in\mathscr{X}\subset\mathbb{R}^n$ is the state, $u\in\mathscr{U}\subset\mathbb{R}^{m_u}$ is the control input and $y\in\mathscr{Y}\subset\mathbb{R}^p$ is the measured output, respectively. The fault signals to detect are represented by the vector $\nu\in$

$\mathscr{F} \subset \mathbb{R}^{m_f}$, while $d \in \mathscr{D} \subset \mathbb{R}^{m_d}$ is the disturbance vector. It is assumed that the system $\Sigma_\rho^u|_{x_0}$ is stabilizable and detectable.

By linearity the available information can be summarized as:

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \Sigma_\rho^u|_{x_0} \\ I \end{pmatrix} u + \begin{pmatrix} \Sigma_\rho^d|_0 \\ 0 \end{pmatrix} d + \begin{pmatrix} \Sigma_\rho^f|_0 \\ 0 \end{pmatrix} \nu \tag{4.3}$$

Applying a left coprime factorization for $x_0 = 0$ one has:

$$\begin{pmatrix} \tilde{M}_\rho y \\ u \end{pmatrix} = \begin{pmatrix} \tilde{N}_\rho^u \\ I \end{pmatrix} u + \begin{pmatrix} \tilde{N}_\rho^d \\ 0 \end{pmatrix} d + \begin{pmatrix} \tilde{N}_\rho^f \\ 0 \end{pmatrix} \nu, \tag{4.4}$$

i.e.,

$$\begin{pmatrix} I & -\tilde{N}_\rho^u \end{pmatrix} \begin{pmatrix} \tilde{M}_\rho y \\ u \end{pmatrix} = \tilde{N}_\rho^d d + \tilde{N}_\rho^f \nu. \tag{4.5}$$

It follows that

$$\underbrace{F_\rho(\, y - \hat{y}\,)}_{\text{implemented filter}} = \underbrace{F_\rho \begin{bmatrix} \tilde{N}_\rho^f & \tilde{N}_\rho^d \end{bmatrix} \begin{bmatrix} \nu \\ d \end{bmatrix}}_{\text{designed residual}} \tag{4.6}$$

settles the possibilities of any FDI scheme, where $y - \hat{y} = \tilde{M}_\rho y - \tilde{N}_\rho^u u$ is the output mismatch.

The stable filter F_ρ is designed in accordance to the desired properties of the FDI filter, i.e., to ensure fault identifiability in presence of disturbances. Such a choice can be performed based on a geometrical design or based on some optimization criteria (norm based design). In the LTI case and in some extent for the qLPV case residual design can be completely decoupled from the design of the control loop provided that there is no uncertainty (nominal case). In the qLPV case the closed loop influences the trajectories of the scheduling parameter ρ, hence, implicitly, the behavior of the plants $\tilde{N}_\rho^f$ and $\tilde{N}_\rho^d$.

It is seldom possible to perfectly decouple the effect of faults from disturbances, i.e., to achieve $F_\rho \tilde{N}_\rho^d = 0$ while $F_\rho \tilde{N}_\rho^f \neq 0$. To achieve robustness in the presence of disturbances and uncertainty, optimization-based FDI schemes have been proposed where an appropriately selected performance index is chosen to enhance sensitivity to the faults (detection performance) and simultaneously attenuate disturbances (disturbance rejection). While traditional methods that try to enhance the robustness of the detection filters against perturbations and model uncertainties via eigenstructure assignment applied in the linear context, e.g., [35] for parity space methods or [10, 14] for observer based methods, are not applicable in the general nonlinear setting. For LPV systems the robustness issue can be handled in certain circumstances using $\mathscr{H}_\infty$ theory, see, e.g., [15, 17, 22, 25, 32, 52, 57].

4.1. **Fault models.** To obtain fault models the types of faults are categorized into actuator faults and sensor faults, which are defined through the directions L_i, J_i, while the associated signals $(\nu_i(t), \mu_i(t))$ are unknown:

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) + \sum_{i=1}^{k} L_i(\rho(t))\nu_i(t) \tag{4.7}$$

$$y(t) = C(\rho(t))x(t) + \sum_{i=1}^{q} J_i(\rho(t))\mu_i(t) \tag{4.8}$$

When no failure or disturbance is present, $\nu_i(t)$ and $\mu_i(t)$ are all, by definition, equal to zero. We refer to the functions $\nu_i(t)$ and $\mu_i(t)$ as failure modes.

Note, that the sensor fault occurs on the output channel of the system, which is inherently difficult to handle. The difficulty arises from the fact that in this case some columns of the observer gain matrix are the fault signatures, hence sensor failure treatment requires special attention. Often it is convenient that sensor failures be treated the same way as actuator failures. One possible approach to overcome this problem is to model the sensor fault as pseudo-actuator fault through appropriate state augmentation. It can be assumed that the unknown function $\mu_i(t)$ is the output of some linear time-invariant system, i.e., we can assume without loss of generality that

$$\dot{\mu}_i(t) = a_i \mu_i(t) + s_i(t) \tag{4.9}$$

for some scalars a_i and unknown functions $s_i(t)$. If the dynamics of the systems generating the sensor failure modes are added to the dynamics of the system, the sensor failures can be represented as actuator failures.

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x(t) \\ \mu \end{bmatrix} + \begin{bmatrix} B(\rho(t)) & L(\rho(t)) & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} u \\ \nu \\ s \end{bmatrix}, \tag{4.10}$$

$$\bar{y} = \begin{bmatrix} C(\rho(t)) & J(\rho(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ \mu \end{bmatrix}. \tag{4.11}$$

In this augmented representation, $s(t)$ appears as a pseudo-actuator failure mode and, consequently, no sensor failure signature will be present.

Another approach is to choose a $C_c(\rho)$ which makes

$$T(\rho) = \begin{bmatrix} C(\rho) \\ C_c(\rho) \end{bmatrix} \tag{4.12}$$

nonsingular and apply the transformation $\tilde{x} = T(\rho)x$, which leads to

$$\dot{\tilde{x}}(t) = \bar{A}(\rho)\tilde{x}(t) + \bar{B}(\rho)u(t) + \bar{L}_\nu(\rho)\nu(t) \tag{4.13}$$

$$y(t) = \bar{C}\tilde{x}(t) + J(\rho(t))\mu, \tag{4.14}$$

with

$$\bar{A}(\rho) = T(\rho)A(\rho)T^{-1}(\rho) + \dot{T}(\rho)T^{-1}(\rho), \quad \bar{B}(\rho) = T(\rho)B(\rho), \quad \bar{L}_\nu(\rho) = T(\rho)L(\rho),$$

and $\bar{C} = \begin{bmatrix} I & 0 \end{bmatrix}$. Then, consider $C(\rho)\ell(\rho) = J(\rho)$, i.e.,

$$y(t) = \bar{C}\tilde{x}(t) + J(\rho)\mu(t) = \bar{C}(\tilde{x}(t) + \ell(\rho)\mu(t)) = \bar{C}\bar{x}(t). \tag{4.15}$$

Such an $\ell(\rho)$ exists, since $\bar{C}$ is assumed to be of full raw rank:

$$\ell(\rho) = [\bar{C}^T(\bar{C}\bar{C}^T))^{-1} + K(\rho)]J(\rho), \quad \text{with any} \quad \bar{C}K(\rho) = 0. \tag{4.16}$$

Thus, with

$$\bar{L}_{\bar{\mu}}(\rho) = \begin{bmatrix} \dot{\ell}(\rho) - \bar{A}(\rho)\ell(\rho) & \ell(\rho) \end{bmatrix}, \qquad \bar{\mu} = \begin{bmatrix} \mu \\ \dot{\mu} \end{bmatrix}, \tag{4.17}$$

we have

$$\dot{\bar{x}}(t) = \bar{A}(\rho)\bar{x}(t) + \bar{B}(\rho)u(t) + \bar{L}_\nu(\rho)\nu(t) + \bar{L}_\mu(\rho)\bar{\mu}(t), \tag{4.18}$$

$$y(t) = \bar{C}\bar{x}(t). \tag{4.19}$$

Note that (4.17) is the generalization of the LTI result, that was formulated in Proposition A.1 of [31], to qLPV systems.

Hence, after a possible reparametrization, all the analysis that follows can use the model

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) + \sum_{i=1}^{k} L_i(\rho(t))\nu_i(t) \tag{4.20}$$

$$y(t) = Cx(t). \tag{4.21}$$

We shall restrict our attention to the case when C is constant, since invariant subspaces, required for geometric LPV FDI approaches, can be calculated for this case only. However, this restriction is not conservative, since in most of the applications many of the states are directly measurable, via time invariant sensor dynamics. It is also assumed that $A(\rho(t))$, $B(\rho(t))$, and $L_i(\rho(t))$ depend affinely on the time varying scheduling parameter $\rho(t)$:

$$A(\rho(t)) = A_0 + \rho_1(t)A_1 + \ldots + \rho_n(t)A_n \tag{4.22}$$

$$B(\rho(t)) = B_0 + \rho_1(t)B_1 + \ldots + \rho_n(t)B_n \tag{4.23}$$

$$L_i(\rho(t)) = L_{i,0} + \rho_1(t)L_{i,1} + \ldots + \rho_n(t)L_{i,n} \tag{4.24}$$

4.2. **Example: pursuit-evasion.** As an example let us consider the linearized pursuit-evasion problem, [31], which can be expressed by the following fault-corrupted linear parameter-varying system

$$\dot{x}(t) = Ax(t) + Bu(t) + Lv(t),$$
$$y(t) = C(\rho)x(t) + e_1\mu(t),$$

where the states are the relative position, relative speed, evader acceleration, and pursuer acceleration. $u(t)$ is the pursuer command, $v(t)$ is the unknown evader command signal, and $\mu(t)$ is the sensor noise on the first measurement, $y_1(t)$.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -ba_{\mathrm{P}}^{\max} \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ 0 \\ aa_{\mathrm{E}}^{\max} \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 \end{bmatrix}, \quad \text{with} \quad a = -\frac{1}{\tau_{\mathrm{E}}}, \quad \text{and} \quad b = -\frac{1}{\tau_{\mathrm{P}}},$$

where τ_{E} and τ_{P} are the time constants and $a_{\mathrm{E}}^{\max}$ and $a_{\mathrm{P}}^{\max}$ are the maximal achievable acceleration of the evader and the pursuer, respectively. The (control and evader) inputs are normalized. Moreover, $\rho = 1/\left(t_f - t\right)$, thus $\dot{\rho} = \rho^2$, where t_f is a given parameter (duration of the endgame engagement).

With the transformation

$$T(\rho)=\begin{bmatrix}1&0&0&0\\ \rho(t)&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix},\quad T^{-1}(\rho)=\begin{bmatrix}1&0&0&0\\ -\rho(t)&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix},\quad \dot{T}(\rho)=\begin{bmatrix}0&0&0&0\\ \rho^2(t)&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix},$$

we get $\bar{C}=\begin{bmatrix}1&0&0&0\\ 0&1&0&0\end{bmatrix}$,

$$\bar{A}(\rho)=\bar{A}_0+\bar{A}_1\rho=\begin{bmatrix}0&1&0&0\\ 0&0&1&-1\\ 0&0&a&0\\ 0&0&0&b\end{bmatrix}+\rho\begin{bmatrix}-1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix}=\begin{bmatrix}-\rho&1&0&0\\ 0&\rho&1&-1\\ 0&0&a&0\\ 0&0&0&b\end{bmatrix},$$

and thus, in (4.16) making the choice $\ell(\rho)=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, the state matrices of (4.18) will be

$$\bar{B}(\rho)=T(\rho)B=B,\quad \bar{L}_\nu(\rho)=T(\rho)L=L,\quad \bar{L}_\mu(\rho)=\begin{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\rho & \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\end{bmatrix}.$$

5. Parameter-varying Invariant Subspaces

For LTI systems the concept of certain invariant subspaces and the corresponding global decompositions of the state equations induced by these invariant subspaces was one of the main thrusts for the development of geometric methods for the solution of a series of relevant problems, see [58]. Nonlinear systems can be studied using tools from differential geometry, when the central role is played by the concept of invariant distributions. From the geometric viewpoint results of classical linear control can be seen as special cases of more general nonlinear results, for details see [27] and [34]. Due to the computational complexity involved, these general nonlinear methods have limited applicability in practice. It has been fruitful to introduce a robust invariant concept which still leads to subspaces whose dimension do not depend on the scheduling variables, hence constant state transformation matrices that reveal the corresponding invariance property in a structural decomposition. As in the LTI case, it turns out that these decompositions considerably facilitate the solution of all kind of qLPV problems.

Restricting the investigations to linear subspaces, as special instances of distributions, then a subspace $\mathscr{V}$ of $\mathbb{R}^n$ will be an invariant distribution for system (3.1) if and only if $A(\rho(t))\mathscr{V}\subset\mathscr{V}$ for all $t\in\mathscr{I}$, where $\mathscr{I}$ is an interval on which the solutions are defined.

This fact motivates the introduction of the following notion for LPV systems: a subspace $\mathscr{V}$ is called parameter-varying invariant subspace for the family of the linear maps $A(\rho)$ (or shortly $\mathscr{A}$-invariant subspace) if

$$A(\rho)\mathscr{V}\subset\mathscr{V}\quad \text{for all } \rho\in\mathscr{P},\ \text{i.e. for all } t\in\mathscr{I}. \tag{5.1}$$

Analogously, let $\mathscr{B}(\rho)$ denote Im $B(\rho)$. Then a subspace $\mathscr{V}$ is called a parameter-varying (A,B)-invariant subspace (or shortly $(\mathscr{A},\mathscr{B})$-invariant subspace) if for all $\rho \in \mathscr{P}$ either of the following equivalent conditions holds :

$$A(\rho)\mathscr{V} \subset \mathscr{V} + \mathscr{B}(\rho); \tag{5.2}$$

there exists a mapping $F \circ \rho : [0,T] \to \mathbb{R}^{m\times n}$ such that:

$$(A(\rho)+B(\rho)F(\rho))\mathscr{V} \subset \mathscr{V}. \tag{5.3}$$

The dual notion is the following: let $\mathscr{C}(\rho)$ denote Ker $C(\rho)$. Then a subspace $\mathscr{W}$ is called a parameter-varying (C,A)-invariant subspace (or shortly $(\mathscr{C},\mathscr{A})$-invariant subspace) if for all $\rho \in \mathscr{P}$ either of the following equivalent conditions holds:

$$A(\rho)(\mathscr{W} \cap \mathscr{C}(\rho)) \subset \mathscr{W}; \tag{5.4}$$

there exists a mapping $G \circ \rho : [0,T] \to \mathbb{R}^{n\times p}$ such that:

$$(A(\rho)+G(\rho)C(\rho))\mathscr{W} \subset \mathscr{W}. \tag{5.5}$$

These definitions are suitable for quasi-LPV (qLPV) systems, too.

Let us denote the maximal $\mathscr{A}$-invariant subspace contained in a constant subspace $\mathscr{K}$ by $\langle \mathscr{K} | A(\rho) \rangle$.

For the LPV case (with constant B matrix) one can get the following definition for the controllability subspace: a subspace $\mathscr{R}$ is called parameter-varying controllability subspace if there exists a constant matrix K and a parameter–varying matrix $F : [0,T] \to \mathbb{R}^{m\times n}$ such that

$$\mathscr{R} = \langle \mathscr{A} + B\mathscr{F} | \text{Im } BK \rangle, \tag{5.6}$$

where $\mathscr{A} + B\mathscr{F}$ denotes the system $A(\rho)+BF(\rho)$. As an alternative characterization $\mathscr{R}$ is a parameter–varying controllability subspace if and only if $\mathscr{R} = \langle \mathscr{A} + B\mathscr{F} | \mathscr{B} \cap \mathscr{R} \rangle$. As in the classical case, it can be seen that the family of controllability subspaces contained in a given subspace $\mathscr{K}$ is closed under subspace addition. Hence this family has a maximal element $\mathscr{R}^*$. The dual notion of parameter-varying controllability subspace is the following: a subspace $\mathscr{S}$ is called an unobservability subspace associated to an LPV system if there exists a constant matrix H and a parameter–varying matrix $G : \mathscr{P} \to \mathbb{R}^{n\times p}$ such that

$$\mathscr{S} = \langle \text{Ker } HC | A(\rho) + G(\rho)C \rangle. \tag{5.7}$$

The family of unobservability subspaces associated to an LPV system containing a given subspace $\mathscr{L}$ is closed under subspace intersection. This family has a minimal element denoted by $\mathscr{S}^*$.

From a practical point of view it is an important question to characterize these subspaces associated to an LPV system by a finite number of conditions. A series of efficient algorithms were derived for the computation of these subspaces, for detail see [2].

If certain conditions are fulfilled, e.g., if the parameter functions are differential algebraically independent, then the parameter invariant subspaces defined above coincide with the corresponding invariant distribution or codistribution, respectively. However, to give sufficient conditions for the solution of certain state feedback and observer filter design problems it is enough that some decompositions of the state equations could be performed. The parameter–varying versions of these invariant spaces are suitable objects to define the required decompositions, therefore they can play the same role in the solution of the fundamental problems, such as disturbance decoupling, unknown input observer design, fault detection (FPRG), as their counterparts in the time invariant context.

5.1. **Detection Filter Design for qLPV systems.** Let us consider to the following LPV system:

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) + L_1(\rho)\nu_1(t) + L_2(\rho)\nu_2(t) \\ y(t) &= Cx(t). \end{aligned} \tag{5.8}$$

The task of designing a residual generator that is sensitive to the fault associated to the L_1 direction and insensitive to the fault associated with the L_2 direction is exactly the FPRG. More precisely, one has to design a residual generator, let us denote its output by r_1, such that if $\nu_1 \neq 0$ then $r_1 \neq 0$ and if $\nu_1 = 0$ then $\lim_{t\to\infty} ||r_1(t)|| = 0$, i.e. a stability condition requirement on the residual generator.

It turns out that the fundamental problem of residual generation has a solution if and only if $\mathscr{S}^* \cap \mathscr{L}_1 = 0$, moreover, if the problem has a solution, the dynamics of the residual generator can be assigned arbitrary.

For LPV systems given in equation (5.8) one can design a – not necessarily stable – residual generator of type

$$\dot{w}(t) = N(\rho)w(t) - G(\rho)y(t) + F(\rho)u(t) \tag{5.9}$$

$$r(t) = Mw(t) - Hy(t), \tag{5.10}$$

if for the smallest unobservability subspace (associated to an LPV system) $\mathscr{S}^*$ containing $\mathscr{L}_2$ one has $\mathscr{S}^* \cap \mathscr{L}_1 = 0$, where $\mathscr{L}_i = \sum_{j=0}^{N} Im L_{i,j}$, $i = 1,2$.

One can compute an acceptable $\hat{G}(\rho)$ as follows: let us consider a splitting of $A(\rho)$ and C according to the projection $P : \mathscr{X} \to \mathscr{X}/\mathscr{S}^*$ and $P^\perp$, respectively, and let us consider $A_{12}(\rho) := PA(\rho)P^\perp$ and $C_{12} := CP^\perp$. Then one has

$$\hat{G}(\rho) = \begin{bmatrix} -A_{12}(\rho)C_{12}^{(-1)} \\ 0 \end{bmatrix},$$

where $C_{12}^{(-1)}$ is a pseudoinverse.

In order to achieve that the condition $\mathscr{S}^* \cap \mathscr{L}_1 = 0$ become necessary and sufficient the subspace $\mathscr{S}^*$ has to be the minimal unobservability codistribution containing $\mathscr{L}_2$. If the components of the parameter function ρ are differential algebraically independent then this property holds.

These ideas can also be applied to nonlinear systems, and a similar condition was obtained for the solvability of the FPRG problem in terms of the observability codistributions, see [13, 26].

One can extend these results to the case of qLPV systems, i.e., for systems with $\rho_i = \rho_i(y)$, see [5, 9, 39].

Recall, that the affine LPV system is said to be quadratically stable if there exists a matrix $P = P^T > 0$ such that

$$A(\rho)^T P + PA(\rho) < 0 \tag{5.11}$$

for all the parameters $\rho \in \mathscr{P}$. A necessary and sufficient condition for a system to be quadratically stable is that the condition in equation (5.11) holds for all the corner points of the parameter space, i.e., one can obtain a finite system of LMI's that has to be fulfilled for $A(\rho)$ with a suitable positive definite matrix P, see [20].

In order to obtain a quadratically stable residual generator one can set $N_{stable}(\rho) = N(\rho) + G(\rho)M$ in equation (5.9), where $N(\rho) = (A(\rho) + \hat{G}(\rho)C)|_{\mathscr{X}/\mathscr{S}^*}$, and $G(\rho) = G_0 + \rho_1 G_1 + \cdots \rho_N G_N$ is determined such that the LMI defined in equation (5.11),i.e.,

$$(N(\rho) + G(\rho)M)^T P + P(N(\rho) + G(\rho)M) < 0$$

holds for suitable $G(\rho)$ and $P = P^T > 0$. By introducing the auxiliary variable $K(\rho) = G(\rho)P$, one has to solve the following set of LMIs on the corner points of the parameter space:

$$N(\rho)^T P + PN(\rho) + M^T K(\rho)^T + K(\rho)M < 0.$$

In certain cases one can find $\operatorname{Ker} C \subset \mathscr{S}^*$. Then one can choose $G(\rho)$ such that the matrix $N(\rho)$ becomes constant, since the equation $G(\rho)CU = UT - A(\rho)U$ has a solution for arbitrary T, where U is the insertion map of $\mathscr{X}/\mathscr{S}^*$. The matrix T is a design parameter that contains the information about the desired poles.

5.2. **Example: pursuit-evasion continuated.** If we consider Example 4.2 with the (aggregated) directions $L_\nu = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $L_\mu = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, respectively, we van compute the corresponding invariant subspaces. It turns out that $\mathscr{S}_\nu^*$ is the entire space while $\mathscr{S}_\mu^* = \operatorname{span} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. This shows that while the FPRG problem cannot be solved for L_μ, it has a solution for L_ν. Based on this information it was possible to provide a solution for a linearized pursuit–evasion problem, for details, see [31].

5.3. **DDPS for qLPV systems.** Let us consider the following LPV system:

$$\begin{aligned} \dot{x} &= A(\rho)x + B(\rho)u + S(\rho)d \\ y &= Cx \end{aligned}$$

where d represents disturbance and the matrix $S(\rho)$ has an affine parameter dependent structure. We would like to design a state feedback gain which depends affinely on ρ in order to remove the effect of the disturbance on the output and which is stabilizing, i.e., the following problem will be considered: find a subspace $\mathscr{V}$ in $\mathbb{R}^n$ which contains $\mathscr{S}$ and $F : [0,T] \to \mathbb{R}^{m\times n}$ such that

$$(A(\rho) + BF(\rho))\mathscr{V} \subset \mathscr{V} \subset \mathscr{C} \quad \text{for all } \rho \in \mathscr{P} \tag{5.12}$$

$$A(\rho) + BF(\rho) \quad \text{stable}\ , \tag{5.13}$$

where $\operatorname{Im} B = \sum_{\rho\in\mathscr{P}} B(\rho)$ and $\operatorname{Im} S = \sum_{\rho\in\mathscr{P}} S(\rho)$.

It is possible to give a sufficient condition for the solvability of the problem using the notion of asymptotically stabilizability property of parameter varying controllability subspaces. One can prove, as in the LTI case, that it is enough to find a subset $\mathscr{V}$ and $F(\rho)$ for which $\mathscr{S} \subset \mathscr{V}$, condition (5.12) holds and $(A(\rho) + BF(\rho))|_{\mathscr{V}}$ is asymptotically stable.

For an asymptotically stabilizable pair $(\mathscr{A}, \mathscr{B})$ and an $(\mathscr{A}, \mathscr{B})$-invariant subspace $\mathscr{V}$ contained in $\mathscr{C}$ such that there is an $F_0(\rho)$ that $(A(\rho) + BF_0(\rho))_{|\mathscr{V}}$ is asymptotically stable there exists an $F(\rho)$ such that $(A(\rho)+BF(\rho))\mathscr{V}\subset\mathscr{V}$ and $A(\rho)+BF(\rho)$ is asymptotically stable, see [45].

This result simplifies the solvability of LPV disturbance decoupling problem with Stability (DDPS): for $\mathscr{V}^*$, the maximal $(\mathscr{A},\mathscr{B})$-invariant subspace in $\mathscr{C}$, a sufficient condition for the solvability of DDPS is simply

$$\mathscr{V}^*\supset\mathscr{S}, \tag{5.14}$$

provided that the pair $(\mathscr{A},\mathscr{B})$ is asymptotically stabilizable. If this is not the case then consider the maximal parameter-varying controllability subspace $\mathscr{R}^*$ in $\mathscr{C}$. Then, there exists $\mathscr{F}_0$ for which $\mathscr{R}^*=\langle\mathscr{A}+B\mathscr{F}_0|\mathscr{B}\cap\mathscr{R}^*\rangle$. With $\mathscr{B}_0=\mathscr{B}\cap\mathscr{R}^*$ and $A_0(\rho)=A(\rho)+BF_0(\rho)|_{\mathscr{R}^*}$, we have $\langle\mathscr{A}_0|\mathscr{B}_0\rangle=\mathscr{R}^*$.

Concerning stability properties, if ρ_i are persistently exciting, see [53], and $\langle\mathscr{A}|\mathscr{B}\rangle=\mathbb{R}^n$, then the pair $(\mathscr{A},\mathscr{B})$ is asymptotically stabilizable and Then there exists $\mathscr{F}$ such that

$$\mathscr{R}^*=\langle\mathscr{A}+B\mathscr{F}|\mathrm{Im}\,BK\rangle=\langle\mathscr{A}+B\mathscr{F}|\mathscr{B}\cap\mathscr{R}^*\rangle$$

and $(\mathscr{A}+B\mathscr{F})|_{\mathscr{R}^*}$ is asymptotically stable.

Thus, if persistence of ρ_is hold, $\langle\mathscr{A}|\mathscr{B}\rangle=\mathbb{R}^n$ and $\mathscr{R}^*\supset\mathscr{S}$ the DDPS for qLPV systems is solvable with asymptotic stability. As far as quadratic stabilizability is concerned we do not have an analogous result.

Example 5.1. As a small example let us consider

$$A(\rho)=\begin{bmatrix}-1&0&0\\0&1&\rho\\\rho&0&-1\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}+\rho\begin{bmatrix}0&0&0\\0&0&1\\1&0&0\end{bmatrix},$$

and

$$B=\begin{bmatrix}0\\0\\1\end{bmatrix}\quad S(\rho)=\begin{bmatrix}0\\1+\rho\\\rho\end{bmatrix}\quad C=\begin{bmatrix}1&0&0\end{bmatrix},$$

where the parameter ρ is varying in the interval $[1,2]$. It can be seen that $\mathscr{C}=\mathrm{Ker}C$ is $\mathscr{A}$ invariant, hence $(\mathscr{A},\mathscr{B})$ is either, i.e., $\mathscr{V}^*=\mathscr{C}$. By solving linear matrix inequalities we get that for the induced pair

$$\tilde{F}(\rho)=[1.8225\ 1.2957]+\rho[-8.0181-2.5257]$$

quadratically stabilizes the system on $\mathscr{V}^*$ and satisfies the decoupling requirements.

5.4. **Inversion of qLPV Systems.** Let us consider the class of linear parameter-varying (LPV) systems of m inputs and p outputs that can be described as:

$$\dot{x}(t)=A(\rho(t))x(t)+B(\rho(t))\nu(t) \tag{5.15}$$
$$y(t)=Cx(t) \tag{5.16}$$

where

$$A(\rho(t))=A_0+\rho_1(t)A_1+\ldots+\rho_N(t)A_N, \tag{5.17}$$
$$B(\rho(t))=B_0+\rho_1(t)B_1+\ldots+\rho_N(t)B_N, \tag{5.18}$$
$$\tag{5.19}$$

and the dimension of the state space is supposed to be n.

It is assumed that each parameter ρ_i ranges between known extremal values $\rho_i(t) \in [\underline{\rho}_i, \overline{\rho}_i]$ and the parameter set that contains all $(\rho_1(t), \cdots, \rho_N(t))$, where $t \in [0, T]$ will be denoted by $\mathscr{P}$. For the sake of notational simplicity the time dependency of the matrices will be omitted $(A(\rho) := A(\rho(t)))$ where it is possible.

It is not hard to figure out that if some technical conditions for the parameter functions (persistence) are fulfilled, then the output nulling manifold is $\mathscr{V}^*$, i.e., the maximal $(\mathscr{A}, \mathscr{B})$-invariant subspace contained in Ker$C$. The invertibility conditions reduce to

$$\dim \mathrm{Im} B = m, \qquad \mathscr{V}^* \cap \mathrm{Im} B = 0.$$

Let us observe, that if these conditions are fulfilled, one can always choose a coordinate transform of the form

$$z = Tx, \text{ where } \quad T = \begin{bmatrix} \mathscr{V}^{*\perp} \\ \Lambda \end{bmatrix}, \Lambda \subset (\mathrm{Im} B)^{\perp}.$$

Accordingly, the system will be decomposed to:

$$\dot{\xi} = A_{11}(t)\xi + A_{12}(t)\eta + B\nu \tag{5.20}$$

$$\dot{\eta} = A_{21}(t)\xi + A_{22}(t)\eta \tag{5.21}$$

$$y = C_1\xi. \tag{5.22}$$

It follows, that applying a suitable feedback

$$\nu = F_2(\rho(t))\eta + v, \tag{5.23}$$

such that $\mathscr{V}^*$ is $(\mathscr{A} + \mathscr{B}F, \mathscr{B})$ invariant, one can obtain the system:

$$\dot{\xi} = A_{11}(t)\xi + Bv \tag{5.24}$$

$$y = C_1\xi. \tag{5.25}$$

By the maximality of $\mathscr{V}^*$ follows that both ξ and v can be expressed as functions of y and its derivatives.

With $\tilde{y} = \mathscr{S}\xi$, where

$$\tilde{y} = \left[y_1, \cdots, y_1^{(\gamma_1)}, \cdots, y_p, \cdots, y_p^{(\gamma_p)} \right]^T$$

one has $v = B^{-1}\mathscr{S}^{-1}(\dot{\tilde{y}} - \dot{\mathscr{S}}\mathscr{S}^{-1}\tilde{y} - \mathscr{S}\bar{A}_{11}\mathscr{S}^{-1}\tilde{y})$, i.e.,

$$\begin{aligned} \dot{\eta} &= A_{22}\eta + A_{21}\xi \\ \nu &= F_2\eta + \bar{B}^{-1}\mathscr{S}^{-1}(\dot{\tilde{y}} - (\dot{\mathscr{S}}\mathscr{S}^{-1} + \mathscr{S}A_{11}\mathscr{S}^{-1})\tilde{y}). \end{aligned}$$

For details, see [48].

In obtaining an inversion based filter for the sampled system the first solution could be a direct application of a general discretization scheme with a suitable choice of the approximation order N_T. This procedure, however, has the big disadvantage that the relative degree decreases with the order of the approximation scheme, hence the size of the zero dynamics increases. Moreover, the obtained discrete time system will be non-minimum phase, in general, even if the continuous time system was minimum phase.

There is only one discretization scheme, that preserves the relative degree (γ_i), i.e., the explicit Euler scheme that corresponds to the choice $N_T = 1$. The numerical properties (accuracy, stability) of the explicit Euler scheme, however, are not advantageous. This fact motivates the introduction of a mixed discretization strategy.

The starting point of the proposed discretization scheme is the observation that unknown input depends on a dynamic component – zero dynamics – and a static one, corresponding to the feedback $\nu = F(\rho)\eta + v$, where the static part v is obtained from the flat system

$$\dot{\xi} = A_{11}(\rho)\xi + \bar{L}\nu, \quad y = C_1\xi. \tag{5.26}$$

For flatness and relative degree; see, e.g., [16, 18]. In order to obtain v, the discretization should maintain the relative degree of this system. Therefore, the Euler discretization will be applied only for (5.26), i.e., after discretization one has the system

$$\zeta_{k+1} = A_d^e(\rho_k)\zeta_k + L_d v_k, \quad y_k = \bar{C}\zeta_k. \tag{5.27}$$

where $A_d^e(\rho_k) = I + T_s A_c(\rho_k)$ with $L_d = T_s\bar{L}$ and $\bar{C} = C_1$.

By a straightforward computation one has that the discrete state ζ_k can be expressed as $\tilde{y}_d(k) = \mathscr{S}_d(\rho,k)\zeta_k$, where

$$\tilde{y}_d(k) = \left[y_1(k), \cdots, y_1(k+r_1), \cdots, y_p(k), \cdots, y_p(k+r_p)\right]^T.$$

The rows of the coordinate transform $\mathscr{S}_d$ can be determined by using the recursion

$$\bar{S}_i^0 = I, \quad \bar{S}_i^{k+1} = \bar{A}_d^e(\rho_{k+1})\bar{S}_i^k,\ S_{d,i}^k = c_i\bar{S}_i^k,\ k \le r_i. \tag{5.28}$$

Hence, the value of the discrete state can be computed as $\zeta_k = \mathscr{S}_d(\rho,k)^{-1}\tilde{y}_d(k)$ while $v_k = L_d^{\{-1\}}(\zeta_{k+1} - A_d^e(\rho_k)\zeta_k)$.

Observe that $\tilde{y}_d(k)$ and $\mathscr{S}_d(\rho,k)$ requires samples forward in time up till $\max r_i$, i.e., the implementation will contain a delay of $\max r_i + 1$. The accuracy of the scheme can be set by a reasonable choice of the sampling time.

Having the approximative values $\xi(kT_s) = \zeta_k$, the state of the zero dynamics can be computed based on a discretization of (5.21). The accuracy of this discretization, however, can be set arbitrary. This freedom can be used to ensure the stability of the discretized dynamics for a fixed sampling time T_s, see [30].

6. NULL-SPACE COMPUTATION

A traditional application of annihilators can be found in the fault detection filter design and the fault tolerant reconfiguration based control design approaches of LTI systems, see, e.g., [36, 55, 56] and [12, 21]. For this class of systems the annihilator is determined based on (both polynomial or rational) matrix pencil methods, see [19, 28] and [54]. Although this approach is computationally efficient, it highly depends on the frequency domain formulation, which prevents its extension to general time varying systems.

While the LPV system is actually a set of time varying systems determined by the assumptions on the parameters, by a slight abuse of the notation, we identify and denote it as $\Sigma(\rho)$:

$$\Sigma(\rho) \sim \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & D(\rho) \end{array}\right], \tag{6.1}$$

i.e., as a linear system that obeys to (3.1)-(3.2) viewed at fixed parameter ρ. In what follows we will use greek upper case letters to denote systems and latin upper case letters for the matrices of a state space representation. Note that we consider the rank of the parameter varying matrices as being independent on the parameter, i.e., we assume tacitly that the rank is constant on $\mathscr{P}$. A standing assumption throughout this paper is that the system is stable. To ease the notation we will suppress the parameter dependence when there is no risk of confusion.

The image and the kernel of a linear map are well-known notions. More formally, given a bounded linear operator $T : U \to Y$ it is standard to introduce the notion of the image-space $\mathscr{R}(T) = \{y \in Y \,|\, \exists u \in U, T(u) = y\}$ and null-space (or kernel) $\mathscr{N}(T) = \{u \in U \,|\, T(u) = 0\}$, respectively. Having an input output description of a linear dynamical system Σ, i.e., a transfer function in the linear time invariant (LTI) context, it is natural to consider the set $\mathscr{N}(\Sigma)$ and to search for a stable linear system Γ such that $\mathscr{R}(\Gamma) = \mathscr{N}(\Sigma)$, i.e., $\Sigma\Gamma = 0$. This system Γ will be called the right-annihilator of Σ. Analogously one can define as $\Lambda\Sigma = 0$ the left-annihilator system associated to Σ. In what follows, we will concentrate on right-annihilators only.

At a fixed parameter ρ we can define the corresponding (right) annihilator system $\Gamma(\rho)$ as $\Sigma(\rho)\Gamma(\rho) = 0$ and the notion can be trivially extended to the LPV system itself. In [49] we already proposed a general method for the computation of the annihilator based on an inversion technique. The core of the algorithm assumes a full row-rank $D(\rho)$ and, if it is not the case, a considerable amount of work is necessary to produce an equivalent system with the desired property by a successive derivation of the outputs. In other words, it is supposed that the LPV system has a well-defined vector relative degree; see, e.g., [17, 27] for the definition.

In contrast to the LTI case, however, the computation of the LPV annihilators is not a straightforward task in general and there is an interest in alternative approaches to comply the task.

Considering finite rank linear maps $P(\Delta)$, i.e., those that can be represented in an upper linear fractional (LFT) form:

$$P(\Delta) = \mathfrak{F}_u(M,\Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \sim \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]^{(\Delta)},$$

where Δ is an element of a given set of linear bounded causal maps $\mathscr{D}$, the problem is to determine the right null space of the system, i.e., a stable (well-defined) system $N(\Delta)$ such that $P(\Delta)N(\Delta) = 0$ for all $\Delta \in \mathscr{D}$. To follow the conventional notation for a state space representation we will use the partitioning $M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and for sake of simplicity we will suppress ρ from the notation wherever is possible. Sometimes it is also convenient to consider also a pure memoryless operator $P(\rho)$. Then $\mathscr{D} = \mathscr{P}$ collects the time-varying parameters (scheduling variables) and M is a constant matrix. As it is well-known, the LFT connection is called well-defined if $I - A(\rho)\Delta$ is boundedly invertible for all $\Delta \in \mathscr{D}$ and $\rho \in \mathscr{P}$ and it is called stable if the corresponding feedback connection is internally stable, i.e., the connection is well-defined and the corresponding inverse is causal. In contrast to the LTI/(q)LPV case, where well-definedness is granted on $\mathscr{D}$ and only stability is an issue, in the memoryless case only well-definedness is nontrivial.

To illustrate the idea of the null space construction algorithm, let us consider the plant where $D = \begin{bmatrix} I & 0 \end{bmatrix}$ and where the inputs, i.e., $B = \begin{bmatrix} B_i & B_s \end{bmatrix}$, are partitioned accordingly. Recall that for an invertible D_a we have

$$\left(\left[\begin{array}{c|c} A & B \\ \hline C & D_a \end{array}\right]^{(\Delta)} \right)^{-1} = \left[\begin{array}{c|c} A - BD_a^{-1}C & BD_a^{-1} \\ \hline -D_a^{-1}C & D_a^{-1} \end{array}\right]^{(\Delta)}. \tag{6.2}$$

This is an easy consequence of the matrix inversion lemma, for the LTI framework see also [60]. Note that the invertibility of an LFT depends only on D, thus an invertible augmentation of the system is given by an invertible augmentation of D, which is a trivial task. In our case let us consider $D_a = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and simply zero out the corresponding lines of the extended readout map, i.e., augment the rows of the plant by the constant system $\begin{bmatrix} 0 & I \end{bmatrix}$. Then we obtain an invertible system with the inverse

$$\left[\begin{array}{c|cc} A - B_i C & B_i & B_s \\ \hline -C & I & 0 \\ 0 & 0 & I \end{array}\right]^{(\Delta)} . \tag{6.3}$$

Thus, a candidate for the null-space of $P = \mathfrak{F}_u(M, \Delta)$ is the system

$$\mathcal{N}_M^0 = \left[\begin{array}{c|c} A - B_i C & B_s \\ \hline -C & 0 \\ 0 & I \end{array}\right]^{(\Delta)} . \tag{6.4}$$

The notation reflects the dependency of this null-space representation from the given realization M of the plant P. The details of construction in the general case can be found in [49, 50].

Example 6.1. Let us illustrate the qLPV case by designing an FDI filter for the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -1 + 0.3\rho_1 \\ 1 & -0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \\ y &= \begin{bmatrix} -1 + \rho_2 & 1 \end{bmatrix} x + u + f, \end{aligned}$$

where $|\rho_i| \le 1$ and f is the fault signal. We denote by $A(\rho)$, $B(\rho)$, $B_f(\rho)$, $C(\rho)$ the state matrices of this system.

In order to construct the filter we need the right null-space of

$$P_\rho = \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & 1 \\ 0 & 1 \end{array}\right],$$

which can be obtain by using the general scheme presented in the paper [49], i.e., by augmenting the D matrix of this system to $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, determining the inverse system applying (6.2) and taking the corresponding rows (since a left null-space is required) of this inverse. Finally, we obtain

$$\left[\begin{array}{c|cc} A(\rho) - B(\rho)C(\rho) & B(\rho) & 0 \\ \hline C(\rho) & -1 & 1 \end{array}\right]$$

for the candidate null-space, which is not quadratically stable. Computing a left coprime factorization with an output injection gain $L = B(\rho)$ the resulting stable left null-space will be given by $\left[\begin{array}{c|cc} A(\rho) & 0 & B(\rho) \\ \hline C(\rho) & -1 & 1 \end{array}\right]$. Augmenting the system with with the additive fault having direction $B_f(\rho)$ and, according to (4.5), taking the computed null-space system as the FDI filter the residual will reflect the signal $\left[\begin{array}{c|c} A(\rho) & B_f(\rho) \\ \hline C(\rho) & 1 \end{array}\right] f$.

6.1. **Geometry based annihilator construction.** In this section we would like to construct an annihilator for systems of the type

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + Bu(t)\\ y(t) &= Cx(t),\end{aligned}$$

where $A(\rho)$ depends affinely on the parameters. Note that the results can be also extended to the case where B is also parameter varying.

Let us denote by $\mathscr{V}^*$ the weakly unobservable subspace, i.e., the set of initial conditions for which there exists an input function such that the ensuing output is identically zero. $\mathscr{V}^*$ is the largest subspace such that there exists a static feedback gain $F(\rho)$ ensuring

$$(A(\rho)+BF(\rho))\mathscr{V} \subset \mathscr{V}, \quad (C+DF(\rho))\mathscr{V} = 0,\ \rho \in \mathscr{P}.$$

These gains F are called friends of $\mathscr{V}$.

Recall that the controllable weakly unobservable subspace $\mathscr{R}^* \subset \mathscr{V}^*$, i.e., the set of initial conditions for which there exists an input function able to steer the state to zero in finite time while keeping the output identically zero, has the same friends as $\mathscr{V}^*$. Moreover, it is known that in the LTI case the spectrum on $\mathscr{R}^*$ is freely assignable.

Let us chose the invertible input mixing map $T_u = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$ such that

$$\mathrm{im} V_1 = B^{(-1)}\mathscr{R}^*, \quad \mathrm{im}\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \ker D$$

and the invertible output mixing map $T_y = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ such that $W_1 D V_3 = 0$ and $W_2 D V_3 = I$, respectively.

Chose a state transformation ($\xi = T^{-1}x$) matrix $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ such that

$$\mathrm{im} T_1 = \mathscr{R}^*, \quad \mathrm{im}\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \mathscr{V}^*.$$

Note that these matrices do not depend on the parameters.

Applying all these transformations one has, see [1, 3]

$$\dot{\xi} = \bar{A}(\rho)\xi + \bar{B}\bar{u}, \quad \bar{u} = T_u^{-1}u, \tag{6.5}$$

$$\bar{y} = \bar{C}\xi + \bar{D}\bar{u}, \quad \bar{y} = T_y y \tag{6.6}$$

with

$$\bar{A}(\rho) = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{bmatrix}(\rho), \quad \bar{B} = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{13} \\ 0 & \bar{B}_{22} & \bar{B}_{23} \\ 0 & \bar{B}_{32} & \bar{B}_{33} \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & \bar{C}_{13} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

One should take a $\bar{F}_1(\rho)$ (e.g., by considering the first block column of a friend of $\mathscr{V}^*$) that fulfils conditions

$$\bar{C}_{21} + \bar{F}_{13}(\rho) = 0, \tag{6.7}$$

$$\bar{A}_{21}(\rho) + \bar{B}_{22}\bar{F}_{12}(\rho) + \bar{B}_{23}\bar{F}_{13}(\rho) = 0, \tag{6.8}$$

$$\bar{A}_{31}(\rho) + \bar{B}_{32}\bar{F}_{12}(\rho) + \bar{B}_{33}\bar{F}_{13}(\rho) = 0, \tag{6.9}$$

and renders the parameter varying matrix $\bar{A}_0(\rho) = \bar{A}_{11}(\rho) + \bar{B}_{11}\bar{F}_{11}(\rho) + \bar{B}_{12}\bar{F}_{12}(\rho) + \bar{B}_{13}\bar{F}_{13}(\rho)$ stable (e.g., quadratically). Note that for strictly proper systems condition (6.7) vanishes.

After all these preparatory steps we are ready to design the LPV annihilator Γ:

$$\dot{\zeta} = \bar{A}_0(\rho)\zeta + \bar{B}_{11}v, \quad \zeta(0) = 0 \tag{6.10}$$

$$u = T_u(\bar{F}_1(\rho)\zeta + Ev), \tag{6.11}$$

with $E = \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T$ and $\bar{F}_1(\rho)$ the first block column of $\bar{F}(\rho)$. Note that while the dimensions of the matrices follows from the dimensions of the invariant subspaces, one can figure out that $v(t) \in \mathbb{R}^{m-p}$, as for the inversion based approach.

One can check that this Γ defined by (6.10) is indeed an annihilator. Plug in (6.11) into (6.5) and take the new state as $e = \begin{bmatrix} \xi_1^T - \zeta^T & \xi_2^T & \xi_3^T \end{bmatrix}^T$ to get

$$\begin{aligned} \dot{e} &= \bar{A}e, \quad e(0) = 0, \\ \dot{\zeta} &= \bar{A}_0\zeta + \bar{B}_{11}v \\ \bar{y} &= \begin{bmatrix} 0 & 0 & \bar{C}_{13} \\ 0 & \bar{C}_{22} & \bar{C}_{23} \end{bmatrix} e, \end{aligned}$$

i.e., $\bar{y} = 0$.

Note that even if $D \neq 0$ one can compute $\mathscr{R}^*$ and $\mathscr{V}^*$ by using an equivalent strictly proper system, [1, 3]. Then $\mathscr{V}^*$ is the maximal $(\mathscr{A}, \mathscr{B})$-invariant subspace contained in $\ker C$ and $\mathscr{R}^*$ is the corresponding maximal controllability subspace that can be computed by applying the algorithms described in [2, 6].

6.2. **Example.** For illustrative purposes we show a simple academic example. Consider an LPV system

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned}$$

with $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ and $|\rho_i| \leq 1$. The state matrices are:

$$A_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Applying the algorithms which compute the parameter-varying invariant subspaces one has

$$\mathscr{V}^* = \mathrm{Im} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

and

$$\mathscr{R}^* = \operatorname{Im}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T,$$

respectively. Then the corresponding state transform and input mixing is given by:

$$T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_u = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Taking $\bar{F}_1 = \begin{bmatrix} -\rho_1 \\ \rho_1 \\ -\rho_1 \end{bmatrix}$ one has the annihilator

$$\begin{aligned} \dot{\zeta} &= -\zeta + v, \\ u &= \begin{bmatrix} -\rho_1 \\ 0 \\ 0 \end{bmatrix} \zeta + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} v. \end{aligned}$$

It is not hard to figure out that by following the inversion based principle one will necessarily have derivatives in that description: after taking the first two derivatives we have $CB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $CA(\rho)B = \begin{bmatrix} 1+\rho_1 & \rho_2 & 1+\rho_1+\rho_2 \\ 0 & -1 & -1 \end{bmatrix}$. Thus, a second derivation is needed ($\ddot{y}_1$ in our case) that will introduce the derivatives of the parameters into the state matrices through $CA^2(\rho) + \frac{d}{dt}CA(\rho)$. Moreover, selecting $\ddot{y}_1$ and $\dot{y}_2$, the final full row rank D matrix will be

$$D = \begin{bmatrix} 1+\rho_1 & \rho_2 & 1+\rho_1+\rho_2 \\ 0 & 1 & 1 \end{bmatrix}$$

which needs some further manipulation to be augmented to an invertible one. Thus even this simple example can show the benefit of the geometric approach for the annihilator computation.

7. Conclusions

The qLPV system representation can include LTI, LTV and certain input affine nonlinear systems. In this paper a fairly complete overview on geometric approaches to design detection filters for unknown signals representing possible changes or faults in a qLPV dynamic system has been given. We provide solutions for the FPRG design, for the DDP problem, dynamic inversion and also null-space based methods, where the system matrix depends affinely from the parameters. The proposed method relies on the concept of parameter varying invariant subspaces and the results of classical geometrical system theory.

Acknowledgements

The research was supported by the European Union within the framework of the National Laboratory for Autonomous Systems (RRF-2.3.1-21-2022-00002).

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