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FIXED POINT THEORY FOR TWO CLASSES OF NONLINEAR MAPPINGS

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Dedicated to Avi Berman on the occasion of his 80th birthday

Abstract. In this survey paper, we discuss our recent fixed point results obtained for uniformly locally contractive mappings and for nonexpansive mappings in metric spaces with graphs.
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1. Properties of Uniformly Locally Contractive Mappings

The fixed points of nonexpansive mappings have a wide real applications in numerous fields. The fixed point theory of nonexpansive mappings has been a rapidly growing area of research since the seminal result of Banach [1]; see, e.g., [3, 6, 7, 10, 12, 13, 14, 15, 17, 21, 27, 28, 29] and the reference therein. Many developments have taken place in the field of nonexpansive mappings, such as the studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [4, 5, 9, 28, 29]. In this paper, we survey our recent fixed point results which were obtained for two classes of nonlinear mappings in [22, 23, 24, 25, 26, 30]. The first class consists of uniformly locally contractive mappings while the second one is the class of nonexpansive mappings in metric spaces with graphs.

In this section, we begin our discussion of uniformly locally contractive mappings. We show that a uniformly locally contractive mapping has a fixed point, the corresponding fixed point problem is well posed and that inexact iterates of such a mapping converge to its unique fixed point, uniformly on bounded sets. Using the porosity notion, we also show that most uniformly locally nonexpansive mappings are, in fact, uniformly locally contractive. We also extend these results to uniformly locally contractive non-self mappings defined on a closed subset of the metric space. The results of this section were obtained in [23].

Assume that (X,ρ) is a complete metric space, $\Delta > 0$, and that the following assumption holds.

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(A) For each $x,y \in X$, there exist an integer $q \geq 1$ and points $x_i \in X$, $i = 0,\dots,q$, such that

$$x_0 = x,\ x_q = y,\ \rho(x_i,x_{i+1}) \leq \Delta,\ i \in \{0,\dots,q\} \setminus \{q\}.$$

For each $x,y \in X$, define

$$\rho_1(x,y) := \inf\{\sum_{i=0}^{q-1} \rho(x_i,x_{i+1}) :\ q \geq 1 \text{ is an integer},$$

$$x_i \in X,\ i = 0,\dots,q,\ x_0 = x,\ x_q = y,\ \rho(x_i,x_{i+1}) \leq \Delta,\ i \in \{0,\dots,q\} \setminus \{q\}\}.$$

It follows from assumption (A) and the above definition that for each $x,y,z \in X$, $\rho_1(x,y)$ is finite,

$$\rho(x,y) \leq \rho_1(x,y) < \infty,$$

$$\text{if } \rho(x,y) \leq \Delta, \text{ then } \rho_1(x,y) = \rho(x,y),$$

$$\rho_1(x,y) = \rho_1(y,x),$$

$$\rho_1(x,x) = 0,$$

$$\text{if } \rho_1(x,y) = 0, \text{ then } \rho(x,y) = 0 \text{ and } x = y,$$

and

$$\rho_1(x,z) \leq \rho_1(x,y) + \rho_1(y,z).$$

It is clear that ρ_1 is a metric on X. This metric plays an important role in our study.

Let $T : X \to X$ be a self-mapping of X. We assume that for each $x,y \in X$ satisfying $\rho(x,y) \leq \Delta$, the inequality

$$\rho(T(x),T(y)) \leq \rho(x,y)$$

holds. It is not difficult to see that, for each $x,y \in X$, $\rho_1(T(x),T(y)) \leq \rho_1(x,y)$.

Such a mapping T is said to be uniformly locally nonexpansive. We remark in passing that the smaller class of uniformly local (strict) contractions was introduced in [8], while the larger class of locally nonexpansive mappings was studied in [2]. The work [8] also contains an example of a uniformly local (strict) contraction which is not nonexpansive. Assume that $\phi : [0,\Delta] \to [0,1]$ is a decreasing function, $\phi(t) < 1$ for all $t \in (0,\Delta]$, and for each $x,y \in X$ satisfying $\rho(x,y) \leq \Delta$, we have

$$\rho(T(x),T(y)) \leq \phi(\rho(x,y))\rho(x,y).$$

The following theorems were obtained in [23]. They are established under all the assumptions made above. The first of them shows the well-posedness of the fixed point problem for the mapping T.

Theorem 1.1. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $x,y \in X$ satisfying*

$$\max\{\rho(T(x),x),\ \rho(T(y),y)\} \leq \delta,$$

the inequality $\rho(x,y) \leq \varepsilon$ holds.

Theorem 1.2. *Let $\varepsilon, M > 0$. Then there exist a number $\delta \in (0,\varepsilon)$ and an integer $n_0 \geq 1$ such that for each integer $n > n_0$ and each finite sequence $\{x_i\}_{i=0}^{n} \subset X$ which satisfies $\rho_1(x_0,x_1) \leq M$ and*

$$\rho(x_{i+1},T(x_i)) \leq \delta,\ i = 0,\dots,n-1,$$

the inequality $\rho(x_{i+1},x_i) \leq \varepsilon$ holds for all $i = n_0,\dots,n-1$.

Theorems 1.1 and 1.2 imply that there exists a unique $x_* \in K$ such that $T(x_*) = x_*$. Note that the existence of the unique fixed point was obtained in [16] (see also [18]). Theorems 1.1 and 1.2 easily imply the following convergence result for inexact iterates of the mapping T.

Theorem 1.3. *Let the point $x_* \in X$ satisfy $T(x_*) = x_*$ and let $\varepsilon, M > 0$. Then there exist a number $\delta > 0$ and an integer $n_0 \geq 1$ such that for each integer $n > n_0$ and each finite sequence $\{x_i\}_{i=0}^{n} \subset X$ which satisfies $\rho_1(x_0, x_1) \leq M$ and*

$$\rho(x_{i+1}, T(x_i)) \leq \delta, \ i = 0, \dots, n-1,$$

the inequality $\rho(x_i, x_*) \leq \varepsilon$ holds for all $i = n_0, \dots, n-1$.

2. The first porosity result

The results of this section were obtained in [23]. Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. Recall that a subset $E \subset Y$ is called porous (with respect to the metric d) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called σ-porous (with respect to d) if it is a countable union of porous subsets of Y.

Using the porosity notion, we now show that most uniformly locally nonexpansive mappings are, in fact, uniformly locally contractive. For mappings which are nonexpansive and contractive, respectively, such a result was obtained in [20].

We continue to use the definitions, notations and assumptions introduced in Section 1. Assume that

$$\operatorname{diam}(X) := \sup\{\rho_1(x, y) : \ x, y \in X\} < \infty.$$

We call a mapping $T : X \to X$ Δ-contractive if there exists a decreasing function

$$\phi : [0, \Delta] \to [0, 1]$$

such that $\phi(t) < 1$, $t \in (0, \Delta]$, and for each $x, y \in X$ satisfying $\rho(x, y) \leq \Delta$,

$$\rho(T(x), T(y)) \leq \phi(\rho(x, y))\rho(x, y).$$

Denote by $\mathscr{F}$ the set of all Δ-contractive self-mappings of X.

Also, denote by $\mathfrak{M}$ the set of all mappings $T : X \to X$ such that for each $x, y \in X$ satisfying $\rho(x, y) \leq \Delta$, $\rho(T(x), T(y)) \leq \rho(x, y)$. Assume that $\theta \in X$ and that for each $\gamma \in (0, 1)$ and each $x \in X$, there is a point $(1-\gamma)x \oplus \gamma\theta \in X$ such that for each $x, y \in X$ and each $\gamma \in (0, 1)$,

$$\rho((1-\gamma)x \oplus \gamma\theta, \ (1-\gamma)y \oplus \gamma\theta) \leq (1-\gamma)\rho(x, y)$$

and

$$\rho((1-\gamma)x \oplus \gamma\theta, x) \leq \gamma\rho(x, \theta).$$

Note that this assumption indeed holds if X is a convex subset of a hyperbolic space in the sense of [10, 21].

For each $S, T \in \mathfrak{M}$, set

$$d(S, T) := \sup\{\rho(S(x), T(x)) : \ x \in X\}.$$

It is not difficult to see that $(\mathfrak{M},d)$ is a complete metric space. The following result was established in [23] under all the assumptions made in this section.

Theorem 2.1. *The set* $\mathfrak{M}\setminus\mathscr{F}$ *is* σ*-porous in* $(\mathfrak{M},d)$.

3. Uniformly locally contractive non-self mappings

Now we extend the results of the previous sections to uniformly locally contractive non-self mappings defined on a closed subset of the metric space. These extensions were obtained in [24].

We use the notations, definitions and assumptions of Section 1. In particular, we assume that assumption (A), which we recall below for the reader's convenience, holds.

Assume that (X,ρ) is a complete metric space, $\Delta>0$, and that the following assumption holds.

(A) For each $x,y\in X$, there exist an integer $q\geq 1$ and points $x_i\in X$, $i=0,\dots,q$, such that

$$x_0=x,\ x_q=y,\ \rho(x_i,x_{i+1})\leq\Delta,\ i\in\{0,\dots,q\}\setminus\{q\}.$$

Next, we also recall that for each $x,y\in X$,

$$\rho_1(x,y):=\inf\{\sum_{i=0}^{q-1}\rho(x_i,x_{i+1}):\ q\geq 1 \text{ is an integer},$$

$$x_i\in X,\ i=0,\dots,q,\ x_0=x,\ x_q=y,\ \rho(x_i,x_{i+1})\leq\Delta,\ i\in\{0,\dots,q\}\setminus\{q\}\}.$$

Clearly, ρ_1 is a metric on X.

Assume that $K\subset X$ is a nonempty closed set and that the following assumption holds.

(B) For each $x,y\in K$,

$$\rho_1(x,y):=\inf\{\sum_{i=0}^{q-1}\rho(x_i,x_{i+1}):\ q\geq 1 \text{ is an integer},$$

$$x_i\in K,\ i=0,\dots,q,\ x_0=x,\ x_q=y,\ \rho(x_i,x_{i+1})\leq\Delta,\ i\in\{0,\dots,q\}\setminus\{q\}\}.$$

Let $T:K\to X$ be a mapping. We assume that for each $x,y\in X$ satisfying $\rho(x,y)\leq\Delta$, the inequality $\rho(T(x),T(y))\leq\rho(x,y)$ holds. In view of (B) and the above relation, for each $x,y\in X$, we have $\rho_1(T(x),T(y))\leq\rho_1(x,y)$. Such a mapping T is said to be uniformly locally nonexpansive. Assume that $\phi:[0,\Delta]\to[0,1]$ is a decreasing function, $\phi(t)<1$ for all $t\in(0,\Delta]$, and that for each $x,y\in K$ satisfying $\rho(x,y)\leq\Delta$, we have

$$\rho(T(x),T(y))\leq\phi(\rho(x,y))\rho(x,y).$$

The following theorems were established in [24] under all the assumptions made in this section. The first of them shows the well-posedness of the fixed point problem for the mapping T.

Theorem 3.1. *Given* $\varepsilon>0$*, there exists* $\delta>0$ *such that for each* $x,y\in K$ *satisfying*

$$\max\{\rho(T(x),x),\ \rho(T(y),y)\}\leq\delta,$$

the inequality $\rho(x,y)\leq\varepsilon$ *holds.*

Theorem 3.2. *Let $\varepsilon, M > 0$ be given. Then there exist a number $\delta \in (0,\varepsilon)$ and an integer $n_0 \geq 1$ such that for each integer $n > n_0$ and each finite sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies*

$$\rho_1(x_0,x_1) \leq M$$

and

$$\rho(x_{i+1},T(x_i)) \leq \delta,\ i = 0,\dots,n-1,$$

the inequality $\rho(x_i,x_{i+1}) \leq \varepsilon$ holds for all $i = n_0,\dots,n-1$.

Theorems 3.1 and 3.2 easily imply the following two results regarding the existence of a unique fixed point and the convergence of inexact iterates.

Theorem 3.3. *Assume that for each integer $n \geq 1$, there exists a finite sequence $\{x^{(n)}\}_{i=0}^n \subset K$ such that*

$$\sup\{\rho_1(x_0^{(n)},x_1^{(n)}) :\ n = 1,2,\dots\} < \infty$$

and

$$\lim_{n\to\infty} \sup\{\rho(x_{i+1}^{(n)},Tx_i^{(n)}) :\ i = 0,\dots,n-1\} = 0.$$

Then there exists a unique point $x_* \in K$ such that $T(x_*) = x_*$.

Theorem 3.4. *Let the point $x_* \in K$ satisfy $T(x_*) = x_*$ and let $\varepsilon, M > 0$ be given. Then there exist a number $\delta > 0$ and an integer $n_0 \geq 1$ such that for each integer $n > n_0$ and each finite sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies $\rho_1(x_0,x_1) \leq M$ and*

$$\rho(x_{i+1},T(x_i)) \leq \delta,\ i = 0,\dots,n-1,$$

the inequality $\rho(x_i,x_*) \leq \varepsilon$ holds for all $i = n_0,\dots,n$.

4. The second porosity result

The theorem of this section was established in [25]. In our second porosity result we consider a complete metric space of uniformly locally nonexpansive self-mappings of a bounded and closed subset of a complete hyperbolic space.

Let (X,ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of R^1 into X if $\rho(c(s),c(t)) = |s-t|$ for all real s and t. The image of R^1 under a metric embedding is called a metric line. The image of a real interval $[a,b] = \{t \in R^1 : a \leq t \leq b\}$ under such a mapping is called a metric segment.

Assume that (X,ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x,y]$. For each $0 \leq t \leq 1$, there is a unique point z in $[x,y]$ such that

$$\rho(x,z) = t\rho(x,y) \text{ and } \rho(z,y) = (1-t)\rho(x,y).$$

This point will be denoted by $(1-t)x \oplus ty$. We say that X, or more precisely (X,ρ,M), is a hyperbolic space if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \leq \frac{1}{2}\rho(y,z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z) \leq \frac{1}{2}(\rho(x,w) + \rho(y,z))$$

for all x,y,z and w in X. A set $K \subset X$ is called ρ-convex if $[x,y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball can be found, for example, in [10, 19, 21].

Let (X,ρ) be a complete hyperbolic space and let K be a nonempty, closed and bounded subset of (X,ρ). Assume that $\Delta > 0$, $\theta \in K$ and that for each $\gamma \in (0,1)$ and each $x \in K$,

$$\gamma\theta \oplus (1-\gamma)x \in K.$$

Set

$$\mathrm{diam}(K) = \sup\{\rho(x,y) : \ x,y \in K\}.$$

It follows from the boundedness of K in (X,ρ) that (A) holds for $X = K$ and that

$$\sup\{\rho_1(x,y) : \ x,y \in K\} < \infty.$$

Denote by $\mathscr{A}$ the set of all mappings $T : K \to X$ such that

$$\rho(T(x),T(y)) \le \rho(x,y) \text{ for each } x,y \in K \text{ satisfying } \rho(x,y) \le \Delta.$$

For each $A,B \in \mathscr{A}$, define

$$d(A,B) := \sup\{\rho(A(x),B(x)) : \ x \in K\}.$$

Since K is bounded in (X,ρ_1), $d(A,B)$ is finite for each $A,B \in \mathscr{A}$.

It is not difficult to see that $(\mathscr{A},d)$ is a complete metric space.

Recall that a mapping $T \in \mathscr{A}$ is a uniformly local contraction if there exists a decreasing function $\phi : [0,\Delta] \to [0,1]$ such that $\phi(t) < 1$ for all $t \in (0,\Delta]$, and that for each $x,y \in K$ satisfying $\rho(x,y) \le \Delta$, we have

$$\rho(T(x),T(y)) \le \phi(\rho(x,y))\rho(x,y).$$

Denote by $\mathscr{F}$ the set of all uniformly local contractions $A \in \mathscr{A}$. The following result was established in [25] under all the assumptions made in this section.

Theorem 4.1. *The set $\mathscr{A} \setminus \mathscr{F}$ is σ-porous.*

5. G-NONEXPANSIVE MAPPINGS

In this section we begin our discussion of nonexpansive mappings in spaces with graphs. Let (X,ρ) be a complete metric space and let G be a (directed) graph. Let $V(G)$ be the set of its vertices and let $E(G)$ be the set of its edges. We identify the graph G with the pair $(V(G),E(G))$.

Denote by $\mathscr{M}_{ne}$ the set of all mappings $T : X \to X$ such that for each $x,y \in X$ satisfying $(x,y) \in E(G)$, we have

$$(T(x),T(y)) \in E(G) \text{ and } \rho(T(x),T(y)) \le \rho(x,y).$$

A mapping $T \in \mathscr{M}_{ne}$ is called G-nonexpansive. If $T \in \mathscr{M}_{ne}$, $\alpha \in (0,1)$ and for each $x,y \in X$ satisfying $(x,y) \in E(G)$, we have $\rho(T(x),T(y)) \le \alpha\rho(x,y)$, then T is called a G-strict contraction. Fix $\theta \in X$. For each $x \in X$ and each $r > 0$, set

$$B_\rho(x,r) := \{y \in X : \ \rho(x,y) \le r\}.$$

We may assume without loss of generality that if $x,y \in X$ satisfies $(x,y) \in E(G)$, then $(y,x) \in E(G)$.

We assume that the following assumption holds.

(P) For each $x,y \in X$, there exist an integer $q \geq 1$ and points $x_i \in X$, $i = 0,\dots,q$, such that

$$x_0 = x,\ x_q = y,\ (x_i,x_{i+1}) \in E(G),\ i = 0,\dots,q-1.$$

Thus, $V(G) = X$ and the graph G is connected.

Let $T \in \mathscr{M}_{ne}$ be a G-strict contraction. It is known [11] that under certain mild assumptions, the mapping T has a unique fixed point which attracts all the iterates of T. We begin our discussion by presenting the results of [22], where we provided a very simple proof of this fact by using a certain metric on X which is defined below. In [22] we also showed that the iterates of T converge uniformly on bounded subsets of X and that this convergence is stable under small perturbations of these iterates.

For each $x,y \in X$, define

$$\rho_1(x,y) := \inf\{\sum_{i=0}^{q-1} \rho(x_i,x_{i+1}) :\ q \geq 1 \text{ is an integer},$$

$$x_i \in X,\ i = 0,\dots,q,\ x_0 = x,\ x_q = y,\ (x_i,x_{i+1}) \in E(G),\ i = 0,\dots,q-1\}.$$

It is easy to see that for each $x,y,z \in X$, $\rho_1(x,y)$ is finite,

$$\rho_1(x,y) \geq \rho(x,y),$$

$$\rho_1(x,y) = \rho_1(y,x),$$

$$\rho_1(x,z) \leq \rho_1(x,y) + \rho_1(y,z),$$

and if $\rho_1(x,y) = 0$, then $x = y$. However, ρ_1 is a metric only if $(x,x) \in E(G)$ for all $x \in X$. This pseudometric ρ_1 plays an important role in our study.

For each $x \in X$ and each $r > 0$, we set $B_{\rho_1}(x,r) := \{y \in X :\ \rho_1(x,y) \leq r\}$.

6. Strict contractions

The results of this section were obtained in [22] under all the assumptions made in Section 5.

Theorem 6.1. *Let $T \in \mathscr{M}_{ne}$, $\alpha \in (0,1)$ and assume that for each $x,y \in X$ satisfying $(x,y) \in E(G)$, the inequality*

$$\rho(T(x),T(y)) \leq \alpha\rho(x,y)$$

holds. Then for each $x,y \in X$, $\rho_1(T(x),T(y)) \leq \alpha\rho_1(x,y)$. If T is continuous as a self-mapping of (X,ρ), then there exists a unique point $x_* \in X$ satisfying $T(x_*) = x_*$ and for each $x \in X$,

$$\lim_{i\to\infty} T^i(x) = x_* \text{ in } (X,\rho).$$

Theorem 6.2. *Let all the assumptions of Theorem 6.1 hold, let the mapping T be continuous as a self-mapping of (X,ρ) and let $x_* \in X$ be as guaranteed by Theorem 6.1 and satisfy $x_* = T(x_*)$. Suppose that the following assumption holds.*

(A1) For each $M_0 > 0$, there exists $M_1 > 0$ such that, for each point $x \in B_\rho(\theta,M_0)$, $\rho_1(x,\theta) \leq M_1$. Then $T^i(x) \to x_*$ as $i \to \infty$ in (X,ρ), uniformly on all bounded subsets of (X,ρ).

Theorem 6.3. *Let all the assumptions of Theorem 6.2 hold and let $x_* \in X$ be as guaranteed by Theorem 6.1 and satisfy $x_* = T(x_*)$. Suppose that T is uniformly continuous and bounded on bounded sets as a self-mapping of (X,ρ) and let $\varepsilon, M > 0$ be given. Then there exist $\delta > 0$ and* *a natural number* n_0 *such that for each integer* $n \geq n_0$ *and every finite sequence* $\{x_i\}_{i=0}^n \subset X$ *satisfying* $x_0 \in B_\rho(\theta, M)$ *and*

$$\rho(x_{i+1}, T(x_i)) \leq \delta,\ i = 0, \dots, n-1,$$

the inequality $\rho(x_i, x_*) \leq \varepsilon$ *holds for all* $i = n_0, \dots, n$.

Denote by $\mathscr{M}$ the set of all mappings $S \in \mathscr{M}_{ne}$ which are uniformly continuous and bounded on bounded sets as self-mappings of (X, ρ). We equip the space $\mathscr{M}$ with the uniformity which has the base

$$\mathscr{E}(\varepsilon, M) = \{(S_1, S_2) \in \mathscr{M} \times \mathscr{M} :\ \rho(S_1(x), S_2(x)) \leq \varepsilon \text{ for all } x \in B(\theta, M)\},$$

where $M, \varepsilon > 0$. It is not difficult to see that the uniform space $\mathscr{M}$ is metrizable and complete (by a metric d).

Theorem 6.4. *Let all the assumptions of Theorem 6.2 hold and let* $x_* \in X$ *be as guaranteed by Theorem 6.1 and satisfy* $x_* = T(x_*)$. *Suppose that* $T \in \mathscr{M}$ *and that* $\varepsilon, M > 0$ *are given. Then there exist a neighborhood* $\mathscr{U}$ *of* T *in* $\mathscr{M}$ *and a natural number* n_0 *such that for each* $C \in \mathscr{U}$, *each* $x \in B_\rho(\theta, M)$ *and each integer* $n \geq n_0$, *we have*

$$\rho(C^n(x), x_*) \leq \varepsilon.$$

7. G-CONTRACTIVE MAPPINGS

Now our goal is to show that the results of the previous sections, which were obtained in [22], also hold for mappings on complete metric spaces with graphs that are merely contractive under certain additional assumptions on the graphs. These results were established in [26].

Let (X, ρ) be a complete metric space and let G be a (directed) graph. Let $V(G)$ be the set of its vertices and let $E(G)$ be the set of its edges. We identify the graph G with the pair $(V(G), E(G))$. Denote by $\mathscr{M}_{ne}$ the set of all mappings $T : X \to X$ such that for each $x, y \in X$ satisfying $(x, y) \in E(G)$, we have

$$(T(x), T(y)) \in E(G),\ \ \rho(T(x), T(y)) \leq \rho(x, y)$$

and such that

$$\{(x, T(x)) :\ x \in X\} \text{ is a closed set in the product space } X \times X.$$

Recall that a mapping $T \in \mathscr{M}_{ne}$ is called G-nonexpansive. If $T \in \mathscr{M}_{ne}$, $\alpha \in (0, 1)$ and for each $x, y \in X$ satisfying $(x, y) \in E(G)$, we have

$$\rho(T(x), T(y)) \leq \alpha \rho(x, y),$$

then T is called a G-strict contraction.

A mapping $T \in \mathscr{M}_{ne}$ is called G-contractive (or G-Rakotch contraction [17]) if there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that $\phi(t) < 1$ $t \in [0, \infty)$, and for each $x, y \in X$ satisfying $(x, y) \in E(G)$, we have

$$\rho(T(x), T(y)) \leq \phi(\rho(x, y))\rho(x, y).$$

In the sequel we assume that the infimum over the empty set is ∞, $\infty + \infty = \infty$, and $a + \infty = \infty$ for each $a \in R^1$.

For each point $x \in X$ and each number $r > 0$, set

$$B_\rho(x,r) := \{y \in X : \rho(x,y) \le r\}.$$

Recall that for each $x,y \in X$,

$$\rho_1(x,y) := \inf\{\sum_{i=0}^{q-1} \rho(x_i,x_{i+1}) : q \ge 1 \text{ is an integer},$$

$$x_i \in X,\ i = 0,\dots,q,\ x_0 = x,\ x_q = y,\ (x_i,x_{i+1}) \in E(G),\ i = 0,\dots,q-1\}.$$

For each point $x \in X$ and each number $r > 0$, we set

$$B_{\rho_1}(x,r) := \{y \in X : \rho_1(x,y) \le r\}.$$

We assume that there exists a point $\bar{x} \in X$ such that the following assumption holds.

(A1) For each $x \in X$, there exist an integer $q \ge 1$ and points $x_i \in X$, $i = 0,\dots,q$, such that

$$x_0 = \bar{x},\ x_q = x,\ (x_i,x_{i+1}) \in E(G),\ i = 0,\dots,q-1.$$

(Note that (A1) holds if and only if $\rho_1(\bar{x},x) < \infty$ for all $x \in X$.) We also assume that there exists a number $\bar{\Delta} > 0$ such that the following assumption holds.

(A2) If $(x_0,x_1),(x_1,x_2) \in E(G)$ satisfy $\rho(x_0,x_1) \le \bar{\Delta}$, $\rho(x_1,x_2) \le \bar{\Delta}$, then $(x_0,x_2) \in E(G)$.

It turns out that these assumptions hold for many important graphs; see, for instance, the examples below.

Example 7.1. Assume that $\Delta > 0$, for each $x,y \in X$, $(x,y) \in E(G)$ if and only if $\rho(x,y) \le \Delta$, and that there exists a point $\bar{x} \in X$ such that for each $x \in X$, there exist an integer $q \ge 1$ and points $x_i \in X$, $i = 0,\dots,q$, satisfying $x_0 = \bar{x}$, $x_q = x$ and $\rho(x_i,x_{i+1}) \le \Delta$, $i = 0,\dots,q$. Clearly, (A1) holds and (A2) holds with $\bar{\Delta} = \Delta/2$. Note that in this case (A1) means that (X,ρ) is Δ-chainable.

Example 7.2. Let X be a closed set in a Banach space $(E,\|\cdot\|)$ ordered by a closed and convex cone E_+ such that $\|x\| \le \|y\|$ for each $x,y \in E_+$ satisfying $x \le y$, $\rho(x,y) = \|x-y\|$, $x,y \in X$, and $(x,y) \in E(G)$ if and only if $x \le y$. Assume that $\bar{x} \in X$ and $\bar{x} \le x$ for each $x \in X$. Clearly, both (A1) and (A2) hold. It is not difficult to see that every partially ordered metric space with a smallest element satisfies both (A1) and (A2).

8. Existence and convergence results

The results of this section were obtained in [26]. Assume that $T \in \mathscr{M}_{ne}$. It is easy to see that for each $x,y \in X$, $\rho_1(T(x),T(y)) \le \rho_1(x,y)$. Note that if $\alpha \in (0,1)$ and for each $x,y \in E(G)$ satisfying $(x,y) \in E(G)$ the inequality

$$\rho(T(x),T(y)) \le \alpha\rho(x,y)$$

holds, then for each $x,y \in X$, $\rho_1(T(x),T(y)) \le \alpha\rho_1(x,y)$. Assume that $\phi : [0,\infty) \to [0,1]$ is a decreasing function satisfying $\phi(t) < 1$, $t \in [0,\infty)$, and that for each $x,y \in X$ satisfying $(x,y) \in E(G)$ the inequality

$$\rho(T(x),T(y)) \le \phi(\rho(x,y))\rho(x,y)$$

holds. In other words, T is G-contractive.

The following results were obtained in [26] under all the assumptions made in the previous section and in this one.

Theorem 8.1. *Assume that $\rho_1(T(\bar{x}),\bar{x}) < \infty$. Then the following assertions hold.*
1. There exists $x_* = \lim_{i\to\infty} T^i(\bar{x})$ in (X,ρ) and $T(x_*) = x_*$.
2. For every $M_0 > 0$, $\lim_{n\to\infty} T^n(x) = x_*$ in (X,ρ), uniformly on $B_{\rho_1}(\bar{x},M_0)$.
3. Assume that for each $M_0 > 0$, there exists $M_1 > 0$ such that

$$B_\rho(\bar{x},M_0) \subset B_{\rho_1}(\bar{x},M_1).$$

Then $\lim_{n\to\infty} T^n(x) = x_*$ in (X,ρ), uniformly on all bounded subsets of (X,ρ).

Theorem 8.2. *Assume that $\rho_1(T(\bar{x}),\bar{x}) < \infty$ and that for each $M_0 > 0$, there exists $M_1 > 0$ such that*

$$B_\rho(\bar{x},M_0) \subset B_{\rho_1}(\bar{x},M_1)$$

and that T is uniformly continuous and bounded on all bounded sets as a self-mapping of (X,ρ). Let $M,\varepsilon > 0$. Then there exist a number $\delta > 0$ and a natural number n_0 such that for each integer $n \ge n_0$ and each sequence $\{x_i\}_{i=0}^n \subset X$ which satisfies $x_0 \in B_\rho(\bar{x},M)$ and

$$\rho(x_{i+1},T(x_i)) \le \delta,\ i = 0,1,\dots,n-1,$$

the inequality $\rho(x_i,x_*) \le \varepsilon$ holds for all $i = n_0,\dots,n$.

This result easily follows from Theorem 8.1 and Theorem 2.65 of [21].

Denote by $\mathscr{M}$ the set of all $S \in \mathscr{M}_{ne}$ which are uniformly continuous and bounded on bounded sets as self-mappings of (X,ρ). Denote by $\mathscr{M}_c$ the set of all continuous mappings $T \in \mathscr{M}_{ne}$. Fix a point $\theta \in X$. The space $\mathscr{M}_{ne}$ is equipped with the uniformity determined by the base

$$\mathscr{E}(\varepsilon,M) = \{(S_1,S_2) \in \mathscr{M}_{ne} \times \mathscr{M}_{ne} :\ \rho(S_1(x),S_2(x)) \le \varepsilon$$
$$\text{for all } x \in B(\theta,M)\},$$

where $M,\varepsilon > 0$. The uniform space $\mathscr{M}_{ne}$ is metrizable (by a metric d) and complete. It is complete if the set $\{(x,y) \in X \times X :\ (x,y) \in E(G)\}$ is closed in the product topology. It is clear that $\mathscr{M}$ is a closed subset of $\mathscr{M}_{ne}$. We equip it with the relative topology. The next result easily follows from Theorem 8.1 and Theorem 2.68 of [21].

We say that a set $C \subset X$ is bounded in (X,ρ_1) if $\sup\{\rho_1(x,y) :\ x,y \in C\} < \infty$.

Theorem 8.3. *Assume that $\rho_1(T(\bar{x}),\bar{x}) < \infty$ and let x_* be as guaranteed by Theorem 8.1. Assume further that each bounded set in (X,ρ) is also bounded in (X,ρ_1) and let $\varepsilon,M > 0$. Then there exist a neighborhood $\mathscr{U}$ of T in $\mathscr{M}$ and a natural number n_0 such that for each $x \in B(\theta,M)$, each integer $n \ge n_0$ and each $\{B_i\}_{i=1}^n \subset \mathscr{U}$, we have*

$$\rho(B_q\cdots,B_1(x),x_*) \le \varepsilon,\ q = n_0,\dots,n.$$

9. Extensions of Theorem 8.1

The results of this section were obtained in [30] Assume that (X,d) is a complete metric space equipped with a graph G. We denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. We assume that $(x,x) \in E(G)$ for any point $x \in X$ and that $\phi : [0,\infty) \to [0,1]$ is a decreasing function such that $\phi(t) < 1$ $t \in [0,\infty)$ and $\phi(0) = 1$. Assume that a mapping $T : X \to X$ satisfies the following assumption.

(A1) For every pair of points $x,y \in X$ satisfying $(x,y) \in E(G)$ the inequality

$$d(T(x),T(y)) \le \phi(d(x,y))d(x,y)$$

holds and

$$(T(x),T(y)) \in E(G).$$

Theorem 8.1 shows the that T possesses a unique fixed point under the assumption that there exists a number $\bar{\Delta} > 0$ such that if $(x_0,x_1),(x_1,x_2) \in E(G)$ satisfy $\rho(x_0,x_1) \leq \bar{\Delta}$, $\rho(x_1,x_2) \leq \bar{\Delta}$, then $(x_0,x_2) \in E(G)$. The following existence results were obtained in [30] under all the assumptions made in this section.

Theorem 9.1. *Assume that $\bar{x} \in X$ and that there exist a natural number q and $M > 1$ such that for each integer $n \geq 1$ there exist $x_i^{(n)} \in X$, $i = 0,\dots,q$, such that*

$$x_0^{(n)} = \bar{x},\ x_q^{(n)} = T^n(\bar{x})$$

and that for each $i \in \{0,\dots,n-1\}$, $d(x_i^{(n)},x_{i+1}^{(n)}) \leq M$ and at least one of the following relations hold:

$$(x_i^{(n)},x_{i+1}^{(n)}) \in E(G),\ (x_{i+1}^{(n)},x_i^{(n)}) \in E(G).$$

Then there exists $x_* = \lim_{i\to\infty} T^i(x)$ and if T is continuous at x_*, then $T(x_*) = x_*$..

Theorem 9.2. *Assume that $x_* \in X$ satisfies $T(x_*) = x_*$, q is a natural number, $M > 0$ and that*

$$X_{M,q} = \{x \in X:\ \text{there exist } x_i \in X,\ i = 0,\dots,q,$$
$$\text{such that } x_0 = x,\ x_q = x_* \text{ and for each } i \in \{0,\dots,q-1\},$$
$$d(x_i,x_{i+1}) \leq M \text{ and at least one of the following relations holds:}$$
$$(x_i,x_{i+1}) \in E(G);\ (x_{i+1},x_i) \in E(G)\}.$$

Then $d(T^n(x),x_*) \to 0$ as $n \to \infty$ uniformly on $X_{q,M}$.

We believe that the results presented in the paper can be extended to classes of quasi-nonexpansive mappings and mappings which map a subset of a complete metric space to the space itself.

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