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MATRICES WHOSE PERMANENT RANK EQUALS HALF THEIR RANK

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Abstract. The permanent rank of an $m \times n$ matrix A over a field $\mathbb{F}$ generalizes the notion of the rank of A and is the largest k such that A has a $k \times k$ submatrix whose permanent is nonzero. In 1999, Yu proved that the permanent rank of a matrix is always at least half the rank. This paper gives an explicit characterization the matrices for which equality holds; and demonstrates that, for characteristic different than 2, fixed m, n and even r with $r \leq \min\{m,n\}$ there is essentially a unique $m \times n$ matrix over $\mathbb{F}$ with rank r and permanent rank $r/2$.
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1. Introduction

Let A be an $m \times n$ matrix over a field $\mathbb{F}$. It is a basic fact that the rank of A is the largest k such that A has a $k \times k$ matrix with nonzero determinant. Analogously, the *permanent rank* of A is defined to be the largest k such that A has a $k \times k$ submatrix with nonzero permanent. We denote the rank (respectively, permanent rank) of A by rank A (respectively, perrank A). The permanent rank was introduced in [2]. The paper resolves a question raised in that research article.

Let B be an $n \times n$ matrix. The *permanental adjoint* of B is denoted by p-adj(B) and is the $n \times n$ matrix whose (i,j)-entry is per$B(j,i)$. If $\mathbf{y}$ is an $n \times 1$ vector, then the Laplace expansion of the permanent implies that the i-th entry of $\mathbf{y}^T$ p-adj(B) is the permanent of the matrix obtained by replacing the i-th row of B by $\mathbf{y}^T$, and the j-th entry of p-adj$(B)\mathbf{y}$ is the permanent of the matrix obtained by replacing the j-th column of B by $\mathbf{y}$. We begin with a result that utilizes these facts about the permanental adjoint.

If $\alpha \subseteq \{1,\ldots,m\}$ and $\beta \subseteq \{1,\ldots,n\}$, then $B[\alpha,\beta]$ denotes the submatrix of B whose row indices lie in α and column indices lie in β. When $m = n$, this is simplified to $B[\alpha]$. When $\alpha = \{1,\ldots,n\}$, respectively $\beta = \{1,\ldots,n\}$ we write $B[:,\beta]$, respectively $B[\alpha,:]$. If $\alpha = \{i\}$ we simply write $B[i,\beta]$. The submatrix of B whose row indices lie outside α and column indices

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lie outside β is denoted by $A(\alpha,\beta)$. Thus $B[\alpha,\beta]$ and $B(\alpha,\beta)$ are complementary submatrices of B.

Lemma 1.1. *Let C be a matrix of the form*

$$\left[\begin{array}{c|c} B & \mathbf{u} \\ \hline \mathbf{v}^T & c \end{array}\right],$$

where B is an $n \times n$ matrix with nonzero permanent, $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ vectors, and $\operatorname{per} C = 0$. *Then*

$$c = -\frac{\mathbf{v}^T \text{p-adj}(B)\mathbf{u}}{\operatorname{per} B}.$$

Proof. By Laplace expansion of the permanent along the last row, followed by Laplace expansion along the last column of each resulting matrices, we have

$$\begin{aligned} \operatorname{per} C &= c \operatorname{per} B + \sum_{i=1}^{n} \sum_{j=1}^{n} v_j u_i \operatorname{per}\, B(i,j) \\ &= c \operatorname{per} B + \sum_{j=1}^{n} \sum_{i=1}^{n} v_j \operatorname{per} B(i,j) u_i \\ &= c \operatorname{per} B + \mathbf{v}^T \text{p-adj}(B)\mathbf{u}. \end{aligned}$$

The result follows by solving for c. □

The following lemma relates the perrank of a matrix to the permanental adjoint of a largest square submatrix with nonzero permanent.

Lemma 1.2. *Let A be an $m \times n$ matrix over a field $\mathbb{F}$ such that $B = A[\{1,\ldots,k\}]$ has nonzero permanent, and $k =$* $\operatorname{perrank}(A)$. *Then A has the form*

$$\left[\begin{array}{c|c} B & X \\ \hline Y & -YPX \end{array}\right],$$

where $P = \frac{\text{p-adj}(B)}{\operatorname{per} B}$.

Proof. By Lemma 1.1,

$$a_{k+i,k+j} = -\frac{\mathbf{y_i}^T \text{p-adj}(B)\mathbf{x_j}}{\operatorname{per} B},$$

where $\mathbf{y_i}^T$ is the i-th row of Y and $\mathbf{x_j}$ is the j-th column of X. Hence

$$A[\{k+1,\ldots,m\},\{k+1,\ldots,n\}] = -YPB.$$

□

The next result gives an upper bound on $\operatorname{perrank} A$ in terms of $\operatorname{rank} A$, and was proven in [2].

Theorem 1.3. *Let A be an $m \times n$ matrix. Then* $\operatorname{perrank}(A) \geq \operatorname{rank}(A)/2$.

Proof. By Lemma 1.2, it suffices to prove the result in the case that A is an $n \times n$ invertible matrix of the form

$$\left[\begin{array}{c|c} B & X \\ \hline Y & -YPX \end{array}\right],$$

where B is a $k \times k$ matrix and $\operatorname{perrank}(A) = k$. Note that if $\mathbf{v}$ is in the nullspace $\operatorname{NS}(X)$ of X, then

$$\begin{bmatrix} \mathbf{0}_{k\times 1} \\ \mathbf{v} \end{bmatrix}$$

is in the nullspace of A. As A is invertible, $\operatorname{NS}(X) = \mathbf{0}$. Hence $k \geq n-k$, which is equivalent to

$$k \geq n/2 = \operatorname{rank}(A)/2.$$

□

Equality can hold in Theorem 1.3. Let P_2 denote the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

For any positive integer k and any field not of characteristic two,

$$\operatorname{perrank}(\oplus_{i=1}^{k} P_2) = k = \frac{\operatorname{rank}(\oplus_{i=1}^{k} P_2)}{2}. \tag{1.1}$$

Additionally, any matrix B obtained from (1.1) by bordering with rows of zeros and columns of zeros has $\operatorname{perrank} B = \frac{\operatorname{rank} B}{2}$. The paper [2] asks whether or not these are the only such matrices whose perrank is half the rank. This papers answers that question in the affirmative in the case the underlying field is not of characteristic 2. When $\mathbb{F}$ has characteristic 2, then the rank and perrank functions coincide, and equality holds in Theorem 1.3 if and only if $A = O$.

2. Structure of Matrices with Perrank Half the Rank

The main result of this paper is that for characteristic not 2, these are essentially the only matrices whose permanent rank is half its rank. The proof of the main result is a sequence of observations that establish increasingly more restrictive constraints on such matrices. We begin with a simple consequence of Theorem 1.3.

Corollary 2.1. *Let A be a $2n \times 2n$ matrix with $\operatorname{rank} A = 2n$ and $\operatorname{perrank} A = n$. Then each column of A has as least two nonzero entries.*

Proof. As A is invertible, each column of A has at least one nonzero entry. Suppose to the contrary that some column, say the first, has exactly one nonzero entry, say its first. Then $A[\{2,\ldots,2n\}]$ has rank $2n-1$. By Theorem 1.3, $\operatorname{perrank} A[\{2,\ldots,2n] \geq \frac{2n-1}{2}$. As the permanent rank is an integer,

$$\operatorname{perrank} A[\{2,\ldots,2n\}] \geq n.$$

Hence, $A[\{2,\ldots,2n\}]$ contains an $n \times n$ submatrix $A[\alpha,\beta]$ with nonzero permanent. This leads to the contradiction that $\operatorname{per} A[\{1\}\cup\alpha,\{1\}\cup\beta] \neq 0$. □

The next result gives much stronger constraints, and holds only for characteristic different than 2.

Corollary 2.2. *Let A be an $m \times n$ matrix over a field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$ of the form*

$$\left[\begin{array}{c|c} B & X \\ \hline Y & Z \end{array}\right],$$

where B is a $p\times p$ matrix, $\operatorname{per} B\neq 0$*,* $p=\operatorname{perrank}(A)$ *and* $\mathbf{e_j}\in CS(X)$*. Here $CS(X)$ denotes the column space of the matrix X. Then the j-th column of $Y\,$*p-adj(B) *has at most one nonzero entry.*

Proof. For $k\neq \ell$, let $\mathbf{y}_k^T$, respectively $\mathbf{y_\ell}^T$, denote the k-th, respectively ℓ-th row of Y. Let $\widehat{B}$ be the matrix obtained from B by deleting its j-th row. Lemma 1.1 implies that

$$\operatorname{per}\left[\begin{array}{c|c} \widehat{B} & \widehat{B}\mathbf{e_i} \\ \hline \mathbf{y_k}^T & -\frac{\mathbf{y_k}^T \text{ p-adj}(B)B\mathbf{e_i}}{\operatorname{per} B} \\ \hline \mathbf{y_\ell}^T & -\frac{\mathbf{y_\ell}^T \text{ p-adj}(B)B\mathbf{e_i}}{\operatorname{per} B} \end{array}\right]=0,$$

for each $i\in\{1,\dots,n-p\}$. Hence, by multilinearity of the permanent,

$$\operatorname{per}\left[\begin{array}{c|c} \widehat{B} & \mathbf{e_j} \\ \hline \mathbf{y_k}^T & -\frac{\mathbf{y_k}^T \text{ p-adj}(B)\mathbf{e_j}}{\operatorname{per} B} \\ \hline \mathbf{y_\ell}^T & -\frac{\mathbf{y_\ell}^T \text{ p-adj}(B)\mathbf{e_j}}{\operatorname{per} B} \end{array}\right]=0.$$

By Laplace expansion along the last column, we obtain

$$\begin{aligned} 0 &= -\frac{\mathbf{y_k}^T \text{ p-adj}(B)\mathbf{e_j}}{\operatorname{per} B}\operatorname{per}\left[\begin{array}{c} \widehat{B} \\ \mathbf{y_\ell}^T \end{array}\right]-\frac{\mathbf{y_\ell}^T \text{ p-adj}(B)\mathbf{e_j}}{\operatorname{per} B}\operatorname{per}\left[\begin{array}{c} \widehat{B} \\ \mathbf{y_k}^T \end{array}\right] \\ &= \frac{-2\cdot \mathbf{y_k}^T \text{ p-adj}(B)\mathbf{e_j}\cdot \mathbf{y_\ell}^T \text{ p-adj}(B)\mathbf{e_j}}{\operatorname{per} B}. \end{aligned}$$

Since char $\mathbb{F}\neq 2$, at least one of $\mathbf{y_k}^T$ p-adj$(B)\mathbf{e_j}$ and $\mathbf{y_\ell}^T$ p-adj$(B)\mathbf{e_j}$ is zero. It follows that at most one entry of the j-th column of $Y\,$p-adj(B) is nonzero. □

An analogous result holds for rows; namely, under the assumption of Corollary 2.2 each row of p-adj$(B)X$ has at most one nonzero entry. Additionally, under the assumption of Corollary 2.2, if Y has rank p, then in each column of $Y\,$p-adj(B) has exactly one nonzero entry.

The leading $2n\times 2n$ principal submatrix of the matrix in the next corollary is permutation similar to the direct sum of n P_2s. This corollary will enable one to complete the characterization from the characterization of $2n\times 2n$ matrices of rank $2n$ and perrank n.

Corollary 2.3. *Let A be $r\times s$ matrix of the form*

$$\left[\begin{array}{c|c|c} I_n & I_n & X \\ \hline I_n & -I_n & Y \\ \hline U & V & W \end{array}\right],$$

where $\operatorname{rank}(A)=2n$ *and* $\operatorname{perrank}(A)=n$*. If char $\mathbb{F}\neq 2$, then each of X, Y, U, V W is a zero matrix.*

Proof. Assume that char $\mathbb{F}\neq 2$. Note $A[\{1,\dots,n\}]$ has nonzero permanent, and $A[\{1,\dots,2n\}]$ has nonzero determinant. By Lemma 1.2, $V=-U$, $Y=-X$, and $W=-UX$. By Corollary 2.2, $U=O$, and $X=O$. Hence each of the claimed matrices is a zero matrix. □

3. More Structure

The next theorem relates the determinant of $n \times n$ submatrix to the permanent of the complementary matrix for $2n \times 2n$ matrices that achieve equality in Theorem 1.3.

Theorem 3.1. *Let A be a $2n \times 2n$ matrix over a field whose characteristic is not 2 such that* $\operatorname{rank} A = 2n$ *and* $\operatorname{perrank} A = n$*, and let $A[\alpha, \beta]$ be an $n \times n$ submatrix of A with nonzero permanent. Then the complementary submatrix $A(\alpha, \beta)$ is invertible.*

Proof. Among all $n \times n$ submatrices $A[\alpha, \beta]$ of A having nonzero permanent, choose one such that the rank of $A(\alpha, \beta)$ is the smallest, say r. Without loss of generality we may assume that $\alpha = \beta = \{1, \ldots, n\}$, and A has the form

$$\left[\begin{array}{c|c} B & X \\ \hline Y & Z \end{array}\right],$$

where B is an $n \times n$ matrix, $\operatorname{per} B \neq 0$, and $\operatorname{rank} Z = r$. By Corollary 2.2, we know that $Z = -YPX$, where

$$P = \frac{\text{p-adj}\, B}{\operatorname{per} B},$$

X and Y are invertible, each column of $-YP$ has exactly one nonzero entry, and each row of $-PX$ has exactly one nonzero entry. Since Z has rank r, both $-YP$ and $-PX$ have rank r. Morever, exactly r rows of $-YP$ are nonzero and exactly r columns of $-PX$ are nonzero. Hence Z has an $r \times r$ submatrix of rank r that contains all the nonzero entries of Z. By permuting the rows of Y and the columns of X we may assume that this submatrix is the leading $r \times r$ submatrix of Z; that is, Z has the form

$$\left[\begin{array}{c|c} W & O \\ \hline O & O \end{array}\right],$$

where W is an invertible $r \times r$ matrix. By Corollary 2.1, each of columns $r+1, \ldots, n$ of X have at least two nonzero entries.

Let i and j be indices such that $i \in \{1, \ldots, r\}$ and the (i, j)-entry of $-YP$ is nonzero. We claim the row vector $X[\{j\}, \{r+1, \ldots, n\}]$ has at most one nonzero entry, and that for at least one such j the row vector has no nonzero entries.

Let $\widehat{A}$ be the matrix obtained from A by interchanging rows j and $n+i$. The (i, j)-entry of $-YP$ being nonzero implies that

$$\operatorname{per} \widehat{A}[\{1, \ldots, n\}, \{1, \ldots, n\}] \neq 0.$$

Hence, by Corollary 2.2, and the choice of B, the matrix

$$\widehat{A}(\{1, \ldots, n\}, \{1, \ldots, n\})$$

has rank $s \geq r$, s nonzero columns, and s nonzero rows. The only possible nonzero rows of this matrix are rows $1, \ldots, r$. Thus $s = r$.

As W is invertible, $W[\overline{\{i\}}, :]$ has a most one zero column. We consider two cases.

Case 1. Each column of the matrix $W[\overline{\{i\}}, :]$ obtained from W by deleting row i is nonzero. Then the nonzero columns of $\widehat{A}(\{1, \ldots, n\}, \{1, \ldots, n\})$ are columns $1, 2, \ldots, r$. In particular, for each j such that the (i, j)-entry of $-YP$ is nonzero, we have that $X[\{j\}, \{r+1, \ldots, n\}]$ is a row of zeros.

Case 2. $W[\overline{\{i\}},:]$ has exactly one zero column, say column k.
Since W is invertible, $w_{i,k} \neq 0$. Additionally, there is an ℓ such that the nonzero columns of $\widehat{A}(\{1,\dots,n\},\{1,\dots,n\})$ are $\{1,\dots,r\}\setminus\{k\}\cup\{\ell\}$. Note ℓ may depend upon j, but k is independent of j. If $k=j$, then $X[\{j\},\{r+1,\dots,n\}]$ is a row of zeros. If $k\neq j$, then $X_{j,k}=0$ and the only nonzero entry of $X[\{j\},\{r+1,\dots,n\}$ is $X_{j,\ell}$.

As row i of $Z=(-YP)X$ is a linear combination of the rows of X indexed by the j such that $(-YP)_{i,j}\neq 0$, and $Z_{i,k}\neq 0$, there is at least one such j with $X[\{j\},\{r+1,\dots,n\}]$ is a row of zeros, and the claim is established.

The claim and Corollary 2.1 imply that $X[:,\{r+1,\dots,n\}]$ has at least r rows of zeros, each of the $n-r$ columns has at least two nonzero entries and the supports of these $n-r$ columns are mutually disjoint. Hence $n\geq r+2(n-r)$, which implies that $r\geq n$. Therefore, $r=n$. □

Theorem 3.1 implies in any $2n\times 2n$ matrix A with $\operatorname{rank}A=2n$ and $\operatorname{perrank}A=n$ the complementary submatrix to any $n\times n$ submatrix with nonzero permanent is invertible. We will repeatedly use the contrapositive of this; namely, for such a matrix A, if $\det A(\alpha,\beta)=0$, then $\operatorname{per}A[\alpha,\beta]=0$.

4. Full Rank Case

In this section we use Theorem 3.1 to characterize the $2n\times 2n$ matrices A for which $\operatorname{rank}A=2n$ and $\operatorname{perrank}A=n$. We begin with a definition of, and two simple results on, generalized permutation matrices.

An $n\times n$ matrix G is a generalized permutation matrix over the field $\mathbb{F}$ provided G has exactly one nonzero entry from $\mathbb{F}$ in each row and column. In other words, a generalized permutation matrix is a permutation matrix without the constraint that each nonzero entry must be 1. We can extend this notion to generalized cycles. A generalized cycle matrix is a specific type of generalized permutation matrix which has zeros on the diagonal, nonzero entries on the superdiagonal, and a single nonzero entry in the first position of the n^{th} row. It is known that every generalized permutation matrix is equivalent to a direct sum of generalized cycles.

Lemma 4.1. *Let C be an $n\times n$ generalized cycle matrix. Then $C-I_n$ has nullity at most 1.*

Proof. Let $C_{n\times n}$ be

$$C=\begin{bmatrix}0 & c_1 & & & \\ & 0 & c_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & c_{n-1}\\ c_n & & & & 0\end{bmatrix}.$$

Then if we remove the first column and last row of $C-I_n$, we have

$$\begin{bmatrix}c_1 & & & \\ -1 & c_2 & & \\ & \ddots & \ddots & \\ & & -1 & c_{n-1}\end{bmatrix}.$$

We note that $A(n,1)$ is a lower-triangular matrix with nonzero entries on the diagonal, so $A(n,1)$ has rank $n-1$. Thus, $C-I_n$ has rank at least $n-1$, implying its nullity is at most 1. □

Lemma 4.2. *Let G be an $n \times n$ generalized permutation matrix over a field $\mathbb{F}$ such that 1 is an eigenvalue of geometric multiplicity at least $n-1$ (that is, null$(G-I) \geq n-1$). Then one of the following holds:*

(a) $G = I_n$;
(b) *G is a diagonal matrix with $n-1$ diagonal entries equal to 1 and the remaining diagonal entry not in $\{0,1\}$; and*
(c) *Up to permutation similarity, G has the form*

$$I_{n-2} \oplus \begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix}$$

for some nonzero $\lambda \in \mathbb{F}$.

Proof. Assume G is as described in the statement of the result. Without loss of generality, G is a direct sum of d generalized cycles $C_1, \ldots, C_d$. By Lemma 4.1, each of $C_i - I_n$ has nullity at most 1. But, by the assumption, $G - I_n$ must have nullity at least $n-1$. Thus, we need at least $n-1$ cycles. Additionally, since the eigenvalue of interest is 1, we require $G = I_{n-2} \oplus C$ up to permutation similarity, where C is a direct sum of generalized cycles and has 1 as an eigenvalue. Thus, C has to have one of the following forms:

Case 1. C is a direct sum of two 1-cycles.
Then at least one of the 1-cycles must be a single 1. Thus, G is either the identity, or G is a diagonal matrix with $n-1$ diagonal entries equal to 1, and the remaining diagonal entry not in $\{0,1\}$.

Case 2. C is a 2-cycle. Then

$$C = \begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix}.$$

Since 1 must be an eigenvalue of C, we require $\lambda\mu = 1$, or $\mu = \lambda^{-1}$. Thus, up to permutation similarity, G has form

$$I_{n-2} \oplus \begin{bmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{bmatrix}.$$

For the converse, since the rank of $\begin{bmatrix} -1 & \lambda \\ \lambda^{-1} & -1 \end{bmatrix}$ is 1, it is easy to see that each of these cases is a matrix with 1 as an eigenvalue with geometric multiplicity at least $n-1$ as desired. □

We next note that Theorem 3.1 allows us to restrict our attention to matrices of a very special form. Let A be an $2n \times 2n$ matrix, with $\operatorname{rank} A = 2n$, $\operatorname{perrank} A = n$, $\operatorname{per} A[\{1,\ldots,n\}] \neq 0$, and assume the characteristic is different than two. By Lemma 1.2, we may assume that A has the form

$$\left[\begin{array}{c|c} B & X \\ \hline Y & -YPX \end{array}\right],$$

where $P = \frac{\text{p-adj}\,B}{\operatorname{per} B}$. By Theorem 3.1, P has rank n, and by Corollary 2.3, $-YP = G$ and $-XP = H$ for some generalized permutation matrices G and H. Without loss of generality, we may assume that $G = I$, and $H = I$ (this can be obtained by pre-multiplying A by $I_n \oplus G^{-1}$ and post-multiplying A by $I_n \oplus H^{-1}$). In particular, this implies that $X = Y$ and hence we may without loss of generality assume that A has the form

$$\left[\begin{array}{c|c} B & X \\ \hline X & X \end{array}\right] \tag{4.1}$$

and $XP = -I = PX$.

Theorem 4.3. *Let A be a $2n \times 2n$ matrix of the form in* (4.1), *where* $\det A \neq 0$, *B is an $n \times n$ matrix with nonzero permanent,* $\operatorname{perrank} A = n$, *and the characteristic is different than* 2. *Then either*

(a) *A is partly decomposable,*
(b) *$n = 1$, or*
(c) *for each $i \in \{1, \ldots, n\}$ there exists nonzero λ_i and indices j_i and k_i, such that $X(\mathbf{e_{k_i}} - \lambda_i \mathbf{e_{j_i}})$ is a nonzero multiple of* $\mathbf{e_i}$.

Proof. Assume that A is fully indecomposable and $n \geq 2$. It suffices to show that (c) holds for $i = 1$.

Let $\widehat{A}$ be the matrix obtained from A by interchanging rows 1 and $n+1$. Note $\widehat{A}$ has the form

$$\left[\begin{array}{c|c} \widehat{B} & X \\ \hline \widehat{X} & X \end{array}\right],$$

where $\widehat{B}$ is obtained from B by replacing its first row by the first row of X, and $\widehat{X}$ is obtained from X by replacing its first row by the first row of B. As $XP = -I$, $\operatorname{per} \widehat{B} \neq 0$, and Corollary 2.2 can be applied to $\widehat{B}$. Let $\widehat{P} = \frac{\text{p-adj}\widehat{B}}{\operatorname{per}\widehat{B}}$. This gives that $-\widehat{X}\widehat{P} = G$ for some generalized permutation matrix G, $-\widehat{P}X = H$ for some generalized permutation matrix H, and $-\widehat{X}\widehat{P}X = X$. The latter implies that $G = I$. The former yields $\widehat{X}H = X$. This implies that the rows other than the first row of $\widehat{X}$ are left eigenvectors of H corresponding to eigenvalue 1. So 1 is an eigenvalue of H of geometric multiplicity at least $n-1$.

By Lemma 4.2, either H is a diagonal matrix having at least $n-1$ diagonal entries equal to 1, or there exist $j < k$ and a nonzero λ such that

$$H = \lambda E_{j,k} + \lambda^{-1} E_{k,j} + \sum_{\ell \notin \{j,k\}} E_{\ell,\ell}. \tag{4.2}$$

We know that $XP = -I$. So $\operatorname{per} \widehat{B} = -\operatorname{per} B$. As the characteristic is not 2, we conclude that the first row of X and the first row of B are different, or equivalently that $X \neq \widehat{X}$. Hence $H \neq I$.

Now consider the case that H is a diagonal matrix with all but exactly one diagonal entry, say the k-th, equal to 1. As $\widehat{X}H = X$ and $\widehat{X}$ and X agree everywhere except the first row, and the k-th diagonal is not 1, the only nonzero entry in the k-th column of X (and similarly the k-th column of B) occurs in row 1. But then A is partly decomposable or $n = 1$ contrary to assumption. Hence this case does not occur.

We conclude that there exist $j < k$ and nonzero λ such that (4.2) holds. The equation $\widehat{X}H = X$ and the fact the i-th rows of $\widehat{X}$ and X are equal for $i = 2, \ldots, n$, then $\lambda X \mathbf{e_j}$ and $X \mathbf{e_k}$ agree on all but their first entries. Hence $X(\lambda \mathbf{e_j} - \mathbf{e_k})$ is a scalar multiple of $\mathbf{e_1}$. As X is invertible and $j \neq k$, $X(\lambda \mathbf{e_j} - \mathbf{e_k})$ is nonzero. $\square$

For the next result, we require some additional background and a basic result on trees and unicyclic graphs. A tree is defined to be a connected graph on n vertices with $n-1$ edges. A unicyclic graph is a connected graph containing exactly one cycle. Another way to view a unicyclic graph is as a tree with one additional edge. An important characterization of unicyclic graphs is the following result.

Lemma 4.4. *A simple graph on n vertices with n edges is connected if and only if it is a unicyclic graph.*

Proof. Assume G is a connected graph on n vertices with n edges. We prove this direction of the result by induction on the number of vertices n. For the base case, we can consider $n = 3$. Then the only connected simple graph on 3 vertices with 3 edges is a 3-cycle, which is unicyclic. Thus, we can assume that our result is true for n vertices, and can consider the graph on $n+1$ vertices with $n+1$ edges. Choose any edge e such that $G \setminus e$ is a single connected set of edges, and remove it. Then we have two cases.

Case 1. $G \setminus e$ is connected.
Then $G \setminus e$ is a simple graph on $n+1$ vertices with n edges, making $G \setminus e$ a tree. Then G is a tree with one additional edge, making it unicyclic.

Case 2. $G \setminus e$ is not connected.
Then $G \setminus e$ has a connected component with n vertices and n edges. Thus, by the induction hypothesis, $G \setminus e$ is unicyclic. Then by reintegrating the edge e, we are simply adding a leaf to a previously disconnected vertex, meaning G is also a unicyclic graph.

For the converse, we assume G is a unicyclic graph on n vertices with n edges. Assume G has k connected components $G_1, \ldots, G_k$. Since G is unicyclic, exactly one of these connected components has a cycle, and the rest are trees. Assume, without loss of generality that G_1 has our cycle. Assume each G_i has v_i number of vertices. Note that $\sum v_i = n$, since G has n vertices. Additionally, we can compute the edges in each component. Since G_1 contains our cycle, it has exactly v_1 edges while every other component has $v_i - 1$ edges. Thus, G has $\sum v_i - (k-1) = n - k + 1$ total edges. However, we assume that G has n edges, so $k = 1$, and G has a singular connected component. □

Corollary 4.5. *Let A be a $2n \times 2n$ matrix of the form in (4.1) where $\det A \neq 0$, B is an $n \times n$ matrix with nonzero permanent, $\operatorname{perrank} A = n$ and the characteristic is not 2. Then A is partly decomposable, $n = 1$, or there exists $d \notin \{0, 1\}$ such that X is generalized permutation equivalent to a matrix of the form*

$$(1-d)^{-1}\begin{bmatrix} 1 & d & d & \cdots & d \\ 1 & 1 & d & & d \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & & 1 & d \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Proof. Assume that A is fully indecomposable and $n \geq 2$. Then (c) of Theorem 4.3 holds. Let N be the $n \times n$ matrix whose i-th column has a 1 in position j_i and $-\lambda_i$ in position k_i and 0s elsewhere for $i = 1, \ldots, n$. Then the conditions in (c) imply that XN is a diagonal matrix. As

X is invertible, and each column of N is nonzero. XN is invertible. Thus N is invertible. Additionally, since $XN = D$ is an invertible diagonal matrix, we must have that X is a generalized permutation matrix as well.

The matrix N determines a multi-graph G with vertex set $\{1,2,\ldots,n\}$ and edges $\{\{j_i,k_i\} : i = 1,\ldots,n\}$. We claim that G is a cycle of length n. As N is invertible, any collection of s edges of G must span at least s vertices. If G is disconnected, then

$$N = \left[\begin{array}{c|c} & O \\ \hline O & \end{array}\right].$$

Note that since X and D are invertible, $X^{-1} = ND^{-1}$. Thus, X must have form

$$X = \left[\begin{array}{c|c} & \\ \hline O & \end{array}\right].$$

So X is partly decomposable. Similarly, if there exists some proper collection of s edges that spans exactly s vertices, then

$$N = \left[\begin{array}{c|c} A & \\ \hline O & \end{array}\right]$$

for some $s \times s$ matrix A. So N, and hence X, must be partly decomposable. In both of these cases, we see that X is partly decomposable, which in turn implies that A is partly decomposable contrary to assumption.

Thus, G is a connected simple graph with n edges, and n vertices such that any subset of $s < n$ edges spans more than s vertices. From our first conclusion, by Lemma 4.4, we have that G is a unicyclic graph. Now assume the single cycle is of length $s < n$. By our second conclusion, the set of edges in the cycle must span more than s vertices. This is a contradiction, and thus, the graph G is necessarily a cycle of length n.

Without loss of generality we may assume that G is the cycle 1–2–$\cdots$–n–1, and thus

$$N = \begin{bmatrix} 1 & 0 & \cdots & 0 & -\lambda_n \\ -\lambda_1 & 1 & 0 & \ddots & 0 \\ 0 & -\lambda_2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda_{n-1} & 1 \end{bmatrix}.$$

There exist invertible diagonal matrices D, E and F such that $\widehat{X} = E^{-1}F^{-1}XD^{-1}$ and $\widehat{N} = DNE$ satisfy $\widehat{X}\widehat{N} = I$, and

$$N = \begin{bmatrix} 1 & 0 & \cdots & 0 & -d \\ -1 & 1 & 0 & \ddots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

for some nonzero d. Thus, we may assume without loss of generality, $\lambda_1 = \cdots = \lambda_{n-1} = 1$, $\lambda_n = d$, N is the matrix in (4) and $XN = I$.

The determinant of N is $1-d$. Since N is invertible, $d \neq 1$. It can be verified that

$$N^{-1} = (1-d)^{-1} \begin{bmatrix} 1 & d & d & \cdots & d \\ 1 & 1 & d & & d \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & & 1 & d \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}. \tag{4.3}$$

As $XN = I$, the result follows. □

Corollary 4.6. *Let A be $2n \times 2n$ matrix with* $\operatorname{rank} A = 2n$, $\operatorname{perrank} A = n$, *and the characteristic is not* 2. *Then $n = 1$ or A is partly decomposable.*

Proof. By Corollary 4.5, we may assume that X is the matrix in (4.3), $\lambda_1 = \cdots = \lambda_{n-1} = 1$ and $\lambda_n = d$. This implies that

$$B = (1-d)^{-1} \begin{bmatrix} \mathbf{d} & \mathbf{1} & d & \cdots & d \\ 1 & \mathbf{d} & \mathbf{1} & & d \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & & \mathbf{d} & \mathbf{1} \\ \mathbf{1/d} & 1 & \cdots & 1 & \mathbf{d} \end{bmatrix}.$$

Here, we have put in boldface the entries of B that differ from the corresponding entries of X.

Observe that by Theorem 3.1, any $n \times n$ submatrix of

$$\begin{bmatrix} B \\ X \end{bmatrix}$$

avoiding the $(n-1)$-st row of B and the $(n-1)$-st row of X has permanent equal to 0. Let $\mathbf{x_i}^T$, respectively, $\mathbf{b_i}^T$, denote the i-th row of X, respectively, B. Then by the above observation, and the multilinearity of the permanent as a function of its rows

$$0 = \operatorname{per} \begin{bmatrix} \mathbf{x_1}^T - \mathbf{b_1}^T \\ \mathbf{x_2}^T - \mathbf{b_2}^T \\ \vdots \\ \mathbf{x_{n-2}}^T - \mathbf{b}_{\mathbf{n-2}}^{\mathbf{T}} \\ \mathbf{x}_{\mathbf{n}}^{\mathbf{T}} \end{bmatrix} = \operatorname{per} \begin{bmatrix} \mathbf{e_1}^T - \mathbf{e_2}^T \\ \mathbf{e_2}^T - \mathbf{e_3}^T \\ \vdots \\ \mathbf{e_{n-2}}^T - \mathbf{e_{n-1}}^T \\ (d-1)^{-1} \quad \cdots \quad (d-1)^{-1} \end{bmatrix}.$$

This latter permanent equals $(d-1)^{-1}\sum_{i=1}^{n}(-1)^n$, which is nonzero if n is odd. Thus n is even, which implies

$$
\begin{aligned}
0 &= \operatorname{per}\left[\begin{array}{c} \mathbf{x_1}^T-\mathbf{b_1}^T \\ \mathbf{x_2}^T-\mathbf{b_2}^T \\ \vdots \\ \mathbf{x_{n-3}}^T-\mathbf{b_{n-3}}^T \\ \hline \mathbf{b_{n-2}}^T \\ \mathbf{x_{n-2}}^T \\ \mathbf{x_n}^T \end{array}\right] \\
&= \operatorname{per}\left[\begin{array}{ccccccc} \multicolumn{7}{c}{\mathbf{e_1}^T-\mathbf{e_2}^T} \\ \multicolumn{7}{c}{\mathbf{e_2}^T-\mathbf{e_3}^T} \\ \multicolumn{7}{c}{\vdots} \\ \multicolumn{7}{c}{\mathbf{e_{n-3}}^T-\mathbf{e_{n-3}}^T} \\ \hline (d-1)^{-1} & (d-1)^{-1} & \cdots & d(d-1)^{-1} & (d-1)^{-1} & d(d-1)^{-1} & d(d-1)^{-1} \\ (d-1)^{-1} & (d-1)^{-1} & \cdots & (d-1)^{-1} & d(d-1)^{-1} & (d-1)^{-1} & d(d-1)^{-1} \\ (d-1)^{-1} & (d-1)^{-1} & \cdots & \cdots & (d-1)^{-1} & (d-1)^{-1} & (d-1)^{-1} \end{array}\right] \\
&= (d-1)^{-1}\operatorname{per}\begin{bmatrix} 0 & d & d \\ d-1 & 1 & d \\ 0 & 1 & 1 \end{bmatrix} \\
&= 2d \\
&\neq 0.
\end{aligned}
$$

This is a contradiction since the characteristic is not 2, and d is nonzero.

Therefore, (c) of Theorem 4.3 does not hold. Hence A is partly decomposable, or $n=1$. □

Lemma 4.7. *Let A be a partly decomposable, $n\times n$ matrix with* $\operatorname{rank} A = 2n$ *and* $\operatorname{perrank} A = n$. *Assume that for matrices $\widehat{A}$ of order $m<2n$ that* $\operatorname{perrank}\widehat{A} = m/2$ *implies that up to generalized permutation equivalence $\widehat{A}$ is a direct sum of P_2s. Then up to generalized permutation equivalence A is a direct sum of P_2s.*

Proof. A partly decomposable 2×2 matrix with nonzero determinant has perrank equal to 2. So $n\geq 2$.

Consider the case $n=2$. By Corollary 2.2, A has at least two nonzeros in each row and column. Since A is partly decomposable, we may assume that A is of the form

$$
\left[\begin{array}{cc|c} \multicolumn{2}{c|}{B} & O \\ \hline a & b & \multirow{2}{*}{C} \\ c & d & \end{array}\right],
$$

where B and C have no zeros. The assumption on the rank and perrank of A imply that B and C each have rank 2 and perrank 1. Hence, using generalized permutation equivalence, we may assume each is P_2. The conditions that $\operatorname{per}A[\{i,3,4\},\{1,2,j\}]=0$ for $i\in\{1,2\}$ and $j\in\{3,4\}$ are now equivalent to

$$
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

The coefficient matrix has determinant 16, which is nonzero since the characteristic is not 2. Hence $a = b = c = d = 0$, and the result holds for $n = 2$.

Now assume $n > 2$, and that

$$A = \begin{bmatrix} A_1 & O \\ B & A_2 \end{bmatrix},$$

where A_1 and A_2 are square, non-vacuous matrices. If A_1 is $k \times k$, then Theorem 1.3 implies that

$$n = \operatorname{perrank} A \geq \operatorname{perrank} A_1 + \operatorname{perrank} A_2 \geq \lceil k/2 \rceil + \lceil (2n-k)/2 \rceil \geq n.$$

Thus equality holds throughout, and k is even. By the hypotheses, we may assume that both A_1 and A_2 are direct sums of P_2s. The argument for $n = 2$ can be used to show $B = O$. Hence A is a direct sum of P_2s. □

We are now ready to prove the main results of the paper. The first characterizes the invertible matrices whose perrank is half its rank.

Theorem 4.8. *Let A be an $m \times m$ matrix with* rank $A = m$ *and* perrank $A = m/2$ *over a field of characteristic not* 2. *Then m is even, and A is generalized permutation equivalent to a direct sum of P_2s.*

Proof. The proof is by induction on m. As perrank$A$ is an integer, m is even, say $m = 2n$.

The base case is $n = 1$. As perrank $= 1$ and rank $= 2$ in this case, A is not partly decomposable. Thus each entry of A is nonzero, and A is permutation equivalent to a matrix of the from

$$\begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}$$

for some a. As $perA = 0$, $a = -1$ and A is permutation equivalent to P_2.

Now assume that $m > 2$ and the result holds for $m - 2$. By Corollary 4.6, A is partly decomposable, and by Lemma 4.7, A is permutation equivalent to a direct sum of P_2s.

Therefore the result follows by induction. □

Finally, we characterize all matrices having permanent rank equal to half the rank.

Theorem 4.9. *Let A be an $m \times n$ matrix over a field whose characteristic is not* 2 *with* rank $A = r$ *and* perrank $A = r/2$. *Then A is generalized permutation equivalent to a matrix of the form*

$$\left[\begin{array}{c|c} \oplus_{i=1}^{r/2} P_2 & O \\ \hline O & O \end{array}\right].$$

Proof. This follows from Theorem 4.8 and Corollary 2.3. □

5. Odd Rank Case

It is also natural to consider $m \times n$ matrices A with odd rank $r \leq \min\{m, n\}$ and permanent rank $(r+1)/2$. From the result of this paper, we have an immediate example. Suppose A is a $2n \times 2n$ matrix, with full rank and permanent rank n. Then A has form

$$A = \left[\oplus_{i=1}^{r/2} P_2\right].$$

Then, from A we construct an $(2n+1)\times(2n+1)$ matrix B by letting $B = I_1 \oplus A$. Then B is the matrix

$$B = \left[\begin{array}{c|c} 1 & O \\ \hline O & \oplus_{i=1}^{r/2} P_2 \end{array}\right].$$

By our result, since A has rank $2n$ and permanent rank n, B has rank $2n+1$ and permanent rank $n+1$. This can be extended to any bordering of an example of even rank that increases the rank simply by our bounds. However, unlike the even case, these are not essentially the only $(2n+1)\times(2n+1)$ matrix that satisfy these conditions. For example, consider the matrix below in $\mathbb{F}_5^{3\times 3}$,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We can calculate that the matrix A has rank 3 and permanent rank 2 in $\mathbb{F}_5$, meeting our bounds tightly. Additionally, A is not similar to any matrix that is a bordering of some P_2, so this is a new example. This single example proves that the odd case cannot be expressed as simply as the even case, and it remains to classify all matrices that meet our bound in the odd case, beyond those already listed.

Author's Note. We dedicate this paper to Avi Berman on the occasion of his 80th birthday. Avi, thanks for your insightful, forward-looking contributions to mathematics. Even more so, thank you for being an incredible role model through your excellence in teaching, research, mentorship of mathematicians, and for your friendship.

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