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SET-VALUED PEROV CONTRACTIVE MAPPINGS

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Dedicated to the memory of Professor Gene H. Golub on occasion of his 90th birthday

Abstract. In this work, we consider set-valued Perov contractive mappings acting in a generalized metric space. Our first result shows that a certain iterative process generates approximate fixed points. In our second result, we prove that an iterative process generates iterates which converge to a fixed point of the mapping.
Keywords. Convergence analysis; Fixed point; Iterate; Perov contraction.
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1. Introduction

For more than sixty years now, there have been considerable research activities regarding the fixed point theory of certain classes of nonlinear mappings; see, e.g, [3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18, 19, 20, 25, 26] and the references therein. These activities mainly stem from Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, for instance, studies of convex feasibility, common fixed point problems, and variational inequalities, which find significant applications in engineering, medical and the natural sciences [2, 5, 21, 22, 23, 25, 26].

The study of a class of mappings of Perov type acting in a generalized metric space is an important topic in the fixed point theory [15, 16, 17, 24, 27, 28]. In this work we consider set-valued Perov contractive mappings acting in a generalized metric space. Our first result shows that a certain iterative process generates approximate fixed points. In our second result we prove that an iterative process generates iterates which converge to a fixed point of the mapping.

We use the following notation. Let R^n be an n-dimensional Euclidean space. In other words,

$$R^n = \{x = (x_1, \dots, x_n) : \ x_i \in R^1, \ i = 1, \dots, n\}.$$

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Let

$$R^n_+ = \{x = (x_1, \dots, x_n) \in R^n : \; x_i \geq 0, \; i = 1, \dots, n\}$$

and $e = (1, 1, \dots, 1) \in R^n$. We say that $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in R^n$ satisfy $x \leq y$ if $x_i \leq y_i$ for all $i = 1, \dots, n$. For each $x = (x_1, \dots, x_n) \in R^n$ set $\|x\|_1 = \sum_{i=1}^n |x_i|$. If $\{u_i\}_{i=1}^\infty \subset R^n_+$ and $\sum_{i=1}^\infty \|u_i\|_1 < \infty$, then

$$\sum_{i=1}^\infty u_i = \lim_{k \to \infty} \sum_{i=1}^k u_i.$$

Let Y be a nonempty set and $S : Y \to Y$. We denote by S^0 the identity mapping in Y, set $S^1 = S$ and for every integer $i \geq 0$ define

$$S^{i+1} = S \circ S^i.$$

We suppose that the sum over an empty set is zero.

2. PRELIMINARIES AND THE FIRST MAIN RESULTS

Assume that X is a nonempty set and a function $d : X \times X \to R^n_+$ satisfies for each $x, y, z \in X$,

$$d(x,y) = 0 \text{ if and only if } x = y,$$

$$d(x,y) = d(y,x),$$

$$d(x,z) \leq d(x,y) + d(y,z).$$

The pair (X, d) is called a generalized metric space and d is called a generalized metric [15, 16, 17, 24].

For all $x, y \in X$ set

$$d(x,y) = (d_1(x,y), \dots, d_n(x,y))$$

and

$$\tilde{d}(x,y) = \|d(x,y)\|_1 = d_1(x,y) + \cdots + d_n(x,y).$$

Clearly, $(X, \tilde{d})$ is a metric space.

For each $x \in X$ and each $r \in R^n_+$ set

$$B(x,r) = \{y \in X : \; d(x,y) \leq r\}.$$

We say that a sequence $\{x_i\}_{i=0}^\infty \subset X$ converges to $x_* \in X$ if

$$\lim_{i \to \infty} \|d(x_i, x_*)\|_1 = 0.$$

We say that $\{x_i\}_{i=0}^\infty \subset X$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that, for each pair of integers $p, m \geq n(\varepsilon)$, $\|d(x_p, x_m)\|_1 \leq \varepsilon$. Clearly, the convergence in (X, d) is equivalent to the convergence in $(X, \tilde{d})$. We assume that the metric space $(X, \tilde{d})$ is complete.

Assume that $A : R^n_+ \to R^n_+$ and that the following properties hold:

(i) $A(0) = 0$ and A is continuous at zero;

(ii) for each $z_1, z_2 \in R^n_+$ satisfying $0 \leq z_1 \leq z_2$, $A(z_1) \leq A(z_2)$;

(iii) for each $z_1, z_2 \in R^n_+$, $A(z_1 + z_2) \leq A(z_1 + z_2)$;

(iv) $A^t(e) \to 0$ as $t \to \infty$;

(v) $A(\lambda z) = \lambda A(z)$ for each $\lambda \geq 0$ and each $z \in R^n_+$.

Assume that $T : X \to 2^X \setminus \{\emptyset\}$ satisfies the following assumption.

(B1) For each $x,y \in X$, each $u \in T(x)$ and each $\varepsilon > 0$ there exists $v \in T(y)$ such that

$$d(u,v) \le A(d(x,y)) + \varepsilon e.$$

Clearly, T is a set-valued analog of a single-valued Perov contraction.

The following theorem is our first main result.

Theorem 2.1. *Let $M > 0$, $\varepsilon \in (0,M)$, $m_0 \ge 2$ be an integer,*

$$\|A^k(Me)\|_1 \le \varepsilon/2 \tag{1}$$

for each integer $k \ge m_0$, $\delta \in (0,\varepsilon)$ and

$$\delta \sum_{i=0}^{2m_0-1} A^i(e) \le 2^{-1}\varepsilon e. \tag{2}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$,

$$d(x_0,x_1) \le Me \tag{3}$$

and that for each integer $i \ge 0$,

$$x_{i+1} \in T(x_i) \tag{4}$$

and

$$d(x_{i+1},x_{i+2}) \le A(d(x_i,x_{i+1})) + \delta e. \tag{5}$$

Then, for each integer $n \ge m_0$, $d(x_n,x_{n+1}) \le \varepsilon e$.

3. An auxiliary result for Theorem 2.1

Lemma 3.1. *Let $M > \varepsilon > 0$, $m_0 \ge 2$ be an integer, (1) hold for each integer $k \ge m_0$, $\delta \in (0,\varepsilon)$ satisfy (2), $\{x_i\}_{i=0}^{2m_0} \subset X$,*

$$d(x_0,x_1) \le Me, \tag{6}$$

for each integer $i \in \{0,\dots,2m_0-1\}$,

$$x_{i+1} \in T(x_i) \tag{7}$$

and for each $i \in \{0,\dots,2m_0-2\}$,

$$d(x_{i+1},x_{i+2}) \le A(d(x_i,x_{i+1})) + \delta e. \tag{8}$$

Then, for each integer $k \in \{m_0,\dots,2m_0-1\}$, $d(x_k,x_{k+1}) \le \varepsilon e$.

Proof. We show by induction that for each $k \in \{1,\dots,2m_0-1\}$,

$$d(x_k,x_{k+1}) \le A^k(d(x_0,x_1)) + \delta \sum_{i=0}^{k-1} A^i(e). \tag{9}$$

In view of (8), equation (9) holds for $k = 1$. Assume that $k \in \{0,\dots,2m_0-2\}$ and (9) holds. It follows from (8), (9) and properties (ii), (iii) and (v),

$$\begin{aligned} d(x_{k+1},x_{k+2}) &\le A(d(x_k,x_{k+1})) + \delta e \\ &\le A^{k+1}(d(x_0,x_1)) + \delta \sum_{i=0}^{k-1} A^{i+1}(e) + \delta e \\ &= A^{k+1}(d(x_0,x_1)) + \delta \sum_{i=0}^{k} A^{i+1}(e) \end{aligned}$$

and our assumption holds for $k+1$ too. Thus by induction we showed that (9) holds for each $k \in \{1, \dots, 2m_0 - 1\}$. By (1), (2), (6) and (9), for each $k \in \{m_0, \dots, 2m_0 - 1\}$,

$$d(x_k, x_{k+1}) \le A^k(d(x_0, x_1)) + \delta \sum_{i=0}^{2m_0-1} A^i(e)$$

$$\le 2^{-1}\varepsilon e + \delta \sum_{i=0}^{2m_0-1} A^i(e) \le \varepsilon e.$$

Lemma 3.1 is proved. □

4. Proof of Theorem 2.1

Lemma 3.1 implies that

$$d(x_k, x_{k+1}) \le \varepsilon \tag{10}$$

for all $k = m_0, \dots, 2m_0 - 1$. Assume that $p \ge m_0$ is an integer and (10) holds for all $k = p, \dots, p + m_0 - 1$. (Note that in view of (10) our assumption holds for $p = m_0$.) We apply Lemma 3.1 to $\{x_i\}_{i=p}^{p+2m_0}$ and obtain that

$$d(x_k, x_{k+1}) \le \varepsilon e$$

for each $k \in \{p, \dots, p + m_0 - 1\}$. Thus if $p \ge m_0$ is an integer and (10) holds for all $k = p, \dots, p+m_0-1$, then (10) holds for all $k = p, \dots, p+2m_0-1$. This implies that $d(x_k, x_{k+1}) \le \varepsilon$ for all integers $k \ge m_0$. Theorem 2.1 is proved.

5. The second main result

By property (iv), there exists $\Delta_0 > 0$ such that

$$A^i(e) \le \Delta_0 e, \ i = 0, 1, \dots. \tag{11}$$

Assume that $T : X \to 2^X \setminus \{\emptyset\}$, $x_* \in X$ satisfies

$$x_* \in T(x_*) \tag{12}$$

and that the following property holds:

(B2) For each $x \in X$ and each $\varepsilon > 0$ there exists $y \in T(x)$ such that

$$d(y, x_*) \le A(d(x, x_*)) + \varepsilon e.$$

The following theorem is our second main result.

Theorem 5.1. *Let $M > 0$. Then for each $x_0 \in X$ satisfying*

$$d(x_0, x_*) \le Me \tag{13}$$

there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ such that

$$x_{i+1} \in T(x_i), \ i = 0, 1, \dots, \tag{14}$$

$$\lim_{i \to \infty} x_i = x_* \tag{15}$$

in (X, d) and that for each $\varepsilon > 0$ there exist an integer $n(\varepsilon) \ge 1$ depending only on ε such that

$$\tilde{d}(x, x_*) \le \varepsilon \text{ for each integer } i \ge n(\varepsilon).$$

Proof. Let $\{\varepsilon_i\}_{i=0}^{\infty} \subset (0,1)$,

$$\Delta_0 \sum_{i=0}^{\infty} \varepsilon_i \leq 2^{-1}M,\ \varepsilon_{i+1} \leq \varepsilon_i,\ i = 0,1,\ldots, \tag{16}$$

$\varepsilon \in (0,1)$, $m_1 \geq 1$ be an integer,

$$MA^t(e) \leq 2^{-1}e \text{ for each integer } t \geq m_1, \tag{17}$$

$$\Delta_0 \sum_{i=m_1} \varepsilon_i < \varepsilon/2. \tag{18}$$

Assume that $x_0 \in X$ satisfies (13) and that $\{x_i\}_{i=1}^{\infty} \subset X$ satisfies for each integer $i \geq 0$,

$$x_{i+1} \in T(x_i) \tag{19}$$

and

$$d(x_*, x_{i+1}) \leq A(d(x_i, x_*)) + \varepsilon_i e. \tag{20}$$

(Note that this sequence does not depend on ε.)

Let $k \geq 0$ be an integer. We show by induction that for each integer $p \geq 1$,

$$d(x_{k+p}, x_*) \leq A^p(d(x_k, x_*)) + \sum_{i=0}^{p-1} A^i(\varepsilon_{k+p-i-1}e). \tag{21}$$

In view of (20) equation (21) holds for $p = 1$. Assume that $p \geq 1$ is an integer and (21) holds. Properties (ii), (iii), (v) and equations (20), (21) imply that

$$d(x_{k+p+1}, x_*) \leq A^p(d(x_{k+p}, x_*)) + \varepsilon_{k+p}e$$

$$\leq A^{p+1}(d(x_{k+p}, x_*)) + \sum_{i=0}^{p-1} A^{i+1}(\varepsilon_{k+p-i-1}e) + \varepsilon_{k+p}e$$

$$= A^{p+1}(d(x_{k+p}, x_*)) + \sum_{i=0}^{p} A^i(\varepsilon_{k+p-i}e).$$

Hence our assumption holds for $p+1$. Therefore we showed by induction that (21) holds for each integer $p \geq 1$. It follows from property (v), (11) and (21) that for each integer $p \geq 1$,

$$d(x_{k+p}, x_*) \leq A^p(d(x_k, x_*)) + \sum_{i=k}^{k+p-1} \varepsilon_i \Delta_0 e. \tag{22}$$

By (18) and (22), with $k = 0$, for each integer $p \geq 1$,

$$d(x_p, x_*) \leq A^p(d(x_0, x_*)) + \sum_{i=0}^{p-1} \varepsilon_i \Delta_0 e$$

$$\leq A^p(d(x_0, x_*)) + 2^{-1}Me. \tag{23}$$

Properties (ii), (v) and (17), (23) imply that for each integer $p \geq m_1$,

$$d(x_p, x_*) \leq Me. \tag{24}$$

Assume that $p \geq 1$ is an integer. By (16), (22) (with $k = m_1$), (24) and properties (ii) and (iii),

$$d(x_{m_1+p}, x_*) \leq A^p(d(x_{m_1}, x_*)) + \sum_{i=m_1}^{m_1+p-1} \varepsilon_i \Delta_0 e \leq MA^p(e) + 2^{-1}\varepsilon e. \tag{25}$$

By property (v), there exists a natural number p_0, such that, for each integer $i \geq p_0$, $MA^i(e) \leq 8^{-1}\varepsilon e$. (Here p_0 depends on ε.) Together with (25) this implies that for each integer $i \geq p_0$, $d(x_{m_1+i}, x_*) \leq \varepsilon e$. Theorem 5.1 is proved.

□

REFERENCES

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133–181.

[2] Y. Censor, M. Zaknoon, Algorithms and convergence results of projection methods for inconsistent feasibility problems: a review, Pure Appl. Func. Anal. 3 (2018), 565–586.

[3] F. S. de Blasi, J. Myjak, Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach, C. R. Acad. Sci. Paris 283 (1976), 185–187.

[4] F. S. de Blasi, J. Myjak, S. Reich, A. J. Zaslavski, Generic existence and approximation of fixed points for nonexpansive set-valued maps, Set-Valued and Variational Analysis 17 (2009), 97–112.

[5] A. Gibali, A new split inverse problem and an application to least intensity feasible solutions, Pure Appl. Funct. Anal. 2 (2017), 243–258.

[6] K. Goebel, W. A. Kirk, Topics in metric fixed point theory, Cambridge University Press, Cambridge, 1990.

[7] K. Goebel, S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Marcel Dekker, New York and Basel, 1984.

[8] J. Jachymski, Extensions of the Dugundji-Granas and Nadler's theorems on the continuity of fixed points, Pure Appl. Funct. Anal. 2 (2017), 657–666.

[9] M. A. Khamsi, W. M. Kozlowski, Fixed point theory in modular function spaces, Birkhäuser/Springer, Cham, 2015.

[10] W. A. Kirk, Contraction mappings and extensions, In: Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 1–34.

[11] W. M. Kozlowski, An introduction to fixed point theory in modular function spaces, In: Topics in Fixed Point Theory, Springer, Cham, 2014, pp. 159–222.

[12] M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Ya. B. Rutitskii, V. Ya. Stetsenko, Approximate solution of operator equations, Wolters-Noordhoff Publishing, Groningen, 1972.

[13] R. Kubota, W. Takahashi, Y. Takeuchi, Extensions of Browder's demiclosedness principle and Reich's lemma and their applications, Pure Appl. Func. Anal. 1 (2016), 63–84.

[14] Z. D. Mitrović, S. Radenović, Reich, A. J. Zaslavski, Iterating nonlinear contractive mappings in Banach spaces, Carpatian J. Math. 36 (2020), 287–294.

[15] A. I. Perov, A multidimensional version of M. A. Krasnosel'skii's generalized contraction principle, Funktsional. Anal. i Prilozhen. 44 (2010), 83–87.

[16] A. I. Perov, A. V. Kibenko, On a certain general method for investigation of boundary value problems, Izv. Akad. Nauk SSSR 30 (1966), 249–264.

[17] A. Petrusel, G. Petrusel, J. C. Yao, Perov type theorems for orbital contractions, J. Nonlinear Convex Anal. 21 (2020), 759–769.

[18] E. Rakotch, A note on contractive mappings, Proc. Amer. Math. Soc. 13 (1962), 459–465.

[19] S. Reich, A. J. Zaslavski, Generic aspects of metric fixed point theory, In: Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 557–575.

[20] S. Reich, A. J. Zaslavski, Genericity in nonlinear analysis, Developments in Mathematics, 34, Springer, New York, 2014.

[21] X. Qin, S. Y. Cho, J.-C. Yao, Weak and strong convergence of splitting algorithms in Banach spaces, Optimization 69 (2020), 243–267.

[22] W. Takahashi, The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces, Pure Appl. Funct. Anal. 2 (2017), 685–699.

[23] W. Takahashi, H. K. Xu, J.-C. Yao, Iterative methods for generalized split feasibility problems in Hilbert spaces, Set-Valued Var. Anal. 23 (2015), 205–221, (2015).

[24] F. Vetro, S. Radenovic, Some results of Perov type in rectangular cone metric spaces, J. Fixed Point Theory Appl. 20 (2018), 16 pp.
[25] A. J. Zaslavski, Approximate solutions of common fixed point problems, Springer Optimization and Its Applications, Springer, Cham, 2016.
[26] A. J. Zaslavski, Algorithms for solving common fixed point problems, Springer Optimization and Its Applications, Springer, Cham, 2018.
[27] A. J. Zaslavski, Existence and convergence results for contractive mappings of Perov type, J. Nonlinear Convex Anal. 23 (2022), 2677–2685.
[28] A. J. Zaslavski, Existence and convergence results for Perov contraction mappings, Fixed Point Theory, 24 (2023), 775–786.