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SOLUTIONS OF TWO CLASSES OF ORDINARY DIFFERENTIAL EQUATIONS BY THE LAPLACE TRANSFORM METHOD VIA H AND T FUNCTIONS

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Abstract. The Mittag-Leffler function is an extension of the exponential function and plays important role in the field of differential equations. The objective of this paper is to find two classes of differential equations whose solutions are functions T_{pj} and H_{pj}, $j=0,\dots,p-1$, which are the special case of Mittag-Leffler functions.
Keywords. Laplace transform; Mittag-Leffler functions; Ordinary differential equations.
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1. INTRODUCTION

The special function

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad z,\alpha\in\mathbb{C},\ \Re(\alpha)>0 \tag{1.1}$$

is called the Mittag-Leffler function, introduced in [13, 14] by Gosta Magnus Mittag-Leffler. The Mittag-Leffler function was introduced to give an answer to a classical question of complex analysis, that is, to describe the procedure of analytic continuation of power series outside the disc of their convergence [15]. The generalization of Mittag-Leffler function

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad z,\alpha,\beta\in\mathbb{C},\ \Re(\alpha),\Re(\beta)>0 \tag{1.2}$$

first appeared in [17, 18]. The functions (1.1) and (1.2) are a simple generalization of the exponential function and their main properties were given in [5, 7]. The Mittag-leffler functions have various applications in natural and engineering sciences, especially in physics, stochastic

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systems, dynamical systems theory, and disordered systems; see, e.g., [10, 11]. The Mittag-Leffler functions can be utilized in fractional calculus and its applications. They appear in the solutions of fractional-order integral equations and fractional-order differential equations, and especially in the investigations of the fractional generalization of the kinetic equation, random walks, Lévy flights, superdiffusive transport and in the study of complex systems [11]. Recently, many researchers defined various generalizations and extensions of the Mittag-Leffler functions, which are valuable for studying fractional calculus; see, e.g., [2, 3, 8, 9, 16, 19, 20].

As stated, the Mittag-Leffler functions often appear in the solutions of fractional differential and integral equations. However, in this paper, we identify certain classes of ordinary differential equations whose solutions involve Mittag-Leffler functions. Specifically, we focus on the functions T_{pj} and H_{pj}, $j = 0,\dots,p-1$, as a special case of Mittag-Leffler functions [1]. Utilizing a Laplace transform [4, 6, 12], we present:

(i) A class of ordinary differential equations of order p (Theorem 3.1), whose solutions are linear combinations of the functions T_{pj} and H_{pj}, $j = 0,1,\dots,p-1$, where $p \geq 2$ is a prime number.

(ii) A class of ordinary differential equations of order $p+q$ (Theorem 3.3), whose solutions are linear combinations of functions T_{pj} and H_{pj}, $j = 0,1,\dots,p-1$, and T_{qj} and H_{qj}, $j = 0,1,\dots,q-1$, where p,q are a prime numbers such that $p+q \geq 4$. In the special case when $p = q$, we obtain an application, Theorem 4.1.

2. Preliminaries

2.1. **Notions.** We use the following notation: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of positive integers, real and complex numbers, respectively. For a given complex number z, we denote with $\Re(z)$ the real and with $\Im(z)$ the imaginary part. A Laplace transform of a function f is denoted by $\mathscr{L}(f(t))(s) = \int_0^{+\infty} f(t)e^{st}dt$, $s \in \mathbb{C}$. The gamma function is denoted $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$, $\Re(z) > 0$.

2.2. **Functions T_{pj} and H_{pj}.** We recall some definitions and assertions from [1].

Definition 2.1. [1] The functions $T_{pj}, H_{pj} : \mathbb{R} \to \mathbb{R}$, $j = 0,1,2,\cdots,p-1$, $p \in \mathbb{N}$, are defined as follows:

$$T_{pj}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!}, \quad H_{pj}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{(pn+j)!}. \tag{2.1}$$

For each $t \in \mathbb{R}$, we have

$$T_{10}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} = e^{-t}, \qquad H_{10}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t,$$

$$T_{20}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = \cos t, \qquad T_{21}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sin t,$$

$$H_{20}(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \cosh t, \qquad H_{21}(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = \sinh t.$$

Lemma 2.2. [1] *Let p be a prime number, $p \geq 2$. Then*

$$\mathscr{L}(T_{pj}(at))(s) = \frac{s^{p-j-1}a^j}{s^p + a^p}, j = 0,1,...,p-1,$$

$$\mathscr{L}(H_{pj}(at))(s)=\frac{s^{p-j-1}a^j}{s^p-a^p}, j=0,1,...,p-1,$$
$$\mathscr{L}(\frac{p}{t}(1-T_{p0}(at)))(s)=\ln(1+\frac{a^p}{s^p}),$$
$$\mathscr{L}(\frac{p}{t}(1-H_{p0}(at)))(s)=\ln(1-\frac{a^p}{s^p}),$$

and

$$\mathscr{L}(tT_{p0}(at))(s)=\frac{s^{2p-2}-(p-1)a^p s^{p-1}}{(s^p+a^p)}.$$

3. Main Results

Theorem 3.1. *Let p be a prime number, $p\geq 2$ and $a>0$. The solution of the equation*

$$y^{(p)}(t)-ay(t)-b=0, \qquad y(0)=a_0\ ,\ y'(0)=a_1,...,\ y^{(p-1)}(0)=a_{p-1} \tag{3.1}$$

is

$$y(t)=\sum_{j=0}^{p-1}\frac{a_j}{\sqrt[p]{a^j}}H_{pj}(\sqrt[p]{a}t)+\frac{b}{a}H_{p0}(\sqrt[p]{a}t)-\frac{b}{a}. \tag{3.2}$$

Proof. Using the Laplace transform on (3.1), we obtain

$$\mathscr{L}(y^{(p)}(t)-ay(t)-b)(s)=s^p\mathscr{L}(y(t))(s)-\sum_{j=0}^{p-1}s^jy^{(p-j-1)}(0)-a\mathscr{L}(y(t))(s)-\frac{b}{s}=0$$

from which it follows

$$\mathscr{L}(y(t))(s)=\frac{\sum\limits_{j=0}^{p-1}a_{p-j-1}s^j+\frac{b}{s}}{s^p-a}.$$

Using a inverse Laplace transform and Lemma 2.2, we obtain

$$\begin{aligned}y(t)&=\sum_{j=0}^{p-1}a_{p-j-1}\mathscr{L}^{-1}(\frac{s^j}{s^p-a})(t)+b\mathscr{L}^{-1}(\frac{1}{s(s^p-a)})(t)\\&=\sum_{j=0}^{p-1}\frac{a_{p-j-1}}{\sqrt[p]{a^{p-j-1}}}H_{pp-j-1}(\sqrt[p]{a}t)+b\mathscr{L}^{-1}(\frac{1}{s(s^p-a)})(t)\\&=\sum_{j=0}^{p-1}\frac{a_{p-j-1}}{\sqrt[p]{a^{p-j-1}}}H_{pp-j-1}(\sqrt[p]{a}t)+\frac{b}{a}H_{p0}(\sqrt[p]{a}t)-\frac{b}{a}.\end{aligned}$$

□

From Theorem 3.1 and Lemma 2.2, we have the following remark.

Remark 3.2. Let p be a prime number, $p\geq 2$, and $a<0$. The solution of the equation

$$y^{(p)}(t)-ay(t)-b=0, \qquad y(0)=a_0,\ y'(0)=a_1,\dots,\ y^{(p-1)}(0)=a_{p-1} \tag{3.3}$$

is

$$y(t)=\sum_{j=0}^{p-1}\frac{a_j}{\sqrt[p]{(-a)^j}}T_{pj}(\sqrt[p]{-a}t)+\frac{b}{a}T_{p0}(\sqrt[p]{-a}t)-\frac{b}{a}. \tag{3.4}$$

Theorem 3.3. *Let p,q be prime numbers, $p+q\geq 4$. The solution of the differential equation*

$$\begin{aligned} &y^{(p+q)}(t)-ay^{(p)}(t)-by^{(p)}(t)+aby(t)-d=0,\\ &y(0)=a_0,\ y'(0)=a_1,\dots,\ y^{(p+q-1)}(0)=a_{p+q-1} \end{aligned} \tag{3.5}$$

where $a,b>0,\ a\neq b$, is

$$\begin{aligned} y(t)=&\sum_{j=0}^{p-1}\frac{A_j}{\sqrt[p]{b^j}}H_{pj}(\sqrt[p]{b}t)+\sum_{j=0}^{q-1}\frac{B_j}{\sqrt[q]{a^j}}H_{qj}(\sqrt[q]{a}t)+d\sum_{j=1}^{p-1}\frac{C_{j-1}}{\sqrt[p]{b^j}}H_{pj}(\sqrt[p]{b}t)\\ &+d\sum_{j=1}^{q-1}\frac{D_{j-1}}{\sqrt[q]{a^j}}H_{qj}(\sqrt[q]{a}t)+\frac{dC_{p-1}}{b}(H_{p0}(\sqrt[p]{b}t)-1)+\frac{dD_{q-1}}{a}(H_{q0}(\sqrt[q]{a}t)-1), \end{aligned}$$

where $A_j,\ j=0,\dots,p-1$ and $B_j,\ j=0,\dots,q-1$ are solutions to the system

$$\begin{aligned} aA_{p-j-1}+bB_{q-j-1}&=aa_{p-j-1}+ba_{q-j-1}-a_{p+q-j-1},\ j=0,\dots q-1,\\ A_{q-j-1}+B_{q-j-1}&=a_{q-j-1},\quad j=0,\dots q-1,\\ A_{p+q-j-1}-aA_{p-j-1}&=a_{p+q-j-1}-aa_{p-j-1},\ j=q,\dots,p-1, \end{aligned} \tag{3.6}$$

and $C_j,\ j=0,\dots,p-1$ and $D_j,\ j=0,\dots,q-1$ are solutions to the system

$$\begin{aligned} aC_{p-1}+bD_{q-1}&=-1\\ aC_{p-j-1}+bD_{q-j-1}&=0,\quad j=1,\dots,q-1\\ C_{q-j-1}+D_{q-j-1}&=0,\quad j=0,\dots,q-1\\ C_{p+q-j-1}-aC_{p-j-1}&=0,\ j=q,\dots,p-1. \end{aligned} \tag{3.7}$$

Proof. Without loos of any generality, we can assume $p>q$. Using the Laplace transform on (3.5), we see that

$$\begin{aligned} &\mathscr{L}(y^{(p+q)}(t)-ay^{(p)}(t)-by^{(q)}(t)+aby(t)-d)(s)\\ &=s^{p+q}\mathscr{L}(y(t))(s)-\sum_{j=0}^{p+q-1}a_{p+q-j-1}s^j-as^p\mathscr{L}(y(t))(s)+a\sum_{j=0}^{p-1}a_{p-j-1}s^j\\ &-bs^q\mathscr{L}(y(t))(s)+b\sum_{j=0}^{q-1}a_{q-j-1}s^j+ab\mathscr{L}(y(t))(s)-\frac{d}{s}=0 \end{aligned}$$

from which it follows that

$$\begin{aligned} \mathscr{L}(y(t))(s)=&\frac{\sum\limits_{j=0}^{q-1}(a_{p+q-j-1}-aa_{p-j-1}-ba_{q-j-1})s^j+\sum\limits_{j=q}^{p-1}(a_{p+q-j-1}-aa_{p-j-1})s^j}{(s^p-b)(s^q-a)}\\ &+\frac{\sum\limits_{j=p}^{p+q-1}a_{p+q-j-1}s^j+\frac{d}{s}}{(s^p-b)(s^q-a)}. \end{aligned}$$

In view of

$$\frac{\sum\limits_{j=0}^{q-1}(a_{p+q-j-1}-aa_{p-j-1}-ba_{q-j-1})s^j+\sum\limits_{j=q}^{p-1}(a_{p+q-j-1}-aa_{p-j-1})s^j+\sum\limits_{j=p}^{p+q-1}a_{p+q-j-1}s^j}{(s^p-b)(s^q-a)}$$

$$= \frac{\sum_{j=0}^{p-1} A_{p-j-1} s^j}{s^p - b} + \frac{\sum_{j=0}^{q-1} B_{p-j-1} s^j}{s^q - a}$$

we have

$$\sum_{j=0}^{q-1} (a_{p+q-j-1} - a a_{p-j-1} - b a_{q-j-1}) s^j + \sum_{j=q}^{p-1} (a_{p+q-j-1} - a a_{p-j-1}) s^j + \sum_{j=p}^{p+q-1} a_{p+q-j-1} s^j$$
$$= (s^q - a) \sum_{j=0}^{p-1} A_{p-j-1} s^j + (s^p - b) \sum_{j=0}^{q-1} B_{p-j-1} s^j$$
$$= -\sum_{j=0}^{q-1} (a A_{p-j-1} + b B_{q-j-1}) s^j + \sum_{j=q}^{p-1} (A_{p+q-j-1} - a A_{p-j-1}) s^j$$
$$+ \sum_{j=p}^{p+q-1} (A_{p+q-j-1} + B_{p+q-j-1}) s^j,$$

from which we have system (3.6). Similarly,

$$\frac{1}{(s^p - b)(s^q - a)} = \frac{\sum_{j=0}^{p-1} C_{p-j-1} s^j}{s^p - b} + \frac{\sum_{j=0}^{q-1} D_{q-j-1} s^j}{s^q - a}$$

implies

$$1 = -\sum_{j=0}^{q-1} (a C_{p-j-1} + b D_{q-j-1}) s^j + \sum_{j=q}^{p-1} (C_{p+q-j-1} - a C_{p-j-1}) s^j$$
$$+ \sum_{j=p}^{p+q-1} (C_{p+q-j-1} + D_{p+q-j-1}) s^j,$$

from which we obtain (3.7). Further,

$$\frac{1}{s(s^p - b)(s^q - a)} = \frac{\sum_{j=0}^{p-1} C_j s^{p-j-1}}{s(s^p - b)} + \frac{\sum_{j=0}^{q-1} D_j s^{q-j-1}}{s(s^q - a)}$$
$$= \frac{\sum_{j=0}^{p-2} C_j s^{p-j-2}}{s^p - b} + \frac{\sum_{j=0}^{q-2} D_j s^{q-j-2}}{s^q - a} + \frac{C_{p-1}}{s(s^p - b)} + \frac{D_{q-1}}{s(s^q - a)}$$
$$= \sum_{j=1}^{p-1} \frac{C_{j-1} s^{p-j-1}}{s^p - b} + \sum_{j=1}^{q-1} \frac{D_{j-1} s^{q-j-1}}{s^q - a} - \frac{C_{p-1}}{bs} + \frac{C_{p-1} s^{p-1}}{b(s^p - b)} - \frac{D_{q-1}}{as} + \frac{D_{q-1} s^{q-1}}{a(s^q - a)}.$$

Using an inverse Laplace transform and Lemma 2.2, we obtain the desired conclusion immediately. □

The following assertion is a consequence of Theorem 3.3 and Lemma 2.2.

Remark 3.4. (i) If $a > 0$, $b < 0$, then the solution of equation (3.5) is

$$y(t) = \sum_{j=0}^{p-1} \frac{A_j}{\sqrt[p]{b^j}} H_{pj}(\sqrt[p]{b}t) + \sum_{j=0}^{q-1} \frac{B_j}{\sqrt[q]{(-a)^j}} T_{qj}(\sqrt[q]{-a}t) + d\sum_{j=1}^{p-1} \frac{C_{j-1}}{\sqrt[p]{(-b)^j}} T_{pj}(\sqrt[p]{-b}t)$$
$$+ d\sum_{j=1}^{q-1} \frac{D_{j-1}}{\sqrt[q]{a^j}} H_{qj}(\sqrt[q]{a}t) + \frac{dC_{p-1}}{b}(T_{p0}(\sqrt[p]{-b}t) - 1) + \frac{dD_{q-1}}{a}(H_{q0}(\sqrt[q]{a}t) - 1),$$

where $A_j,\ j = 0,\dots,p-1$ and $B_j,\ j = 0,\dots,q-1$ are solutions to system (3.6), since C_j, $j = 0,\dots,p-1$, and D_j, $j = 0,\dots,q-1$ are solutions to system (3.7).

(ii) If $a < 0$, $b > 0$, then the solution of equation (3.5) is

$$y(t) = \sum_{j=0}^{p-1} \frac{A_j}{\sqrt[p]{(-b)^j}} T_{pj}(\sqrt[p]{-b}t) + \sum_{j=0}^{q-1} \frac{B_j}{\sqrt[q]{a^j}} H_{qj}(\sqrt[q]{a}t) + d\sum_{j=1}^{p-1} \frac{C_{j-1}}{\sqrt[p]{b^j}} H_{pj}(\sqrt[p]{b}t)$$
$$+ d\sum_{j=1}^{q-1} \frac{D_{j-1}}{\sqrt[q]{(-a)^j}} T_{qj}(\sqrt[q]{-a}t) + \frac{dC_{p-1}}{b}(H_{p0}(\sqrt[p]{b}t) - 1) + \frac{dD_{q-1}}{a}(T_{q0}(\sqrt[q]{-a}t) - 1),$$

where $A_j,\ j = 0,\dots,p-1$, and $B_j,\ j = 0,\dots,q-1$ are solutions to system (3.6), since C_j, $j = 0,\dots,p-1$ and D_j, $j = 0,\dots,q-1$ are solutions of the system (3.7).

(iii) If $a < 0$, $b < 0$, then the solution of equation (3.5) is

$$y(t) = \sum_{j=0}^{p-1} \frac{A_j}{\sqrt[p]{(-b)^j}} T_{pj}(\sqrt[p]{-b}t) + \sum_{j=0}^{q-1} \frac{B_j}{\sqrt[q]{(-a)^j}} T_{qj}(\sqrt[q]{-a}t) + d\sum_{j=1}^{p-1} \frac{C_{j-1}}{\sqrt[p]{(-b)^j}} T_{pj}(\sqrt[p]{-b}t)$$
$$+ d\sum_{j=1}^{q-1} \frac{D_{j-1}}{\sqrt[q]{(-a)^j}} T_{qj}(\sqrt[q]{-a}t) + \frac{dC_{p-1}}{b}(T_{p0}(\sqrt[p]{-b}t) - 1) + \frac{dD_{q-1}}{a}(T_{q0}(\sqrt[q]{-a}t) - 1),$$

where $A_j,\ j = 0,\dots,p-1$ and $B_j,\ j = 0,\dots,q-1$ are solutions to system (3.6), since C_j, $j = 0,\dots,p-1$ and D_j, $j = 0,\dots,q-1$ are solutions to system (3.7).

4. Applications

In this section, we give some applications of Theorem 3.3 from Section 3.

Theorem 4.1. *Let p be a prime number, $p \geq 2$. The solution of the differential equation*

$$\begin{aligned} &y^{(2p)}(t) - ay^{(p)}(t) + by(t) - d = 0, \\ &y(0) = a_0,\ y'(0) = a_1, \dots,\ y^{(2p-1)}(0) = a_{2p-1} \end{aligned} \tag{4.1}$$

where $a = m + n$, $b = m \cdot n$, $m > n > 0$, is

$$y(t) = \frac{1}{m-n} \sum_{j=0}^{p-1} \Big(\frac{a_{p+j} - na_j}{\sqrt[p]{m^j}} H_{pj}(\sqrt[p]{m}t) + \frac{ma_j - a_{p+j}}{\sqrt[p]{n^j}} H_{pj}(\sqrt[p]{n}t) \Big)$$
$$+ \frac{d}{b}\Big(1 + \frac{nH_{p0}(\sqrt[p]{m}t) - mH_{p0}(\sqrt[p]{n}t)}{m-n}\Big).$$

Proof. From Theorem 3.3, we find the desired conclusion immediately. □

From Theorem 4.1 and Lemma 2.2, one has the following observations.

Remark 4.2. (i) If $m > 0 > n$, then the solution of equation (4.1) is

$$y(t) = \frac{1}{m-n}\sum_{j=0}^{p-1}\Big(\frac{a_{p+j}-na_j}{\sqrt[p]{m^j}}H_{pj}(\sqrt[p]{m}t) + \frac{ma_j - a_{p+j}}{\sqrt[p]{(-n)^j}}T_{pj}(\sqrt[p]{-n}t)\Big) + \frac{d}{b}\big(1 + \frac{nH_{p0}(\sqrt[p]{m}t) - mT_{p0}(\sqrt[p]{-n}t)}{m-n}\big).$$

(ii) If $0 > m > n$, then the solution of equation (4.1) is

$$y(t) = \frac{1}{m-n}\sum_{j=0}^{p-1}\Big(\frac{a_{p+j}-na_j}{\sqrt[p]{(-m)^j}}T_{pj}(\sqrt[p]{-m}t) + \frac{ma_j - a_{p+j}}{\sqrt[p]{(-n)^j}}T_{pj}(\sqrt[p]{-n}t)\Big) + \frac{d}{b}\big(1 + \frac{nT_{p0}(\sqrt[p]{-m}t) - mT_{p0}(\sqrt[p]{-n}t)}{m-n}\big).$$

5. Examples

In this section, we present some examples, which indicate how our results can be applied to concrete problems.

Example 5.1. Using Remark 3.2, one can see that the solution of the differential equation

$$y'''(t) + 2y(t) - 3 = 0, \quad y(0) = 1,\ y'(0) = 2,\ y''(0) = 4$$

is

$$y(t) = -\frac{1}{2}T_{30}(\sqrt[3]{2}t) + \frac{2}{\sqrt[3]{2}}T_{31}(\sqrt[3]{2}t) + \frac{4}{\sqrt[3]{4}}T_{32}(\sqrt[3]{2}t) + \frac{3}{2}.$$

Example 5.2. Using Theorem 4.1, one can see that the solution of the differential equation

$$\begin{aligned} &y^{(6)}(t) - 3y^{(3)}(t) + 2y(t) - 5 = 0,\\ &y(0) = 1,\ y'(0) = 2,\ y''(0) = 3,\ y'''(0) = 4,\ y^{(4)}(0) = 5,\ y^{(5)}(0) = 6 \end{aligned} \tag{5.1}$$

is

$$y(t) = \frac{11}{2}H_{30}(\sqrt[3]{2}t) + \frac{3}{\sqrt[3]{2}}H_{31}(\sqrt[3]{2}t) + \frac{3}{\sqrt[3]{4}}H_{32}(\sqrt[3]{2}t) - 7H_{30}(t) - H_{31}(t) + \frac{5}{2}.$$

Example 5.3. Using Theorem 3.3, one can see that the solution of the differential equation

$$\begin{aligned} &y^{(5)}(t) - 3y^{(3)}(t) - 2y^{(2)}(t) + 6y(t) - 4 = 0,\\ &y(0) = 1,\ y'(0) = 2,\ y''(0) = 2,\ y'''(0) = 4,\ y^{(4)}(0) = 5 \end{aligned} \tag{5.2}$$

is

$$y(t) = -\frac{2}{23}H_{30}(\sqrt[3]{2}t) + \frac{14}{23\sqrt[3]{2}}H_{31}(\sqrt[3]{2}t) + \frac{17}{23\sqrt[3]{4}}H_{32}(\sqrt[3]{2}t) + \frac{29}{69}H_{20}(\sqrt{3}t) + \frac{32}{23\sqrt{3}}H_{21}(\sqrt{3}t) + \frac{46}{69},$$

where $A_0 = \frac{16}{23}$, $A_1 = \frac{26}{23}$, $A_2 = \frac{25}{23}$, $B_0 = \frac{7}{23}$, $B_1 = \frac{20}{23}$, $C_0 = -\frac{3}{23}$, $C_1 = -\frac{2}{23}$, $C_2 = -\frac{9}{23}$, $D_0 = \frac{3}{23}$, and $D_1 = \frac{2}{23}$.

6. Conclusion

The aim of this paper is to solve linear constant coefficient differential equations of forms:

$$y^{(np)}(t)+c_{n-1}y^{(n-1)p}(t)+\ldots+c_1y^{(p)}(t)+c_0y(t)=c, \quad n=1,2 \tag{6.1}$$

and

$$y^{(p+q)}(t)-ay^{(p)}(t)-by^{(q)}(t)+aby(t)-d=0, \tag{6.2}$$

where the a,b,c,d, and c_i are constants, $p,q\in\mathbb{N}$, and $p,q\geq 2$. Applying the Laplace transform on (6.1) and (6.2), we obtained their solutions as a linear combination of functions $T_{pj},H_{pj}:\mathbb{R}\to\mathbb{R},\ j=0,1,2,\ldots,p-1,p\in\mathbb{N}$, defined by (2.1). We believe that the methods introduced in this paper offer valuable opportunities for further research in the field of ordinary differential equations.

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