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ON THE OPTIMAL NON-AUTONOMOUS LIMIT CYCLE IN THE CO2 EMISSIONS CONTROL PROBLEM WITH SMOOTH SEASONAL FLUCTUATIONS

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Dedicated to Professor Tapan Mitra, who had made outstanding contributions to theoretical economics.

Abstract. We study a CO_2 emissions control problem with smooth seasonal fluctuations of the CO_2 reduction rate due to photosynthesis, where the dynamics of a state variable is generated by a non-autonomous ordinary differential equation. The current value Hamiltonian of the problem is linear in the state variable, and so the dynamics of its co-state variable is de-coupled from that of the state variable. We show that the co-state variable exhibits periodic oscillations, that a unique nontrivial optimal solution does exists under some additional inequality constraint, and that under this constraint the state variable converges to a non-autonomous limit cycle globally on the unique optimal path that exhibits the turnpike property.
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1. INTRODUCTION

An unbalance between carbon dioxide (CO_2) emissions and the natural absorption of CO_2 through photosynthesis deteriorates a natural environment. Therefore the balance between these two factors must be established in order to keep the natural environment from deteriorations. In economic terms CO_2 emissions are *necessary bads* as oppose to necessary goods. Without economic activities generating CO_2, the economy cannot be sustained. From the economic viewpoint, therefore, to keep the natural environment from deteriorations due to CO_2 emissions is to solve an optimal control problem within the framework in which the volume of necessary bads (CO_2 emissions) is continuously reduced by an outside mechanism (photosynthesis). The

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photosynthesis process of natural absorption of CO_2 is not only continuos but also inherently cyclical or fluctuant. This is partly because the photosynthesis process is heavily influenced by sunshine, the volume of which is subject to seasonal change. To understand how to avoid the unbalance between CO_2 emissions and the natural absorption of CO_2, therefore, it is of the primary importance to incorporate this continuously changing exogenous factor into an optimal control problem. In the existing literature this has not been attempted yet.

This study intends to build the first of such an approach for a CO_2 emissions control. We present a simple CO_2 emissions control problem in which CO_2 is released by economic activities and in which CO_2 is continuously absorbed by cyclical photosynthesis. Our main result is that the over-time process of optimal CO_2 emissions eventually synchronizes with the exogenous process of cyclical CO_2 absorption.

In the literature on dynamic optimization this result may be thought of as a contribution to turnpike theory.[1] A turnpike theorem implies that the optimal programs from different initial states converge to one another. Dealing with an oscillating model with a parameter the value of which follows a sinusoidal wave,[2] it implies that the optimal programs converge to a program that also follows a sinusoidal wave. Although this might appear obvious, it is not. While a continuous time autonomous optimal growth model with only one state variable exhibits the turnpike property when there exists a unique steady state, Benhabib and Nishimura [1] has shown that the turnpike property does not necessarily hold in a continuous time autonomous optimal growth model with many state variables, even if there exists a unique steady state. And in a discrete time optimal growth model such that a reduced form utility function, a discount factor, and the law of motion of a state variable are all time invariant, even if the number of state variable is only one, it has been known that the dynamic pattern of an optimal process and the exogenous dynamic structure of a model may not synchronize. More specifically it has been shown that an optimal process can be chaotic.[3] Our model is a continuous time model that includes one state variable the law of motion of which is described by a *non-autonomous* ordinary differential equation. As is case with a continuous time autonomous optimal growth model, the turnpike property of our model might result from the fact that it is a continuous time model with only one state variable.[4] Hartle [8] shows that in one state variable *autonomous* infinite horizon optimal control problems the optimal trajectory of the state variable must always be *monotonic* if the optimal state trajectory is *unique*. Hartle [8] also shows that in such a problem, if the optimal state trajectory is *not unique*, some of optimal state trajectories might exhibit *damped periodic oscillations*. In contrast the present paper treats a one state variable *non-autonomous* infinite horizon optimal control problems such that the optimal state trajectory is *unique* and that this unique optimal state trajectory exhibits *persistent undamped periodic oscillations*.

[1]See Dorfman, Samuelson, and Solow [5], Khan and Zaslavski [10], McKenzie [12], and Yano [18].

[2]See the specification (2.1) given below.

[3]The existence of a chaotic optimal path is shown by Deneckere and Pelikan [3], Boldrin and Montrucchio [2], and Mitra and Sorger [13]. Nishimura and Yano [14, 15, 16], and Khan and Mitra [9] build a structural model that has a chaotic optimal path. See Deng, Khan, and Mitra [4], and Yano and Furukawa [19] for more recent applications of chaotic dynamics to economics.

[4]Gromov, Bondarev, and Gromova [6], and Gromov, Shigoka, and Bondarev [7] also deal with environmental issues like the present paper. And the model due to Gromov et al. [6] and Gromov et al. [7] has the structure quite similar to that of our model and exhibits the turnpike property.

The rest of the paper is composed of the following four sections. Section 2 formulates a CO_2 emissions control problem and applies the maximum principle to this problem. Section 3 is composed of a series of preliminary considerations. Section 4 states the main result. Section 5 briefly concludes.

2. THE MODEL AND ITS UNIQUE OPTIMAL SOLUTION

In Section 2.1 we formulate the model that is the CO_2 emissions control problem (P1). In Section 2.2 we show that the Hamiltonian dynamics prescribes a unique optimal solution to this problem (P1). In Section 2.3 we prove Lemma 1 that is stipulated in Section 2.2.

2.1. **The model.** We shall specify a CO_2 emissions control problem with smooth seasonal fluctuations of the CO_2 reduction rate due to photosynthesis. Let $z(t) \geq 0$ be a state variable in our control problem, which represents the level of cumulative CO_2 on a country that belongs to the temperate zone on the earth. Let $v(t) \geq 0$ be a control input representing the macroeconomic activity that produces ξv_t units of CO_2, where $\xi > 0$ is a positive constant. The state variable $z(t) = z_t \geq 0$ obeys $\dot{z}_t = \xi v_t - \delta(t) z_t$, where $\delta(t)$ is the rate of CO_2 reduction that is attributed to photosynthesis. Let $z(0) = z_0 \geq 0$. We assume that this reduction rate, $\delta(t)$, seasonally fluctuates so that we have $\delta(t+T) = \delta(t)$ for some $T > 0$ and for each $t \geq 0$. More specifically the functional form of $\delta(\cdot)$ is assumed to be given by

$$\delta(t) := \bar{\delta} + a\cos(\frac{2\pi}{T}(t-\theta)), \tag{2.1}$$

where $\bar{\delta}$, a, and T are positive constants with $\bar{\delta} > a > 0$, and where θ is a constant with $0 \leq \theta < T$. Let p be a positive constant that is the price index, let $b > 0$ be a positive constant, and let $pv(t)(b - \frac{1}{2}v(t))$ be the gross domestic product (GDP) of the country that macroeconomic activity $v(t)$ produces. The aggregate production function is given by the quadratic function $v(t)(b - \frac{1}{2}v(t))$. We assume that the domain of this quadratic function is the largest non-negative interval of $\mathbb{R}$ on which the function is monotone non-decreasing. Thus the domain of this function is given by the closed interval $[0, b]$ such that $0 \leq v(t) \leq b$. Moreover let $qz(t)$ be the value of external diseconomies created by CO_2, where $q > 0$ is a positive constant. The net GDP is the GDP minus the value of external diseconomies. The GDP is always non negative, whereas the net GDP could be negative for some time. The objective function of the optimizing agent is given by $\int_0^\infty e^{-rt} \left[pv_t \left(b - \frac{1}{2}v_t\right) - qz_t\right] dt$, where $r > 0$ is a positive constant that is the discount rate.

Then we have the following CO_2 emissions control problem, where $C(\mathbb{R}_+,[0,b])$ denotes the set of all continuous functions from $\mathbb{R}_+$ to $[0,b]$.[5]

$$\max_{v(\cdot)} \int_0^\infty e^{-rt}\left[pv_t\left(b-\frac{1}{2}v_t\right)-qz_t\right]dt \text{ subject to} \tag{P}$$

$$v(\cdot)\in C(\mathbb{R}_+,[0,b])\wedge$$

$$\dot{z}_t=\xi v_t-\delta(t)z_t\wedge z(0)=z_0\geq 0\wedge z_t\geq 0\wedge$$

$$\delta(t)=\bar{\delta}+a\cos(\frac{2\pi}{T}(t-\theta)),$$

where $r,p,b,q,\xi,\bar{\delta},a,T$, and θ are given control parameters that satisfy

$$r,p,b,q,\xi,\bar{\delta},a,T\in\mathbb{R}_{++}\wedge\bar{\delta}-a>0\wedge 0\leq\theta<T.$$

For the sake of tractability we normalize the CO_2 emissions control problem (P) in the following way. Let $u(t), x(t)$, and β be defined as

$$u(t):=\frac{v(t)}{b}\in[0,1]\ \wedge x(t):=\frac{qz(t)}{pb^2}\in\mathbb{R}_+\wedge\beta:=\frac{\xi q}{pb}, \tag{2.2}$$

respectively. Then the intertemporal optimization problem (P) is normalized into the following intertemporal optimization problem, where $C(\mathbb{R}_+,[0,1])$ denotes the set of all continuous functions from $\mathbb{R}_+$ to $[0,1]$.[6]

$$\max_{u(\cdot)} \int_0^\infty e^{-rt}\left[u_t\left(1-\frac{1}{2}u_t\right)-x_t\right]dt \text{ subject to} \tag{P1}$$

$$u(\cdot)\in C(\mathbb{R}_+,[0,1])\wedge$$

$$\dot{x}_t=\beta u_t-\delta(t)x_t\wedge x(0)=x_0\geq 0\wedge x_t\geq 0\wedge$$

$$\delta(t)=\bar{\delta}+a\cos(\frac{2\pi}{T}(t-\theta)),$$

where $r,\beta,\bar{\delta},a,T$, and θ are given control parameters that satisfy the following condition.

$$r,\beta,\bar{\delta},a,T\in\mathbb{R}_{++}\wedge\bar{\delta}-a>0\wedge 0\leq\theta<T. \tag{2.3}$$

Note that both $u_t(1-\frac{1}{2}u_t)-x_t$ and $\beta u_t-\delta(t)x_t$ are linear in x_t, which implies that the dynamics of a co-state variable is de-coupled from that of the state variable x_t. This in turn simplifies our analysis of the CO_2 emissions control problem (P1).

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[5]This specification follows that of a pollution control problem due to Gromov et al. [6] and Gromov et al. [7] closely, although the functional form of $\delta(\cdot)$ due to Gromov et al. [6] and Gromov et al. [7] is quite different from that of $\delta(\cdot)$ in our model.

Let $\mathbb{N}$ be the set of all natural numbers that includes 0. Both Gromov et al. [6] and Gromov et al. [7] assume that the functional form of $\delta(\cdot)$ is given by

$$\delta(t):=\begin{cases}\delta_1>0, & t\in[kT,kT+\alpha T),\\ \delta_2>0, & t\in[kT+\alpha T,(k+1)T),\end{cases}$$

where δ_1, δ_2, T, and α are positive constants with $\delta_1\neq\delta_2$ and with $0<\alpha<1$, and where $k\in\mathbb{N}$. Note that this function satisfies $\delta(t+T)=\delta(t)$ for this given $T>0$ and for each $t\geq 0$.

[6]We have followed Gromov et al. [6] and Gromov et al. [7] in this process of normalization.

2.2. **The Hamiltonian dynamics and a unique optimal solution.** In the present section we show that the Hamiltonian dynamics prescribes a unique optimal solution to the CO_2 emissions control problem (P1). Let $V : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be defined as

$$V(x,u) := u(1-\frac{1}{2}u) - x.$$

The objective function of the CO_2 emissions control problem (P1) is given by

$$\int_0^\infty V(x_t,u_t)e^{-rt}dt.$$

Let $\psi : \mathbb{R} \to [0,1]$ be defined as

$$\psi(\lambda) := \begin{cases} 0, & \lambda \in (-\infty, -\frac{1}{\beta}), \\ 1+\beta\lambda, & \lambda \in [-\frac{1}{\beta}, 0], \\ 1, & \lambda \in \mathbb{R}_{++}. \end{cases} \tag{2.4}$$

Let $H(\delta(t)) : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $H^*(\delta(t)) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be defined as

$$H(x,u,\lambda : \delta(t)) := V(x,u) + \lambda(\beta u - \delta(t)x),$$

$$H^*(x,\lambda : \delta(t)) := V(x,\psi(\lambda)) + \lambda(\beta\psi(\lambda) - \delta(t)x),$$

where $\delta(\cdot)$ is given by the specification (2.1), i.e.,

$$\delta(t) := \bar{\delta} + a\cos(\frac{2\pi}{T}(t-\theta)),$$

and where $r, \beta, \bar{\delta}, a, T$, and θ are given control parameters that satisfy the condition (2.3). Then by construction we have the following relation.

$$H^*(x,\lambda : \delta(t)) = \max_u H(x,u,\lambda : \delta(t)) \text{ subject to } 0 \leq u \leq 1.$$

$H(\delta(t)) = H(x,u,\lambda : \delta(t))$ and $H^*(\delta(t)) = H^*(x,\lambda : \delta(t))$ are the current value Hamiltonian and the current value maximized Hamiltonian of the intertemporal optimization problem (P1), respectively, where x is a state variable, u is a control variable, and λ is a co-state variable, respectively. $H(\delta(t)) = H(x,u,\lambda : \delta(t))$ is concave in $(x,u) \in \mathbb{R}_+ \times [0,1]$ so that it satisfies Mangasarian's condition. It is *strictly concave* in $u \in [0,1]$ and linear in $x \in \mathbb{R}_+$. $H^*(\delta(t)) = H^*(x,\lambda : \delta(t))$ is linear and thus concave in $x \in \mathbb{R}_+$ so that it satisfies Arrow's condition. Therefore we should be able to prescribe a unique optimal solution to the CO_2 emissions control problem (P1) by appealing to the maximum principle.

Before being able to do so we need preliminary considerations on the dynamics of the co-state variable $\lambda = \lambda(t)$ and that of the state variable $x = x(t)$. The dynamics of the co-state variable is de-coupled from that of the state variable because $H^*(\delta(t)) = H^*(x,\lambda : \delta(t))$ is linear in $x \in \mathbb{R}_+$. Since

$$\dot{\lambda}(t) = r\lambda(t) - \frac{\partial H^*}{\partial x}(x(t),\lambda(t) : \delta(t)) = \lambda(t)(r+\delta(t)) + 1,$$

the dynamics of the co-state variable $\lambda = \lambda(t)$ is given by

$$\dot{\lambda}(t) = \lambda(t)(r+\delta(t)) + 1. \tag{2.5}$$

Lemma 1. Non-autonomous ordinary differential equation (2.5) has a unique solution that satisfies $\exists K > 0 : \forall t \geq 0 : |\lambda(t)| < K$. This solution is given by

$$\lambda(t) = -\int_t^{\infty} \exp(-[r+\bar{\delta}](\tau - t) + \frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(t-\theta)) - \sin(\frac{2\pi}{T}(\tau-\theta))])d\tau. \qquad (2.6)$$

K is given by $\frac{\exp(\frac{aT}{\pi})}{r+\bar{\delta}} + 1$.

Proof of Lemma 1. See Section 2.3. $\square$

Consider the following dynamics.

$$\dot{X} = \beta - (\bar{\delta} - a)X.$$

Note that $\beta > 0 \wedge \bar{\delta} - a > 0$ by the condition (2.3). Let $\bar{X}$ be defined as

$$\bar{X} := \frac{\beta}{\bar{\delta} - a}.$$

Let $\bar{x}(x_0)$ be defined as $\bar{x}(x_0) := \max\{\bar{X}, x_0\}$. For any $u(\cdot) \in C(\mathbb{R}_+, [0,1])$ let $x(t) = x_t$ be a solution of the differential equation $\dot{x}(t) = \beta u(t) - \delta(t)x(t)$ with the initial condition $x(0) = x_0$. From the specification (2.1) of $\delta(\cdot)$ we have, for each $t \in \mathbb{R}_+$, $x(t) \in [0, \bar{x}(x_0)]$. Let $x^*(t) = x_t^*$ be a solution of the differential equation $\dot{x}^*(t) = \beta\psi(\lambda(t)) - \delta(t)x^*(t)$ with the initial condition $x^*(0) = x_0$. Then by construction we have, for each $t \in \mathbb{R}_+$, $x^*(t) \in [0, \bar{x}(x_0)]$. Note that $|V(x_t, u_t)| \leq 1 \cdot (1 - \frac{1}{2}) + |\bar{x}(x_0)| < +\infty$ so that $\int_0^{\infty} |V(x_t, u_t)|e^{-rt}dt < +\infty$. Note also that $|V(x_t^*, \psi(\lambda_t))| \leq 1 \cdot (1 - \frac{1}{2}) + |\bar{x}(x_0)| < +\infty$ so that $\int_0^{\infty} |V(x_t^*, \psi(\lambda_t))|e^{-rt}dt < +\infty$. Therefore both $V(x_t, u_t)e^{-rt}$ and $V(x_t^*, \psi(\lambda_t))e^{-rt}$ are Lebesgue integrable in t over $\mathbb{R}_+$. We have sufficient preparations for stipulating the main result of the present section.

Proposition 1. Consider the CO_2 emissions control problem (P1), where $r, \beta, \bar{\delta}, a, T$, and θ are given control parameters that satisfy the condition (2.3). Then the followings hold respectively.

(1) The following Hamiltonian dynamics prescribes an optimal solution of this problem.[7]

$$\lambda(t) = -\int_t^{\infty} \exp(-[r+\bar{\delta}](\tau - t) + \frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(t-\theta)) - \sin(\frac{2\pi}{T}(\tau-\theta))])d\tau, \qquad (2.7)$$

$$u(\cdot) = (\psi \circ \lambda)(\cdot) \in C(\mathbb{R}_+, [0,1]), \qquad (2.8)$$

$$\dot{x}^*(t) = \frac{\partial H^*}{\partial \lambda}(x^*(t), \lambda(t) : \delta(t)), \qquad (2.9)$$

$$x^*(0) = x_0 \geq 0. \qquad (2.10)$$

(2) An optimal solution of this problem is unique.

Proof of Proposition 1. (1) The boundedness of $\lambda(t)$, $x(t)$, and $x^*(t)$ guarantees the transversality condition $\lim_{t\to\infty} e^{-rt}\lambda(t)(x(t) - x^*(t)) = 0$ for any admissible solution $x(t)$. The current value Hamiltonian $H(\delta(t)) = H(x, u, \lambda : \delta(t))$ satisfies the Mangasarian's condition and

[7]The condition (2.7) satisfies $[\dot{\lambda}(t) = r\lambda(t) - \frac{\partial H^*}{\partial x}(x^*(t), \lambda(t) : \delta(t))] \wedge [\exists K > 0 : \forall t \geq 0 : |\lambda(t)| < K]$ by Lemma 1.

thus the transversality condition together with the Lebesgue integrability of $V(x_t,u_t)e^{-rt}$ and $V(x_t^*,\psi(\lambda_t))e^{-rt}$ in t over $\mathbb{R}_+$ guarantees

$$\int_0^\infty V(x_t^*,\psi(\lambda_t))e^{-rt}dt \geq \int_0^\infty V(x_t,u_t)e^{-rt}dt.$$

See Theorem 3.13 in Seierstad and Sydsæter [17] and Theorem 9.3.1 in Léonard and Long [11].

(2) Note that by Proposition 1.(1) there exists at least one optimal solution. Note also that for any given two feasible paths a convex combination of these paths is also a feasible path. The current value Hamiltonian $H(\delta(t)) = H(x,u,\lambda:\delta(t))$ is *strictly concave* in $u\in[0,1]$ and concave in $x\in\mathbb{R}_+$. And $u(\cdot)\in C(\mathbb{R}_+,[0,1])$. Therefore an optimal solution of the problem (P1) is unique. See the paragraph preceding Theorem 4.6.1 in Léonard and Long [11]. □

The above proof is essentially the same as the proof given by Appendix 1 of Gromov et al. [7], although Lemma 1 and the condition (2.7) are due to the present study.

2.3. **The proof of Lemma 1.** The present section proves Lemma 1 stipulated in the previous section. The co-state variable would be given by

$$\lambda(t) = -\int_t^\infty \exp\left(\int_t^\tau -[r+\delta(s)]ds\right)d\tau + C\exp\left(\int_0^t [r+\delta(\tau)]d\tau\right) \tag{2.11}$$

for some constant $C\in\mathbb{R}$, if the limit

$$\lim_{S\to+\infty}\int_t^S \exp\left(\int_t^\tau -[r+\delta(s)]ds\right)d\tau$$

would exist. This is because the relation (2.11) is a general solution of the following non-autonomous ordinary differential equation if the limit does exist.

$$\frac{d}{dt}\lambda(t) = 1+[r+\delta(t)]\lambda(t).$$

Here

$$\begin{aligned}\int_t^\tau [r+\delta(s)]ds &= \int_t^\tau [r+\bar{\delta}+a\cos(\frac{2\pi}{T}(s-\theta))]ds\\ &= [r+\bar{\delta}](\tau-t)+\frac{a}{\frac{2\pi}{T}}[\sin(\frac{2\pi}{T}(s-\theta))|_t^\tau]\\ &= [r+\bar{\delta}](\tau-t)+\frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(\tau-\theta))-\sin(\frac{2\pi}{T}(t-\theta))].\end{aligned}$$

Hence

$$\begin{aligned}&\int_t^S \exp\left(\int_t^\tau -[r+\delta(s)]ds\right)d\tau\\ &= \int_t^S \exp\left(-[r+\bar{\delta}](\tau-t)-\frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(\tau-\theta))-\sin(\frac{2\pi}{T}(t-\theta))]\right)d\tau\\ &\leq \int_t^S \exp\left(-[r+\bar{\delta}](\tau-t)+\frac{aT}{\pi}\right)d\tau\\ &\leq \lim_{S\to+\infty}\int_t^S \exp\left(-[r+\bar{\delta}](\tau-t)+\frac{aT}{\pi}\right)d\tau\\ &= \lim_{S\to+\infty}\exp(\frac{aT}{\pi})\int_t^S \exp(-[r+\bar{\delta}](\tau-t))d\tau = \frac{\exp(\frac{aT}{\pi})}{r+\bar{\delta}}.\end{aligned}$$

Hence

$$\int_t^S \exp(\int_t^\tau -[r+\delta(s)]ds)d\tau \leq \frac{\exp(\frac{aT}{\pi})}{r+\bar{\delta}}.$$

The left hand side of the above inequality is monotone increasing in $S \in (t, +\infty)$ and bounded above. Therefore the limit

$$\lim_{S\to+\infty} \int_t^S \exp(\int_t^\tau -[r+\delta(s)]ds)d\tau$$

does exist. And if we set

$$\lambda^*(t) = -\int_t^\infty \exp(-[r+\bar{\delta}](\tau-t) + \frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(t-\theta)) - \sin(\frac{2\pi}{T}(\tau-\theta))])d\tau,$$

then we have from the relation (2.11) for each $t \geq 0$

$$\lambda(t) = \lambda^*(t) + C\exp(\int_0^t [r+\delta(\tau)]d\tau).$$

By construction

$$|\lambda^*(t)| < \frac{\exp(\frac{aT}{\pi})}{r+\bar{\delta}} + 1$$

for each $t \geq 0$, and by the condition (2.3) $r+\delta(\tau) \geq r+\bar{\delta}-a > 0$ for each $\tau \geq 0$. Thus if $|\lambda(t)| < K$ for some $K > 0$ and for each $t \geq 0$, then we should have $C = 0$ so that $\lambda(t) = \lambda^*(t)$. In other words if $\lambda = \lambda(t)$ satisfies the condition $\exists K > 0 : \forall t \geq 0 : |\lambda(t)| < K$, we should have

$$\lambda(t) = -\int_t^\infty \exp(-[r+\bar{\delta}](\tau-t) + \frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(t-\theta)) - \sin(\frac{2\pi}{T}(\tau-\theta))])d\tau. \quad (2.12)$$

K is given by $\frac{\exp(\frac{aT}{\pi})}{r+\bar{\delta}} + 1$. Thus Lemma 1 has been proved. □

3. The Dynamics of The Co-State Variable and A Unique Nontrivial Optimal Solution

In Section 2.3 that is the proof of Lemma 1 stipulated in Section 2.2 we have initiated to characterize the dynamics of the co-state variable of the CO_2 emissions control problem (P1). In Section 3.1 we further characterize the dynamics of the co-state variable of the problem (P1). In Section 3.2 based on the result obtained in Section 3.1 we show that we need some additional inequality constraint in order to obtain a nontrivial optimal solution. In Section 3.2 we also introduce some auxiliary variables by means of which we construct some inequality constraints either to obtain a nontrivial optimal solution or to avoid a corner solution. In Section 3.3 we show that there exists a unique nontrivial optimal solution to the problem (P1) under either of these inequality constraints.

3.1. **The dynamics of the co-state variable.** In the present section we further characterize the dynamics of the co-state variable of the problem (P1). Let $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined as

$$\varphi(t,\tau) := \frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(t-\theta)) - \sin(\frac{2\pi}{T}(\tau-\theta))].$$

Then from the relation (2.12) we have for each $t \geq 0$

$$\lambda(t) = -\int_t^\infty \exp(-[r+\bar{\delta}](\tau-t) + \varphi(t,\tau))d\tau.$$

Let $\mathbb{Z}$ be the set of all integers. By construction for any m,n in $\mathbb{Z}$

$$\varphi(t,\tau)=\varphi(t+mT,\tau+nT).$$

Thus we have for $t\geq 0$

$$\lambda(t)=-\int_t^{\infty}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau$$

$$=-\int_{\tau-T=t}^{\infty}\exp(-[r+\bar{\delta}](\tau-T-t)+\varphi(t,\tau-T))d[\tau-T]$$

$$=-\int_{\tau=T+t}^{\infty}\exp(-[r+\bar{\delta}](\tau-T-t)+\varphi(t,\tau-T))d\tau$$

$$=-\int_{t+T}^{\infty}\exp(-[r+\bar{\delta}](\tau-T-t)+\varphi(t,\tau-T))d\tau$$

$$=-\int_{t+T}^{\infty}\exp(-[r+\bar{\delta}](\tau-t-T)+\varphi(t+T,\tau))d\tau=\lambda(t+T).$$

And we also have for $t\geq 0$

$$\lambda(t)=-\int_t^{\infty}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau$$

$$\begin{aligned}=\ &-\int_t^{t+T}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau\\&-\int_{t+T}^{\infty}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau\end{aligned}$$

$$\begin{aligned}=\ &-\int_t^{t+T}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau\\&+\exp(-[r+\bar{\delta}]T)[-\int_{t+T}^{\infty}\exp(-[r+\bar{\delta}](\tau-t-T)+\varphi(t,\tau))d\tau]\end{aligned}$$

$$\begin{aligned}=\ &-\int_t^{t+T}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau\\&+\exp(-[r+\bar{\delta}]T)[-\int_{t+T}^{\infty}\exp(-[r+\bar{\delta}](\tau-t-T)+\varphi(t+T,\tau))d\tau]\end{aligned}$$

$$=-\int_t^{t+T}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau+\exp(-[r+\bar{\delta}]T)\lambda(t+T).$$

Since $\lambda(t)=\lambda(t+T)$ for each $t\geq 0$, we have for $t\geq 0$

$$\lambda(t)=-\int_t^{t+T}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau+\exp(-[r+\bar{\delta}]T)\lambda(t).$$

In other words, we have for $t\geq 0$

$$\lambda(t)=\frac{-1}{1-\exp(-[r+\bar{\delta}]T)}\int_t^{t+T}\exp(-[r+\bar{\delta}](\tau-t)+\varphi(t,\tau))d\tau.$$

Thus we have obtained $\lambda(t)=$

$$\frac{-1\times\int_t^{t+T}\exp(-[r+\bar{\delta}](\tau-t)+\frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(t-\theta))-\sin(\frac{2\pi}{T}(\tau-\theta))])d\tau}{1-\exp(-[r+\bar{\delta}]T)}\tag{3.1}$$

for $t \geq 0$, where $r, \bar{\delta}, a, T$, and θ are control parameters that satisfy the condition (2.3). Figure 1 depicts $\lambda = \lambda(t)$ for $t \geq 0$ and for $(r, \bar{\delta}, a, T, \theta) = (1, 2.5, 2, 2, 1)$, where the horizontal axis is the $t-$axis and the vertical axis is the $\lambda-$axis, respectively.

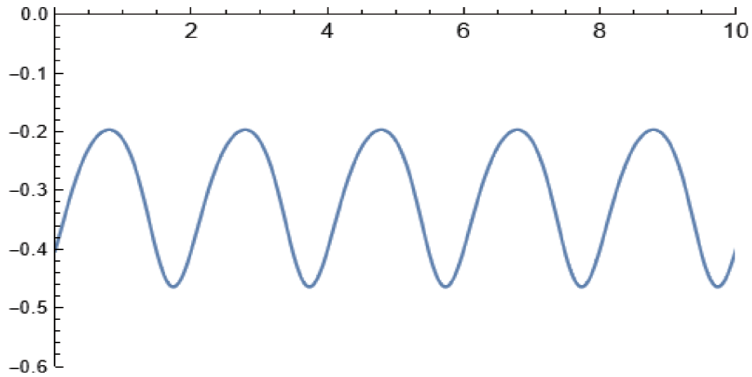

FIGURE 1. Figure 1

3.2. **Inequality constraints.** Since by the relation (2.8) for $t \geq 0$

$$u(t) = \begin{cases} 0, & 1 + \beta\lambda(t) < 0, \\ 1 + \beta\lambda(t), & 0 \leq 1 + \beta\lambda(t) \leq 1, \\ 1, & 1 < 1 + \beta\lambda(t), \end{cases}$$

and since as shown by Figure 1 $\lambda = \lambda(t)$ is uniformly bounded away from and below the $t-$axis, the optimal level of the normalized macroeconomic activity $u(t)$ is always equal to 0, if $\lambda(t) \leq -\frac{1}{\beta}$ for each $t \geq 0$. In order to avoid such a case as well as a corner solution we introduce auxiliary variables and inequality constraints.

Let $\mathbb{R}_{--}$ be defined as $\mathbb{R}_{--} := \{x \in \mathbb{R} : x < 0\}$ and let $l(r, \bar{\delta}, a, T, \theta) : \mathbb{R}_{+} \to \mathbb{R}_{--}$ be defined as $l(t : r, \bar{\delta}, a, T, \theta) :=$

$$\frac{-1 \times \int_t^{t+T} \exp(-[r+\bar{\delta}](\tau - t) + \frac{aT}{2\pi}[\sin(\frac{2\pi}{T}(t-\theta)) - \sin(\frac{2\pi}{T}(\tau-\theta))])d\tau}{1 - \exp(-[r+\bar{\delta}]T)},$$

where $r, \bar{\delta}, a, T$, and θ are control parameters that satisfy the condition (2.3). Note that by construction for each $t \in \mathbb{R}_{+}$ $l(t : r, \bar{\delta}, a, T, \theta) = \lambda(t) \wedge l(t : r, \bar{\delta}, a, T, \theta) = l(t + T : r, \bar{\delta}, a, T, \theta)$.

Let $\lambda_{\max}(r, \bar{\delta}, a, T, \theta)$ and $\lambda_{\min}(r, \bar{\delta}, a, T, \theta)$ be defined as

$$\lambda_{\max}(r, \bar{\delta}, a, T, \theta) := \max_{t \in [0,T]} l(t : r, \bar{\delta}, a, T, \theta), \tag{A}$$

$$\lambda_{\min}(r, \bar{\delta}, a, T, \theta) := \min_{t \in [0,T]} l(t : r, \bar{\delta}, a, T, \theta). \tag{B}$$

Then by construction we have for each $t \in \mathbb{R}_{+}$

$$\lambda_{\max}(r, \bar{\delta}, a, T, \theta) \geq \lambda(t) \geq \lambda_{\min}(r, \bar{\delta}, a, T, \theta).$$

If $-\frac{1}{\beta} \geq \lambda_{\max}(r, \bar{\delta}, a, T, \theta)$, $u(t) = 0$ for all $t \geq 0$. If $\lambda_{\max}(r, \bar{\delta}, a, T, \theta) > -\frac{1}{\beta} \geq \lambda_{\min}(r, \bar{\delta}, a, T, \theta)$, $u(t) > 0$ for some $t \geq 0$, and $u(t) = 0$ for other $t \geq 0$. If $\lambda_{\min}(r, \bar{\delta}, a, T, \theta) > -\frac{1}{\beta}$, $u(t) > 0$ for all $t \geq 0$. Figure 2 depicts $\lambda = \lambda(t)$ and $\lambda = -\frac{1}{\beta}$, where $(r, \bar{\delta}, a, T, \theta) = (1, 2.5, 2, 2, 1)$ so that

$\lambda_{\max}(r,\bar{\delta},a,T,\theta) = -0.196558215152\cdots$ and $\lambda_{\min}(r,\bar{\delta},a,T,\theta) = -0.464503169994\cdots$, and where $\beta = 10$ for Figure 2.a, $\beta = 3$ for Figure 2.b, and $\beta = 1.85$ for Figure 2.c, respectively. In each of Figures 2.a, 2.b, and 2.c the horizontal axis is the $t-$axis and the vertical axis is the $\lambda-$axis, respectively.

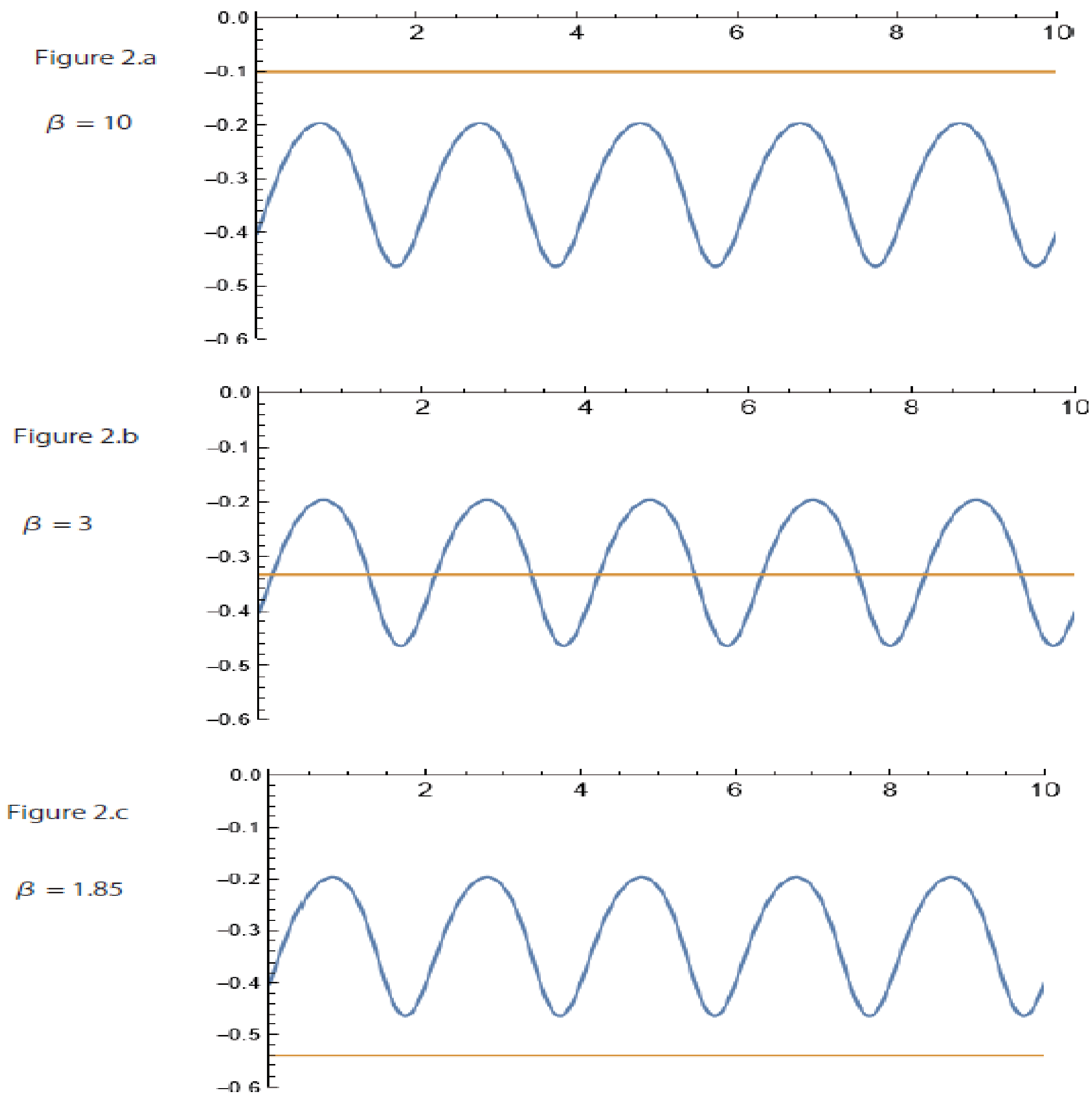

FIGURE 2. Figure 2

Note that if we assume $\lambda_{\max}(r,\bar{\delta},a,T,\theta) > -\frac{1}{\beta}$, $u(t) > 0$ for some t, and that if we assume $\lambda_{\min}(r,\bar{\delta},a,T,\theta) > -\frac{1}{\beta}$, $u(t) > 0$ for all t. Thus we introduce the following assumptions.

Assumption 3.1. Let $r,\beta,\bar{\delta},a,T$, and θ be given control parameters that satisfy the condition (2.3). Let $\lambda_{\max}(r,\bar{\delta},a,T,\theta)$ be the constant given by the definition (A). It holds that $\lambda_{\max}(r,\bar{\delta},a,T,\theta) > -\frac{1}{\beta}$.

Assumption 3.2. Let $r, \beta, \bar{\delta}, a, T$, and θ be given control parameters that satisfy the condition (2.3). Let $\lambda_{\min}(r, \bar{\delta}, a, T, \theta)$ be the constant given by the definition (B). It holds that $\lambda_{\min}(r, \bar{\delta}, a, T, \theta) > -\frac{1}{\beta}$.

See Table 1 for parametric examples of $\lambda_{\max}(r, \bar{\delta}, a, T, \theta)$ and $\lambda_{\min}(r, \bar{\delta}, a, T, \theta)$ for some given $(r, \bar{\delta}, a, T, \theta)$. In the rest of the paper we always assume the inequality constraint of Assumption 3.1 so that the CO_2 emissions control problem (P1) has a unique *nontrivial* optimal solution. We shall hypothetically strengthen Assumption 3.1 with Assumption 3.2 at the ends of Section 3.3 and Section 4. For the latter case the optimal level of the normalized macroeconomic activity is given by $u(t) = 1 + \beta\lambda(t)$ that is always positive. The latter assumption is more natural as an assumption made for such a macroeconomic model as the CO_2 emissions control problem (P1) than the former assumption.

$(r, \bar{\delta}, a, T, \theta)$	$\lambda_{\max}(r, \bar{\delta}, a, T, \theta)$	$\lambda_{\min}(r, \bar{\delta}, a, T, \theta)$
$(1, 2.5, 2, 2, 1)$	$-0.196558215152\cdots$	$-0.464503169994\cdots$
$(1, 2.5, 2, 2.5, 1)$	$-0.191375355265\cdots$	$-0.492562121558\cdots$
$(1, 3.5, 3, 2, 1)$	$-0.140287851077\cdots$	$-0.430861424554\cdots$
$(1.5, 2.5, 2, 2, 1)$	$-0.176551065630\cdots$	$-0.391208652324\cdots$
$(1.5, 3.5, 3, 2, 1)$	$-0.130015449367\cdots$	$-0.368861047174\cdots$
$(1.5, 3.5, 3, 2.5, 1)$	$-0.128005225452\cdots$	$-0.389669900863\cdots$

Table 1

Without Assumption 3.1 the optimal level of the normalized macroeconomic activity $u(t)$ could always be equal to 0 due to $-\frac{1}{\beta} \geq \lambda(t)$ for each $t \geq 0$. In order to obtain a nontrivial optimal solution and a nontrivial non-autonomous limit cycle we explicitly assume Assumption 3.1 in the rest of the paper, and we shall construct a series of arguments in order to exclude such a trivial solution as $u(t) = 0$ for all $t \geq 0$ as well as such a trivial $\omega-$limit set as a set composed of a single point. If Assumption 3.1 is satisfied, but if Assumption 3.2 is not satisfied, then $u(t) > 0$ for some $t \geq 0$, but $u(t) = 0$ for other $t \geq 0$. In contrast if Assumption 3.2 is satisfied, then $u(t) > 0$ for each $t \geq 0$. In order to avoid a corner solution we need Assumption 3.2. As mentioned above we always assume Assumption 3.1 in the rest of paper, and we shall argue at the ends of Section 3.3 and Section 4 that if we strengthen Assumption 3.1 with Assumption 3.2, we can exclude a corner solution. As mentioned above Assumption 3.2 is a more natural assumption than Assumption 3.1 from the view point of economics. However we shall pay due attentions to Assumption 3.1 in order to clarify the logical relationship of a hypothesis to an obtained solution in the intertemporal optimization problem (P1).

3.3. **A unique nontrivial optimal solution.** We have sufficient preparations for showing that the unique optimal solution in Proposition 1 is a nontrivial solution under Assumption 3.1.

Proposition 2. Assume that Assumption 3.1 is satisfied. For some $t_0 \in [0, T)$ and for each $n \in \mathbb{N}$

$$u(t_0) = u(t_0 + T) = \cdots = u(t_0 + nT) = \cdots > 0 \tag{3.2}$$

so that the unique optimal solution of the problem (P1) is a nontrivial solution.

Proof of Proposition 2. Let $t_0 \in [0,T)$ be a solution of the following equation.

$$\lambda(t_0) = \lambda_{\max}(r, \bar{\delta}, a, T, \theta), \tag{3.3}$$

where $\lambda_{\max}(r, \bar{\delta}, a, T, \theta)$ is the constant given by the definition (A). Recall $\lambda(T) = \lambda(0)$ by construction. Therefore by the definition (A) the equation (3.3) has a solution t_0 in $[0,T)$. And by Assumption 3.1 we have

$$\lambda(t_0) > -\frac{1}{\beta}. \tag{3.4}$$

And we have $\lambda(t_0) = \lambda(t_0 + nT)$ for each $n \in \mathbb{N}$. Thus we obtain the relation (3.2) from the relations (2.8) and (3.4). Therefore there exists a unique nontrivial optimal solution to the problem (P1) under Assumption 3.1. □

Proposition 2 is due to the present study. Under Assumption 3.1, i.e., $\lambda_{\max}(r, \bar{\delta}, a, T, \theta) > -\frac{1}{\beta}$ alone we might have $u(t) = \psi(\lambda(t)) = 0$ for some $t \in \mathbb{R}_+$. If we strengthen this assumption by Assumption 3.2, i.e., $\lambda_{\min}(r, \bar{\delta}, a, T, \theta) > -\frac{1}{\beta}$, we should have $u(t) = \psi(\lambda(t)) > 0$ for all $t \in \mathbb{R}_+$ by construction.

4. The Dynamics of The State Variable On The Unique Optimal Path

In Section 3.1 we have obtained the relation (3.1) for $\lambda = \lambda(t)$, $t \geq 0$. And we also have

$$\lambda(t) = \lambda(t+T).$$

In the present section we assume Assumption 3.1, and characterize the state variable $x^*(t) = x_t^*$ on the unique optimal path of the CO_2 emissions control problem (P1). We shall show that the state variable converges to a non-autonomous limit cycle globally on the optimal path of this problem (P1) and that the unique optimal path exhibits the turnpike property such that the non-autonomous limit cycle constitutes the turnpike of the problem (P1).

Under this assumption by Proposition 2 we have for some $t_0 \in [0,T)$ and for each $n \in \mathbb{N}$

$$u(t_0) = u(t_0+T) = \cdots = u(t_0+nT) = \cdots > 0.$$

Let $\psi : \mathbb{R} \to [0,1]$ be the function given by the definition (2.4). Then the dynamics of the state variable $x^*(t) = x_t^*$ on the unique optimal path is given by the following *non-autonomous* ordinary differential equation together with the initial condition.

$$\dot{x}_t^* = \beta\psi(\lambda(t)) - \delta(t)x_t^* \wedge x^*(0) = x_0 \in \mathbb{R}_+, \tag{4.1}$$

where $\lambda = \lambda(t)$, $t \geq 0$, is given by the relation (3.1) with $\beta \in \mathbb{R}_{++}$, and where

$$\delta(t) = \bar{\delta} + a\cos(\frac{2\pi}{T}(t-\theta))$$

with $\bar{\delta}, a, T \in \mathbb{R}_{++} \wedge 0 \leq \theta < T$ and with

$$\bar{\delta} - a > 0,$$

which implies that

$$\delta(t) \geq \bar{\delta} - a > 0.$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as

$$f(t) := \exp(-\int_0^t \delta(s)ds)\int_0^t \beta\psi(\lambda(\tau))\exp(\int_0^\tau \delta(s)ds)d\tau.$$

The function $f = f(t)$ satisfies

$$\frac{d}{dt}f(t) = \beta\psi(\lambda(t)) - \delta(t)f(t) \wedge f(0) = 0.$$

Let $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be defined as the solution of the initial value problem of the non-autonomous ordinary differential equation (4.1). Then by construction we have

$$\phi(t,x_0) = x_0 \exp(-\int_0^t \delta(s)ds) + f(t) \wedge \phi(0,x_0) = x_0.$$

And we have the following result.

Theorem 1. Assume that Assumption 3.1 is satisfied so that the CO_2 emissions control problem (P1) has a unique nontrivial optimal solution.

(1) There exists a unique point x_{eq} in $\mathbb{R}_+$ such that

$$\phi(T,x_{eq}) = x_{eq} \wedge \exists t' \in (0,T) : \phi(t',x_{eq}) \neq x_{eq}.$$

(2) For each $n \in \mathbb{N}$, for each $t \in [0,T)$, and for each $x_0 \in \mathbb{R}_+$

$$\phi(nT+t,x_{eq}) = \phi(t,x_{eq}) \wedge \lim_{n\to\infty} \phi(nT+t,x_0) = \phi(t,x_{eq}).$$

In other words there exists a unique non-autonomous limit cycle to which the state variable globally converges on the unique optimal path of the CO_2 emissions control problem (P1), and the unique optimal path exhibits the turnpike property in the sense that the optimal programs from different initial states converge to one another, where the unique non-autonomous limit cycle constitutes the turnpike of the problem (P1).

Proof of Theorem 1. (1) Note that $\delta(t) \geq \bar{\delta} - a > 0$, and we obtain the following bound for any two initial conditions x_0' and x_0'' in $\mathbb{R}_+$.

$$|\phi(t,x_0') - \phi(t,x_0'')| = |x_0' - x_0''| \exp(-\int_0^t \delta(s)ds) \leq |x_0' - x_0''| \exp(-[\bar{\delta} - a]t).$$

Note that $\mathbb{R}_+$ is a complete metric space with the metric $d(x',x'') := |x' - x''|$ for each x' and x'' in $\mathbb{R}_+$. Let $S : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping defined as$S(x) := \phi(T,x)$. Then, for each x' and x'' in $\mathbb{R}_+$,

$$|S(x') - S(x'')| \leq |x' - x''| \exp(-[\bar{\delta} - a]T).$$

Since $0 < \exp(-[\bar{\delta} - a]T) < 1$, the mapping $S : \mathbb{R}_+ \to \mathbb{R}_+$ is a contraction mapping with $0 < \exp(-[\bar{\delta} - a]T) < 1$. Therefore by the contraction mapping fixed point theorem the mapping $S(\cdot)$ has a unique fixed point in $\mathbb{R}_+$. See Theorem 8.1 in Kolmogorov and Fomin (1970) for the contraction mapping fixed point theorem. We shall denote this unique fixed point of $S(\cdot)$ by $x_{eq} \in \mathbb{R}_+$. Then by construction we have $\phi(T,x_{eq}) = x_{eq}$. By Assumption 3.1 $\psi(\lambda(t))$ exhibits oscillations with the period $T > 0$, and the solution of the following initial value problem of the non-autonomous differential equation

$$\frac{\partial}{\partial t}\phi(t,x_{eq}) = \beta\psi(\lambda(t)) - \delta(t)\phi(t,x_{eq}) \wedge \phi(0,x_{eq}) = x_{eq} \in \mathbb{R}_+,$$

i.e., $\phi = \phi(t, x_{eq})$ could not be constant on the interval $[0, T)$. This is because if $\phi = \phi(t, x_{eq})$ could be constant and so equal to x_{eq} on the interval $[0, T)$, then we should have for all $t \in [0, T)$

$$\frac{\psi(\lambda(t))}{\delta(t)} = \frac{x_{eq}}{\beta}.$$

And also because the following claim holds. Therefore we should have

$$\exists t' \in (0, T) : \phi(t', x_{eq}) \neq x_{eq}.$$

Claim. $\frac{\psi(\lambda(t))}{\delta(t)}$ could not be constant on the interval $[0, T)$.

Proof of Claim.

(1) Suppose that $\lambda_{\max}(r, \bar{\delta}, a, T, \theta) > -\frac{1}{\beta} \geq \lambda_{\min}(r, \bar{\delta}, a, T, \theta)$. In this case $\psi(\lambda(t)) > 0$ for some $t \in [0, T)$, and $\psi(\lambda(t)) = 0$ for other $t \in [0, T)$. On the other hand $\delta(t) > 0$ for all $t \in [0, T)$. Therefore $\frac{\psi(\lambda(t))}{\delta(t)}$ could not be constant on $t \in [0, T)$.

(2) Suppose that $\lambda_{\min}(r, \bar{\delta}, a, T, \theta) > -\frac{1}{\beta}$. In this case $\psi(\lambda(t)) = 1 + \beta\lambda(t) > 0$, where $\lambda(t)$ is the right hand side of the relation (3.1). Suppose that $\frac{1+\beta\lambda(t)}{\delta(t)}$ is constant for each $t \in [0, T)$. Then we should have for each $t \in [0, T)$

$$\frac{\beta \frac{d}{dt}\lambda(t)}{1 + \beta\lambda(t)} = \frac{\frac{d}{dt}\delta(t)}{\delta(t)},$$

which implies that for each $t \in [0, T)$

$$\frac{\beta + \beta[r + \delta(t)]\lambda(t)}{1 + \beta\lambda(t)} = \frac{-\frac{2\pi a}{T}\sin(\frac{2\pi}{T}(t - \theta))}{\bar{\delta} + a\cos(\frac{2\pi}{T}(t - \theta))},$$

which in turn implies that for each $t \in [0, T), \lambda(t) =$

$$\frac{-\frac{2\pi a}{T}\sin(\frac{2\pi}{T}(t-\theta)) - \beta[\bar{\delta} + a\cos(\frac{2\pi}{T}(t-\theta))]}{\beta[\bar{\delta} + a\cos(\frac{2\pi}{T}(t-\theta))][r + \bar{\delta} + a\cos(\frac{2\pi}{T}(t-\theta))] + \beta\frac{2\pi a}{T}\sin(\frac{2\pi}{T}(t-\theta))}.$$

This relation contradicts with the relation (3.1). Therefore we have for some $t' \in [0, T)$

$$\frac{\beta \frac{d}{dt}\lambda(t')}{1 + \beta\lambda(t')} \neq \frac{\frac{d}{dt}\delta(t')}{\delta(t')},$$

which implies that $\frac{1+\beta\lambda(t)}{\delta(t)}$ could not be constant on the interval $[0, T)$. □

Note that if Assumption 3.1, i.e., $\lambda_{\max}(r, \bar{\delta}, a, T, \theta) > -\frac{1}{\beta}$ would not be satisfied, we should have $\psi(\lambda(t)) = 0$ for all $t \in \mathbb{R}_+$, which implies both $x_{eq} = 0$ and $\phi(t, 0) = 0$ for all $t \in (0, T)$. In other words Assumption 3.1 guaranties $\exists t' \in (0, T) : \phi(t', x_{eq}) \neq x_{eq}$ and the existence of a nontrivial cyclical solution.

(2) Let $L(t)$ be a set defined as for each $t \in [0, T)$

$$L(t) := \{\phi(t, x) \in \mathbb{R}_+ : x \in \mathbb{R}_+ \wedge \phi(t, x) := x\exp(-\int_0^t \delta(s)ds) + f(t)\}.$$

By construction we have for each $x', x'' \in \mathbb{R}_+$ and for each $t \in [0,T)$

$$|\phi(t,x') - \phi(t,x'')| = |x'-x''| \exp(-\int_0^t \delta(s)ds) \le |x'-x''| \exp(-[\bar{\delta}-a]t).$$

Let $\Phi(n,t) : L(t) \to \mathbb{R}_+$ be a mapping defined as for each $x \in \mathbb{R}_+$, for each $n \in \mathbb{N}$, and for each $t \in [0,T)$

$$\Phi(\phi(t,x) : n,t) := \phi(nT+t,x).$$

Then by $\lambda(t+T) = \lambda(t) \wedge \delta(t+T) = \delta(t)$ for each $t \in \mathbb{R}_+$, by the relation (4.1), and by $\phi(T,x_{eq}) = x_{eq}$ we have for each $n \in \mathbb{N}$ and for each $t \in [0,T)$

$$\Phi(\phi(t,x_{eq}) : n,t) = \phi(nT+t,x_{eq}) = \phi(t,x_{eq}),$$

and we also have for each $x', x'' \in \mathbb{R}_+$, for each $n \in \mathbb{N}$, and for each $t \in [0,T)$

$$\begin{aligned} &\left|\Phi(\phi(t,x') : n,t) - \Phi(\phi(t,x'') : n,t)\right| \\ &= |x'-x''| \exp(-\int_0^{nT+t} \delta(s)ds) \le |x'-x''| \exp(-[\bar{\delta}-a] \times [nT+t]). \end{aligned}$$

Therefore we have for each $x_0 \in \mathbb{R}_+$, for each $n \in \mathbb{N}$, and for each $t \in [0,T)$

$$\begin{aligned} \left|\phi(nT+t,x_0) - \phi(t,x_{eq})\right| &= \left|\Phi(\phi(t,x_0) : n,t) - \Phi(\phi(t,x_{eq}) : n,t)\right| \\ &\le \left|x_0 - x_{eq}\right| \exp(-[\bar{\delta}-a] \times [nT+t]). \end{aligned}$$

And since $0 < \exp(-[\bar{\delta}-a]) < 1$, we have for each $n \in \mathbb{N}$, for each $t \in [0,T)$, and for each $x_0 \in \mathbb{R}_+$

$$\phi(nT+t,x_{eq}) = \phi(t,x_{eq}) \wedge \lim_{n\to\infty} \phi(nT+t,x_0) = \phi(t,x_{eq}). \qquad \square$$

The above proof that $\phi(T,x_{eq}) = x_{eq} \wedge \lim_{n\to\infty} \phi(nT,x_0) = x_{eq}$ for $n \in \mathbb{N}$ and for $x_0 \in \mathbb{R}_+$ is essentially the same as the proof given by Appendix 2 of Gromov et al. [7], although our fixed point argument is more streamlined than that of Gromov et al. [7]. Further arguments given above are due to the present study. As mentioned immediately after the proof of Claim Assumption 3.1 guaranties $\exists t' \in (0,T) : \phi(t',x_{eq}) \ne x_{eq}$ and the existence of a nontrivial non-autonomous limit cycle. And the unique optimal path exhibits the turnpike property in the sense that the optimal programs from different initial states converge to one another, where the mapping $\mathbb{R}_+ \ni t \to \phi(t,x_{eq}) \in \mathbb{R}_+$ constitutes the turnpike of the intertemporal optimization problem (P1). Under Assumption 3.1 alone we might have $\psi(\lambda(t)) = 0$ for some $t \in \mathbb{R}_+$. If we strengthen this assumption by Assumption 3.2, i.e., $\lambda_{\min}(r,\bar{\delta},a,T,\theta) > -\frac{1}{\beta}$, we should have $\psi(\lambda(t)) > 0$ for all $t \in \mathbb{R}_+$ by construction.

5. Conclusion

We considered the intertemporal optimization problem (P1) that is intended to constitute a CO_2 emissions control problem with smooth seasonal fluctuations of the CO_2 reduction rate $\delta(t)$ due to photosynthesis, where u, $u(1-\frac{1}{2}u)$, and x are *proportional to* the macroeconomic activity, the GDP, and the level of cumulative CO_2 on some country that belongs to the temperate zone on the earth, respectively. See the definition (2.2). The current value Hamiltonian

$$H(x,u,\lambda : \delta(t)) = u\left(1-\frac{1}{2}u\right) - x + \lambda(\beta u - \delta(t)x)$$

is linear in the state variable x, and thus the dynamics of its co-state variable λ is de-coupled from that of the state variable x. First we have prescribed a unique optimal solution to the problem (P1) by appealing to the maximum principle, and next we have characterized the dynamics of co-state variable and have shown that we can not exclude a trivial optimal solution without some additional inequality constraint. Thus we have also introduced the inequality constraint stipulated in Assumption 3.1 in order to avoid such a trivial solution as $u(t) = 0$ for all $t \in \mathbb{R}_+$ as well as such a trivial $\omega-$limit set as a set composed of a single point. Next we have shown that under Assumption 3.1 there exists a unique *nontrivial* optimal solution to the CO_2 emissions control problem (P1). Finally we have shown that under Assumption 3.1 there exists a unique non-autonomous limit cycle to which the state variable globally converges on the unique optimal path of the CO_2 emissions control problem (P1), where the unique optimal path exhibits the turnpike property in the sense that the optimal programs from different initial states converge to one another. The unique non-autonomous limit cycle constitutes the turnpike of the intertemporal optimization problem (P1).

Under Assumption 3.1 alone aggregate variables such as $u(t)$ and $u(t)(1 - \frac{1}{2}u(t))$ could be equal to 0 for some $t \in \mathbb{R}_+$. On the other hand under Assumption 3.2 these aggregate variables are strictly positive for all $t \in \mathbb{R}_+$. Therefore from the view point of economics Assumption 3.2 is a more natural assumption to make than Assumption 3.1. In spite of the fact that Assumption 3.1 is slightly unrealistic as an assumption made for such a macroeconomic model as the CO_2 emissions control problem (P1), we have paid due attentions to this assumption until now in order to clarify the logical relationship of a hypothesis to an obtained solution in the intertemporal optimization problem (P1).

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