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ON BETTER APPROXIMATION ORDER FOR THE NONLINEAR BASKAKOV OPERATOR OF MAXIMUM PRODUCT KIND

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Abstract. The nonlinear Baskakov operator of the max-product type, which uses the maximum instead of the sum, was introduced by Bede et al. [8]. Our aim in this paper is to obtain a better approximation order for this operator. In [8], the approximation order for this operator was found to be $\frac{\sqrt{x(1+x)}}{\sqrt{n}}$ with the help of the classical modulus of continuity, and it was claimed that this approximation order cannot be improved except for some subclasses of functions. Contrary to this claim, under some circumstances, we show that a better order of approximation can be obtained with the help of classical and weighted modulus of continuities.

Keywords. Approximation operators; Modulus of continuity; Nonlinear Baskakov operator of maximum product kind.

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1. Introduction

In response to the question of whether all approximation operators need to be linear, Bede et al. presented nonlinear operators for the open problem in [27]. They defined nonlinear maximum product operators by taking the maximum (or supremum) instead of the sum. These operators provide the property of pseudo-linearity, which is weaker than linearity. Studies of nonlinear maximum product operators started with Shepard operators (see [13]), and the maximum product type of many operators was introduced. The order of approximation was obtained with the help of the classical modulus of continuity:

$$\omega\left(f,\delta\right)=\max\left\{\left|f\left(x\right)-f\left(y\right)\right|;\ x,y\in I,\ \left|x-y\right|\leq\delta\right\} \tag{1.1}$$

and shape preserving properties were examined [6, 7, 8, 9, 10, 11]. Also, some statistical approximation properties of max-product type operators were given by Duman in [18].

The nonlinear max-product type Baskakov operator, which is the main subject of this study, was examined in [8]. The order of approximation for the max-product type Baskakov operator

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can be found by means of the modulus of continuity as $\omega\left(f;\sqrt{x(1+x)/n}\right)$. Additionally, Bede et al. indicated that the order of approximation under the modulus was $\sqrt{x(1+x)/n}$, and it could not be improved except for some subclasses of functions (see [8] for details).

Contrary to this claim, under some circumstances, we will show that a better order of approximation can be obtained with the help of classical and weighted modulus of continuities. Notice that, in [24, 25, 26], we showed that the rates of approximation of some max-product type operators can be improved.

2. The Concept of Nonlinear Max-Product Operators

We recall the basic definitions and theorems about nonlinear operators given in [11] and [12].

Let us consider together the set of non-negative real numbers, $\mathbb{R}_+$, with the operations $\vee$ (maximum) and $\cdot$ (product). Then $(\mathbb{R}_+,\vee,\cdot)$ has a semi-ring structure called a maximum product algebra.

Let $I\subset\mathbb{R}$ be bounded or unbounded interval, and

$$CB_+(I)=\{f:I\to\mathbb{R}_+ : f \text{ continuous and bounded on } I\}.$$

Let us take the general form of $L_n : CB_+(I)\to CB_+(I)$, as

$$L_n(f)(x)=\bigvee_{i=0}^{n}K_n(x,x_i)f(x_i)$$

or

$$L_n(f)(x)=\bigvee_{i=0}^{\infty}K_n(x,x_i)f(x_i),$$

where $n\in\mathbb{N}$, $f\in CB_+(I)$, $K_n(.,x_i)\in CB_+(I)$ and $x_i\in I$, for all i. These operators are nonlinear, positive operators and moreover they satisfy the following pseudo-linearity condition of the form

$$L_n(\alpha f\vee\beta g)(x)=\alpha L_n(f)(x)\vee\beta L_n(g)(x),$$

where $\forall\alpha,\beta\in\mathbb{R}_+$, $f,g\in CB_+(I)$.

In this section, we present some general results on these kinds of operators which will be used later.

Lemma 2.1. [11] *Let $I\subset\mathbb{R}$ be bounded or unbounded interval,*

$$CB_+(I)=\{f:I\to\mathbb{R}_+ : f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I)\to CB_+(I)$, $n\in\mathbb{N}$ be a sequence of operators satisfying the following properties:

(i) If $f,g\in CB_+(I)$ satisfy $f\le g$ then $L_n(f)\le L_n(g)$ for all $n\in\mathbb{N}$.
(ii) $L_n(f+g)\le L_n(f)+L_n(g)$ for all $f,g\in CB_+(I)$.
Then, for all $f,g\in CB_+(I)$, $n\in\mathbb{N}$ and $x\in I$,

$$|L_n(f)(x)-L_n(g)(x)|\le L_n(|f-g|)(x).$$

Corollary 2.2. [11] *Let $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the conditions $(i), (ii)$ in Lemma 2.1 and in addition being positive homogenous. Then, for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$,*

$$|L_n(f)(x) - f(x)| \leq \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega(f,\delta) + f(x) \, |L_n(e_0)(x) - 1|,$$

where $\omega(f,\delta)$ is the classical modulus of continuity defined by (1.1), $\delta > 0$, $e_0(t) = 1$, $\varphi_x(t) = |t-x|$ for all $t \in I$, $x \in I$, and if I is unbounded then we suppose thet there exists $L_n(\varphi_x)(x) \in \mathbb{R}_+ \cup \{\infty\}$, for any $x \in I$, $n \in \mathbb{N}$.

A consequence of Corollary 2.2, we have the following result.

Corollary 2.3. [11] *Suppose that in addition to the conditions in Corollary 2.2, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in \mathbb{N}$. Then, for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$,*

$$|L_n(f)(x) - f(x)| \leq \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] \omega(f,\delta)$$

where $\omega(f,\delta)$ is the classical modulus of continuity defined by (1.1) and $\delta > 0$.

3. The Nonlinear Baskakov Operator of Max-Product Kind

For $f \in C[0,\infty)$, the classical Baskakov operators is given in [5] as

$$V_n(f;x) = \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right),$$

where $x \in [0,\infty)$, $n \in \mathbb{N}$ and $b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$.

Since $V_n(e_0;x) = \sum_{k=0}^{\infty} b_{n,k}(x) = 1$, we can also write the classical Baskakov operator as

$$V_n(f;x) = \frac{\sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\sum_{k=0}^{\infty} b_{n,k}(x)}.$$

Now, if we replace the sum operator $\sum$ by the supremum operator $\bigvee$, we obtain the nonlinear Baskakov operator:

$$V_n^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{\infty} b_{n,k}(x) \, f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} b_{n,k}(x)}, \tag{3.1}$$

where $b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$, $f \in C[0,\infty)$, $x \in [0,\infty)$, $n \in \mathbb{N}$ as introduced by Bede et al. in [8].

Remark 3.1. [8] 1) It is easy to see that the nonlinear Baskakov max-product operator satisfy the conditions (i), (ii) of Lemma 2.1. In fact, instead of (i) it also satisfies the following stronger condition:

$$V_n^{(M)}(f \vee g)(x) = V_n^{(M)}(f)(x) \vee V_n^{(M)}(g)(x),$$

where $f, g \in CB_+(I)$, $I = [0,\infty)$. Indeed, taking into consideration of the equality above, for $f \leq g$, $f, g \in CB_+(I)$, it easily follows

$$V_n^{(M)}(f)(x) \leq V_n^{(M)}(g)(x).$$

2) In addition to this, it is immadiate that the nonlinear Baskakov max-product operator is positive homogenous, that is $V_n^{(M)}(\lambda f) = \lambda V_n^{(M)}(f)$ for all $\lambda \geq 0$.

Lemma 3.2. [8] *For any arbitrary bounded function $f : [0,\infty) \to \mathbb{R}_+$, max-product operator $V_n^{(M)}(f)(x)$ is positive, bounded, continuous and satisfies $V_n^{(M)}(f)(0) = f(0)$, for all $n \in \mathbb{N}$, $n \geq 3$.*

4. Auxiliary Results

In this section, we present the statements and the lemmas given in [8] for the approximation theorem without proof, except for Lemma 4.5. We revise Lemma 4.5, which allows us to improve the order of approximation of the operator, and different from [8], we prove it by induction method again.

Lemma 4.1. [8] *Let $n \in \mathbb{N}$, $n \geq 2$. Then,*

$$\bigvee_{k=0}^{\infty} b_{n,k}(x) = b_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right],\ j = 0,1,2,...$$

where $b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$.

We define the following expression similar to [8].

For each $n \in \mathbb{N}, n \geq 2$, $k, j \in \{0,1,2,...\}$ and $x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right]$, $x > 0$,

$$m_{k,n,j}(x) \;:=\; \frac{b_{n,k}(x)}{b_{n,j}(x)} = \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}} \frac{x^k}{(1+x)^{n+k}} \frac{(1+x)^{n+j}}{x^j} = \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}} \left(\frac{x}{1+x}\right)^{k-j}.$$

For $x = 0$ let us denote $m_{0,n,0}(x) = 1$ and $m_{k,n,0}(x) = 0$ for all $k \in \{1,2,...\}$.

Lemma 4.2. [8] *Let $n \in \mathbb{N}$, $n \geq 2$. For all $k, j \in \{0,1,2,...\}$ and $x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right]$, $m_{k,n,j}(x) \leq 1$.*

Remark 4.3. [8] From Lemmas 3.2, 4.1, and 4.2, it is clear that $V_n^{(M)}(f)(x)$ satisfies, for all $n \in \mathbb{N}$, $n \geq 2$, all the hypothesis in Lemma 2.1, Corollary 2.2 and Corollary 2.3 for $I = [0,\infty)$.

From Lemma 3.2, we have $V_n^{(M)}(f)(0) - f(0) = 0$ for all $n \geq 3$, so, in this part, we consider $x > 0$ in the notations, proofs and statements of the all approximation results. Again, let us define the following expressions similar to [8]. For each $n \in \mathbb{N}$, $n \geq 3$, $k, j \in \{0,1,2,...\}$ and $x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right]$,

$$M_{k,n,j}(x) := m_{k,n,j}(x) \left|\frac{k}{n} - x\right|.$$

It is clear that if $k \geq \frac{n}{n-1}(j+1)$, then

$$M_{k,n,j}(x) = m_{k,n,j}(x) \left(\frac{k}{n} - x\right)$$

and if $k \leq \frac{n}{n-1} j$, then

$$M_{k,n,j}(x) = m_{k,n,j}(x) \left(x - \frac{k}{n}\right).$$

Also, for each $n\in\mathbb{N}$, $n\geq 3$, $k,j\in\mathbb{N}$, $k\geq\frac{n}{n-1}(j+1)$ and $x\in\left[\frac{j}{n-1},\frac{j+1}{n-1}\right]$

$$\overline{M}_{k,n,j}(x):=m_{k,n,j}(x)\left(\frac{k}{n-1}-x\right)$$

and for each $n\in\mathbb{N}$, $n\geq 3$, $k,j\in\mathbb{N}$, $k\leq\frac{n}{n+1}j$ and $x\in\left[\frac{j}{n-1},\frac{j+1}{n-1}\right]$

$$\underline{M}_{k,n,j}(x):=m_{k,n,j}(x)\left(x-\frac{k}{n-1}\right).$$

At this point, let us recall the following lemma.

Lemma 4.4. [8] *Let* $x\in\left[\frac{j}{n-1},\frac{j+1}{n-1}\right]$ *and* $n\in\mathbb{N}$, $n\geq 3$.

(i) *For all* $k,j\in\{0,1,2,...\}$ *with* $k\geq\frac{n}{n-1}(j+1)$, $M_{k,n,j}(x)\leq\overline{M}_{k,n,j}(x)$.

(ii) *For all* $k,j\in\mathbb{N}$ *with* $k\geq\frac{n}{n-2}(j+1)$, $\overline{M}_{k,n,j}(x)\leq 2M_{k,n,j}(x)$.

(iii) *For all* $k,j\in\mathbb{N}$ *with* $k\leq\frac{n}{n+1}j$, $\underline{M}_{k,n,j}(x)\leq M_{k,n,j}(x)\leq 2\underline{M}_{k,n,j}(x)$.

Now, we give our first main result of this part which is proven by not only using the proof techniques given in [8] but also using the induction method.

Lemma 4.5. *Let* $x\in\left[\frac{j}{n-1},\frac{j+1}{n-1}\right]$ *and* $n\in\mathbb{N}$, $n\geq 3$ *and* $\alpha\in\{2,3,...\}$.

(i) *If* $j\in\{0,1,2,...\}$ *is such that* $k\geq\frac{n}{n-1}(j+1)$ *and* $(k-j)^{\alpha}\geq\frac{(n+j)(k+1)}{(n-1)}$, *then* $\overline{M}_{k,n,j}(x)\geq\overline{M}_{k+1,n,j}(x)$.

(ii) *If* $k\in\{1,2,...,j\}$ *is such that* $k\leq\frac{n}{n+1}j$ *and* $(j-k)^{\alpha}\geq\frac{k(n+j-1)}{(n-1)}$, *then*

$$\underline{M}_{k,n,j}(x)\geq\underline{M}_{k-1,n,j}(x).$$

Proof. (i) From the case (i) of Lemma 3.2 in [8], we can write

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)}\geq\frac{k+1}{n+k}\frac{n+j}{j+1}\frac{k-j-1}{k-j}.$$

After this point we use a different proof technique from [8]. By using the induction method, let us show that the following inequality

$$\frac{k+1}{n+k}\frac{n+j}{j+1}\frac{k-j-1}{k-j}\geq 1\tag{4.1}$$

holds for $(k-j)^{\alpha}\geq\frac{(n+j)(k+1)}{(n-1)}$. For $\alpha=2$, condition $(k-j)^{2}\geq\frac{(n+j)(k+1)}{(n-1)}$ is equivalent to $(n-1)(k-j)^{2}-(n+j)(k+1)\geq 0$ or
$n\left[(k-j)^{2}-(k+1)\right]+kj-j^{2}-k^{2}-j\geq 0$. Therefore, this condition is the same as in (i) of Lemma 3.2 in [8], and inequality (4.1) holds for $\alpha=2$.

Now, we assume that inequality (4.1) is provided for $\alpha-1$. It follows $\frac{k+1}{n+k}\frac{n+j}{j+1}\frac{k-j-1}{k-j}\geq 1$ when $(k-j)^{\alpha-1}\geq\frac{(n+j)(k+1)}{(n-1)}$. Since $k\geq\frac{n}{n-1}(j+1)$, we have $nk-k\geq nj+n$, and clearly it follows $n(k-j)\geq n+k$ or $(k-j)\geq 1+\frac{k}{n}\geq 1$, and $\alpha=2,3,...$. Then we can write $(k-j)^{\alpha}\geq(k-j)^{\alpha-1}\geq\frac{(n+j)(k+1)}{(n-1)}$. It is true for α; hence, for arbitrary $\alpha=2,3,...$ the inequality (4.1) is provided when $(k-j)^{\alpha} \geq \frac{(n+j)(k+1)}{(n-1)}$. So we obtain

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} \geq \frac{k+1}{n+k}\frac{n+j}{j+1}\frac{k-j-1}{k-j} \geq 1.$$

(ii) From the case (ii) of Lemma 3.2 in [8], we can write

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} \geq \frac{n+k-1}{k}\frac{j}{n+j-1}\frac{j-k}{j-k+1}.$$

After this point, we use the our proof technique again. By using the induction method, let us show that the following inequality

$$\frac{n+k-1}{k}\frac{j}{n+j-1}\frac{j-k}{j-k+1} \geq 1 \tag{4.2}$$

holds for $(j-k)^{\alpha} \geq \frac{k(n+j-1)}{(n-1)} = k\left(\frac{j}{n-1}+1\right)$. For $\alpha = 2$, similarly previous case (i), the condition $(j-k)^2 \geq \frac{k(n+j-1)}{(n-1)}$ is equivalent to $(j-k)^2(n-1) - k(n+j-1) \geq 0$ or $n\left[(j-k)^2 - k\right] + kj - j^2 - k^2 - j \geq 0$. Therefore, this condition is the same as in (ii) of Lemma 3.2 in [8], and the inequality (4.2) holds for $\alpha = 2$.

Now, we assume that (4.2) is correct for $\alpha - 1$. Hence, $\frac{n+k-1}{k}\frac{j}{n+j-1}\frac{j-k}{j-k+1} \geq 1$ is provided when $(j-k)^{\alpha-1} \geq \frac{k(n+j-1)}{(n-1)}$ and $k \leq \frac{n}{n+1}j$. For $k = j$, conditions $k \leq \frac{n}{n+1}j$ and $(j-k)^{\alpha} \geq \frac{k(n+j-1)}{(n-1)}$ are not satisfied, so we can take $k \leq j-1$. Therefore, since $1 \leq j-k$ and $\alpha = 2,3,\ldots$ it follows that $(j-k)^{\alpha} \geq (j-k)^{\alpha-1} \geq \frac{k(n+j-1)}{(n-1)}$. It is true for $\alpha = 2,3,\ldots$ and then the desired inequality is provided for $(j-k)^{\alpha} \geq \frac{k(n+j-1)}{(n-1)}$. So we obtain

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} \geq \frac{n+k-1}{k}\frac{j}{n+j-1}\frac{j-k}{j-k+1} \geq 1,$$

which gives the desired result. □

5. Pointwise Rate of Convergence

Our aim in this section is to improve the order of approximation for the operators $V_n^{(M)}(f)(x_0)$. And our motivation is that when approximating the nonlinear max-product operators by the classical modulus of continuity, the rate of approximation does not have to be square-root since the Cauchy-Schwarz inequality is not used. From the following theorem, we can say that the order of the pointwise approximation can be improved when α is big enough. Furthermore, if $\alpha = 2$ then these approximation results become the results in [8].

Theorem 5.1. *Let $f : [0,\infty) \to \mathbb{R}_+$ be bounded and continuous. Then, for any fixed point x_0 on the interval $[0,\infty)$, which also satisfy $x_0^{\alpha-2} \leq n-1$, we have the following order of approximation for the operators (3.1) to the function f by means of the modulus of continuity, for all $n \in \mathbb{N}$, $n \geq 4$,*

$$\left|V_n^{(M)}(f)(x_0) - f(x_0)\right| \leq \left(1 + 6\left[x_0(1+x_0)\right]^{\frac{1}{\alpha}}\right)\,\omega\left(f; \frac{1}{(n-1)^{1-\frac{1}{\alpha}}}\right),$$

where $\omega(f;\delta)$ is the classical modulus of continuity defined by (1.1) and $\alpha = 2,3,\ldots$.

Proof. Since nonlinear max-product Baskakov operators satisfy the conditions in Corollary 2.3, for any $x_0 \in [0,\infty)$, and using the properties of $\omega(f;\delta)$, we obtain

$$\left|V_n^{(M)}(f)(x_0) - f(x_0)\right| \le \left[1 + \frac{1}{\delta_n} V_n^{(M)}(\varphi_{x_0})(x_0)\right] \omega(f,\delta), \tag{5.1}$$

where $\varphi_{x_0}(t) = |t - x_0|$. At this point, let us define the following

$$E_n(x_0) := V_n^{(M)}(\varphi_{x_0})(x_0) = \frac{\bigvee_{k=0}^{\infty} b_{n,k}(x_0)\left|\frac{k}{n} - x_0\right|}{\bigvee_{k=0}^{\infty} b_{n,k}(x_0)},\ x_0 \in [0,\infty).$$

Let $x_0 \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right]$, where $j \in \{0,1,...\}$ is fixed arbitrarily. By Lemma 4.1 We easily find

$$E_n(x_0) = \max_{k=0,1,...}\left\{M_{k,n,j}(x_0)\right\}.$$

Firstly let us check for $j = 0$,

$$\begin{aligned} M_{k,n,0}(x_0) &= \frac{\binom{n+k-1}{k}}{\binom{n+0-1}{0}} \left(\frac{x_0}{1+x_0}\right)^{k-0} \left|\frac{k}{n} - x_0\right| \\ &= \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x_0}{1+x_0}\right)^{k} \left|\frac{k}{n} - x_0\right| \end{aligned}$$

where $x_0 \in \left[0, \frac{1}{n-1}\right]$ and $\alpha = 2,3,\ldots$.

For $k = 0$, we have

$$M_{0,n,0}(x_0) = x_0 = x_0^{\frac{1}{\alpha}} x_0^{1-\frac{1}{\alpha}} \le \frac{x_0^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}.$$

For $k = 1$, we obtain

$$M_{1,n,0}(x_0) = \binom{n}{1}\left(\frac{x_0}{1+x_0}\right)\left|\frac{1}{n} - x_0\right| \le n\frac{x_0}{1+x_0}\frac{1}{n} \le x_0 \le \frac{x_0^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}.$$

Now suppose that $k \ge 2$. For $j = 0$ and $k \ge 2$, we see that all of the Lemma 4.4 (i)'s hypotheses are satisfied. Thus we get $M_{k,n,0}(x_0) \le \overline{M}_{k,n,0}(x)$. Also by Lemma 4.5 (i), for $j = 0$ it follows that $\overline{M}_{k,n,0}(x) \ge \overline{M}_{k+1,n,0}(x)$ for every $k \ge 2$ such that $k^\alpha \ge \frac{n(k+1)}{n-1}$ then we have $(n-1)k^\alpha - nk - n \ge 0$, when $\alpha = 2,3,\ldots$. Let's define the function γ as $\gamma(t) := (n-1)t^\alpha - nt - n$, $t \ge 1$. $\gamma(t)$ is nondecreasing for $t \ge 1$, since $\gamma'(t) = \alpha(n-1)t^{\alpha-1} - n \ge 0$. Since

$$\gamma\left(n^{\frac{1}{\alpha}}\right) = (n-1)n - nn^{\frac{1}{\alpha}} - n = n\left(n - 2 - n^{\frac{1}{\alpha}}\right) \ge 0,\ n \ge 4,$$

it follows $\overline{M}_{k,n,0}(x) \ge \overline{M}_{k+1,n,0}(x)$ for every $k \in \mathbb{N}$, $k \ge n^{\frac{1}{\alpha}}$. Let us denote

$$A_\alpha = \left\{k \in \mathbb{N},\ 2 \le k \le n^{\frac{1}{\alpha}} + 1\right\}$$

and let $k\in A_{\alpha}$. Since $4\leq n$ then $0\leq k\left(n-3\right)$ it follows that $2nk\leq 3nk-3k$ or $\frac{k}{n-1}\leq\frac{3k}{2n}$. By Lemma 4.2, we obtain

$$
\begin{aligned}
\overline{M}_{k,n,0}(x_0) &= \binom{n+k-1}{k}\left(\frac{x_0}{1+x_0}\right)^k\left(\frac{k}{n-1}-x_0\right)\\
&\leq \binom{n+k-1}{k}\left(\frac{x_0}{1+x_0}\right)^k\frac{k}{n-1}\\
&\leq \binom{n+k-1}{k}\left(\frac{x_0}{1+x_0}\right)^k\frac{3k}{2n}\\
&= \frac{(n+k-1)!}{(k-1)!n!}\frac{3}{2}\left(\frac{x_0}{1+x_0}\right)^k\\
&= \frac{3}{2}\binom{n+k-1}{k-1}\left(\frac{x_0}{1+x_0}\right)^{k-1}\frac{x_0}{1+x_0}\\
&= \frac{3}{2}\binom{n+k-1}{k-1}\left(\frac{\frac{1}{n}}{1+\frac{1}{n}}\right)^{k-1}\left(\frac{x_0}{1+x_0}\frac{1+\frac{1}{n}}{\frac{1}{n}}\right)^{k-1}\frac{x_0}{1+x_0}\\
&= \frac{3}{2}m_{k-1,n+1,0}\left(\frac{1}{n}\right)\left(\frac{(n+1)\,x_0}{1+x_0}\right)^{k-1}\frac{x_0}{1+x_0}\\
&\leq \frac{3}{2}\left(\frac{(n+1)\,x_0}{1+x_0}\right)^{k-1}\frac{x_0}{(1+x_0)},
\end{aligned}
$$

taking into account that $\frac{1}{n+1}=\frac{\frac{1}{n}}{1+\frac{1}{n}}$. If we denote

$$\zeta_{n,k}\left(x_0\right):=\left(\frac{(n+1)\,x_0}{1+x_0}\right)^{k-1},$$

then we see that the function $\zeta_{n,k}\left(x_0\right)$ is nondecreasing on the interval $\left[0,\frac{1}{n-1}\right]$. Observe that

$$
\begin{aligned}
\zeta_{n,k}'\left(x_0\right) &= (k-1)\left(\frac{(n+1)\,x_0}{1+x_0}\right)^{k-2}\left(\frac{(n+1)\left(1+x_0\right)-(n+1)\,x_0}{\left(1+x_0\right)^2}\right)\\
&= (k-1)\left(\frac{(n+1)\,x_0}{1+x_0}\right)^{k-2}\frac{n+1}{\left(1+x_0\right)^2}\geq 0.
\end{aligned}
$$

Using the above property, and $x_0\leq\frac{1}{n-1}$, we obtain

$$\zeta_{n,k}\left(x_0\right)\leq\zeta_{n,k}\left(\frac{1}{n-1}\right)=\left(\frac{\frac{n+1}{n-1}}{1+\frac{1}{n-1}}\right)^{k-1}=\left(\frac{n+1}{n}\right)^{k-1},$$

for all $x_0 \in \left[0, \frac{1}{n-1}\right]$. Then

$$\begin{aligned}
\overline{M}_{k,n,0}(x_0) &\leq \frac{3}{2}\left(\frac{n+1}{n}\right)^{k-1}\frac{x_0}{(1+x_0)} \\
&< \frac{3}{2}\left(\frac{n+1}{n}\right)^{n^{\frac{1}{\alpha}}}\frac{x_0}{(1+x_0)} \\
&\leq \frac{3}{2}\left(\frac{n+1}{n}\right)^{n}\frac{x_0}{(1+x_0)} \\
&< \frac{3}{2}e\frac{x_0}{1+x_0} < \frac{3e}{2}x_0 \\
&\leq 5\frac{x_0^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}
\end{aligned}$$

taking into account that $k+1 \leq n^{\frac{1}{\alpha}}$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and $\frac{3}{2}e < 5$. So, we find an upper estimate for any $k = 0, 1, 2, ...$

$$\begin{aligned}
E_n(x_0) &= \max_{k=0,1,...}\left\{M_{k,n,0}(x_0)\right\} \\
&\leq \max\left\{M_{0,n,0}(x_0), M_{1,n,0}(x_0), \max_{k=2,3,...}\left\{\overline{M}_{k,n,0}(x_0)\right\}\right\} \\
&= \max\left\{M_{0,n,0}(x_0), M_{1,n,0}(x_0), \max_{k\in A}\left\{\overline{M}_{k,n,0}(x_0)\right\}\right\} \\
&< 5\frac{x_0^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}} \leq 5\frac{\left[x_0(1+x_0)\right]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}},
\end{aligned}$$

when $j = 0$. As a result, it remains to find an upper estimate for each $M_{k,n,j}(x_0)$ when $j = 1, 2, ...$ is fixed, $x_0 \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right]$, $k \in \{0, 1, ...\}$ and $\alpha = 2, 3, ...$ In fact, we will show that

$$M_{k,n,j}(x_0) \leq 6\frac{\left[x_0(1+x_0)\right]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}} \tag{5.2}$$

for all $x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right]$, $k = 0, 1, 2, ...$ which implies that directly

$$E_n(x_0) \leq 6\frac{\left[x_0(1+x_0)\right]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}, \text{ for all } x_0 \in [0,\infty) \text{ and } n \in \mathbb{N},\, n \geq 4.$$

Taking $\delta_n = \frac{1}{(n-1)^{1-\frac{1}{\alpha}}}$ in (5.1), we received the estimate in the statement immediately. So, we consider the following cases to complete the proof of (5.2):

1) $\frac{n}{n+1}j \leq k \leq \frac{n}{n-1}(j+1)$;
2) $k > \frac{n}{n-1}(j+1)$;
3) $k < \frac{n}{n+1}j$.

Case 1). We have

$$\frac{k}{n}-x_0\leq\frac{\frac{n}{n-1}\left(j+1\right)}{n}-x_0\leq\frac{j+1}{n-1}-\frac{j}{n-1}=\frac{1}{n-1}\leq\frac{2x_0}{n-1}+\frac{1}{n-1}.$$

On the other hand,

$$\begin{aligned}\frac{k}{n}-x_0 &\geq \frac{\frac{n}{n+1}j}{n}-x_0\geq\frac{j}{n+1}-\frac{j+1}{n-1}\\ &= \frac{nj-j-nj-n-j-1}{\left(n-1\right)\left(n+1\right)}=\frac{-2j-n-1}{\left(n-1\right)\left(n+1\right)}\\ &= \frac{-2j}{\left(n-1\right)\left(n+1\right)}-\frac{1}{n-1}\geq-\frac{2x_0}{n+1}-\frac{1}{n-1}\\ &\geq -\frac{2x_0}{n-1}-\frac{1}{n-1}.\end{aligned}$$

As a result, we obtain$\left|\frac{k}{n}-x_0\right|\leq\frac{2x_0}{n-1}+\frac{1}{n-1}$. Since $x_0^{\alpha-2}\leq n-1$ and $n\geq 4$ from the hypothesis, we obtain

$$\frac{x_0}{n-1}\leq\frac{\left[x_0\left(1+x_0\right)\right]^{\frac{1}{\alpha}}}{\left(n-1\right)^{1-\frac{1}{\alpha}}}$$

for all $x>0$ and $\alpha=2,3,...$ And also

$$\begin{aligned}\frac{1}{n-1} &= \left(\frac{1}{n-1}\right)^{\frac{1}{\alpha}}\left(\frac{1}{n-1}\right)^{1-\frac{1}{\alpha}}\leq\left(\frac{j}{n-1}\right)^{\frac{1}{\alpha}}\frac{1}{\left(n-1\right)^{1-\frac{1}{\alpha}}}\\ &\leq \frac{\left(x_0\right)^{\frac{1}{\alpha}}}{\left(n-1\right)^{1-\frac{1}{\alpha}}}\leq\frac{\left[x_0\left(1+x_0\right)\right]^{\frac{1}{\alpha}}}{\left(n-1\right)^{1-\frac{1}{\alpha}}}.\end{aligned}$$

Hence, it follows that $M_{k,n,j}(x)=m_{k,n,j}(x)\left|\frac{k}{n}-x_0\right|\leq 3\frac{\left[x_0(1+x_0)\right]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}$.

Case 2). Subcase a). Let us $k>\frac{n}{n-1}\left(j+1\right)$ and assume first that $\left(k-j\right)^{\alpha}<\frac{(n+j)(k+1)}{(n-1)}$. If we denoting $k=j+\beta$, where $\beta\geq 1$, the condition becomes

$$\begin{aligned}&\beta^{\alpha}<\tfrac{(n+j)(j+\beta+1)}{(n-1)}\\ &\beta^{\alpha}\left(n-1\right)-\left(n+j\right)\left(j+\beta+1\right)<0\\ &\beta^{\alpha}\left(n-1\right)-\beta\left(n+j\right)-\left(n+j\right)\left(j+1\right)<0.\end{aligned}$$

Let us define the function $\eta_{\alpha}\left(t\right):=t^{\alpha}\left(n-1\right)-t\left(n+j\right)-\left(n+j\right)\left(j+1\right),\ t\in\mathbb{R}$. We claim that

$$\eta_{\alpha}\left(\left[\frac{3\left(j+1\right)\left(n+j\right)}{n-1}\right]^{\frac{1}{\alpha}}\right)>0,$$

which implies $k-j=\beta<\left[\frac{3(j+1)(n+j)}{n-1}\right]^{\frac{1}{\alpha}}$. After simple calculation, we have

$$\begin{aligned}
&\eta_\alpha\left(\left[\frac{3(j+1)(n+j)}{n-1}\right]^{\frac{1}{\alpha}}\right)\\
&=\frac{3(j+1)(n+j)}{n-1}(n-1)-\left[\frac{3(j+1)(n+j)}{n-1}\right]^{\frac{1}{\alpha}}(n+j)-(n+j)(j+1)\\
&=2(n+j)(j+1)-\left[\frac{3(j+1)(n+j)}{n-1}\right]^{\frac{1}{\alpha}}(n+j)\\
&=(n+j)(j+1)^{\frac{1}{\alpha}}\left\{2(j+1)^{1-\frac{1}{\alpha}}-\left(3+\frac{3j+3}{n-1}\right)^{\frac{1}{\alpha}}\right\}\\
&\geq(n+j)(j+1)^{\frac{1}{\alpha}}\left\{2(j+1)^{1-\frac{1}{\alpha}}-\left(\frac{3j+9}{2}\right)^{\frac{1}{\alpha}}\right\}>0.
\end{aligned}$$

In the above, we have considered the following: It is clear that $2^{\alpha+1}(j+1)^{\alpha-1}>3j+9$ for all $\alpha\geq 2$ and $j\geq 1$. It follows that $2^{\alpha}(j+1)^{\alpha-1}>\frac{3j+9}{2}$ and we obtain

$$2(j+1)^{1-\frac{1}{\alpha}}>\left(\frac{3j+9}{2}\right)^{\frac{1}{\alpha}}.$$

Based on the above findings, we have

$$\begin{aligned}
\overline{M}_{k,n,j}(x_0) &= m_{k,n,j}(x_0)\left(\frac{k}{n-1}-x_0\right)\leq\frac{k}{n-1}-x_0\\
&\leq \frac{k}{n-1}-\frac{j}{n-1}=\frac{k-j}{n-1}=\frac{\beta}{n-1}\\
&< \frac{\left[\frac{3(j+1)(n+j)}{n-1}\right]^{\frac{1}{\alpha}}}{n-1}=\frac{\left[3(j+1)(n+j)\right]^{\frac{1}{\alpha}}}{(n-1)^{1+\frac{1}{\alpha}}}\leq\frac{\left[6j(n+j)\right]^{\frac{1}{\alpha}}}{(n-1)^{1+\frac{1}{\alpha}}}\\
&= \frac{1}{(n-1)^{1-\frac{1}{\alpha}}}\left(\frac{6j}{n-1}\right)^{\frac{1}{\alpha}}\left(\frac{n+j-1}{n-1}\right)^{\frac{1}{\alpha}}\left(\frac{n+j}{n+j-1}\right)^{\frac{1}{\alpha}}\\
&\leq \frac{1}{(n-1)^{1-\frac{1}{\alpha}}}(6x_0)^{\frac{1}{\alpha}}(1+x_0)^{\frac{1}{\alpha}}\left(\frac{4}{3}\right)^{\frac{1}{\alpha}}\\
&= 2^{\frac{3}{\alpha}}\frac{\left[x_0(1+x_0)\right]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}},
\end{aligned}$$

where we used that $\frac{n+j-1}{n-1}=1+\frac{j}{n-1}\leq 1+x_0$ and $\frac{n+j}{n+j-1}\leq\frac{4}{3}$ since $j\geq 1$, $n\geq 4$.

Subcase b). Assume now that $(k-j)^{\alpha}\geq\frac{(n+j)(k+1)}{(n-1)}$, which means

$$(k-j)^{\alpha}(n-1)-(n+j)(k+1)\geq 0.$$

Because n and j are fixed, we can define the real function

$$\lambda_{n,j}(t) := (t-j)^{\alpha}(n-1) - (n+j)(t+1),$$

for all $t \in \mathbb{R}$. For $t \geq \frac{n}{n-1}(j+1)$, $\lambda_{n,j}(t)$ is nondecreasing on the interval $\left[\frac{n}{n-1}(j+1), \infty\right)$. Observe that

$$\begin{aligned} \lambda'_{n,j}(t) &\geq \alpha\left(\frac{n}{n-1}(j+1)-j\right)^{\alpha-1}(n-1)-n-j \\ &= \alpha\left(\frac{n+j}{n-1}\right)^{\alpha-1}(n-1)-n-j \\ &= (n+j)\left[\alpha\left(\frac{n+j}{n-1}\right)^{\alpha-2}-1\right] > 0. \end{aligned}$$

In the inequality above, we take into account that $\alpha \geq 2$, and $n+j > n-1$ i.e. $\frac{n+j}{n-1} > 1$, $(\frac{n+j}{n-1})^{\alpha-2} > \frac{1}{\alpha}$. Since $\lim_{t\to\infty} \lambda_{n,j}(t) = \infty$, by the monotonicity of $\lambda_{n,j}$ too, it follows that there exists $\bar{k} \in \mathbb{N}$, $\bar{k} > \frac{n}{n-1}(j+1)$ of minimum value, satisfying the inequality

$$\lambda_{n,j}(\bar{k}) = (\bar{k}-j)^{\alpha}(n-1) - (n+j)(\bar{k}+1) \geq 0.$$

Denote $k_1 = \bar{k}+1$, where evidently $k_1 \geq j+1$. If $k_1 \geq \frac{n}{n-1}(j+1)$, then from the properties of $\lambda_{n,j}$ and by the way we choose $\bar{k}$ it results that $\lambda_{n,j}(k_1) < 0$. If $k_1 < \frac{n}{n-1}(j+1)$, then $j < k_1 < \frac{n}{n-1}(j+1)$. Now, let us prove

$$\lambda_{n,j}\left(\frac{n}{n-1}(j+1)\right) < 0.$$

Observe that

$$\begin{aligned} \lambda_{n,j}\left(\frac{n}{n-1}(j+1)\right) &= \left(\frac{n}{n-1}(j+1)-j\right)^{\alpha}(n-1)-(n+j)\left(\frac{n}{n-1}(j+1)+1\right) \\ &= \left(\frac{n+j}{n-1}\right)^{\alpha}(n-1)-(n+j)\left(\frac{nj+2n-1}{n-1}\right) \\ &= (n+j)\left[\left(\frac{n+j}{n-1}\right)^{\alpha-1}-\frac{nj+2n-1}{n-1}\right] < 0. \end{aligned}$$

We were able to write the last inequality above because, firstly, it holds for $\alpha = 2$. Secondly, for $\alpha > 2$, we have $x_0^{\alpha-2} \leq n-1$ from the hypothesis, and since also we have $x_0 \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right]$, this give us that $\left(\frac{j}{n-1}\right)^{\alpha-2} \leq n-1$ or equivalently $j \leq (n-1)^{1+\frac{1}{2-\alpha}}$. Since $\lambda_{n,j}$ is a polynomial function and because $\lambda_{n,j}(j) < 0$ and $\lambda_{n,j}\left(\frac{n}{n-1}(j+1)\right) < 0$ we immediately obtain the same conclusion as in the previous case, which is $\lambda_{n,j}(k_1) < 0$ or equivalently $\beta^{\alpha}(n-1)-(n+j)(j+\beta+1) < 0$, where $k_1 = j+\beta$. Using the same method as in subcase

a), we have $k_1 - j < \left[\frac{3(j+1)(n+j)}{n-1}\right]^{\frac{1}{\alpha}}$. Then

$$\begin{aligned}\overline{M}_{\bar{k},n,j}(x_0) &= m_{\bar{k},n,j}(x_0)\left(\frac{\bar{k}}{n-1} - x_0\right) \leq \frac{\bar{k}}{n-1} - x_0 \\ &\leq \frac{\bar{k}}{n-1} - \frac{j}{n-1} = \frac{\bar{k}-j}{n-1} = \frac{k_1-j}{n-1} + \frac{1}{n-1} \\ &\leq 2^{\frac{3}{\alpha}}\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}} + \frac{1}{n-1} \\ &\leq 2^{\frac{3}{\alpha}}\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}} + \frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}} \\ &\leq 4\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}},\end{aligned}$$

taking into account that $2^{\frac{3}{\alpha}} + 1 \leq 4$. By Lemma 4.5 (i), it follows that $\overline{M}_{\bar{k},n,j}(x_0) \geq \overline{M}_{\bar{k}+1,n,j}(x_0) \geq \ldots$. Therefore, $\overline{M}_{\bar{k},n,j}(x_0) < 4\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}$ for any $k \in \left\{\bar{k}, \bar{k}+1, \ldots\right\}$. As a result, in both subcases, by Lemma 4.4 (i), we have

$$M_{k,n,j}(x_0) < 4\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}.$$

Case 3). Subcase a). Assume first that $(j-k)^{\alpha} < \frac{k(n+j-1)}{(n-1)}$. If we denoting $k = j - \beta$, where $\beta \geq 1$, the condition becomes

$$\begin{aligned}&\beta^{\alpha} < \tfrac{(j-\beta)(n+j-1)}{(n-1)} \\ &\beta^{\alpha}(n-1) - (j-\beta)(n+j-1) < 0.\end{aligned}$$

Let us define the function

$$\phi_{n,j}(t) = t^{\alpha}(n-1) - (j-t)(n+j-1),\ t \in \mathbb{R}.$$

We claim that $\phi_{n,j}\left(\left[\frac{j(n+j-1)}{n-1}\right]^{\frac{1}{\alpha}}\right) > 0$ which implies $j - k = \beta < \left[\frac{j(n+j-1)}{n-1}\right]^{\frac{1}{\alpha}}$. After simple calculation, we have

$$\begin{aligned}\phi_{n,j}\left(\left[\frac{j(n+j-1)}{n-1}\right]^{\frac{1}{\alpha}}\right) &= \frac{j(n+j-1)}{n-1}(n-1) - \left(j - \left[\frac{j(n+j-1)}{n-1}\right]^{\frac{1}{\alpha}}\right)(n+j-1) \\ &= j(n+j-1) - j(n+j-1) + \left[\frac{j(n+j-1)}{n-1}\right]^{\frac{1}{\alpha}}(n+j-1) \\ &= \left[\frac{j(n+j-1)}{n-1}\right]^{\frac{1}{\alpha}}(n+j-1) > 0.\end{aligned}$$

Then we obtain

$$\begin{aligned}
\underline{M}_{k,n,j}(x_0) &= m_{k,n,j}(x_0)\left(x_0-\frac{k}{n-1}\right)\le\frac{j+1}{n-1}-\frac{k}{n-1}\\
&= \frac{j-k}{n-1}+\frac{1}{n-1}<\frac{\left[\frac{j(n+j-1)}{n-1}\right]^{\frac{1}{\alpha}}}{n-1}+\frac{1}{n-1}\\
&= \frac{1}{(n-1)^{1-\frac{1}{\alpha}}}\left(\frac{j(n+j-1)}{n-1}\right)^{\frac{1}{\alpha}}\left(\frac{1}{n-1}\right)^{\frac{1}{\alpha}}+\frac{1}{n-1}\\
&= \frac{1}{(n-1)^{1-\frac{1}{\alpha}}}\left(\frac{j}{n-1}\right)^{\frac{1}{\alpha}}\left(1+\frac{j}{n-1}\right)^{\frac{1}{\alpha}}+\frac{1}{n-1}\\
&\le \frac{1}{(n-1)^{1-\frac{1}{\alpha}}}(x_0)^{\frac{1}{\alpha}}(1+x_0)^{\frac{1}{\alpha}}+\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}\\
&= 2\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}.
\end{aligned}$$

Subcase b). Assume now that $(j-k)^{\alpha}\ge\frac{k(n+j-1)}{(n-1)}$ it means $(j-k)^{\alpha}(n-1)-k(n+j-1)\ge 0$. Since n and j are fixed, we can define the real function

$$\psi_{n,j}(t):=(j-t)^{\alpha}(n-1)-t(n+j-1),$$

for all $t\in\mathbb{R}$. For $t\le\frac{n}{n+1}j$, $\psi_{n,j}(t)$ is nonicreasing on the interval $\left[0,\frac{n}{n+1}j\right]$. Really since

$$\psi'_{n,j}(t)=-\alpha(j-t)^{\alpha-1}(n-1)-(n+j-1)<0.$$

Let's we observe $\psi_{n,j}\left(\frac{nj}{n+1}\right)<0$:

$$\begin{aligned}
\psi_{n,j}\left(\frac{nj}{n+1}\right) &= \left(j-\frac{nj}{n+1}\right)^{\alpha}(n-1)-\frac{nj}{n+1}(n+h-1)\\
&= j^{\alpha}\frac{(n-1)}{(n+1)^{\alpha}}-\frac{nj}{n+1}(n+h-1)<0.
\end{aligned}$$

We were able to write the last inequality above because of $x_0^{\alpha-2}\le n-1$ for the same reasons as Case 2) subcase b). Considering these results, that we find above, and by the monotonicity of $\psi_{n,j}$ too, it follows that there exists $\tilde{k}\in\mathbb{N}$, $\tilde{k}<\frac{nj}{n+1}$ of maximum value, such that $\psi_{n,j}\left(\tilde{k}\right)=\left(j-\tilde{k}\right)^{\alpha}(n-1)-\tilde{k}(n+j-1)\ge 0$. Denoting $k_2=\tilde{k}+1$ and reasoning as in case 2), subcase b) we have $\psi_{n,j}(k_2)<0$. Furter, reasoning as in case 3), subcase a) we have $j-k_2<\left[\frac{j(n+j-1)}{n-1}\right]^{\frac{1}{\alpha}}$. It follows

$$\begin{aligned}
\underline{M}_{\tilde{k},n,j}(x_0) &= m_{\tilde{k},n,j}(x_0)\left(x_0-\frac{\tilde{k}}{n-1}\right)\le\frac{j+1}{n-1}-\frac{\tilde{k}}{n-1}\\
&= \frac{j-k_2}{n-1}+\frac{2}{n-1}<3\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}.
\end{aligned}$$

In light of Lemma 4.5, (ii), it follows that $\underline{M}_{\tilde{k},n,j}(x_0) \geq \underline{M}_{\tilde{k}-1,n,j}(x_0) \geq ... \geq \underline{M}_{0,n,j}(x_0)$. Thus we obtain

$$\underline{M}_{k,n,j}(x_0) < 3\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}$$

for any $k \in \{0,1,...,\tilde{k}\}$. In both subcases, Lemma 4.4 (iii), we have

$$M_{k,n,j}(x_0) < 6\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}.$$

Taking into consideration the fact that

$$\max\left\{ 3\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}, 4\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}, 5\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}, 6\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}} \right\} \leq 6\frac{[x_0(1+x_0)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}},$$

we have desired result immediately. □

6. Weighted Rate of Convergence

We see that previous results works for a fixed x_0 point or finite intervals. If we want to obtain a uniform approximation order on infinite intervals, then we should use weighted modulus of continuities. Before giving useful properties about these type of modulus of continuities, let us recall the following spaces and norm (see, for instance, [20] and [21])

$$\begin{aligned} B_\rho(\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{R} |\ \text{a constant } M_f \text{ depending on } f \text{ exists} \\ &\quad \text{such that } |f| \leq M_f \rho\}, \end{aligned}$$

$$C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R}) |\ f \text{ continuous on } \mathbb{R}\},$$

endowed with the norm:

$$\|f\|_\rho = \sup_{x\geq 0\geq} \frac{|f(x)|}{\rho(x)}.$$

In order to obtain rate of weighted approximation of the positive linear operators defined on infinite intervals, various weighted modulus of continuities are introduced. Some of them include term h in the denominator of the supremum expression. In the chronological order, let us refer to some related papers as [1, 4, 16, 19, 22, 23, 28, 30]. The weighted modulus defined in [1], in order to obtain weighted approximation properties of some linear positive operators on $\mathbb{R}_+$. In [23], the second author together with Gadjieva introduced the following modulus of continuity:

$$\Omega(f;\delta) = \sup_{0\leq x,|h|\leq\delta} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}. \tag{6.1}$$

There are some papers including rates of weighted approximation with the help of $\Omega(f;\delta)$. (see, for instance, [3], [14], [17], and [29]). In [16], second author defined the following modulus of continuity:

$$\omega_\rho(f;\delta) = \sup_{0\leq x,|h|\leq\delta} \frac{|f(x+h)-f(x)|}{\rho(x+h)} \tag{6.2}$$

where $\rho(x) \geq \max(1,x)$.

In [16], the author introduced a generalization of the Gadjiev-Ibragimov operators which includes many well-known operators and obtain its rate of weighted convergence with the help of $\omega_\rho(f;\delta)$ defined in (6.2). In [30], Moreno introduced another type of modulus of continuity in (6.2) as follows

$$\overline{\Omega}_\alpha(f;\delta)=\sup_{0\le x,|h|\le\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^\alpha}.$$

In [22], Gadjiev and Aral defined the following modulus of continuity:

$$\widetilde{\Omega}_\rho(f;\delta)=\sup_{x,t\in\mathbb{R}_+,|\rho(t)-\rho(x)|\le\delta}\frac{|f(t)-f(x)|}{(|\rho(t)-\rho(x)|+1)\,\rho(x)}$$

where $\rho(0)=1$ and $\inf_{x\ge0}\rho(x)\ge1$. It is obvious that by choosing $\alpha=2$, in the definition of $\overline{\Omega}_\alpha(f;\delta)$, then we obtain $\overline{\Omega}_2(f;\delta)=\omega_{\rho_0}(f;\delta)$ for $\rho_0(x)=1+x^2$, and if we choose $\alpha=2+\lambda$ in the definition of $\overline{\Omega}_\alpha(f;\delta)$, then we obtain

$$\widehat{\Omega}_{\rho\lambda}(f;\delta)=\sup_{0\le x,|h|\le\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^{2+\lambda}}$$

(see [2]). Finally, in [28], Holhoş defined a more general weighted modulus of continuity as

$$\omega_\varphi(f;\delta)=\sup_{0\le x\le y,|\varphi(y)-\varphi(x)|\le\delta}\frac{|f(x)-f(y)|}{\rho(x)+\rho(y)}$$

such that, for $\varphi(x)=x$, this modulus of continuity is equivalent to $\Omega(f;\delta)$ defined in (6.1). Also, let $C^0_\rho(\mathbb{R})$ be the subspace of all functions in $C_\rho(\mathbb{R})$ such that $\lim_{|x|\to\infty}\frac{f(x)}{\rho(x)}$ exists finitely. Notice also that some remarkable properties about these type of modulus of continuities can be found in [15]. In light of these definitions, we can give the following theorem.

Theorem 6.1. *Let $f:[0,\infty)\to\mathbb{R}_+$ be continuous. Then for all $x\in[0,\infty)$, which also satisfy $x_0^{\alpha-2}\le n-1$ and $n\ge4$, we have the following order of approximation for the operators (3.1) to the function f by means of the weighted modulus of continuity defined in (6.2). Then for each $f\in C^0_{\rho_0}(\mathbb{R}_+)$, we have*

$$\frac{\left|V_n^{(M)}(f)(x)-f(x)\right|}{(\rho_0(x))^2}\le\frac{\left[\ (1+9x^2)\left(1+6\,[x(1+x)]^{\frac{1}{\alpha}}\right)\ \right]}{(1+x^2)^2}\,\omega_{\rho_0}\left(f;\frac{1}{(n-1)^{1-\frac{1}{\alpha}}}\right),\qquad(6.3)$$

for all $n\in\mathbb{N}$, $n\ge4$, where $\rho_0(x)=1+x^2$ and $\alpha=2,3,\ldots$.

Proof. By using the properties of $\omega_{\rho_0}(f;\delta)$, (see [30]), we can write

$$\left|V_n^{(M)}(f)(x)-f(x)\right|\le\left[\ \left(1+(2x+V_n^{(M)}(e_1)(x))^2\right)\left(\tfrac{1}{\delta}V_n^{(M)}(\varphi_x)(x)+1\right)\ \right]\omega_{\rho_0}(f;\delta).\quad(6.4)$$

In the proof of Theorem 5.1, for all $n\in\mathbb{N}$, $n\ge4$ and $x\in[0,\infty)$, $x_0^{\alpha-2}\le n-1$, we obtain

$$V_n^{(M)}(\varphi_x)(x)\le6\frac{[x(1+x)]^{\frac{1}{\alpha}}}{(n-1)^{1-\frac{1}{\alpha}}}.\qquad(6.5)$$

On the other hand, we have

$$
\begin{aligned}
V_n^{(M)}(e_1)(x) &= \frac{\bigvee_{k=0}^{\infty} \frac{(n+k-1)!}{k!(n-1)!}\frac{x^k}{(1+x)^{n+k}}\frac{k}{n}}{\bigvee_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}} = \frac{x\bigvee_{k=1}^{\infty}\binom{n+k-1}{k-1}\frac{x^{k-1}}{(1+x)^{n+k}}}{\bigvee_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}} \\
&= \frac{x\bigvee_{k=0}^{\infty}\binom{n+k}{k}\frac{x^k}{(1+x)^{n+k+1}}}{\bigvee_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}}.
\end{aligned}
$$

Thus

$$V_n^{(M)}(e_1)(x) \leq x. \tag{6.6}$$

Substituting inequalities (6.5) and (6.6) into (6.4) and choosing

$$\delta = \frac{1}{(n-1)^{1-\frac{1}{\alpha}}},$$

we obtain the desired conclusion immediately. □

This theorem allows us to express the following weighted uniform approximation result.

Theorem 6.2. *Let $f:[0,\infty)\to\mathbb{R}_+$ be continuous. Then, for all $x\in[0,\infty)$, which also satisfy $x_0^{\alpha-2}\leq n-1$ and $n\geq 4$, we have the following order of approximation for the operators (3.1) to the function f by means of the weighted modulus of continuity defined in (6.2). Then, for each $f\in C_{\rho_0}^0(\mathbb{R}_+)$,*

$$\left\| V_n^{(M)}(f)(x) - f(x) \right\|_{\rho_0^2(x)} \leq 70\ \omega_{\rho_0}\left(f;\frac{1}{(n-1)^{1-\frac{1}{\alpha}}}\right), \tag{6.7}$$

for all $n\in\mathbb{N}$, $n\geq 4$, where $\rho_0(x)=1+x^2$ and $\alpha=2,3,\ldots$.

Proof. From $\frac{1}{1+x^2}\leq 1$, $\frac{x^2}{1+x^2}\leq 1$, and $\frac{[x(x+1)]^{\frac{1}{\alpha}}}{1+x^2}\leq 1$, we have

$$\frac{(1+9x^2)\left(1+6\left[x(1+x)\right]^{\frac{1}{\alpha}}\right)}{(1+x^2)^2} \leq 70. \tag{6.8}$$

With the aid of (6.8) in (6.3), we obtain desired result easily. □

Remark 6.3. Theorem 5.1, Theorem 6.1 and Theorem 6.2 show that the orders of pointwise approximation, weighted approximation and weighted uniform approximation are $1/(n-1)^{1-\frac{1}{\alpha}}$. For big enough α , $1/(n-1)^{1-\frac{1}{\alpha}}$ tends to $1/(n-1)$. As a result, since $1-\frac{1}{\alpha}\geq\frac{1}{2}$ for $\alpha=2,3,\ldots$ this selection of α improves the order of approximation.

Dedication

This paper is dedicated to o Professor Vladimir Tikhomirov on the occasion of his 90th birthday, with high esteem.

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