Commun. Optim. Theory 2025 (2025) 50 https://doi.org/10.23952/cot.2025.50

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

THE MONOTONIC PATH AND ITS VALUE LOSS WHEN AN OPTIMAL PATH IS NON-MONOTONIC

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Abstract. Under standard economic assumptions, the optimal paths in optimal growth models can be non-monotonic and, at times, extremely complex. In contrast, real-world policies are typically based on the assumption of a monotonic progression towards objectives. To address this discrepancy, this study investigates the characteristics and value loss associated with an alternative monotonic path when the optimal path is non-monotonic in discrete-time, one-state-variable optimal growth models. We assume that the planner selects the best path from a class of monotonic paths (i.e., either monotonically increasing or decreasing paths). We show that if the optimal path is increasing (or decreasing), the corresponding monotonic path will also be increasing (or decreasing). Monotonic paths generically encounter time inconsistency when reaching their steady states. If the monotonic path is revised at this point, the transition from increasing to decreasing, or vice versa, in the monotonic path occurs in tandem with a similar transition in the associated optimal path. Distinct features of the monotonic paths compared to the optimal paths include time inconsistency and the finite time to reach the steady state. Moreover, the monotonic path with revision exhibits differences in the local stability of the common interior steady state compared to the optimal policy. Regarding value loss, in three models demonstrating chaotic optimal paths, the study finds that the upper bounds of the value loss ratios incurred by adopting monotonic paths without revision range from 10^{-5} to 10^{-13} relative to the optimal value function. We argue the potential generality of this marginal value loss. Furthermore, we discuss several implications of these findings, including a possible rationale for why complex solutions to optimization problems can describe human behavior that is not universally optimal.

Keywords. Business Cycle; Chaotic optimal path; Monotonic path; Value loss.

2020 MSC. 91B62.

1. INTRODUCTION

Since the 1980s, it has been known that an optimal path in a standard optimal growth model can exhibit complex dynamics [9]. The central implication is that various cyclical or non-cyclical fluctuations may be endogenously generated as optimal paths in competitive economies. For a policymaker, this suggests that achieving the optimal path may be highly complex and delicate. In practice, many plans made by governments, firms, and households aim to set a goal,

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Received May 7, 2024; Accepted September 22, 2024.

steadily approach that goal, and try to achieve it within a finite period. If the optimal path is non-monotonic, adopting such monotonic paths may be suboptimal. However, in everyday life, it is not always easy to pursue complex plans that follow the optimal path. If the outcomes from adopting monotonic paths are not significantly worse compared to optimal policies, people may opt for adopting monotonic paths instead of strictly adhering to optimal ones.

Motivated by this gap between theoretical complexity and practical difficulty, this study examines the consequences of adopting monotonic paths when the optimal path is not monotonic. We analyze a discrete-time optimal growth model with a single state variable, assuming that the model has a unique interior steady state and its optimal policy function is unimodal, generating non-monotonic optimal paths. We assume that planners must choose a state path from the monotonic class (monotonically increasing or decreasing paths) at the initial time point.

The study first examines the properties of monotonic paths when the optimal path is non-monotonic. We show that when the optimal path is increasing (or decreasing), the monotonic path will also be increasing (or decreasing). Due to the non-recursive nature of the optimization problem for monotonic paths, the monotonic path is generically time-inconsistent. This time inconsistency arises at the steady state of the monotonic path, where the optimal path transitions from increasing to decreasing or vice versa. Consequently, if the planner revises the monotonic path, it will change from increasing to decreasing or vice versa, depending on the optimal path. Additionally, the period until the monotonic path encounters time inconsistency aligns with the period during which the optimal path shifts from increasing to decreasing or vice versa. Therefore, the monotonic path is similar to the optimal path and very similar when it is revised when it faces time inconsistency.

One feature of the monotonic path that is different from the non-monotonic optimal path is that the monotonic path generically faces time inconsistency. Another distinct feature is that the monotonic path reaches a steady state in finite time. Furthermore, there is a difference in the local stability of their common interior steady state: while the stability of the interior steady state of the optimal path implies the stability of the steady state of the monotonic path, the converse does not hold. Therefore, it is possible for the steady state of the optimal path to be unstable while it remains stable for the monotonic path.

The monotonic path described above bears similarities to real-world planning in the following manner. Real-world plans typically aim to steadily achieve objectives within a specific period. Similarly, the monotonic path monotonically approaches the steady state and reaches it in finite time. Real-world plans often create a new plan at the end of the current plan. Likewise, a monotonic path switches to a new monotonic path when the current monotonic path reaches its steady state.

Following the characterization of the monotonic path, the study evaluates the magnitude of value loss resulting from adopting the monotonic path without revision. We measure value loss using the ratio of the value functions for the optimal path and the monotonic path. Since the value loss ratio varies with the level of initial capital stock, we seek the upper bound of the value loss ratio. We then present examples of optimal growth models with chaotic optimal paths, as discussed by Boldrin and Montrucchio [2], Deneckere and Pelikan [3], and Nishimura and Yano [11]. The results indicate that the upper bounds of the value loss ratio are on the order of magnitude of at most 10^{-5}, showing that the suboptimality of the monotonic path is marginal. We discuss that this low-value loss is generally expected to hold, not just in the examples presented.

The similarity between the optimal path and the monotonic path, along with the minimal value loss incurred by adopting the monotonic path instead of the optimal path, provides intriguing insights from both descriptive and normative perspectives. From a descriptive standpoint, the similarity between the optimal path and the monotonic path helps explain why human behavior, which is not universally optimal, can sometimes be well described by complex solutions to optimization problems. From a normative standpoint, the minimal value difference between the optimal and monotonic paths explains why humans (despite not being universally optimal), can achieve results comparable to optimal plans. The final section of this paper will further discuss the implications of these findings.

The remainder of this paper is structured as follows: Section 2 describes the model and assumptions. Section 3 characterizes monotonic paths. Section 4 formulates the value loss ratio, examines its properties, and calculates the upper bound of the value loss ratio for the economic model examples. Finally, Section 5 discusses the implications of the results. All proofs are provided in the Appendix.

2. MODEL

2.1. **Optimal growth model and policy function.** Let $t = 0, 1, 2, \ldots$ denote discrete time and consider an economy described by (Γ, u, ρ), where $\Gamma : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}_+$ is a correspondence that represents the production set for each period, i.e., output y can be produced at the end of the period from the input x at the beginning of the period if $y \in \Gamma(x)$, $u : \mathbb{R}^2_+ \twoheadrightarrow \mathbb{R}_+$ is a one period utility function, and $\rho \in (0,1)$ is a discount factor. An optimal growth model is defined by the following problem:

$$V_c(x) = \sup_{\{x_t\}_{t\geq 0}} \sum_{t=1}^{\infty} \rho^{t-1} u(x_{t-1}, x_t)$$
$$\text{subject to } x_t \in \Gamma(x_{t-1}),\ t = 1, 2, \ldots,\ x_0 = x \geq 0. \tag{2.1}$$

We call $\{x_t\}_{t\geq 0}$ that satisfies the above condition a feasible path from x. We make the following assumption for Γ and u:

- $(A.\Gamma 1)$ Γ is nonempty, compact, convex-valued, and continuous.
- $(A.\Gamma 2)$ $\Gamma(0) = \{0\}$.
- $(A.\Gamma 3)$ If $0 \leq x \leq x'$, then $\Gamma(x) \subset \Gamma(x')$.
- $(A.\Gamma 4)$ There exists $x^\# > 0$ such that if $x < x^\#$, then there exists $y > x$ such that $y \in \Gamma(x)$, and if $x > x^\#$, then there exists $\gamma < 1$ such that if $y \in \Gamma(x)$, then $y < \gamma x$.
- $(A.U1)$ $u(x,y)$ is continuous, concave, increasing in x, and strictly decreasing in y, and jointly strictly increasing, i.e., $u(x,x) < u(y,y)$ if $x < y$, for any $(x, \Gamma(x)) \subset \mathbb{R}^2_+$.

Due to $(A.\Gamma 4)$, any feasible path enters $[0, x^\#]$ in a finite time and stays there. Then, we may restrict the state space to this interval. Furthermore, we normalize the maximum stock level as $x^\# = 1$. We denote the production possibility set with $D := \{(x,y) \in [0,1] \times [0,1] | y \in \Gamma(x)\}$.

Then, we rewrite the above problem as:

$$\text{(P)} \quad V_c(x) = \sup_{\{x_t\}_{t\geq 0}} \sum_{t=1}^{\infty} \rho^{t-1} u(x_{t-1}, x_t)$$
$$\text{subject to } (x_{t-1}, x_t) \in D,\ t = 1, 2, \ldots,\ x_0 = x \in [0,1].$$

With these assumptions, $V_c(x)$ is continuous and satisfies the following Bellman equation, and conversely, no function other than $V_c(x)$ satisfies this functional equation:

$$\text{(P')} \quad V_c(x) = \sup_{y \in \Gamma(x)} u(x,y) + \rho V_c(y)$$

(See, for example, Stokey and Lucas [13]). Since a solution for (P') exists for all $x \in [0,1]$, the optimal path for problem (P) exists for all $x \in [0,1]$.

To distinguish (P) and (P') from similar problems discussed below, we refer to them as the complete optimization problem. Similarly, we call $V_c : [0,1] \to \mathbb{R}_+$ the completely optimal value function and a path $\{x_t^*\}_{t \geqq 0}$ achieving $V_c(x)$ the completely optimal path. The following results are readily derived from the assumptions or are well-known[1]:

- $(F.1)$ $V_c(x)$ is continuous, concave, and strictly increasing.
- $(F.2)$ The optimal policy correspondence defined by

$$H_c(x) := \{y \in \Gamma(x) | u(x,y) + \rho V_c(y) = V_c(x)\}$$

  is non-empty, compact valued, and upper hemi-continuous.

If u is strictly concave, H_c is a singleton at each point. Thus, H_c becomes a function $h_c(x)$, which is referred to as the optimal policy function. However, despite u being just concave, there exists an economic model where the optimal policy correspondence becomes a function (Nishimura and Yano [11]). This paper does not assume strict concavity of u but assumes the existence of the optimal policy function $h_c(x)$ for problem (P). Furthermore, it is assumed that $h_c(x)$ possesses the following property:

- $(A.H)$ $H_c(x) = \{h_c(x)\}$ for all $x \in [0,1]$. The optimal policy function $h : [0,1] \to [0,1]$ is continuous[2]. There are x^p and x^0, such that $0 < x^p < x^0 < 1$ and $h(x)$ is strictly increasing in $[0, x^p)$ and strictly decreasing in $(x^p, 1]$ with $h_c(x) > x$ for $x \in (0, x^0)$ and $h_c(x) < x$ for $x \in (x^0, 1)$.

This assumption ensures that the completely optimal path from almost every initial stock is not monotonic.

2.2. **Monotonic path.** This subsection defines the monotonic path problem that this paper investigates. Define first the following two correspondences $\Gamma_i : [0,1] \twoheadrightarrow [0,1], i = a, d$ by:

$$\Gamma_a(x) := \{y \in [0,1] | y \in \Gamma(x) \text{ and } y \geq x\},$$
$$\Gamma_d(x) := \{y \in [0,1] | y \in \Gamma(x) \text{ and } y \leq x\}.$$

Denote the associated production possibility sets by:

$$D_i := \{(x,y) \in [0,1] \times [0,1] | y \in \Gamma_i(x)\}, i = a, d.$$

[1]Refer to Stokey and Lucas [13] (Theorem 3.6) for (F.2).

[2]The continuity is a result rather than an assumption since H is upper hemi-continuous and it holds that $\{\lim_{x \nearrow z} h(x), \lim_{x \searrow z} h(x)\} \in H(z) = \{h(z)\}$ for any $z \in [0,1]$.

We refer to the following two problems as the ascending problem (AP) and the descending problem (DP), respectively.

$$\text{(AP)} \quad V_a(x) = \sup_{\{x_t\}_{t\geq 0}} \sum_{t=1}^{\infty} \rho^{t-1} u(x_{t-1}, x_t)$$
$$\text{subject to } (x_{t-1}, x_t) \in D_a, t = 1, 2, \dots, x_0 = x \in [0,1].$$

$$\text{(DP)} \quad V_d(x) = \sup_{\{x_t\}_{t\geq 0}} \sum_{t=1}^{\infty} \rho^{t-1} u(x_{t-1}, x_t)$$
$$\text{subject to } (x_{t-1}, x_t) \in D_d, t = 1, 2, \dots, x_0 = x \in [0,1].$$

By a similar argument to the above, these problems have an optimal path referred to as an ascending optimal path and a descending optimal path, respectively, and these optimal paths satisfy the associated Bellman equations:

$$\text{(AP')} \quad V_a(x) = \sup_{y \in \Gamma_a(x)} u(x,y) + \rho V_a(y),$$
$$\text{(DP')} \quad V_d(x) = \sup_{y \in \Gamma_d(x)} u(x,y) + \rho V_d(y).$$

We denote the solution set of the above problems, i.e., the ascending and descending optimal correspondence, by $H_a(x)$ and $H_d(x)$, respectively, and represent the element by $h_a(x)$ and $h_d(x)$. That is,

$$h_i(x) \in H_i(x) := \left\{ \arg\max_{y \in \Gamma_i(x)} u(x,y) + \rho V_i(y) \right\}, i = a, d.$$

By a similar argument as above, these correspondences are nonempty, compact-valued, and upper hemicontinuous. Now, we define the monotonic problem and the optimal paths.

Definition 2.1 (The monotonic path)**.** The monotonic path, referred to as $\{x_t^m\}_{t\geq 0}$, is a solution to the following problem:

$$\text{(MP)}\ V_m(x) := \sum_{t=1}^{\infty} \rho^{t-1} u(x_{t-1}^m, x_t^m) = \max\{V_a(x), V_d(x)\}.$$

Although the function "max" is convex, we will see that V_m is a concave function like V_i $(i = c, a, d)$ at Proposition 3.5. In addition, we will see that a monotonic path is time inconsistent for almost every x in $[0,1]$, since the problem is not recursive, unlike the Bellman equations for V_i $(i = c, a, d)$.

We refer to $V_m(x)$ as the monotonic value function. We denote the associated policy correspondence by $H_m(x)$ and represent the elements by $h_m(x)$, i.e.,

$$x_t^m = h_m(x_{t-1}^m) \in H(x_{t-1}^m).$$

3. Characterization of the Monotonic Path

This section examines the properties of the ascending and descending policies and then characterizes the monotonic path. The following sequence of capital stocks $\{x^i\}_{i\geq 0}$ plays a crucial role in their characterizations.

Definition 3.1. x^i $(i = 0, 1, 2, \dots)$ satisfies $x^{i+1} < x^i$ and $h_c(x^{i+1}) = x^i$, where x^0 is the unique interior steady state of $h_c(x)$ defined in $(A.H)$.

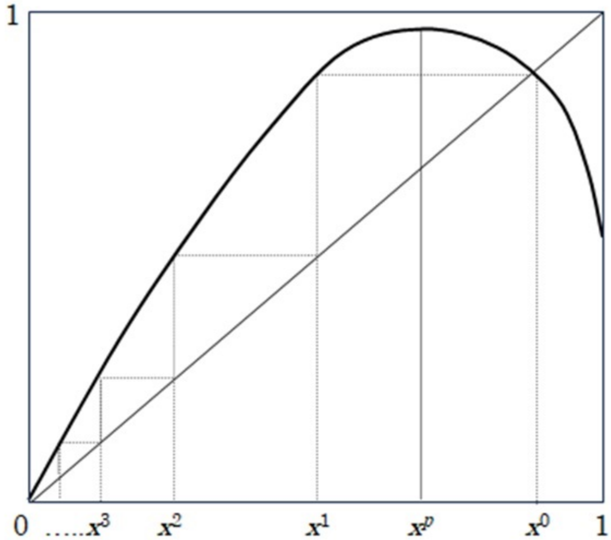

FIGURE 1. Optimal policy function $h_f(x)$ and x^is

Figure 1 illustrates a typical policy function we are considering with some x^is. As seen from this, if the initial stock is x^i $(i = 1,2,\dots)$, the completely optimal path is monotonically increasing and reaches x^0 at $t = i$. In this case, a completely optimal path and an ascending optimal path coincide. We note this as a proposition.

Proposition 3.2. *$V_c(x) = V_a(x) = V_m(x)$ if $x = 0$ or x^i $(i = 0,1,2,\dots)$.*

The proposition below shows further results regarding an ascending optimal path.

Proposition 3.3. *The ascending optimal correspondence H_a satisfies the following properties.*

- **(a):** *$H_a(x) = \{h_c(x)\}$ for $x \in \{0, x^0, x^1, \dots\}$.*
- **(b):** *$H_a((x^{i+1}, x^i)) \subset [x^i, x^{i-1}]$ for $i \geq 1$ and $H_a((x^1, x^0)) \subset [x^0, 1]$.*
- **(c):** *$H_a(x) = \{x\}$ for $x \in [x^0, 1]$.*
- **(d):** *An ascending optimal path from $x \in (0,1]$ reaches a steady state in $[x^0, 1]$ in a finite time.*

Proposition 3.4 shows a descending optimal path.

Proposition 3.4. *The descending optimal correspondence H_d satisfies the following properties.*

- **(a):** *$H_d(x) \subset [0, x^0)$ for $x \in (x^0, 1]$.*
- **(b):** *$x \in H_d(x)$ for $x \in [0, x^0]$.*
- **(c):** *A descending path reaches its steady state in $[0, x^0]$ at least one period later.*

Using these results, we can show some properties of the ascending, descending, and monotonic value functions.

Proposition 3.5. *The following hold for the ascending value function $V_a(x)$, the descending value function $V_d(x)$, and the monotonic value function V_m:*

(a): *If $x < h_c(x)$, i.e., $x \in (0, x_0)$, $V_a(x) > V_d(x)$. If $x > h_c(x)$,i.e., $x \in (x_0, 1]$, $V_a(x) < V_d(x)$. Therefore,*

$$V_m(x) = \begin{cases} V_a(x) \\ V_d(x) \end{cases} \text{if } x \begin{matrix} \leq \\ > \end{matrix} x^0 .$$

(b): *$V_m(x)$ is strictly increasing and concave.*

As an immediate consequence of Propositions 3.3, 3.4, and 3.5 is:

Corollary 3.6. *The monotonic path is time-inconsistent, except when the initial state x_0 satisfies $x_0 = 0$ or x^i for $i = 0, 1, 2, \dots$.*

As the last topic characterizing the monotonic path, we consider the "stability" property of the interior steady state x^0. The question is whether a monotonic path from a neighborhood of x^0 approaches or gets away from x^0 if the planner revises the monotonic path when facing time inconsistency. For this analysis, we impose an additional assumption:

$(A.U2)$ $u(x,y)$ is twice continuously differentiable and $u_{12}(x,y) := \partial^2 u(x,y)/\partial x \partial y \neq 0$ at (x^0, x^0).

Proposition 3.7. *Assume $(A.U2)$ additionally to the assumptions in Section 2. (a) The policy function for the monotonic paths $h_m(x)$ exists in a neighborhood of x^0: $H_m(x) = \{h_m(x)\}$ for $x \in (x^0 - \varepsilon, x^0 + \varepsilon)$ for a small positive ε. (b) x^0 is locally asymptotically stable steady state of the monotonic path problem (MP) with revision, i.e., $|x - x^0| > |h_m(x) - x^0|$ for x in a neighborhood of x^0, if and only if*

$$\rho u_{11}(x^0, x^0) + (3\rho - 1) u_{12}(x^0, x^0) + u_{22}(x^0, x^0) < 0. \tag{3.1}$$

(c) (3.1) holds if $\rho \geq 1/9$ or x^0 is an asymptotically stable steady state of the completely optimal problem (P).

Therefore, roughly speaking, the monotonic path with revision is more likely to converge to a steady state than the optimal policy.

4. Value Loss

4.1. **Lower Bound of Value Ratio.** In this section, we investigate the extent of loss in value incurred by adopting a monotonic path instead of following the optimal policy. Here, "value" can refer to various meanings, such as social welfare, household utility, or corporate profits. To measure the loss in value, we introduce the value ratio $(V_m(x)/V_c(x))$. To ensure that each term in the ratio is non-negative, we define the value ratio as:

$$R(x) := \frac{V_m(x) - u(0,0)/(1-\rho)}{V_c(x) - u(0,0)/(1-\rho)}, \quad x \in (0,1].$$

Note that the range of $R(x)$ is $(0,1]$ since $u(x,x) > u(x,0)$ for all $x > 0$, and thus

$$V_c(x) \geq V_m(x) \geq u(x,x)/(1-\rho) > u(0,0)/(1-\rho).$$

Moreover, note that $R(x)$ is invariant up to a positive affine transformation of u. Hence, we assume that, without loss of generality,

$(A.U3)$ $u(0,0) = 0$,

throughout this section. With this assumption, $V_m(x)$ and $V_c(x)$ are positive for $x \in (0,1]$ and the value ratio becomes

$$R(x) = \frac{V_m(x)}{V_c(x)}.$$

Proposition 4.1. *There is $x \in (x^1, 1]$ that minimizes $R(x)$ over $(0,1]$.*

4.2. **An Example of the Upper Bound of the Value Loss Ratio.** Since the lower bound of the value ratio ($R(x)$) or the upper bound of the value loss ratio ($1-R(x)$) varies depending on the model (Γ, u, ρ), we consider specific models and present the results of their upper bounds of the value loss ratio. The models include those of Deneckere and Pelikan [3], Boldrin and Montrucchio [2], and Nishimura and Yano [11]. All of these models feature optimal policy functions that generate ergodic chaos[3]. As shown below, these studies provide their utility functions $u(x,y)$ and policy functions $h_c(x)$. The optimal value functions $V_c(x)$ are either provided (Denecker and Pelikan [3]) or derived as shown below. The values of the monotonic value function $V_m(x)$ for $[x^1, 1]$ is calculated as:

$$V_m(x) = \max_y u(x,y) + \frac{\rho}{1-\rho} u(y,y)$$
$$\text{subject to} \begin{cases} y \geq x^0 \\ y \leq x^0 \end{cases} \text{for} \quad \begin{matrix} x^1 \leq x \leq x^0 \\ x^0 < x \leq 1 \end{matrix} \quad .$$

Deneckere and Pelikan [3] provide their utility function

$$u(x,y) = xy - x^2y - (1/3)y - (75/1000)y^2 + (100/3)x - 7x^2 + 4x^3 - 2x^4,$$

discount factor 0.01, optimal policy function $h_c(x) = 4x(1-x)$, and the associated value function

$$V_c(x) = -5x^2 + (100/3)x.$$

Boldrin and Montrucchio [2] provide their utility function

$$u(x,y) = -0.0857845y^4 + 0.171569y\hat{}^3 - 0.3285y^2 - 1.61y + 4xy(1-x) - 24x^2 + 150x,$$

discount factor 0.0107231, and optimal policy function that is the same logistic map as one of Deneckere and Pelikan [3]. The value function is

$$V_c(x) = (1/2)(h(x))^2 - (47.9871/2)x^2 + 149.994x,$$

which is derived from the identity $V_c(x) - \{u(x,h(x)) + \rho V_c(h(x))\} = 0$, where the left-hand side is a polynomial.

Nishimura and Yano [11] provide a pair of utility function and discount factor, with which

$$h_c(x) = \begin{cases} \mu x \\ -(\mu/(\mu-\alpha))(\alpha x - 1) \end{cases} \text{if} \quad \begin{matrix} 0 \leq x \leq 1/\mu \\ 1/\mu < x \leq 1 \end{matrix}$$

is rationalized as the optimal policy function. It is assumed that $1 < \mu < 2\alpha < 2$ for the tent map to be well-defined in the domain $[0,1]$ and expansive. Additionally, it is specified that the support of the limit distribution of $(h_c)^n(x)$ is $[h_c(1), 1]$ by assuming that the parameters are

[3]In the notation used in this paper, ergodic chaos implies that the limit distribution of $((h_c)^n(x)$ $(n \to \infty)$ is a uniform distribution, identical for almost every $x \in [0,1]$, and the support has positive measures. As a result, almost every optimal path visits each point on the support infinitely many times, exhibiting complicated state paths.

chosen to satisfy $(h_c)^{3*2^n}(1) = 1$ for nonnegative integer n. Once n is fixed, then the utility function

$$u(x,y) = \begin{cases} x - (1/\mu)y & \text{for } 0 \le y \le -(\alpha\mu/(\mu-\alpha))(\alpha x - 1) \\ 1/\alpha/(1-y) & -(\alpha\mu/(\mu-\alpha))(\alpha x - 1) < y \le 1 \end{cases}$$

and the range of discount factor ρ are determined. Although Nishimura and Yano [11] do not refer to the optimal value function, it is derived as follows: For $0 \le x \le 1/\mu$,

$$V_c(x) = u(x, \mu x) + \rho V_c(\mu x) = \rho V_c(\mu x).$$

Thus, the value function is homogeneous in degree $-\log(\rho)/\log(\mu)$ over the state space. It is identified with the information on the value at the interior steady state x^0,

$$V_c(h^{-1}(x^0)) = \rho V_c(x^0) = (\rho/(1-\rho))u(x^0, x^0).$$

The table below shows the maximum value loss ratios for the three studies[4]. For the Nishimura and Yano [11] model, we present nine cases of periodic solutions starting from $x = 1$ with periods ranging from 3 ($= 3 \cdot 2^0$) to $3 \cdot 2^8$, where the discount factor ranges from approximately 0.366 to approximately 0.996.

Table 1 shows the maximum value loss ratio incurred by adopting monotonic policies instead of completely optimal policies is marginal, at most on the order of 10^{-5}, in these optimal growth models. This means that, for instance, when annual rents of $1,000,000 are obtained on the completely optimal path, the value loss incurred by adopting monotonic policies would be at most a few tens of dollars per year, sometimes much less than $1.00. Furthermore, if policy revisions are allowed, the value loss incurred by adopting monotonic policies will be even smaller.

Table 1 Maximum value loss ratio

		Maximum value loss ratio	Maximum point	Steady state	Range of support	Discount factor
Deneckere and Pelikan (1986)		1.54869E-05	0.892066861	0.75	(0 , 1)	0.01
Boldrin & Montrucchio (1986)		1.48268E-05	0.891652521	0.75	(0 , 1)	0.0107231
Nishimura and Yano (1995)	0	4.05524E-08	0.633980327	0.633974529	(0.133974 , 1)	0.366025
	1	9.96713E-09	0.796842907	0.796839848	(0.571078 , 1)	0.614019
	2	2.35344E-10	0.889364656	0.889364498	(0.771403 , 1)	0.785600
	3	3.47722E-13	0.942458809	0.942458771	(0.883217 , 1)	0.886569
	4	1.06929E-10	0.970582605	0.970582597	(0.940712 , 1)	0.941611
	5	3.21062E-10	0.985130106	0.985130073	(0.970149 , 1)	0.970371
	6	1.08786E-11	0.992523154	0.992523146	(0.985018 , 1)	0.985074
	7	1.99907E-10	0.996257839	0.996251107	(0.992495 , 1)	0.992509
	8	3.41123E-10	0.998122913	0.996247588	(0.996244 , 1)	0.996248

5. Discussion

At the end of this paper, we discuss the implications derived from the qualitative results of monotonic policies and the numerical results of the value loss incurred by following a monotonic path when an optimal path is non-monotonic.

[4]The calculations were performed using Mathematica. The codes used for the calculations are available upon request from the author.

Although we presented numerical examples of the value loss for only a few models, it is reasonable to expect that a similar order of maximum value loss ratio would appear in many models with complex non-monotonic optimal paths. This expectation is based on the following two factors: First, studies on economic models that produce complex equilibrium paths (Denecker and Pelikan [3]; Boldrin and Montrucchio [2]; and Sorger [12]) required heavy discounting to generate such paths. Mitra [6] and Nishimura and Yano [10] prove that the discount factor must be less than $\left[\left(\sqrt{5}-1\right)/2\right]^2 \approx 0.3819$ for the model (Γ, u, ρ) to have a period-three cycle optimal path, which implies that the optimal policy function is chaotic in the sense of Li and Yorke [4]. Heavy discounting diminishes the impact of the outcomes of future choices. Therefore, choosing a suboptimal monotonic path in the future is likely to have a small impact on value. This is the reason why the maximum value loss ratios are very small, with the order of 10^{-5} to 10^{-8} in the heavy discounting case on the table ($\rho = 0.01$ to 0.37).

Heavy discounting implies that the model's unit period is long. For instance, with a discount factor of 0.3819 and assuming a standard annual discount rate of 0.03, one unit period would amount to about 33 years. Therefore, economic models employing heavy discounting are deemed useful for examining long-term business cycles, such as Kuznets cycles or Kondratiev waves. However, models that generate non-monotonic optimal paths with lighter discounting are required to explore shorter-term economic fluctuations. As a result, developments in this field have shifted towards creating complex optimal paths using lighter discounting. Examples from Nishimura and Yano [11] and Wan [14, 15, 16] and the data demonstrated in Table 1 show that the smaller the discount rate adopted, the narrower the range of the support on which an optimal path fluctuates becomes. This more general possibility is suggested by the neighborhood turnpike theorem (see, e.g., McKenzie [5]) and is expected from the results of Mitra and Sorger [7]. However, if the range of the support becomes narrower, the impact of choosing a suboptimal monotonic path over an optimal non-monotonic path within that range would also decrease. This is the second reason why the maximum value loss ratios are expected to be very small. Despite light discounting, it occurs due to the narrow range of oscillation. Table 1 demonstrates that for the examples of Nishimura and Yano [11], they are of the order of about 10^{-10}.

The fact that adopting a suboptimal monotonic path results in a small value loss actually suggests that monotonic paths may become optimal with slight modifications to the model. If economic frictions, such as resizing of firms, scrapping and rebuilding of capital during economic downturns, layoffs of employees, and increases in unemployment rates, are not reflected in the model as costs incurred by economic fluctuations, and if these costs are incorporated to make the model more realistic, a monotonic path may emerge as an optimal policy, rather than a complicated one.

Contrary to this argument, when complex policies are indeed optimal, the results obtained in this paper can lead to the following interpretation: Real people may not necessarily be able to effectively follow a complex optimal path with an infinite planning horizon. Hence, they may adopt a monotonic path. Alternatively, regardless of human capability, policymakers and managers may choose monotonic policies because they are tasked with achieving specific goals, such as economic recovery, eliminating fiscal deficits, expanding market share, and streamlining unprofitable sectors within a finite period. In either case, these economic agents appear different from the economically rational agents assumed by economic models. However, the results of this paper suggest that their paths and outcomes may not differ significantly. The same holds for the monotonic path if the state variable increases or decreases along the optimal path. In other words, the paths followed by both types of agents who can and cannot follow the complex policies coincide in terms of whether the state variable increases or decreases in each period. Furthermore, the differences in outcomes may be marginal, as indicated by Table 1 and discussed above. This may explain why, even if real humans are not necessarily economically rational, optimal growth models assuming such behavior can still effectively represent real behavior and outcomes.

Acknowledgements
This study, dedicated to Professor Tapan Mitra, is a substantial revision of a Japanese paper by Akao [K. Akao, Tancho Seisaku To Kachi Sonshitsu, in R. Nagata, (ed.), Keizai Gaku No Suuri To Ronri, Waseda University Press, 2001. (in Japanese)]. Significant changes have been made to the analysis and presentation of the results to reflect new developments in the research. I thank the referee for their valuable comments. This work was supported by JSPS KAKENHI Grant Numbers 23K20604 and 18H00833.

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6. APPENDIX

6.1. **Proposition 3.3.** (a) It follows from the definition of x^i and Assumption $(A.\Gamma 2)$. (b) Take $i \geq 0$, $z \in (x^{i+1}, x^i)$, and $h_a(z) \in H_a(z)$. We prove this by contradiction. Assume first that $h_a(z) < x^i$. Then, since $h_a(z) \neq h_c(z) \in (x^i, x^{i-1})$, by $(A.H)$,

$$V_a(z) < V_c(z) = u(z, h_c(z)) + \rho V_c(h_c(z)). \tag{A.1}$$

Since $D_a \subset D$, one has

$$V_a(z) = u(z, h_a(z)) + \rho V_a(h_a(z)) \leq u(z, h_a(z)) + \rho V_c(h_a(z)). \tag{A.2}$$

Since $h_a(z) < x^i < h_c(z)$, there exists $\lambda \in (0,1)$ such that $\lambda h_a(z) + (1-\lambda)h_c(z) = x^i$. By the concavities of u, V_c, and V_a, (A.1) and (A.2) yield

$$\begin{aligned} V_a(z) &< \lambda V_a(z) + (1-\lambda)V_c(z) \\ &\leq \lambda \{u(z, h_a(z)) + \rho V_c(h_a(z))\} + (1-\lambda)\{u(z, h_c(z)) + \rho V_c(h_c(z))\} \\ &\leq u(z, x^i) + \rho V_c(x^i) \\ &= u(z, x^i) + \rho V_a(x^i), \end{aligned}$$

where the last equality follows from Proposition 3.2. Since $(z, x^i) \in D_a$, we have a contradiction. Next, assume that $h_a(z) > x^{i-1}$ for $i \geq 1$. Since $h_a(z) > x^{i-1} > h_c(z)$, there is $\lambda \in (0,1)$ such that $\lambda h_a(z) + (1-\lambda)h_c(z) = x^{i-1}$. By a similar argument to the above, we have

$$V_a(z) < \lambda V_a(z) + (1-\lambda)V_c(z) \leq u(z, x^{i-1}) + \rho V_c(x^{i-1}) = u(z, x^{i-1}) + \rho V_a(x^{i-1}),$$

a contradiction.

(c) $H_a(x^0) = \{x^0\}$ is obvious. Take $z \in (x^0, 1]$ and $h_a(z) \in H_a(z)$. Since $h_c(z) < x^0 < z \leq h_a(z)$, by a similar argument to the above, we have, for some $\lambda \in (0,1)$ such that $\lambda h_c(z) + (1-\lambda)h_a(z) = x^0$,

$$V_a(z) < \lambda V_a(z) + (1-\lambda)V_c(z) \leq u(z, x^0) + \rho V_a(x^0). \tag{A.3}$$

If $z < h_a(z)$, there exists $\mu \in (0,1)$ such that $\mu h_a(z) + (1-\mu)x^0 = z$, and (A.2) and (A.3) imply that

$$\begin{aligned} V_a(z) &< \mu \{u(z, h_a(z)) + \rho V_a(h_a(z))\} + (1-\mu)\left\{u(z, x^0) + \rho V_a(x^0)\right\} \\ &\leq u(z, z) + \rho V_a(z), \end{aligned}$$

which is a contradiction.

(d) From (a) - (c) in this proposition, any $x > 0$ reaches $[x^0, 1]$ in a finite time and stay there.

6.2. **Proposition 3.4.** (a) We prove by contradiction. Assume that there are $h_d(x) \in H_d(x)$ and $z \in (x^0, 1]$ such that $h_d(z) > x^0$. Since $h_c(z) < x^0 < h_d(z)$, it holds that $V_d(z) < V_c(z)$ and $x^0 = \lambda h_d(z) + (1-\lambda)h_c(x)$ with some $\lambda \in (0,1)$. Then we have a contradiction

$$\begin{aligned} V_d(z) &< \lambda V_d(z) + (1-\lambda)V_c(z) \\ &\leq \lambda \{u(z, h_d(z)) + \rho V_c(h_d(z))\} + (1-\lambda)\{u(z, h_c(z)) + \rho V_c(h_c(z))\} \\ &\leq u(z, \lambda h_d(z) + (1-\lambda)h_c(z)) + \rho V_c(\lambda h_d(z) + (1-\lambda)h_c(z)) \\ &= u(z, x^0) + \rho V_c(x^0) \\ &= u(z, x^0) + \rho V_d(x^0). \end{aligned}$$

Note that $x^0 \in \Gamma_d(z)$.

(b) The proof follows a series of lemmas below:

Lemma 6.1. *A concave function $f : \mathbb{R} \to \mathbb{R}$ satisfies $\partial f(x) \geq \partial f(y)$ for $x < y$, where $\partial f(x)$ is a subgradient of f at x.*

Proof. Since $f(x)$ is a concave function, we have, for $x < y$, $\partial f(x)(y-x) \geq f(y) - f(x) \geq \partial f(y)(y-x)$. □

Lemma 6.2. *If there exists $p > 0$ such that $u(x_0,x_0) - px_0 + \rho p x_0 \geq u(x_{t-1},x_t) - px_{t-1} + \rho p x_t$ holds for any descending path $\{x_t\}_{t\geq 0}$ starting from x_0, then x_0 is a steady state for the descending problem (DP).*

Proof.

$$\sum_{t=1}^{\infty} \rho^{t-1} \{u(x_0,x_0) - u(x_{t-1},x_t)\} \geq \sum_{t=1}^{\infty} \rho^{t-1} \{px_0 - \rho p x_0 - px_{t-1} + \rho p x_t\} = 0.$$

□

Lemma 6.3. *At the unique steady state of the completely optimization problem x^0, there exists a subgradient of $u(x^0,x^0)$, $\left(\partial_1^0 u(x^0,x^0), \partial_2^0 u(x^0,x^0)\right)$, such that $\rho \partial_1^0 u(x^0,x^0) + \partial_2^0 u(x^0,x^0) = 0$.*

Proof. Consider $u(x^0,y) + \rho u(y,x^0)$ as a function of y. If this is not the case, all subgradients $\partial_2 u(x^0,x^0) + \rho \partial_1 u(x^0,x^0)$ at $y = x^0$ are either positive or negative. Assume that they are positive. Then, by the upper hemi-continuity of subgradients of a concave function, there exists a subgradient such that $\partial_2 u(x^0,x^0+\varepsilon) + \rho \partial_1 u(x^0+\varepsilon,x^0) > 0$ for some small positive ε. However, this leads to a contradiction:

$$\begin{aligned} 0 &> \left\{u(x^0,x^0+\varepsilon) + \rho u(x^0+\varepsilon,x^0)\right\} - \left\{u(x^0,x^0) + \rho u(x^0,x^0)\right\} \\ &\geq \left\{\partial_2 u(x^0,x^0+\varepsilon) + \rho \partial_1 u(x^0+\varepsilon,x^0)\right\} \varepsilon \\ &> 0. \end{aligned}$$

A parallel argument can be applied in the case that all subgradients are negative. □

Proof of Proposition 3.4 (b)

For $x = 0$ and x^0, $H_d(x) = \{x\}$ is obvious. Let $x_0 \in (0,x^0)$ and $p = \partial_1 u(x_0,x_0)$. Take any descending path from x_0, $\{x_t\}_{t\geq 0}$. Then

$$\begin{aligned} &[u(x_0,x_0) - px_0 + \rho p x_0] - [u(x_{t-1},x_t) - px_{t-1} + \rho p x_t] \\ &\geq [\rho \partial_1 u(x_0,x_0) + \partial_2 u(x_0,x_0)]\,(x_0 - x_t) \\ &\geq 0, \end{aligned}$$

where the last inequality follows from

$$[\rho \partial_1 u(x_0,x_0) + \partial_2 u(x_0,x_0)] \geq \rho \partial_1^0 u(x^0,x^0) + \partial_2^0 u(x^0,x^0) = 0$$

by Lemmas 6.1 and 6.3. Then, x_0 is a steady state of the descending optimal problem by Lemma 6.2.

(c) For $x_0 \in [0,x^0]$, it is a steady state by (b) of this proposition. For $x_0 \in (x^0,1]$, $H^d(x^0) \subset [0,x^0)$ and it reaches a steady state after one period by (a) of this proposition.

6.3. **Proposition 3.5.** (a) They follow from Propositions 3.3 and 3.4. That is, for $x \in (0, x^0)$, $\Gamma_a(x) \supset \{x\} = H_d(x)$, but $\{x\} \notin H_a(x)$, and thus $V_a(x) > V_d(x)$. Similarly, for $x \in (x^0, 1]$, $\Gamma_d(x) \supset \{x\} = H_a(x)$, but $\{x\} \notin H_a(x)$, and thus $V_a(x) < V_d(x)$.

(b) We first show the concavity. The completely optimal problem, the ascending problem, and the descending problem are all concave problems. Hence, the associated value functions are concave, and we need to check the concavity of $V_m(x)$ around x^0 where the maximized value switches from $V_a(x)$ to $V_d(x)$.

Consider the case of $z \in (0, x^0]$. Take $x_a \in (0, z)$ and $x_d \in (x^0, 1]$. Let $\lambda \in (0, 1)$ satisfy $\lambda x_a + (1-\lambda)x_d = z$. Define function

$$v_a(x) := \begin{cases} V_a(x) & \text{if } x \in [0, x^0) \\ \partial V_c(x^0)(x - x^0) + V_c(x^0) & \text{if } x \in (x^0, 1] \end{cases},$$

where $\partial V_c(x^0)$ is a common subgradient of V_i, $i = c, a, d$, at x^0. Since $v_a(x)$ is a concave function by construction and $v_a(x_d) \geq V_d(x_d)$, we have

$$\begin{aligned} V_m(z) = V_a(z) = v_a(z) &\geq \lambda v_a(x_a) + (1-\lambda) v_a(x_d) \\ &\geq \lambda V_a(x_a) + (1-\lambda) V_d(x_d) = \lambda V_m(x_a) + (1-\lambda) V_m(x_d). \end{aligned}$$

A parallel argument is applied to the case of $z \in (x^0, 1)$: Take $x_a \in (0, x^0)$ and $x_d \in (z, 1]$. Let $\lambda \in (0, 1)$ satisfy $\lambda x_a + (1-\lambda)x_d = z$. Using a concave function,

$$v_d(x) := \begin{cases} \partial V_c(x^0)(x - x^0) + V_c(x^0) & \text{if } x \in [0, x^0) \\ V_d(x) & \text{if } x \in (x^0, 1] \end{cases},$$

we have

$$\begin{aligned} V_m(z) = V_d(z) = v_d(z) &\geq \lambda v_d(x_a) + (1-\lambda) v_d(x_d) \\ &\geq \lambda V_a(x_a) + (1-\lambda) V_d(x_d) = \lambda V_m(x_a) + (1-\lambda) V_m(x_d). \end{aligned}$$

Next, we prove that $V_m(x)$ is a strictly increasing function. Define $f(x) := u(x, x)/(1-\rho)$. Take $z \in (x^0, 1]$. Since $V_m(x)$ is concave and $V_m(z) > V_a(z)$, we have

$$V_m(x^0) - V_a(z) > V_m(x^0) - V_m(z) \geq \partial V_m(x^0)(x^0 - z).$$

Then we have

$$\partial V_m(x^0) > \frac{V_m(x^0) - V_a(z)}{x^0 - z} = \frac{f(x^0) - f(z)}{x^0 - z} > 0.$$

The last inequality follows from Assumption $(A.U1)$. Using Lemma 6.1 in the proof of Proposition 3.4, $\partial V_m(x) \geq \partial V_m(x^0) > 0$ for $x \in (0, x^0]$. Therefore, $V_m(x)$ is strictly increasing on $(0, x^0]$.

Regarding the interval $(x^0, 1]$, note that $V_m(z) = u(z, h_d(z)) + (\rho/(1-\rho))u(h_d(z), h_d(z))$ where $h_d(z) \in H_d(z)$. Since $h_d(z) < z$, the subgradient of the first argument of $u(z, h_d(z))$ satisfies

$$\partial_1 u(z, h_d(z)) \geq \frac{u(z, z) - u(z, h_d(z))}{z - h_d(z)} > 0,$$

where the last strict inequality follows from Assumption $(A.U1)$. $\partial_1 u(z, h_d(z)) > 0$ implies $u(z, h_d(z)) < u(z + \varepsilon, h_d(z))$ for any small positive ε. Then, we have,

$$V_m(z) < u(z + \varepsilon, h_d(z)) + \frac{\rho}{1-\rho}(u(h_d(z), h_d(z)) \leq V_m(z + \varepsilon).$$

6.4. **Proposition 3.7.** (a) $u_{12}(x^0,x^0) \neq 0$ implies $u_{22}(x^0,x^0) < 0$ since $u_{11}u_{22} \geq (u_{12})^2$ by the concavity of u. Hence, this implies strict concavity of the right-hand side of

$$V_m(x) = \max_y u(x,y) + (\rho/(1-\rho))\, u(y,y)$$

in y for $x \in (x^0 - \varepsilon, x^0 + \varepsilon)$ with a small positive ε by Berge's theorem of the maximum. (Note that the right-hand side follows from Propositions 3.3, 3.4, and 3.5.) Therefore, we have $H_m(x) = \{h_m(x)\}$ in a neighborhood of x^0.

(b) $u_{12}(x^0,x^0) \neq 0$ also implies $u_{12}(x^0,x^0) < 0$ since the policy function $h_c(x)$ is strictly decreasing at x^0 by Assumption $(A.H)$ (see Mitra, Nishimura, and Sorger [8, Theorem 6.3.2] and also Benhabib and Nishimura [1]).

Note that $h_m(x) = \arg\max_y u(x,y) + (\rho/(1-\rho)u(y,y)$, and it holds that

$$u_2(x,h_m(x)) + \frac{\rho}{1-\rho}\{u_1(h_m(x),h_m(x)) + u_2(h_m(x),h_m(x))\} = 0,$$

where $u_1(x,y) := \partial u/\partial x$ and $u_2(x,y) := \partial u/\partial y$. From this first order condition, we have

$$\frac{dh_m(x)}{dx} = \frac{-(1-\rho)\,u_{12}(x,h_m(x))}{(1-\rho)\,u_{22}(x,h_m(x)) + \rho\,\{u_{11}(h_m(x),h_m(x)) + 2u_{12}(h_m(x),h_m(x)) + u_{22}(h_m(x),h_m(x))\}}.$$

At $x = x^0$, $h_m(x^0) = x^0$, and thus

$$\frac{dh_m(x^0)}{dx} = \frac{-(1-\rho)\,u_{12}(x^0,x^0)}{\rho u_{11}(x^0,x^0) + 2\rho u_{12}(x^0,x^0) + u_{22}(x^0,x^0)} (< 0).$$

The condition for the local stability of the steady state is given by $dh_m(x^0)/dx > -1$. It is equivalent to

$$\rho u_{11}(x^0,x^0) + (3\rho - 1)\,u_{12}(x^0,x^0) + u_{22}(x^0,x^0) < 0. \tag{A.5}$$

(c) Since

$$\begin{aligned}
&\rho u_{11}(x^0,x^0) - 2\sqrt{\rho}u_{12}(x^0,x^0) + u_{22}(x^0,x^0)\\
&< \rho u_{11}(x^0,x^0) + 2\sqrt{\rho u_{11}(x^0,x^0)u_{22}(x^0,x^0)} + u_{22}(x^0,x^0)\\
&= -\left(\sqrt{|\rho u_{11}(x^0,x^0)|} - \sqrt{|u_{22}(x^0,x^0)|}\right)^2 < 0,
\end{aligned}$$

Inequality (A.5) holds if

$$(3\rho + 2\sqrt{\rho} - 1)\,u_{12}(x^0,x^0) = (\sqrt{\rho}+1)\,(3\sqrt{\rho}-1)\,u_{12}(x^0,x^0) \leq 0,$$

i.e., $\rho \geq 1/9$. If

$$u_{11}(x^0,x^0) + (-\rho - 1)\,u_{12}(x^0,x^0) + u_{22}(x^0,x^0) < 0, \tag{A.6}$$

the optimal policy $h_c(x)$ is asymptotically stable at x^0 (Mitra et al., 2006, Theorem 6.3.3 and Equation (6.2)). It is readily confirmed that (A.6) implies (A.5).

6.5. **Proposition 4.1.** Since $V_m(x) = V_a(x)$ for $x \in [0, x^0]$, the statement is true if we show that the minimizer of V_a/V_c locates in the interval (x^1, x^0). Note that, for $x > 0$,

$$\frac{V_a(x)}{V_c(x)} \geq \frac{u(x, h_c(x)) + \rho V_a(h_c(x))}{u(x, h_c(x)) + \rho V_c(h_c(x))} \geq \frac{V_a(h_c(x))}{V_c(h_c(x))}, \tag{A.6}$$

where the second inequality follows from the facts that $u(x, h_c(x)) > 0$ and $V_c(h_c(x)) - V_a(h_c(x) \geq 0$. Define $r(i)$ $(i = 0, 1, \dots)$ by

$$r(i) := \min \frac{V_a(x)}{V_c(x)} \text{ subject to } x \in [x^{i+1}, x^i].$$

Then, for $x(i+1)$ which attains $r(i+1)$,

$$r(i+1) = \frac{V_a(x(i+1))}{V_c(x(i+1))} \geq \frac{V_a(h_c(x(i+1)))}{V_c(h_c(x(i+1)))} \geq r(i),$$

where the first inequality follows from (A.6), and the last inequality follows from $h_c(x(i+1)) \in [x^{i+1}, x^i]$ and the definition of $r(i)$.

Differentiate function $R(u) = (a+u)/(b+u)$: $R'(u) = \frac{b-a}{(b+u)^2} \geq 0$ if $b \geq a$.