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SOME SIMULTANEOUS GENERALIZATIONS OF CELEBRATED FIXED POINT THEOREMS INDUCED BY THE (RC)-CONDITION

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Abstract. In this paper, we introduce the concept of (RC)-condition and present some sufficient conditions for the (RC)-condition. We show that a sequence in a metric space satisfying the (RC)-condition is Cauchy. Some new fixed point theorems and new simultaneous generalizations of celebrated fixed point theorems for (RC)-condition are established.

Keywords. Caristi's fixed point theorem; Kannan's fixed point theorem; Mizoguchi-Takahashi fixed point theorem; Nadler's fixed point theorem.

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1. INTRODUCTION AND PRELIMINARIES

Let (X,d) be a metric space. For $x \in X$ and a subset A of X, define $d(x,A) = \inf_{y\in A} d(x,y)$. Denote by $\mathscr{N}(X)$ the family of all nonempty subsets of X and $\mathscr{CB}(X)$ the class of all nonempty closed and bounded subsets of X. A function $\mathscr{H} : \mathscr{CB}(X) \times \mathscr{CB}(X) \to [0,\infty)$ defined by

$$\mathscr{H}(A,B) = \max\left\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\right\}$$

is said to be the Hausdorff metric on $\mathscr{CB}(X)$ induced by the metric d on X. A point v in X is said to be a fixed point of of a mapping T if $v \in Tv$ (when $T : X \to \mathscr{N}(X)$ is a multivalued mapping) or $Tv = v$ (when $T : X \to X$ is a single-valued mapping). The set of fixed points of T is denoted by $\mathscr{F}(T)$. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively. An extended real valued function $f : X \to (-\infty,\infty]$ is said to be lower semicontinuous (in short $l.s.c.$) (resp. upper semicontinuous, in short $u.s.c.$) at $v \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to v$, we have $f(v) \le \liminf_{n\to\infty} f(x_n)$ (resp. $f(v) \ge \limsup_{n\to\infty} f(x_n)$). The function f is called to be l.s.c. (resp. u.s.c.) on X if f is l.s.c. (resp. u.s.c.) at every point of X. The function f is said to be proper if $f \not\equiv \infty$. An extended real valued function $g : X \to (-\infty,\infty]$ is said to be

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(i) *lower semicontinuous from above* (abbreviated as lsca) at $v \in X$ [4, 27] if for any sequence $\{x_n\}$ in X with $x_n \to v$ and $g(x_n) \geq g(x_{n+1})$ for all $n \in \mathbb{N}$ imply that $g(v) \leq \lim_{n\to\infty} g(x_n)$;

(ii) *upper semicontinuous from below* (abbreviated as uscb) at $v \in X$ [27] if for any sequence $\{x_n\}$ in X with $x_n \to v$ and $g(x_n) \leq g(x_{n+1})$ for all $n \in \mathbb{N}$ imply that $g(v) \geq \lim_{n\to\infty} g(x_n)$.

It is obvious that a function is l.s.c. (resp. u.s.c.) then it is lsca (resp. uscb), but the reverse is not true (see [4, Example 1.3]).

Theorem 1.1 (**Banach contraction principle** [1])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. Assume that there exists a nonnegative number $\gamma < 1$ such that $d(T(x),T(y)) \leq \gamma d(x,y)$ for all $x,y \in X$. Then T admits a unique fixed point in X.*

Since the establishment of the famous Banach contraction principle, fixed point theory and its applications have developed rapidly in the past one hundred years, spawning many academic papers to study its promotion and application in nonlinear analysis, applied mathematics and other fields. A large number of authors devoted their attention to investigating generalizations in various different directions of the well-known fixed point theorems; see [2–32] and references therein.

In 1969, Kannan [24] established his interesting fixed point theorem as follows:

Theorem 1.2 (**Kannan's fixed point theorem** [24])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping on X. Suppose that there exists $\gamma \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx,Ty) \leq \gamma(d(x,Tx)+d(y,Ty)) \quad \text{for all } x,y \in X .$$

Then T admits a unique fixed point in X.

In 1972, Chatterjea established so-called the Chatterjea's fixed point theorem [3] as follows:

Theorem 1.3 (**Chatterjea's fixed point theorem** [3])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping on X. Suppose that there exists $\gamma \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx,Ty) \leq \gamma(d(x,Ty)+d(y,Tx)) \text{ for all } x,y \in X.$$

Then T admits a unique fixed point in X.

Nadler's fixed point theorem [29] is a well-known generalization of Banach contraction principle, which extends Banach contraction principle from single-valued mappings to multivalued mappings.

Theorem 1.4 (**Nadler's fixed point theorem** [29])**.** *Let (X,d) be a complete metric space and $T : X \to \mathscr{CB}(X)$ be a k-contraction; that is, there exists a nonnegative number $k < 1$ such that $\mathscr{H}(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. Then $\mathscr{F}(T) \neq \emptyset$.*

Definition 1.1 (see [9, Definition 1.1])**.** A function $\varphi : [0,\infty) \to [0,1)$ is said to be a $\mathscr{MT}$-function (or $\mathscr{R}$- function) if $1 > \limsup_{s\to t^+} \varphi(s) := \inf_{\varepsilon>0} \sup_{t<s<t+\varepsilon} \varphi(s)$ for all $t \in [0,\infty)$.

Clearly, if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is a $\mathscr{MT}$-function. So the set of $\mathscr{MT}$-functions is a rich class.

In 2012, Du [9] established the following characterizations of $\mathscr{MT}$-functions.

Theorem 1.5 (**Du** [9, Theorem 2.1]). *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*

(a) φ is an $\mathscr{MT}$-function.
(b) For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.
(c) For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.
(d) For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.
(e) For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.
(f) For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.
(g) φ is a function of contractive factor; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

In 1989, Mizoguchi and Takahashi [28] proved the following famous generalization of Nadler's fixed point theorem.

Theorem 1.6 (**Mizoguchi-Takahashi's fixed point theorem** [28]). *Let (X,d) be a complete metric space, $\varphi : [0,\infty) \to [0,1)$ be a $\mathscr{MT}$-function and $T : X \to \mathscr{CB}(X)$ be a multivalued mapping. Assume that $\mathscr{H}(Tx,Ty) \le \varphi(d(x,y))d(x,y)$ for all $x,y \in X$. Then $\mathscr{F}(T) \neq \emptyset$.*

In 2017, Du [13] present the following fixed point theorem which simultaneously generalizes and improves Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem.

Theorem 1.7 (**Du** [13, Theorem 1.7]). *Let (X,d) be a complete metric space and $T : X \to \mathscr{CB}(X)$ be a multivalued mapping. Suppose that there exists an $\mathscr{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that*

$$\mathscr{H}(Tx,Ty) \le \varphi(d(x,y)) \max\left\{ d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2} \right\}$$

for all $x,y \in X$. Then T admits a fixed point in X.

Caristi's fixed point theorem [2] is undoubtedly one of the most valuable generalizations of Banach's contraction principle. It is well-known that the Caristi's fixed point theorem is equivalent to the Ekeland's variational principle, to the Takahashi's nonconvex minimizationth eorem, to the Daneš' drop theorem, to the petal theorem, and to the Oettli-Théra's theorem; see, e.g., [10, 12, 16, 22, 23, 25–27, 30, 31] and references therein for more details. In 2016, Du [11] gave a new, simple and direct proof of Caristi's fixed point theorem without using Zorn's lemma, transfinite induction and any well-known principle. For more detailed information, the interested readers can refer to [11].

Theorem 1.8 (**Caristi's fixed point theorem** [2]). *Let (X,d) be a complete metric space and $f : X \to \mathbb{R}$ be a lower semicontinuous and bounded below function. Suppose that T is a Caristi* *type mapping on X dominated by f; that is, $T : X \to X$ satisfies $d(x,Tx) \leq f(x) - f(Tx)$ for each $x \in X$. Then T has a fixed point in X.*

In this work, we introduce the concept of (RC)-condition and present some sufficient conditions for the (RC) -condition in section 2. We show that a sequence in a metric space satisfies (RC)-condition, then it is Cauchy. Some new fixed point theorems and new simultaneous generalizations of well-known fixed point theorems for (RC)-condition are established in section 3.

2. THE CONCEPT OF (RC)-CONDITION AND RELATED RESULTS

In this paper, we introduce the concept of (RC)-condition.

Definition 2.1. Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is said to have regulated contractility condition (abbreviated as (RC)-condition), if there exist a nonnegative real number $\lambda < 1$ and a proper function $h : X \to [0,\infty]$, such that

$$d(x_{n+2},x_{n+1}) \leq \lambda d(x_{n+1},x_n) + h(x_{n+1}) - h(x_{n+2}) \text{ for all } n \in \mathbb{N},$$

where $x_1, x_2 \in Dom(h) := \{x \in X : h(x) < \infty\}$.

The following results will play vital roles as examples of sufficient conditions for the (RC) -condition.

Theorem 2.1. *Let (X,d) be a metric space, $T : X \to X$ be a selfmapping, $f : X \to (-\infty,\infty]$ be a proper bounded below function and $\kappa : (-\infty,\infty] \to [0,\infty)$ be a nondecreasing function. If T is Caristi type with respect to κ, that is*

$$d(x,Tx) \leq \kappa(f(x))(f(x) - f(Tx)) \quad \textit{for all } x \in X. \tag{2.1}$$

Then the following hold:

(a) Define a set-valued mapping $\Gamma : X \to 2^X$ (the power set of X) by

$$\Gamma(x) = \{y \in X : d(x,y) \leq \kappa(f(x))(f(x) - f(y))\} \textit{ for } x \in X.$$

Then for any $u \in Dom(f)$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_1 = u$ and $x_{n+1} \in \Gamma(x_n)$ satisfies (RC)-condition.

(b) For any $u \in Dom(f)$, the sequence $\{T^n u\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies (RC)-condition (here, $T^0 = I$ is the identity mapping).

Proof. We first prove the conclusion (a) holds. Clearly, $\Gamma(x) \neq \emptyset$ for all $x \in X$. Let $u \in Dom(f)$ be given and let the sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfy $x_1 = u$ and $x_{n+1} \in \Gamma(x_n)$. Let $\lambda = 0$ and define a proper function $h : X \to [0,\infty]$ by

$$h(x) = \kappa(f(u)) \left(f(x) - \inf_{x\in X} f(x) \right).$$

Since $x_{n+1} \in \Gamma(x_n)$, we have $f(x_{n+1}) \leq f(x_n)$ for each $n \in \mathbb{N}$ which means that $\{f(x_n)\}$ is nonincreasing. For any $n \in \mathbb{N}$, since $x_{n+2} \in \Gamma(x_{n+1})$ and κ is nondecreasing, the inequality (2.1) implies

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\leq \kappa(f(x_{n+1}))(f(x_{n+1}) - f(x_{n+2})) \\ &\leq \lambda d(x_{n+1},x_n) + h(x_{n+1}) - h(x_{n+2}). \end{aligned}$$

So, $\{x_n\}$ satisfies (RC)-condition. Similarly, we can show that the conclusion (b) is true. □

Theorem 2.2. *Let (X,d) be a metric space, $T : X \to \mathscr{CB}(X)$ be a multivalued mapping, $f : X \to (-\infty,\infty]$ be a proper bounded below function and $D,S : X\times X \to [0,\infty)$ be two functions. Suppose that there exists an MT-function $\varphi : [0,\infty) \to [0,1)$ such that for each $x,y \in X$, it holds*

$$D(x,y) \le \varphi(\ell(x,y))S(x,y) + f(x) - f(z) \ \text{ for all } z \in Tx, \tag{2.2}$$

where $\ell(x,y) = d(x,y) + f(x) - \inf_{x\in X} f(x)$ for $x,y \in X$. If a sequence $\{x_n\}$ in X satisfies $x_1 \in Dom(f)$, $x_{n+1} \in Tx_n$, $D(x_{n+1},x_n) = d(x_{n+2},x_{n+1})$ and $S(x_{n+1},x_n) \le d(x_{n+1},x_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ satisfies (RC)-condition.

Proof. Define a proper function $h : X \to [0,\infty]$ by $h(x) = f(x) - \inf_{x\in X} f(x)$. Then $\ell(x,y) = d(x,y) + h(x)$ and inequality (2.2) implies that, for each $x,y \in X$,

$$D(x,y) \le \varphi(\ell(x,y))S(x,y) + h(x) - h(z) \ \text{ for all } z \in Tx. \tag{2.3}$$

Since $\varphi(t) < 1$ for all $t \in [0,\infty)$, we can define an $\mathscr{MT}$-function $\tau : [0,\infty) \to (0,1)$ by

$$\tau(t) = \frac{1}{2}(1+\varphi(t)) \quad \text{for all } t \in [0,\infty).$$

Clearly, $0 \le \varphi(t) < \tau(t) < 1$ for all $t \in [0,\infty)$. Let $\{x_n\}$ in X satisfy $x_1 \in Dom(f) = Dom(h)$, $x_{n+1} \in Tx_n$, $D(x_{n+1},x_n) = d(x_{n+2},x_{n+1})$ and $S(x_{n+1},x_n) \le d(x_{n+1},x_n)$ for all $n \in \mathbb{N}$. Clearly, $h(x_n) < \infty$ for all $n \in \mathbb{N}$. Let

$$\xi_n = d(x_{n+1},x_n) + h(x_{n+1}),\ n \in \mathbb{N}.$$

By (2.3), we have

$$d(x_{n+2},x_{n+1}) < \tau(\xi_n)d(x_{n+1},x_n) + h(x_{n+1}) - h(x_{n+2}) \ \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

By (2.4), we have $(1-\tau(\xi_n))d(x_{n+1},x_n) \le \xi_n - \xi_{n+1}$, which deduces $0 \le \xi_{n+1} \le \xi_n$ for all $n \in \mathbb{N}$. Hence the sequence $\{\xi_n\}_{n=1}^{\infty}$ is nonincreasing in $[0,\infty)$. Since τ is an $\mathscr{MT}$-function, by applying Theorem 1.5, we have $0 \le \sup_{n\in\mathbb{N}} \tau(\xi_n) < 1$. Let $\lambda := \sup_{n\in\mathbb{N}} \tau(\xi_n)$. So $\lambda \in [0,1)$. By (2.3), we get

$$d(x_{n+2},x_{n+1}) \le \lambda d(x_{n+1},x_n) + h(x_{n+1}) - h(x_{n+2}) \quad \text{for all } n \in \mathbb{N}.$$

Hence $\{x_n\}$ satisfies (RC)-condition. The proof is completed. □

If we take $f(x) = 0$ for all $x \in X$ in Theorem 2.2, we obtain the following result immediately.

Corollary 2.1. *Let (X,d) be a metric space, $T : X \to \mathscr{CB}(X)$ be a multivalued mapping and $D,S : X\times X \to [0,\infty)$ be two mappings. Suppose that there exists an $\mathscr{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that for each $x,y \in X$, it holds$D(x,y) \le \varphi(d(x,y))S(x,y)$. If a sequence $\{x_n\}$ in X satisfies $x_1 \in X$ is arbitrary, $x_{n+1} \in Tx_n$, $D(x_{n+1},x_n) = d(x_{n+2},x_{n+1})$ and $S(x_{n+1},x_n) \le d(x_{n+1},x_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ satisfies (RC)-condition.*

The following result can be derived immediately from Theorem 2.2 with $\varphi(x) = 0$ for all $x \in X$.

Corollary 2.2. *Let (X,d) be a metric space, $T : X \to \mathscr{CB}(X)$ be a multivalued mapping, $f : X \to (-\infty,\infty]$ be a proper bounded below function and $D : X\times X \to [0,\infty)$ be a mapping. Suppose that for each $x,y \in X$, it holds $D(x,y) \le f(x) - f(z)$ for all $z \in Tx$. If a sequence $\{x_n\}$ in X satisfies $x_1 \in Dom(f)$, $x_{n+1} \in Tx_n$, $D(x_{n+1},x_n) = d(x_{n+2},x_{n+1})$, then $\{x_n\}$ satisfies (RC)-condition.*

The following result is a single-valued version of Theorem 2.2 and follows directly from Theorem 2.2.

Theorem 2.3. *Let (X,d) be a metric space, $T : X \to X$ be a self-mapping, $f : X \to (-\infty,\infty]$ be a proper bounded below function and $D,S : X \times X \to [0,\infty)$ be two mappings. Suppose that there exists an $\mathscr{M}\mathscr{T}$-function $\varphi : [0,\infty) \to [0,1)$ such that for each $x,y \in X$, it holds*

$$D(x,y) \le \varphi(\ell(x,y))S(x,y) + f(x) - f(Tx),$$

where $\ell(x,y) = d(x,y) + f(x) - \inf_{x\in X} f(x)$ for $x,y \in X$. If a sequence $\{x_n\}$ in X satisfies $x_1 \in Dom(f)$, $x_{n+1} = Tx_n$, $D(x_{n+1},x_n) = d(x_{n+2},x_{n+1})$ and $S(x_{n+1},x_n) \le d(x_{n+1},x_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ satisfies (RC)-condition.

Corollary 2.3. *Let (X,d) be a metric space, $T : X \to X$ be a self-mapping and $D,S : X \times X \to [0,\infty)$ be two mappings. Suppose that there exists an $\mathscr{M}\mathscr{T}$-function $\varphi : [0,\infty) \to [0,1)$ such that for each $x,y \in X$, it holds $D(x,y) \le \varphi(d(x,y))S(x,y)$. If a sequence $\{x_n\}$ in X satisfies $x_1 \in X$ is arbitrary, $x_{n+1} = Tx_n$, $D(x_{n+1},x_n) = d(x_{n+2},x_{n+1})$ and $S(x_{n+1},x_n) \le d(x_{n+1},x_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ satisfies (RC)-condition.*

Corollary 2.4. ***Corollary 2.4.*** *Let (X,d) be a metric space, $T : X \to X$ be a self-mapping, $f : X \to (-\infty,\infty]$ be a proper bounded below function and $D : X \times X \to [0,\infty)$ be a mappings. Suppose that for each $x,y \in X$, it holds $D(x,y) \le f(x) - f(Tx)$. If a sequence $\{x_n\}$ in X satisfies $x_1 \in Dom(f)$, $x_{n+1} = Tx_n$, $D(x_{n+1},x_n) = d(x_{n+2},x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\}$ satisfies (RC)-condition.*

Theorem 2.4. *Let (X,d) be a metric space and $T : X \to X$ be a self-mapping. Suppose that*

(H) there exists an $\mathscr{M}\mathscr{T}$-function $\varphi : [0,\infty) \to [0,1)$ such that

$$\min\{d(Tx,Ty),d(x,Tx)\} \le \varphi(d(x,y)) \max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

for all $x,y \in X$.

Then for any $u \in X$, the sequence $\{T^n u\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies (RC)-condition (here, $T^0 = I$ is the identity mapping).

Proof. Define two mappings $D,S : X \times X \to [0,\infty)$ by

$$D(x,y) = \min\{d(Tx,Ty),d(x,Tx)\}$$

and

$$S(x,y) = \max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

respectively. Then, by (H), we obtain

$$D(x,y) \le \varphi(d(x,y))S(x,y) \text{ for all } x,y \in X.$$

Let $u \in X$ be given. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_1 = u$ and $x_{n+1} = Tx_n = T^n u$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

$$D(x_{n+1},x_n) = d(x_{n+2},x_{n+1}) \quad \text{for all } x,y \in X. \tag{2.5}$$

and

$$S(x_{n+1},x_n) = \max\left\{d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1}),\frac{d(x_n,x_{n+2})\}}{2}\right\}. \tag{2.6}$$

Assume there exists $j \in \mathbb{N}$ such that $d(x_{j+1},x_j) < d(x_{j+2},x_{j+1})$. Then, by (H), (2.5) and (2.6), we have

$$d(x_{j+2},x_{j+1}) = D(x_{j+1},x_j) < S(x_{j+1},x_j) = d(x_{j+2},x_{j+1}),$$

a contradiction. Hence it must be $d(x_{n+2},x_{n+1}) \le d(x_{n+1},x_n)$ and hence we have $S(x_{n+1},x_n) \le d(x_{n+1},x_n)$ for all $n \in \mathbb{N}$. By Corollary 2.3, $\{x_n\}$ satisfies (RC)-condition. The proof is completed. □

Remark 2.1. Let T be a self-mapping on a metric space (X,d). Then, by Theorem 2.4, for any $u \in X$, the sequence $\{T^n u\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies (RC)-condition if T is Banach type or Kannan type or Chatterjea type.

3. Some fixed point theorems for (RC)-condition

The following is one of main results in this paper.

Theorem 3.1. *Let (X,d) be a metric space. If a sequence $\{x_n\}$ in X satisfies (RC)-condition, then $\{x_n\}$ is Cauchy.*

Proof. Since $\{x_n\}$ satisfies (RC)-condition, there exist a nonnegative real number $\lambda < 1$ and a proper function $h : X \to [0,\infty]$ with $x_1 \in Dom(h)$, such that

$$d(x_{n+2},x_{n+1}) \le \lambda d(x_{n+1},x_n) + h(x_{n+1}) - h(x_{n+2}) \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Clearly, $h(x_n) < \infty$ for all $n \in \mathbb{N}$. Let $\xi_n = d(x_{n+1},x_n) + h(x_{n+1})$, $n \in \mathbb{N}$. By (3.1), we get

$$(1-\lambda)d(x_{n+1},x_n) \le d(x_{n+1},x_n) + h(x_{n+1}) - [d(x_{n+2},x_{n+1}) + h(x_{n+2})], \tag{3.2}$$

which deduces $0 \le \xi_{n+1} \le \xi_n$ for all $n \in \mathbb{N}$. Hence the sequence $\{\xi_n\}_{n=1}^{\infty}$ is nonincreasing in $[0,\infty)$ and

$$\ell := \lim_{n\to\infty} \xi_n = \inf_{n\in\mathbb{N}} \xi_n \ \text{ exists.}$$

By (3.2), we have

$$d(x_n,x_{n+1}) \le \frac{1}{1-\lambda}(\xi_n - \xi_{n+1}) \ \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

For m, $n \in \mathbb{N}$ with $m > n$, we have form (3.3) that

$$d(x_n,x_m) \le \sum_{j=n}^{m-1} d(x_j,x_{j+1}) = \frac{1}{1-\lambda}(\xi_n - \xi_m).$$

Since $\lim\limits_{n\to\infty} \xi_n = \ell$, we have $\lim_{n\to\infty} \sup\{d(x_n,x_m) : m > n\} = 0$. This prove that $\{x_n\}$ is a Cauchy sequence in X. □

Applying Theorem 3.1, we present the following new fixed point theorem for Caristi type mappings and lsca functions.

Theorem 3.2. *Let (X,d) be a complete metric space, $f : X \to (-\infty,\infty]$ a proper lsca and bounded below function and $\kappa : (-\infty,\infty] \to (0,\infty)$ a nondecreasing function. Suppose that $T : X \to X$ is a self-mapping satisfying*

$$d(x,Tx) \le \kappa(f(x))(f(x) - f(Tx)) \ \text{ for each } x \in X. \tag{3.4}$$

Then $\mathcal{F}(T) \neq \emptyset$. Moreover, there exists a sequence $\{x_n\}$ in X such that it satisfies (RC)-condition and converges to a fixed point v of T with $f(v) < \infty$.

Proof. Define a set-valued mapping $\Gamma : X \to 2^X$ by

$$\Gamma(x) = \{y \in X : d(x,y) \leq \kappa(f(x))(f(x) - f(y))\} \text{ for } x \in X.$$

Since $Tx \in \Gamma(x)$, we have $\Gamma(x) \neq \emptyset$ for all $x \in X$. First, we claim that for each $y \in \Gamma(x)$, we have $f(y) \leq f(x)$ and $\Gamma(y) \subseteq \Gamma(x)$. Let $y \in \Gamma(x)$ be given. Thus $d(x,y) \leq \kappa(f(x))(f(x) - f(y))$ and hence $f(y) \leq f(x)$. Let us verify $\Gamma(y) \subseteq \Gamma(x)$. Given $z \in \Gamma(y)$. Then $d(y,z) \leq \kappa(f(y))(f(y) - f(z))$. So $f(z) \leq f(y) \leq f(x)$. Let $u \in Dom(f)$. We shall construct a sequence $\{x_n\}$ in X by induction, starting with $x_1 = u$. Suppose that $x_n \in X$ is known. Then choose $x_{n+1} \in \Gamma(x_n)$ such that

$$f(x_{n+1}) \leq \inf_{z \in \Gamma(x_n)} f(z) + \frac{1}{n} \quad \text{for } n \in \mathbb{N}. \tag{3.5}$$

Applying Theorem 2.1 (a), we know that $\{x_n\}$ satisfies (RC)-condition. Hence, by Theorem 3.1, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. We now verify $v \in \mathscr{F}(T)$. For any $n \in \mathbb{N}$, since $x_{n+1} \in \Gamma(x_n)$, we have

$$d(x_n, x_{n+1}) \leq \kappa(f(x_n))(f(x_n) - f(x_{n+1}), \tag{3.6}$$

and hence

$$f(x_{n+1}) \leq f(x_n). \tag{3.7}$$

Since f is bounded below,

$$\xi := \lim_{n\to\infty} f(x_n) = \inf_{n \in \mathbb{N}} f(x_n) \quad \text{exists.} \tag{3.8}$$

Since f is lsca, by taking into account (3.7) and (3.8), we get

$$f(v) \leq \lim_{n\to\infty} f(x_n) = \xi \leq f(x_j) < \infty \quad \text{for all } j \in \mathbb{N}. \tag{3.9}$$

We claim that $\bigcap_{n=1}^{\infty} \Gamma(x_n) = \{v\}$. Let $n \in \mathbb{N}$ be fixed. For $m > n$ with $m, n \in \mathbb{N}$, since κ is nondecreasing, by (3.6), (3.7) and (3.9), we obtain

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \leq \kappa(f(x_n))(f(x_n) - f(v)). \tag{3.10}$$

Since $x_m \to v$ as $m \to \infty$, by (3.10), we obtain $d(x_n, v) \leq \kappa(f(x_n))(f(x_n) - f(v))$ for all $n \in \mathbb{N}$, which shows $v \in \bigcap_{n=1}^{\infty} \Gamma(x_n)$. Hence $\bigcap_{n=1}^{\infty} \Gamma(x_n) \neq \emptyset$ and $\Gamma(v) \subseteq \cap_{n=1}^{\infty} \Gamma(x_n)$. For any $w \in \bigcap_{n=1}^{\infty} \Gamma(x_n)$, taking into account (3.5) and (3.7), we have

$$\begin{aligned} d(x_n, w) &\leq \kappa(f(x_n))(f(x_n) - f(w)) \\ &\leq \kappa(f(x_1)) \left(f(x_n) - \inf_{z \in \Gamma(x_n)} f(z) \right) \\ &\leq \kappa(f(x_1)) \left(f(x_n) - f(x_{n+1}) + \frac{1}{n} \right) \end{aligned}$$

for all $n \in \mathbb{N}$. Let $\beta_n = \kappa(f(x_1)) \left(f(x_n) - f(x_{n+1}) + \frac{1}{n} \right)$, $n \in \mathbb{N}$. So $\lim_{n\to\infty} \beta_n = 0$ and hence $\lim_{n\to\infty} d(x_n, w) = 0$. This shows that $x_n \to w$ as $n \to \infty$. By the uniqueness of limit of a sequence, we have $w = v$. So we prove $\bigcap_{n=1}^{\infty} \Gamma(x_n) = \{v\}$. Since $\Gamma(v) \neq \emptyset$ and

$$\Gamma(v) \subseteq \bigcap_{n=1}^{\infty} \Gamma(x_n) = \{v\}$$

we get $\Gamma(v) = \{v\}$. Since $Tv \in \Gamma(v)$, it must be $Tv = v$. Therefore T has a fixed point v in X. Clearly, $f(v) < \infty$. The proof is completed. □

Now, we establish the following new fixed point theorem which generalizes and improves Mizoguchi-Takahashi's fixed point theorem, Kannan's fixed point theorem and Chatterjea's fixed point theorem simultaneously.

Theorem 3.3. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Define two mapping $D,S : X \times X \to [0,\infty)$ by*

$$D(x,y) = \min\{\mathscr{H}(Tx,Ty), d(x,Tx)\}$$

and

$$S(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \quad \text{for } x,y \in X.$$

respectively. Suppose that there exists an $\mathscr{M}\mathscr{T}$-function $\varphi : [0,\infty) \to [0,1)$ such that

$$D(x,y) \le \varphi(d(x,y))S(x,y) \quad \text{for all } x,y \in X. \tag{3.11}$$

Then T admits a fixed point in X. Moreover, there exists a sequence $\{x_n\}$ in X such that it satisfies (RC)-condition and converges to a fixed point v of T.

Proof. Since $\varphi(t) < 1$ for all $t \in [0,\infty)$, we can define a function $\tau : [0,\infty) \to (0,1)$ by

$$\tau(t) = \frac{1}{2}(1+\varphi(t)) \quad \text{for all } t \in [0,\infty).$$

Clearly, $0 \le \varphi(t) < \tau(t) < 1$ for all $t \in [0,\infty)$. Thus, by (3.11), we obtain

$$D(x,y) < \tau(d(x,y))S(x,y) \quad \text{for all } x,y \in X.$$

Let $z \in X$ be given. Take $x_1 = z \in X$ and choose $x_2 \in Tx_1$. If $x_2 = x_1$, then $x_1 \in Tx_1$ and we are done. Otherwise, if $x_2 \neq x_1$, then $d(x_2,x_1) > 0$. Since $D(x_2,x_1) = d(x_2,Tx_2)$ and

$$\begin{aligned} S(x_2,x_1) &= \max\left\{d(x_2,x_1), \frac{d(x_2,Tx_2)+d(x_1,Tx_1)}{2}, \frac{d(x_1,Tx_2)}{2}\right\} \\ &\le \max\left\{d(x_2,x_1), \frac{d(x_2,Tx_2)+d(x_1,x_2)}{2}, \frac{d(x_1,Tx_2)}{2}\right\}, \end{aligned}$$

by (3.11), we have

$$d(x_2,Tx_2) < \tau(d(x_2,x_1)) \max\left\{d(x_2,x_1), \frac{d(x_2,Tx_2)+d(x_1,x_2)}{2}, \frac{d(x_1,Tx_2)}{2}\right\}.$$

Hence there exists $x_3 \in Tx_2$ such that

$$d(x_2,x_3) < \tau(d(x_2,x_1)) \max\left\{d(x_2,x_1), \frac{d(x_2,x_3)+d(x_1,x_2)}{2}, \frac{d(x_1,x_3)}{2}\right\}. \tag{3.12}$$

Assume $d(x_2,x_1) < d(x_3,x_2)$. Thus we have

$$\max\left\{d(x_2,x_1), \frac{d(x_2,x_3)+d(x_1,x_2)}{2}, \frac{d(x_1,x_3)}{2}\right\} = \frac{d(x_2,x_3)+d(x_1,x_2)}{2}. \tag{3.13}$$

So, by (3.12) and (3.13), we get

$$d(x_2,x_3) < \frac{1}{2}(d(x_2,x_3)+d(x_1,x_2))$$

which implies $d(x_2,x_3) < d(x_2,x_1)$, a contradiction. Hence it must be $d(x_3,x_2) \leq d(x_2,x_1)$ and (3.12) deduces $d(x_2,x_3) < \tau(d(x_2,x_1))d(x_2,x_1)$. If $x_3 = x_2$, then $x_2 \in Tx_2$ and the desired conclusion is proved. Assume $x_3 \neq x_2$. By (3.11) again, we obtain

$$d(x_3,Tx_3) < \tau(d(x_3,x_2)) \max\left\{d(x_3,x_2), \frac{d(x_3,Tx_3)+d(x_2,x_3)}{2}, \frac{d(x_2,Tx_3)}{2}\right\}.$$

So there exists $x_4 \in Tx_3$ such that

$$d(x_3,x_4) < \tau(d(x_3,x_2)) \max\left\{d(x_3,x_2), \frac{d(x_3,x_4)+d(x_2,x_3)}{2}, \frac{d(x_2,x_4)}{2}\right\}.$$

Following a similar argument as above, we obtain $d(x_3,x_4) < \tau(d(x_3,x_2))d(x_3,x_2)$. Hence, by induction, we can obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying the following: for each $n \in \mathbb{N}$,

(i) $x_{n+1} \in Tx_n$ with $x_n \neq x_{n-1}$;

(ii) $d(x_{n+2},x_{n+1}) < \tau(d(x_{n+1},x_n))d(x_{n+1},x_n)$.

Since $\tau(t) < 1$ for all $t \in [0,\infty)$, by (ii), we know that the sequence $\{d(x_{n+1},x_n)\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0,\infty)$. Since φ is an $\mathscr{MT}$-function, by applying Theorem 1.5, we have

$$0 \leq \sup_{n\in\mathbb{N}} \varphi(d(x_{n+1},x_n)) < 1$$

and hence deduces

$$0 < \sup_{n\in\mathbb{N}} \tau(d(x_{n+1},x_n)) = \frac{1}{2}\left[1+\sup_{n\in\mathbb{N}} \varphi(d(x_{n+1},x_n))\right] < 1.$$

Let $\lambda := \sup_{n\in\mathbb{N}} \tau(d(x_{n+1},x_n))$. So $\lambda \in (0,1)$. Define a proper function $h : X \to [0,\infty]$ by $h(x) = 0$ for all $x \in X$. Then for any $n \in \mathbb{N}$, by (ii) again, we have

$$\begin{aligned} d(x_{n+2},x_{n+1}) &< \tau(d(x_{n+1},x_n))d(x_{n+1},x_n) \\ &\leq \gamma d(x_{n+1},x_n) + h(x_{n+1}) - h(x_{n+2}) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Hence $\{x_n\}$ satisfies (RC)-condition. Using Theorem 3.1, we know that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. In order to finish the proof, it is sufficient to show $v \in \mathscr{F}(T)$. For any $n \in \mathbb{N}$, by (3.11), we have

$$\begin{aligned} & \min\{d(x_{n+1},Tv), d(v,Tv)\} \\ \leq\ & \min\{\mathscr{H}(Tx_n,Tv), d(v,Tv)\} \\ \leq\ & \varphi(d(x_n,v)) \max\left\{d(x_n,v), \frac{d(x_n,x_{n+1})+d(v,Tv)}{2}, \frac{d(x_n,Tv)+d(v,x_{n+1})}{2}\right\} \end{aligned}$$

for all $n \in \mathbb{N}$. Since the function $x \longmapsto d(x,Tv)$ is continuous and $x_n \to v$ as $n \to \infty$, by taking the limit as $n \to \infty$ on both sides of the last inequality, we acquire

$$d(v,Tv) \leq \frac{1}{2} d(v,Tv)$$

which implies $d(v,Tv) = 0$. Therefore we obtain $v \in \mathscr{F}(T)$. The proof is completed. □

Remark 3.1. Theorem 1.7 is a special case of Theorem 3.3.

The following new fixed point theorem is a simultaneous generalization of Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem.

Theorem 3.4. *Let (X,d) be a complete metric space and $T:X\to X$ be a self-mapping. Define two mapping $D,S:X\times X\to[0,\infty)$ by*

$$D(x,y)=\min\{d(Tx,Ty),d(x,Tx)\}$$

and

$$S(x,y)=\max\left\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

respectively. Suppose that there exists an $\mathscr{MT}$-function $\varphi:[0,\infty)\to[0,1)$ such that

$$D(x,y)\le\varphi(d(x,y))S(x,y)\quad \textit{for all } x,y\in X. \tag{3.14}$$

Then $\mathscr{F}(T)\neq\emptyset$. Moreover, for any $u\in X$, the sequence $\{T^n u\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies (RC)-condition and converges to a fixed point of T.

Proof. Let $u\in X$ be given. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_1=u$ and $x_{n+1}=Tx_n=T^n u$ for all $n\in\mathbb{N}$. Clearly, the condition (H) in Theorem 2.4 holds from (3.14). By Theorem 2.4, the sequence $\{T^n u\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies (RC)-condition. Applying Theorem 3.1, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $v\in X$ such that $x_n\to v$ as $n\to\infty$. Now, we verify that $v\in\mathscr{F}(T)$. For any $n\in\mathbb{N}$, by (3.14), we have

$$\begin{aligned}&\min\{d(Tv,x_{n+1}),d(v,Tv)\}\\&\le\varphi(d(v,x_n))\max\left\{d(v,x_n),\frac{d(v,Tv)+d(x_n,x_{n+1})}{2},\frac{d(v,x_{n+1})+d(x_n,Tv)}{2}\right\}.\end{aligned}$$

Since $x_n\to v$ as $n\to\infty$, by taking the limit as $n\to\infty$ on both sides of the last inequality, we get

$$d(v,Tv)\le\frac{1}{2}d(v,Tv)$$

which implies $d(v,Tv)=0$. Therefore we obtain $v\in\mathscr{F}(T)$. The proof is completed. □

Theorem 3.5. *Let (X,d) be a complete metric space and $T:X\to X$ be a self-mapping. Suppose that there exists an $\mathscr{MT}$-function $\varphi:[0,\infty)\to[0,1)$ such that*

$$d(Tx,Ty)\le\varphi(d(x,y))\max\left\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\right\} \tag{3.15}$$

for all $x,y\in X$. Then T admits a unique fixed point v in X. Moreover, for any $u\in X$, the sequence $\{T^n u\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies (RC)-condition and converges to v.

Proof. Clearly, (3.15) implies (3.14). Applying Theorem 3.4, we have $\mathscr{F}(T)\neq\emptyset$. We claim that $\mathscr{F}(T)$ is a singleton set. Assume there exist $w,v\in\mathscr{F}(T)$ with $w\neq v$. Thus $d(w,v)>0$. By (3.15), we have $d(w,v)=d(Tw,Tv)\le\varphi(d(w,v))d(w,v)<d(w,v)$, a contradiction. Hence we prove that $\mathscr{F}(T)$ is a singleton set, say $\mathscr{F}(T)=\{v\}$. Therefore T has a unique fixed point v in X. Moreover, for any $u\in X$, by using Theorem 3.4 again, the sequence $\{T^n u\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies (RC)-condition and converges to v. The proof is completed. □

4. Conclusions

For more than a century, fixed point theory has been a fascinating theory in various fields including linear and nonlinear analysis, optimization, differential equations, economics, game theory, dynamical systems theory, control theory, signal and image processing, and so forth. In this paper, the concept of (RC)-condition and its sufficient conditions are studied. We show that a sequence in a metric space satisfies (RC)-condition, then it is Cauchy. We establish some new fixed point theorems and new simultaneous generalizations of well-known fixed point theorems for (RC)-condition.

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