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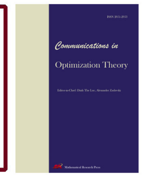

FAST STABILITY TEST OF LINEAR TIME-DELAY SYSTEMS WITH REPEATED CRITICAL IMAGINARY ROOTS SIMPLY BY INTEGRAL EVALUATION

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Abstract. Complete stability analysis of a linear time-delay system with parameters in given interval/region is not a new topic, yet a hard problem in general. In the application of the widely used methods or algorithms, even for a system with a single delay, it is necessary to use transformation, to solve nonlinear equation, to determine crossing direction of characteristic root, or to calculate Puiseux series expansion, and so on, so as to get to know properties of the characteristic quasi-polynomial required in stability analysis. This paper shows that the complete stability of a time-delay system with repeated critical imaginary roots can be carried out, directly and simply by integral evaluation and effectively, with low computational cost. It does not need any special knowledge of the quasi-polynomial such as the critical delay values and the branches of Puiseux series expansion near a repeated critical imaginary root, and it does not impose restriction on the number of delays.

Keywords. Puiseux series; Integral evaluation; Repeated root; Stability switch; Time delay.

2020 Mathematics Subject Classification. 34D20, 34K20, 93D20.

1. INTRODUCTION

In control applications, a time delay is the time that takes to perceive, measure, formulate, act, and so on, and it exists commonly in controllers, filters and actuators. Without a control design, a time delay may lead to undesirable outcomes of a delayed system, even makes the controlled system unstable. Thus, stability analysis of time-delay systems (TDSs for short) has been a major topic in control applications over the past few decades, see [1, 2, 3, 4, 5, 6, 7, 8, 9] for example. An equilibrium of a retarded TDS is asymptotical stable if all the characteristic roots have negative real parts, and it is unstable if there is a characteristic root with positive real part, where a characteristic root with positive real part is usually called unstable root (UR for short). The characteristic quas-polynomial $p(\lambda)$ of a TDS is analytic in the whole complex plane, it has finite number of roots in any bounded region. Thus, let $\kappa = \max\{\Re(\lambda): \ p(\lambda) = 0\}$, where $\Re(z)$ is the real part of complex number z, then κ is well-defined, and it is usually called spectral abscissa. An equilibrium of a retarded TDS is asymptotical stable if $\kappa < 0$, and it is unstable

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if $\kappa > 0$. In a sense, therefore, stability analysis of TDSs can be considered as an optimization problem.

In many applications, time delays are usually hard to be identified exactly, thus they are frequently taken as parameters varing in given intervals. Then it is essentially to know for what delay combination the TDS is asymptotically stable or unstable, namely to find out the stable regions and unstable regions in the parameter space. This is the complete stability problem [10], which is divided into two problems: Problem 1), an exhaustive detection (if any) of the critical imaginary roots (CIRs for short); Problem 2), the asymptotic behavior analysis of the CIRs, for which lots of calculation are necessary. For a TDS with a single delay or with commensurate delays, the complete stability problem can be solved by using the method of stability switch [2, 3]. This method works only if the CIRs are simple characteristic roots. A frequency sweeping method [11] and a geometric method [12] were proposed to draw the critical curves in the plane of two arbitrary selected delays. When the characteristic quasi-polynomial has repeated CIRs, the method based on Puiseux series expansion [10, 13, 14] can be very useful and very effective but maybe not for TDSs with multiple-delays. Actually, as pointed out in [15], the complete stability problem of a TDS can be NP-hard when the delays are unbounded. Thus, more strightforward methods or algorithms with less computation are still welcomed for the complete stability analysis of TDSs.

Based on the Argument Principle or equivalently the Cauchy Theorem, the number of URs of a given TDS with single delay or multi-delays can be calculated directly by using the characteristic quasi-polynomial and its derivative. The stability criteria derived in this way can be very effective, as done in Nyquist criterion [16], Mihkalov criterion [2], Stepan criterion [1] and Hassard criterion [17] for retarded TDSs. These criteria have been generalized for fractional-delay systems [18, 19] and for neutral TDSs [20, 21] under the condition of strong stability. The Difinite Integral Evaluation Method [19, 20, 21] (DIEM for short) works in particularly effective, it provides a simple method to find a delay-free upper limit of the test integral, and thus it simplifies the calculation of the test definite integral over the right half infinite interval to the one over a delay free finite interval. In many applications, the upper limits can be very small, which reduces the computational time substantially. In addition, DIEM works too for neutral TDSs with accumulation points [22].

The aim of this paper is to show with four examples that DIEM [19, 20, 21] works effectively for the complete stability problem of TDSs with repeated CIRs. Compared with the previous commonly used methods or algorithms, the application of DIEM is more straightforward, less computation and easier understandable, and it works also for the complete stability problem with multiple (delay or non-delay) parameters.

2. Problem Statement

For simplicity in presentation, this paper focuses on the stability of a linear retarded time-delay system described by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_0\boldsymbol{x}(t) + \sum_{k=1}^{m} \boldsymbol{A}_k\boldsymbol{x}(t-\tau_k), \quad (\boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{A}_0, \boldsymbol{A}_k \in \mathbb{R}^{n\times n}), \tag{2.1}$$

where the positive numbers $\tau_1,\ \tau_2,\ \cdots,\ \tau_m$ are the time delays. Let $\boldsymbol{I}_n$ be the identity matrix of order n, then the characteristic function

$$p(\lambda) \equiv \det(\lambda \boldsymbol{I}_n - \boldsymbol{A}_0 - \sum_{k=1}^{m} \boldsymbol{A}_k \mathrm{e}^{-\lambda \tau_k})$$

is a quasi-polynomial in the form of

$$p(\lambda) = \lambda^n + \sum_{i=1}^{n} a_i(\mathrm{e}^{-\lambda\tau_1}, \mathrm{e}^{-\lambda\tau_2}, \cdots, \mathrm{e}^{-\lambda\tau_m})\lambda^{n-i}, \tag{2.2}$$

where the coefficients $a_i(z_1, z_2, \cdots, z_m)$ are polynomials with respect to $z_1, z_2, \cdots, z_m$.

For simplicity and comparison, the paper is devoted to the case of single delay $\tau = \tau_1 = \cdots = \tau_m$ only. As τ increases in a given interval, the stability of an equilibrium may change from stable to unstable, or from unstable to stable, probably change repeatedly many times, which is a phenomenum called stability switch. The problem of stability switch were studied intensively in [2, 3]. Two major steps are required. One is to find all the critical delay values for which $p(\lambda)$ has a CIR, say, assume that at $\tau = \tau_0$ one has a CIR $\lambda = \mathrm{i}\omega_0$. The other is to determine

$$S = \mathrm{sgn}\left(\Re \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big|_{(\tau,\lambda)=(\tau_0,\mathrm{i}\omega_0)}\right), \tag{2.3}$$

As τ passes through τ_0 from the left to the right, the TDS increases (decreases) a pair of conjugate URs if $S > 0$ ($S < 0$). Then the complete stability can be obtained by checking the number of URs in each open interval between two adjacent cririical delay values. This method works only if the corresponding derivative in (2.3) exists, which is true if $\lambda = \mathrm{i}\omega_0$ is simple.

When $\lambda = \mathrm{i}\omega_0$ is not simple, with multiplicity larger than 1, the derivative given in (2.3) does not exist, the method of stability switch widely used in [2, 3] does not work. In this case, the Puiseux series can be used as done in [10, 13, 14]. In [10], by using Puiseux series, a new frequency-sweeping framework from an analytic curve perspective was established, with which the asymptotic behavior of TDSs can be classified into the regular and the singular cases, and consequently the complete stability problem was fully determined. In these works, the Taylor series approximation of the characteristic quasi-polynomial $p(\lambda)$ at $(\tau, \lambda) = (\tau_0, \mathrm{i}\omega_0)$ is required, and then the Puiseux series expansion near $(\tau, \lambda) = (\tau_0, \mathrm{i}\omega_0)$, in the form of

$$\lambda = \mathrm{i}\omega_0 + c_1(\tau - \tau_0)^{\xi} + c_2(\tau - \tau_0)^{2\xi} + \cdots, \quad (\xi = \frac{1}{d}), \tag{2.4}$$

is found to determine the crossing direction of the characteristic root passing through $\lambda = \mathrm{i}\omega_0$, where $d \geq 2$ is the multiplicity of $\lambda = \mathrm{i}\omega_0$. Usually a very careful analysis on the multiplicity of $p(\lambda)$ with respect to λ and τ is required in the calculation of the Puiseux series expansion.

In this paper, DIEM is firstly extended and then is used to simplify the stability analysis for TDSs with repeated CIRs, without a need of the information of Problem 1 and Problem 2, and it can be also used for the stability of TDSs with multiple delays.

3. The Crossing Direction Determined by Using the DIE Method

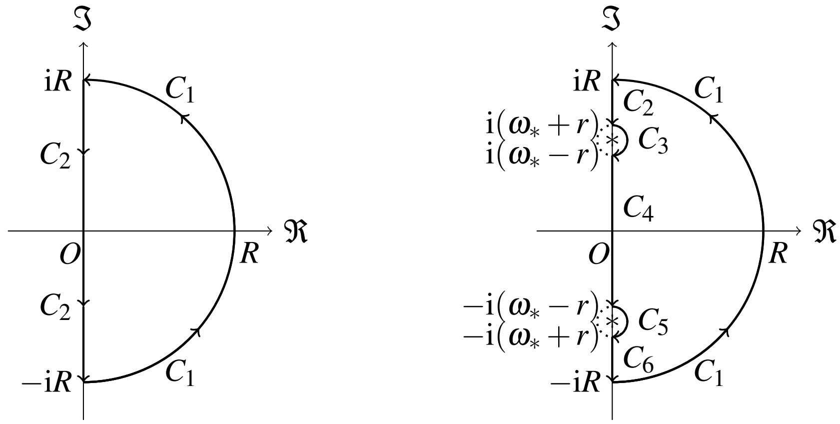
Under the assumption that $p(\lambda)$ has no roots on the imaginary axis, by using the Argument Principle or the equivalent Cauchy Theorem, say as done in [19, 20, 21], the number of URs of

$p(\lambda)$ defined in (2.2) can be calculated from

$$\mathscr{N} = \lim_{R\to+\infty} \frac{\Delta_C \arg p(\lambda)}{2\pi} = \lim_{R\to+\infty} \frac{1}{2\pi \mathrm{i}} \oint_C \frac{p'(\lambda)}{p(\lambda)} \mathrm{d}\lambda, \tag{3.1}$$

where C is the contour depicted in Fig.1(a). The TDS (2.1) is asymptotically stable if and only if $\mathscr{N} = 0$, and it is unstable if $\mathscr{N} > 0$.

The integral formula is more convenient in applications. A key step in simplifying the calcu-



$(a)\ C = C_1 + C_2$ $\qquad$ $(b)\ \tilde{C} = C_1 + C_2 + C_3 + C_4 + C_5 + C_6$

FIGURE 1. The contour C and $\tilde{C}$ for contour integrals, $C = \tilde{C}$ if $r \to 0$.

lation of $\mathscr{N}$ is to evaluate the integrals along C_1 and C_2 respectively, which yields to

$$\mathscr{N} = \frac{n}{2} - \frac{1}{\pi} \int_0^{+\infty} \Re\left(\frac{p'(\mathrm{i}\omega)}{p(\mathrm{i}\omega)}\right) \mathrm{d}\omega. \tag{3.2}$$

The number $\mathscr{N}$ of URs takes non-negative integers only, one can further prove that there exists a sufficient large number $T > 0$ such that

$$\mathscr{N} = \mathrm{round}\left(\frac{n}{2} - \frac{1}{\pi} \int_0^{T} \Re\left(\frac{p'(\mathrm{i}\omega)}{p(\mathrm{i}\omega)}\right) \mathrm{d}\omega\right). \tag{3.3}$$

Once the upper limit T is known, $\mathscr{N}$ can be simply and easily figured out by using the available algorithms for numerical integration, without a need of preliminary knowledge of $p(\lambda)$.

Assume that $p(\lambda)$ has a CIR $\lambda = \pm \mathrm{i}\omega_0$ at $\tau = \tau_0$, then in a small ε-neighborhood of τ_0, as τ passes through τ_0 from the left to the right, the change of $\mathscr{N}$ can be calculated from

$$\Delta\mathscr{N} \equiv \mathscr{N}(\tau_2) - \mathscr{N}(\tau_1). \tag{3.4}$$

It does not need to know any information about the critical values τ_0 and the CIRs, and it does not to calculate a derivative similar to that given in (2.3).

3.1. **Determination of an uniform upper limit.** In the applications of DIEM [3], a key step is to find an upper limit T for the evalation of $\mathscr{N}$. It is done by using the Hassard'technique [4]. Separating the real and imaginary parts of $\mathrm{i}^{-n} p(\omega \mathrm{i})$ gives

$$\alpha(\omega) + \mathrm{i}\beta(\omega) = \mathrm{i}^{-n} p(\mathrm{i}\omega). \tag{3.5}$$

The real part $\alpha(\omega)$ can be expressed as a polynomial with respect to $\sin(\omega\tau)$, $\cos(\omega\tau)$, $\sin(2\omega\tau)$, $\cos(2\omega\tau)$, $\cdots$, $\sin(q\omega\tau)$ and $\cos(q\omega\tau)$, it depends on the delays. When the delay-dependent functions $\sin(\omega\tau_i)$ and $\cos(\omega\tau_i)$ in $\alpha(\omega)$ were replaced with 1 or -1 respectively, a lower bound function $\underline{\alpha}(\omega)$ independent of τ is easily obtained to satisfy

$$\alpha(\omega) \geq \underline{\alpha}(\omega). \tag{3.6}$$

The maximal zero T_0 of the polynomial $\underline{\alpha}(\omega)$ can be easily obtaioned (if it exists). In addition, it is easy to prove that

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\arctan\frac{\beta(\omega)}{\alpha(\omega)} = \frac{\beta'(\omega)\alpha(\omega)-\alpha'(\omega)\beta(\omega)}{\alpha^2(\omega)+\beta^2(\omega)} = \Re\left(\frac{p'(\mathrm{i}\omega)}{p(\mathrm{i}\omega)}\right). \tag{3.7}$$

Thus, the following lemma is proved true [3], not only for retarded TDSs, but also for neutral TDSs under the strong stability condition.

Lemma 3.1. *Assume that $p(\lambda)$ has no roots on the imaginary axix. Let T_0 be the maximal positive root of $\underline{\alpha}(\omega)$ (if exists), then for a $T > T_0$, $\mathscr{N}$ can be calculated from*

$$\mathscr{N} = \mathrm{round}\left(\frac{n}{2} - \frac{1}{\pi}\int_0^T \frac{\beta'(\omega)\alpha(\omega)-\alpha'(\omega)\beta(\omega)}{\alpha^2(\omega)+\beta^2(\omega)}\mathrm{d}\omega\right). \tag{3.8}$$

If $\underline{\alpha}(\omega)$ has no positive root, one can choose $T_0 = 0$ and find $\mathscr{N} = \mathrm{round}(n/2)$.

In many applications, T_0 can be a small positive number, thus calculation of $\mathscr{N}$ with an definite integral over unbounded interval $[0, +\infty)$ can be simplified to that with an definite integral over a finite interval $[0, T]$ with $T > T_0$. Thus, the stability test could be effective.

3.2. **A generalization of DIEM.** It is worthy of pointing out that Eq. (3.8) works too if $p(\lambda)$ has repeated CIRs. For simplication in presentation, assume that $p(\lambda)$ exactly a pair of conjugate pure imaginary roots $\pm\mathrm{i}\omega_0$ with multiplicity $\gamma \geq 1$, then the contour C shown in Fig.1(a) should be replaced with the one shown in Fig.1(b), where $C_3: \lambda = \mathrm{i}\omega_0 + re^{\mathrm{i}\theta}$ with θ varies from $\pi/2$ to $-\pi/2$, and $C_5: \lambda = -\mathrm{i}\omega_0 + re^{\mathrm{i}\theta}$ with θ varies from $\pi/2$ to $-\pi/2$, then it is required to calculate

$$\lim_{r\to 0}\oint_{C_3}\frac{p'(\lambda)}{p(\lambda)}\mathrm{d}\lambda, \quad \lim_{r\to 0}\oint_{C_5}\frac{p'(\lambda)}{p(\lambda)}\mathrm{d}\lambda,$$

where the integrand has singular points $\pm\mathrm{i}\omega_0$. In a small neighborhood of $\lambda_0 = \mathrm{i}\omega_0$ or $\lambda_0 = -\mathrm{i}\omega_0$, the function $p(\lambda)$ can be approximated in the form of

$$p(\lambda) = a(\lambda - \lambda_0)^{\gamma} + \mathrm{h.o.t}$$

by using Taylor expansion, where h.o.t stands for higher order terms, a is a constant, thus, the Laurant series of the operand $p'(\lambda)/p(\lambda)$ is in the form of

$$\frac{p'(\lambda)}{p(\lambda)} = \frac{\gamma}{\lambda - \lambda_0} + \mathrm{h.o.t.}$$

It follows that

$$\lim_{r\to 0}\frac{1}{2\pi\mathrm{i}}\oint_{C_3}\frac{p'(\lambda)}{p(\lambda)}\mathrm{d}\lambda = \frac{\gamma}{2\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\gamma}{2}, \quad \lim_{r\to 0}\frac{1}{2\pi\mathrm{i}}\oint_{C_5}\frac{p'(\lambda)}{p(\lambda)}\mathrm{d}\lambda = \frac{\gamma}{2}. \tag{3.9}$$

Consequently when $p(\lambda)$ has exactly a pair of CIRs $\pm \mathrm{i}\omega_0$ with multiplicity $\gamma \geq 1$, the integral

$$\frac{1}{2\pi}\int_{+\infty}^{-\infty}\frac{p'(\mathrm{i}\omega)}{p(\mathrm{i}\omega)}\mathrm{d}\omega \equiv \lim_{r\to 0}\frac{1}{2\pi \mathrm{i}}\oint_{C_2+C_3+C_4+C_5+C_6}\frac{p'(\lambda)}{p(\lambda)}\mathrm{d}\lambda$$

can be used to calculate $\mathscr{N}$ as done in (3.8).

Theorem 3.2. *Assume that $p(\lambda)$ has a CIR at $\tau = \tau_0$ with multiplicity $\gamma \geq 1$, then in a small ε-neighborhood of τ_0, as τ passes through τ_0 from the left to the right, the change of the number of URs, $\mathscr{N}$, can be calculated simply from*

$$\Delta\mathscr{N} = \mathscr{N}(\tau_2) - \mathscr{N}(\tau_1). \tag{3.10}$$

4. Four Illustrative Examples

Only TDSs with a single delay are considered for simplicity in this section. Four examples studied in [10, 13, 14] are given to illstrate the applications of Theorem 3.2, which are validated with the combined use of the method of stability switch and the method of Puiseux series expansion. As we will see, the proposed method is much more straightforward and simple.

4.1. **Example 1.** Consider a TDS with the following characteristic quasi-polynomial

$$p(\lambda,\,\tau) = \lambda^2 + 1 + \frac{2}{3\pi} + \frac{2}{3\pi}(\lambda+2)\mathrm{e}^{-\lambda\tau} + \frac{2}{3\pi}(\lambda+1)\mathrm{e}^{-2\lambda\tau}.$$

To calculate $\mathscr{N}$, we firstly compute the integrand and the upper limit of the test integral. By separating the real and imaginary parts of $\mathrm{i}^{-2}p(\mathrm{i}\omega,\,\tau) = 0$, one has

$$\alpha(\omega) \equiv \omega^2 - 1 - \frac{2}{3\pi} - \frac{4}{3\pi}\cos(\omega\tau) - \frac{2}{3\pi}\omega\sin(\omega\tau) - \frac{2}{3\pi}\cos(2\omega\tau) - \frac{2}{3\pi}\omega\sin(2\omega\tau) = 0,$$
$$\beta(\omega) \equiv -\frac{2}{3\pi}\omega\cos(\omega\tau) + \frac{4}{3\pi}\sin(\omega\tau) - \frac{2}{3\pi}\omega\cos(2\omega\tau) + \frac{2}{3\pi}\sin(2\omega\tau) = 0,$$

respectively, and

$$\alpha(\omega) \geq \underline{\alpha}(\omega) \equiv \omega^2 - 1 - \frac{8}{3\pi} - \frac{4}{3\pi}\omega.$$

Then it is easy to find that $\alpha(\omega) \geq \underline{\alpha}(\omega) > 0$ if $\omega > 1.5884$. Thus, the upperlimit of the test integral can be any number larger than 1.5884, here let $T = 2.0$. For any $\tau > 0$, the unmber of unstable characteristic roots, $\mathscr{N}$, can be calculated by using

$$\mathscr{N}(\tau) = \mathrm{round}\left(\frac{2}{2} - \frac{1}{\pi}\int_0^{2.0}\frac{\beta'(\omega)\alpha(\omega) - \alpha'(\omega)\beta(\omega)}{\alpha^2(\omega)+\beta^2(\omega)}\mathrm{d}\omega\right). \tag{4.1}$$

With such a so small upper limit, the computation of $\mathscr{N}(\tau)$ can be very effective.

A broad view of $\mathscr{N}$ in $\tau \in [0, 20]$ is given in Fig.2, which tells that the TDS is asymptotically stable for $\tau \in [0, 0.3743) \cup (4.5291, 4.7039)$ and it unstable if $\tau > 4.7039$.

Below let us validate the above-DIEM-derived results shown in Fig.2 alternatively by using the method of stability switch and the method of Puiseux series expansion.

Actually, by separating the real and imaginary parts of $p(\mathrm{i}\omega,\,\tau)\mathrm{e}^{\mathrm{i}\omega\tau} = 0$, one has

$$\Re(p(\mathrm{i}\omega,\,\tau)\mathrm{e}^{\mathrm{i}\omega\tau}) \equiv -\frac{1}{3\pi}(3\pi\cos(\omega\tau)\omega^2 - 3\pi\cos(\omega\tau) - 2\omega\sin(\omega\tau) - 4\cos(\omega\tau) - 4) = 0,$$
$$\Im(p(\mathrm{i}\omega,\,\tau)\mathrm{e}^{\mathrm{i}\omega\tau}) \equiv -\frac{1}{3\pi}(3\pi\omega^2\sin(\omega\tau) - 3\pi\sin(\omega\tau) - 2\cos(\omega\tau)\omega - 2\omega) = 0.$$

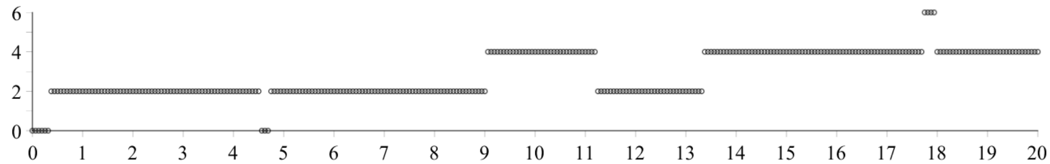


FIGURE 2. The number $\mathscr{N}(\tau)$ with respect to the change of $\tau \in [0, 20]$ for Example 1.

It follows that

$$\sin(\omega\tau) = \frac{6\omega\pi(\omega^2-1)}{9\pi^2\omega^4-(18\pi^2+12\pi+4)\omega^2+9\pi^2+12\pi}, \tag{4.2}$$

$$\cos(\omega\tau) = \frac{4((3\pi+1)\omega^2-3\pi)}{9\pi^2\omega^4-(18\pi^2+12\pi+4)\omega^2+9\pi^2+12\pi}. \tag{4.3}$$

Because $\sin^2(\omega\tau)+\cos^2(\omega\tau)=1$, it is necessary to have

$$27\pi^2(\omega-1)^2(\omega+1)^2(3\pi^2\omega^4-(6\pi^2+8\pi+4)\omega^2+3\pi^2+8\pi)=0, \tag{4.4}$$

which has exactly three positive roots $\omega_1=1,\ \omega_2=0.9369,\ \omega_3=1.4512.$

When $\omega=\omega_1=1$, Eqs.(4.2-4.3) give $\sin\tau=0$ and $\cos\tau=-1$, which result in the first group of critical delay values

$$\tau_{1,k}=(2k+1)\pi\approx 3.142,\ 9.426,\ 21.99,\ \cdots,\quad (k=0,1,\cdots).$$

When $\omega=\omega_2$, one has $\cos(0.9369\tau)=-0.4525$ and $\sin(0.9369\tau)=-0.8907$, which give the second group of critical delay values

$$\tau_{2,k}=\frac{4.2433+2k\pi}{\omega_2}\approx 4.529+\frac{2k\pi}{\omega_2}\approx 4.529,\ 11.24,\ 17.94,\ 24.65,\ \cdots,\quad (k=0,1,\cdots).$$

When $\omega=\omega_3$, one has $\cos(1.4512\tau)=0.8562$ and $\sin(1.4512\tau)=0.5169$, which give the third group of critical delay values

$$\tau_{3,k}=\frac{0.5432+2k\pi}{\omega_3}\approx 0.3743+\frac{2k\pi}{\omega_3}\approx 0.3744,\ 4.705,\ 9.035,\ 13.36,\ 17.70,\ 22.03,\ \cdots$$

Within $\tau\in[0, 20]$, the critical delay values are arranged from small to big as follows

$$0<\tau_{3,0}<\tau_{1,0}<\tau_{2,0}<\tau_{3,1}<\tau_{3,2}<\tau_{1,1}<\tau_{2,1}<\tau_{3,3}<\tau_{1,2}<\tau_{3,4}<\tau_{2,2}<20. \tag{4.5}$$

Strightforward calculation shows that

$$\Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})\Big|_{(\lambda,\tau)=(\mathrm{i},\tau_{1,0}),\,(\mathrm{i},\tau_{1,2})}=0,\ \Re(\frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2})\Big|_{(\lambda,\tau)=(\mathrm{i},\tau_{1,0})}>0,\quad \Re(\frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2})\Big|_{(\lambda,\tau)=(\mathrm{i},\tau_{1,2})}<0,$$

$$\Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})\Big|_{(\lambda,\tau)=(\pm 0.9369\mathrm{i},\tau_{2,k})}<0,\quad (k=0,1,2,\cdots),$$

$$\Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})\Big|_{(\lambda,\tau)=(\pm 1.4512\mathrm{i},\tau_{3,k})}>0,\quad (k=0,1,2,\cdots).$$

Hence, as τ passes $\tau_{1,0}$ or $\tau_{1,2}$ from the left to the right, the TDS keeps the number of URs unchanged; as τ passes $\tau_{2,k}$ from the left to the right, the TDS decreases a pair of conjugate URs; and as τ passes $\tau_{3,k}$ from the left to the right, the TDS increases a pair of conjugate URs.

Special attention should be paid to $\tau=\tau_{1,1}=3\pi$, where $\lambda=\mathrm{i}$ is a repeated CIR with multiplicity 2. At $\tau=\tau_{1,1}$, $\mathrm{d}\lambda/\mathrm{d}\tau$ does not exist, and Puiseux series expansion is required. When $\tau-\tau_{1,1}$ is small, two branches of the Puiseux series expansion are found to be

$$\lambda=\mathrm{i}+(\pm0.0385\pm0.0698\mathrm{i})(\tau-\tau_{1,1})^{1/2}+o((\tau-\tau_{1,1})^{1/2})$$

one branch has positive real part $0.0385(\tau-\tau_{1,1})^{1/2}$ and the other branch has negative real part $-0.0385(\tau-\tau_{1,1})^{1/2}$. Thus, $\Delta\mathscr{N}=\mathscr{N}(\tau_{1,1}+\varepsilon)-\mathscr{N}(\tau_{1,1}-\varepsilon)=0$ for small $\varepsilon>0$.

Because $p(\lambda,0)$ has a pair of conjugate complex zeros $-0.0303\pm1.0585\mathrm{i}$, $\mathscr{N}=0$ when $\tau=0$. Thus, the value of $\mathscr{N}$ in the intervals $[0,\tau_{3,0})$, $(\tau_{3,0},\tau_{1,0})$, $(\tau_{1,0},\tau_{2,0})$, $(\tau_{2,0},\tau_{3,1})$, $(\tau_{3,1},\tau_{3,2})$, $(\tau_{3,2},\tau_{1,1})$, $(\tau_{1,1},\tau_{2,1})$, $(\tau_{2,1},\tau_{3,3})$, $(\tau_{3,3},\tau_{1,2})$, $(\tau_{1,2},\tau_{3,4})$, $(\tau_{3,4},\tau_{2,2})$, $(\tau_{2,2},20]$ are

$$0,\quad 2,\quad 2,\quad 0,\quad 2,\quad 4,\quad 4,\quad 2,\quad 4,\quad 4,\quad 6,\quad 4,$$

respectively, and 8 jumps of $\mathscr{N}(\tau)$ happen at $\tau=\tau_{2,k}$, $(k=0,1,2)$ and $\tau=\tau_{3,k}$, $(k=0,1,2,3,4)$ accordingly. This confirms the results shown in Fig.2.

4.2. **Example 2.** Let us consider a TDS with the following characteristic quasi-polynomial

$$p(\lambda,\tau)=\lambda^4+2\lambda^2+3\mathrm{e}^{-\lambda\tau}-3\mathrm{e}^{-2\lambda\tau}+\mathrm{e}^{-3\lambda\tau}.$$

By separating the real and imaginary parts of $\mathrm{i}^{-4}p(\mathrm{i}\omega,\tau)=0$, one has

$$\alpha(\omega)=\omega^4-2\omega^2+3\cos(\omega\tau)-3\cos(2\omega\tau)+\cos(3\omega\tau)=0,$$
$$\beta(\omega)=-3\sin(\omega\tau)+3\sin(2\omega\tau)-\sin(3\omega\tau)=0,$$

respectively, and

$$R(\omega)\geq\underline{R}(\omega)\equiv\omega^4-2\omega^2-7.$$

Then $R(\omega)\geq\underline{R}(\omega)>0$ if $\omega>1.9566$. Thus, for any $\tau>0$, the unmber of unstable characteristic roots, $\mathscr{N}$, can be calculated by using

$$\mathscr{N}(\tau)=\mathrm{round}\left(\frac{4}{2}-\frac{1}{\pi}\int_0^{2.0}\frac{\beta'(\omega)\alpha(\omega)-\alpha'(\omega)\beta(\omega)}{\alpha^2(\omega)+\beta^2(\omega)}\mathrm{d}\omega\right). \tag{4.6}$$

The plot of $\mathscr{N}$ in $\tau\in[0,20]$ is given in Fig.3, satisfying $\mathscr{N}\geq2$, thus the time-delay system is unstable for all $\tau\in[0,20]$, which can be validated as done in Example 1.

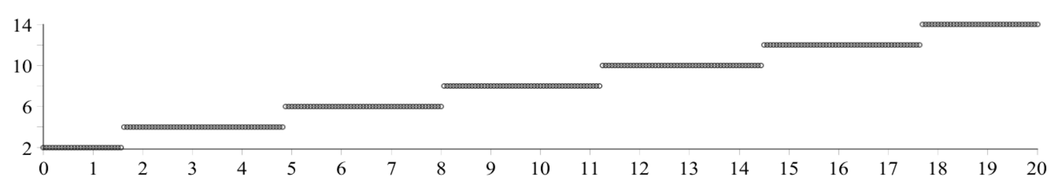


FIGURE 3. The number $\mathscr{N}(\tau)$ with respect to the change of $\tau\in[0,20]$ for Example 2.

In fact, $p(\mathrm{i}\omega,\tau)=0$ if and only if $\alpha(\omega)=\beta(\omega)=0$, where

$$\alpha(\omega)=\omega^4-2\omega^2-6\cos^2(\tau\omega)+3+4\cos^3(\tau\omega),$$
$$\beta(\omega)=-2\sin(\tau\omega)(2\cos(\tau\omega)-1)(\cos(\tau\omega)-1).$$

It follows that $\sin(\tau\omega)=0$, $\cos(\tau\omega)=1$, $\omega^4-2\omega^2+1=0$; or $\sin(\tau\omega)=0$, $\cos(\tau\omega)=-1$, $\omega^4-2\omega^2-7=0$. The first three conditions give $\omega=\pm 1$, $\tau_{1,k}=2k\pi$ $(k=0,1,2,\cdots)$ and $\lambda=\pm\mathrm{i}$ are a pair of repeated conjugate imaginary roots with multiplicity 2; and the second three conditions give $\omega=\pm 1.9566$, $\tau_{2,k}=1.6056+\frac{2k\pi}{1.9566}$ $(k=0,1,2,\cdots)$ and $\lambda=\pm 1.9566\mathrm{i}$ are a pair of conjugate simple imaginary roots.

At $\tau=\tau_{2,k}$ where $\lambda=\pm 1.9566\mathrm{i}$ are simple characteristic roots, it is easy to know that

$$\Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})\Big|_{(\lambda,\,\tau)=(\pm 1.9566\mathrm{i},\,\tau_{2,k})}>0,\quad (k=0,1,2,\cdots).$$

This means that as τ passes $\tau_{2,k}$ from the left to the right, the TDS increases a pair of conjugate unstable characteristic roots. While at $\tau_{1,k}$ where $\lambda=\pm\mathrm{i}$ are not simple, Puiseux series approximation is needed to dtermine the crossing direction. At $(\lambda,\tau)=(\mathrm{i},\tau_{1,k})$ and $(\lambda,\tau)=(-\mathrm{i},\tau_{1,k})$, the Puiseux series has the following approximation

$$\begin{aligned}\lambda &= \mathrm{i}\pm(0.3536+0.3536\mathrm{i})(\tau-\tau_{1,k})^{3/2}+o((\tau-\tau_{1,k})^{3/2}),\\ \lambda &= -\mathrm{i}\pm(-0.3536+0.3536\mathrm{i})(\tau-\tau_{1,k})^{3/2}+o((\tau-\tau_{1,k})^{3/2}),\end{aligned}$$

respectively. It means that as τ passes $\tau_{1,k}$ from the left to the right, the croosing increases two URs and decreases two URs simultaneously, $\mathscr{N}(\tau)$ is kept unchanged. Thus, $\mathscr{N}(\tau)$ can be changed only when τ passes $\tau_{2,k}$ from the left to the right of the complex plane, 2 increased at each $\tau_{2,k}$. Let the critical delay values are arranged in order:

$$0=\tau_{1,0}<\tau_{2,0}<\tau_{2,1}<\tau_{1,1}<\tau_{2,2}<\tau_{2,3}<\tau_{1,2}<\tau_{2,4}<\tau_{2,5}<\tau_{1,3}<20.$$

As a result, the values of $\mathscr{N}(\tau)$ in intervals $[\tau_{1,0},\tau_{2,0})$, $(\tau_{2,0},\tau_{2,1})$, $(\tau_{2,1},\tau_{1,1})$, $(\tau_{1,1},\tau_{2,2})$, $(\tau_{2,2},\tau_{2,3})$, $(\tau_{2,3},\tau_{1,2})$, $(\tau_{1,2},\tau_{2,4})$, $(\tau_{2,4},\tau_{2,5})$, $(\tau_{2,5},\tau_{1,3})$, $(\tau_{1,3},20)$ are

$$2,\quad 4,\quad 6,\quad 6,\quad 8,\quad 10,\quad 10,\quad 12,\quad 14,\quad 14,$$

respectively, and 6 jumps of $\mathscr{N}(\tau)$ happen at $\tau=\tau_{2,k}$, $(k=0,1,2,3,4,5)$ accordingly. The results shown in Fig.3 have been fully confirmed by the above analysis.

4.3. **Example 3.** Let us consider a TDS with the following characteristic quasi-polynomial

$$p(\lambda,\tau)=\lambda^5-a_4\lambda^4-a_3\lambda^3-a_2\lambda^2-a_1\lambda-a_0-(b_4\lambda^4+b_3\lambda^3+b_2\lambda^2+b_1\lambda+b_0)\mathrm{e}^{-\lambda\tau},$$

where the coefficients are $a_0=\pi/2-\pi^2/8-1$, $a_1=\pi/2-2$, $a_2=\pi-\pi^2/4-10$, $a_3=\pi/2-3$, $a_4=\pi/2-\pi^2/8-8$, $b_0=b_1=b_3=-1$, $b_2=-10$ and $b_4=-8$. Let $\alpha(\omega)$ and $\beta(\omega)$ be the real and imaginary parts of $\mathrm{i}^{-5}p(\mathrm{i}\omega,\tau)$ respectively, then one finds that $\alpha(\omega)>0$ if $\omega>8.1451$ for all $\tau>0$. Thus, $\mathscr{N}$ can be calculated from

$$\mathscr{N}(\tau)=\mathrm{round}\left(\frac{5}{2}-\frac{1}{\pi}\int_0^{9.0}\frac{\beta'(\omega)\alpha(\omega)-\alpha'(\omega)\beta(\omega)}{\alpha^2(\omega)+\beta^2(\omega)}\mathrm{d}\omega\right). \tag{4.7}$$

The plot of $\mathscr{N}(\tau)$ in $\tau\in[0,20]$ is given in Fig.4, which can be validated as done above.

Actually, $p(\lambda,\tau)$ has three groups of critical delays:

$\tau_{1,k}=(2k+1)\pi$, $(k=0,1,2,\cdots)$, satisfying $p(\pm\mathrm{i},\tau_{1,k})=0$,

$\tau_{2,k}=1.9457/0.3339+2k\pi/0.3339$, $(k=0,1,\cdots)$, satisfying $p(\pm 0.3339\mathrm{i},\tau_{2,k})=0$,

$\tau_{3,k}=2.8081/2.2421+2k\pi/2.2421$, $(k=0,1,\cdots)$, satisfying $p(\pm 2.2421\mathrm{i},\tau_{3,k})=0$.

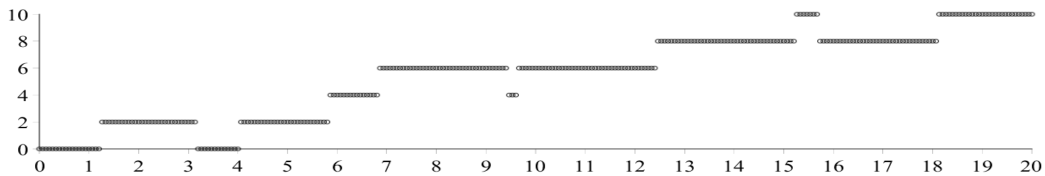


FIGURE 4. The number $\mathcal{N}(\tau)$ with respect to the change of $\tau \in [0, 20]$ for Example 3.

Except $\tau_{1,0} = \pi$ for which $\lambda = \pm\mathrm{i}$ are characteristic roots with multiplicity 2, all the critical characteristic roots at other critical delay values are simple. Straightforward calculation gives

$$\Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})\Big|_{(\lambda,\,\tau)=(\pm 1.4512\mathrm{i},\,\tau_{3,k})} > 0, \quad \Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})\Big|_{(\lambda,\,\tau)=(\pm 0.9369\mathrm{i},\,\tau_{2,k})} > 0, \quad (k = 0, 1, 2, \cdots).$$

This means that as τ passes through every $\tau_{2,k}$ or $\tau_{3,k}$, the TDS increases two UCRs. In addition, in a neighborhood of $(\lambda,\,\tau) = (\mathrm{i},\,\tau_{1,0})$, the Puiseux series has the following approximation

$$\lambda = \mathrm{i} \pm 0.1468\mathrm{i}(\tau - \tau_{1,0})^{1/2} + (-0.0033 - 0.1473\mathrm{i})(\tau - \tau_{1,0}) + o(\tau - \tau_{1,0}),$$

and in a neighborhood of $(\lambda,\,\tau) = (-\mathrm{i},\,\tau_{1,0})$, the Puiseux series has the following approximation

$$\lambda = -\mathrm{i} \pm 0.1468\mathrm{i}(\tau - \tau_{1,0})^{1/2} + (-0.0033 + 0.1473\mathrm{i})(\tau - \tau_{1,0}) + o(\tau - \tau_{1,0}).$$

Thus, as τ passes through $\tau_{1,0}$, the TDS descreses two UCRs, a case unlike in the above two examples where the unmber of UCRs is kept unchanged. While for $k = 1, 2 \cdots$, one has

$$\Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})\Big|_{(\lambda,\,\tau)=(\mathrm{i},\,\tau_{1,k})} = 0, \;\; \Re(\frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2})\Big|_{(\lambda,\,\tau)=(\mathrm{i},\,\tau_{1,k})} = 0, \;\; \Re(\frac{\mathrm{d}^3\lambda}{\mathrm{d}\tau^3})\Big|_{(\lambda,\,\tau)=(\mathrm{i},\,\tau_{1,k})} < 0.$$

This means that as τ passes through every $\tau_{1,1},\,\tau_{1,2},\,\cdots$, the TDS decreases two UCRs. A jump occurs at $\tau_{1,0},\,\tau_{1,1},\,\tau_{1,2};\,\tau_{2,0},\,\tau_{2,1};\,\tau_{3,0},\,\tau_{3,1},\,\cdots,\,\tau_{3,5}$, totally 11 times in $\tau \in [0, 20]$. The TDS is asymptotically stable for $\tau \in [0,\,\tau_{3,0}) \cup (\tau_{1,0},\,\tau_{3,1}) = [0,\,1.2524) \cup (\pi,\,4.0548)$ and it is unstable for other delay values. Thus, the results shown in Fig.4 have been fully confirmed.

4.4. **Example 4.** Finally, consider a time-delay system with quasi-polynomial of the form

$$p(\lambda,\,\tau) = a_0(\lambda) + \sum_{k=1}^{4} a_k(\lambda)\mathrm{e}^{-k\lambda\tau},$$

where

$$a_0 = \frac{15\pi^2}{8}\lambda^6 + (\frac{11\pi}{4} - \frac{15\pi^2}{8})\lambda^4 + \frac{9\pi}{2}\lambda^3 + (1 + \frac{\pi}{2} - \frac{75\pi^2}{8})\lambda^2 + (3 + \frac{9\pi}{2})\lambda + 1 - \frac{9\pi}{4} - \frac{45\pi^2}{8},$$
$$a_1 = \frac{5\pi}{4}\lambda^5 + \frac{11\pi}{2}\lambda^4 + (1 + \frac{7\pi}{2})\lambda^3 + (\pi + 7)\lambda^2 + (11 + \frac{9\pi}{4})\lambda + 4 - \frac{9\pi}{2},$$
$$a_2 = \frac{5\pi}{4}\lambda^5 + \frac{11\pi}{4}\lambda^4 + (3 - \pi)\lambda^3 + (13 + \frac{\pi}{2})\lambda^2 + (15 - \frac{9\pi}{4})\lambda + 6 - \frac{9\pi}{4},$$
$$a_3 = 3\lambda^3 + 9\lambda^2 + 9\lambda + 4, \quad a_4 = \lambda^3 + 2\lambda^2 + 2\lambda + 1.$$

Let $\alpha(\omega)$ and $\beta(\omega)$ be the real and imaginary parts of $\mathrm{i}^{-6}p(\mathrm{i}\omega,\,\tau)$ respectively, then one finds that $\alpha(\omega) > 0$ if $\omega > 1.5662$ for all $\tau > 0$. Thus, $\mathscr{N}$ can be calculated from

$$\mathscr{N}(\tau) = \mathrm{round}\left(\frac{6}{2} - \frac{1}{\pi}\int_0^{2.0} \frac{\beta'(\omega)\alpha(\omega) - \alpha'(\omega)\beta(\omega)}{\alpha^2(\omega)+\beta^2(\omega)}\mathrm{d}\omega\right). \tag{4.8}$$

The plot of $\mathscr{N}$ in $\tau \in [0,\,20]$ is given in Fig.5. Here, at $\tau = \pi,\,3\pi,\,5\pi$, the multiplicity of $\lambda = \pm\mathrm{i}$

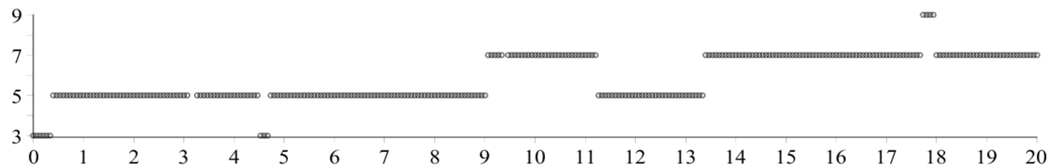


FIGURE 5. The number $\mathscr{N}(\tau)$ with respect to the change of $\tau \in [0,\,20]$ for Example 4.

is 2, 3, 4 respectively, the corresponding approximated Puiseux series are given in [14]. As a direct application of the above extended DIEM, one finds $\mathscr{N}(\pi - 0.05) = \mathscr{N}(\pi + 0.07) = 5$, $\mathscr{N}(3\pi \pm 0.04) = 7$, and $\mathscr{N}(5\pi \pm 0.01) = 7$, which imply that as τ passes through $\tau = \pi,\,3\pi,\,5\pi$ respectively, the integer $\mathscr{N}(\tau)$ is kept unchanged. Thus, the time-delay system is unstable for all $\tau \in [0,\,20]$. The results shown in Fig.5 can be confirmed, and the points where a jump of $\mathscr{N}(\tau)$ occurs can be found, as done in the above three examples.

It is worthy of pointing out that for TDSs with multiple delays (or non-delay parameters), one can firstly mesh the delay intervals and then calculate $\mathscr{N}$ at each node of the meshing network, and thus all the regions with $\mathscr{N} = 0$ can be determined, as done above.

5. CONCLUSION

In this paper, the DIEM that requires integral evaluation only is extended for the fast stability test of linear TDSs with repeated CIRs. As shown in the four illustrative examples, the complete stability analysis based on DIEM for TDSs with a single delay can be very simple and straightforward. The merits of the proposed method includes: 1) the integration is evaluated over a very short delay-free interval, rather than over $[0,\,+\infty)$, thus the computational cost for the complete stability analysis could be very low; 2) the evaluation does not need priliminary knowledge of the critical delay values and the corresponding charcteristic roots; 3) the method works not only for TDSs with a single delay, but also for the ones with multiple delays.

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