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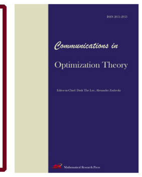

SPECTRAL RIGIDITY FOR INFINITESIMAL GENERATORS OF REPRESENTATIONS OF TWISTED $S_\nu U(2)$

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Abstract. We prove spectral rigidity theorems for infinitesimal generators of representations of twisted $S_\nu U(2)$ and for representations of $\mathfrak{sl}(2)$.

Keywords. Determinantal manifold; Matrix pseudogroup; Projective joint spectrum; Representation.

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1. Introduction and Statement of Results

Given a tuple of $N\times N$ matrices $A_1,\dots,A_n$, the *determinantal manifold (determinantal hypersurface)* $\sigma(A_1,\dots,A_n)$ is defined by

$$\sigma(A_1,\dots,A_n)=\Big\{[x_1:\dots:x_n]\in\mathbb{CP}^{n-1}:\ det\big(x_1A_1+\dots+x_nA_n)=0\big)\Big\}. \tag{1.1}$$

In [25], Yang introduced an analog of the notion of "determinantal manifold" for an arbitrary tuple of bounded operators acting on a Hilbert space H and called it *projective joint spectrum* of the tuple. It is defined as follows:

Definition 1.1. Let $A_1,...,A_n$ be bounded operators acting on a Hilbert space H. The *projective joint spectrum* of the tuple $A_1,...,A_n$ is

$$\sigma(A_1,...,A_n)=\{[x_1,...,x_n]\in\mathbb{CP}^{n-1}:x_1A_1+\dots+x_nA_n \text{ is not invertible}\}. \tag{1.2}$$

If the dimension of H is finite, (1.1) and (1.2) coincide, so in the finite dimensionaln case, and this is the case we concentrate on in the present paper, we use the terminology "determinantal manifold" and "projective joint spectrum" interchangeably. Determinantal manifolds of matrix pencils have been under scrutiny for more than hundred years. Notably, the study of group determinants led Frobenius to laying out the foundation of representation theory. There is an extensive literature on the question when an algebraic manifold of codimension 1 in the projective space admits a determinantal representation. Without trying to give an exhaustive account

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of the references on this topic, we just mention [5, 6, 7, 8, 9, 16, 23] and also the monograph [10].

To avoid trivial redundancies, it is common to assume that at least one of the operators in the tuple (usually the last one) is invertible, and, therefore, can be taken to be the identity (or rather -I). In what follows, we always assume that this is the case and write $\sigma(A_1,...,A_{n-1})$ instead of $\sigma(A_1,...,A_{n-1},-I)$. Also, it was found to be useful to deal with the part of the projective joint spectrum that lies in the chart $\{x_n\neq 0\}$. This part is called the *proper projective joint spectrum* and is denoted by $\sigma_p(A_1,...,A_{n-1})$,

$$\begin{gathered}\sigma_p(A_1,...,A_{n-1})\\ =\{(x_1,...,x_{n-1})\in\mathbb{C}^{n-1}: x_1A_1+\cdots+x_{n-1}A_{n-1}-I \text{ is not invertible}\}.\end{gathered}$$

In the last decade, projective joint spectra of operator and matrix tuples were intensively investigated (see [1]-[4], [11]-[18], [20]-[22], and [25]) from the following angle: what does the geometry of the projective joint spectrum tell us about the relations between operators in the tuple? In particular, [4], [15], and [19] established spectral characterization of representations of infinite dihedral group, of non-special finite Coxeter groups, and of some subgroups of the permutation group related to the Hadamard matrices of Fourier type. In [12], a spectral aspect of representations of simple Lie algebras and certain representations of $\mathfrak{sl}(2)$ were investigated. Somewhat more general type of results, which we call *spectral rigidity results* are statements of the inverse type: "*if counting multiplicities the projective joint spectra of two operator or matrix tuples are the same, then (perhaps under some mild conditions) the tuples are equivalent.*" Results of this type were established for representations of Coxeter groups, [4], [20], and for subgroups of the permutation group associated with Hadamard matrices, [19]. The goal of this paper is to start the investigation of spectral rigidity for representation of Lie algebras, or related objects.

To see the difference between spectral characterization of representations and spectral rigidity, let us look at $\mathfrak{sl}(2)$. The Lie algebra $\mathfrak{sl}(2)$ is generated by

$$e=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ f=\begin{bmatrix}0&0\\1&0\end{bmatrix},\ \ h=\begin{bmatrix}1&0\\0&-1\end{bmatrix}. \tag{1.3}$$

These generators satisfy the commutation relations

$$[h,e]=2e,\ [h,f]=-2f,\ [e,f]=h. \tag{1.4}$$

It is well-known that for each $n\geq 2$ there is a unique up to an equivalence n-dimensional irreducible representation of $\mathfrak{sl}(2)$ generated by the $n\times n$ matrices E_n,F_n,H_n which, as operators on $\mathbb{C}^n$, act in the following way on the standard basis $\zeta_0,...,\zeta_{n-1}$:

$$\begin{aligned}F_n\zeta_j=\zeta_{j+1},\ j=0,\dots,n-2,\ F_n\zeta_{n-1}=0,\\ E_n\zeta_0=0,\ E_n\zeta_j=j(n-j)\zeta_{j-1},\ j=1,\dots,n-1,\\ H_n\zeta_j=(n-1-2j)\zeta_j,\ j-0,\dots,n-1.\end{aligned} \tag{1.5}$$

This obviously implies that the joint spectrum of the images of the generators of $\mathfrak{sl}(2)$ under a finite-dimensional representation determines the representation up to an equivalence.

An explicit expression of the projective joint spectrum $\sigma(H_n, E_n.F_n)$ can be found in [12]. In particular, for $n = 3$

$$E_3 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},\ F_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ H_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

and

$$\sigma(H_3, E_3, F_3) = \left\{[x : y : z : t] \in \mathbb{CP}^3 :\ t(4x^2 + 4yz - t^2) = 0\right\}.$$

Now, let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ satisfy $\alpha\gamma = \beta\delta = 2$,

$$A_1 = H_3,\ A_2 = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 & 0 \\ \gamma & 0 & \delta \\ 0 & 0 & 0 \end{bmatrix}, .$$

Then $\sigma(A_1, A_2, A_3) = \sigma(H_3, E_3, F_3)$, but

$$[A_2, A_3] = \begin{bmatrix} 2 & 0 & \alpha\delta \\ 0 & -4 & 0 \\ \beta\gamma & 0 & 2 \end{bmatrix} \neq A_1,$$

that is, the joint spectrum of the tuple does not determine commutation relations, and, therefore, is not enough for spectral rigidity.

To remedy the situation, we look at the twisted $S_\nu U(2)$ matrix pseudogroup introduced by Woronovicz [24]. As $\nu \to 1$, infinitesimal generators of a finite dimensional representation of $S_\nu U(2)$ approach generators of a representation of $\mathfrak{sl}(2)$. As shown in [24], the infinitesimal generators determine a representation of $S_\nu U(2)$ up to an equivalence, and a Theorem of Woronowicz (Theorem 2.3 below) states that, similar to the situation with $\mathfrak{sl}(2)$, for each n there is only 1 up to an equivalence irreducible n-dimensional representation of $S_\nu U(2)$. The definition of $S_\nu U(2)$ via a C^*-algebra approach suggests including in the tuple not only images of generators, but also their adjoints, and this leads to spectral rigidity results for infinitesimal generators of $S_\nu U(2)$ obtained in this paper. The limit case $\nu = 1$ turns into spectral rigidity for $\mathfrak{sl}(2)$. The basic definitions of $S_\nu U(2)$ and infinitesimal generators of its representations are recalled in the next section. For $\nu \in [-1, 1] \setminus \{0\}$ we denote the infinitesimal generators of the (unique up to an equivalence) irreducible self-adjoint n-dimensional representation of $S_\nu U(2)$ by $E_{n,\nu}, F_{n,\nu}$ and $H_{n,\nu}$. As operators on $\mathbb{C}^n$ they act in the standard orthonormal basis $e_0, ..., e_{n-1}$ according to (2.4) in Theorem 2.3, namely

$$\begin{gathered} F_{n,\nu} e_k = -c_{k+1}(\nu) e_{k+1},\ H_{n,\nu} e_k = \tfrac{\nu^2}{1-\nu^2}(\nu^{2(n-2k-1)} - 1) e_k, \\ E_{n,\nu} e_k = \nu c_k(\nu) e_{k-1}, \end{gathered} \tag{1.6}$$

where

$$c_k(\nu) = \frac{\nu}{1-\nu^2}\left[\left(\nu^{n-2k-1} - \nu^{n-1}\right)\left(\nu^{1-n} - \nu^{n-2k+1}\right)\right]^{1/2}. \tag{1.7}$$

Remark that the difference in exponents for ν between the ones in (1.6) and (1.7) and the corresponding expressions in Theorem 2.3 is caused by the difference in indexing: in (1.6) and (1.7) the dimension of the representation is n, with n being an integer, while in Theorem 2.3 it is $2n+1$ with n being a half-integer. Also in (1.6) and (1.7) indeces for basic vectors vary from 0 to $n-1$, and in Theorem 2.3 - from $-n$ to n.

Our main result is the following.

Theorem 1.2. *Let $\nu \in [-1,1]\setminus\{0\}$. Suppose that A_1, A_2, A_3 are n-dimensional matrices such that*

(a) *A_1 is normal.*
(b) *The following equalities for pairwise projective joint spectra take place:*

$$\sigma_p(A_1, A_2A_2^*) = \sigma_p(H_{n,\nu}, E_{n,\nu}E_{n,\nu}^*);\ \sigma_p(A_1, A_2^*A_2) = \sigma_p(H_{n,\nu}, E_{n,\nu}^*E_{n,\nu});$$
$$\sigma_p(A_1, A_3A_3^*) = \sigma_p(H_{n,\nu}, F_{n,\nu}F_{n,\nu}^*);\ \sigma_p(A_1, A_3^*A_3) = \sigma_p(H_{n,\nu}, F_{n,\nu}^*F_{n,\nu});$$
$$\sigma_p(A_1, A_2A_3) = \sigma_p(H_{n,\nu}, E_{n,\nu}F_{n,\nu}).$$

Then $\Big(A_1, A_2, A_3\Big)$ is unitary equivalent to $\Big(H_{n,\nu}, E_{n,\nu}, F_{n,\nu}\Big)$.

The following corollary is straightforward, since the coincidence of joint spectra of a tuple implies the coincidence of the pairwise joint spectra.

Corollary 1.3. *Let $\nu \in [-1,1]\setminus\{0\}$. Suppose that A_1, A_2, A_3 are n-dimensional matrices such that*

(a) *A_1 is normal.*
(b)

$$\sigma_p\Big(A_1, A_2A_2^*, A_2^*A_2, A_3A_3^*, A_3^*A_3, A_1A_3\Big)$$
$$= \sigma_p\Big(H_{n,\nu}, E_{n,\nu}E_{n,\nu}^*, E_{n,\nu}^*E_{n,\nu}, F_{n,\nu}F_{n,\nu}^*, F_{n,\nu}^*F_{n,\nu}, E_{n,\nu}F_{n,\nu}\Big)$$

Then $\Big(A_1, A_2, A_3\Big)$ is unitary equivalent to $\Big(E_{n,\nu}, H_{n,\nu}, F_{n,\nu)}\Big)$.

We remark that as $\nu \to 1$, $c_k(\nu) \to \sqrt{k(n-k)}$, so that the infinitesimal generators $E_{n,\nu}, F_{n,\nu}$ and $H_{n,\nu}$ converge to

$$\tilde{F}_n = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\sqrt{n-1} & 0 & \ldots & 0 \\ 0 & -\sqrt{2(n-1)} & \cdots & 0 \\ . & . & . & . \\ 0 & \cdots & -\sqrt{n-1} & 0 \end{bmatrix},$$
$$\tilde{H}_n = \begin{bmatrix} 1-n & 0 & \cdots & 0 \\ 0 & 3-n & \cdots & 0 \\ . & . & . & . \\ 0 & \cdots & 0 & n-1 \end{bmatrix}, \qquad (1.8)$$
$$\tilde{E}_n = \begin{bmatrix} 0 & \sqrt{n-1} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2(n-2)} & \cdots & 0 \\ . & . & . & . & . \\ 0 & 0 & \cdots & 0 & \sqrt{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Even though the tuples $(\tilde{F}_n, \tilde{H}_n, \tilde{E}_n)$ and $(-F_n, -H_n, E_n)$ are equivalent, they are not unitary equivalent. Since the statements of Theorem 1.2 and Corollary 1.3 include adjoints of $\tilde{E}_n$ and $\tilde{F}_n$, a similar rigidity result for the representation (1.5) does not follow automatically. Nevertheless, a result similar to the one of Theorem 1.2 is true.

Theorem 1.4. *Suppose that E_n, F_n, and H_n are given by* (1.5) *and A_1, A_2, A_3 are $n \times n$ matrices such that the following are true:*

a) *A_1 is normal;*
b) *Either*

$$\sigma_p(A_1, A_2A_2^*) = \sigma_p(H_n, E_nE_n^*);\ \sigma_p(A_1, A_2^*A_2) = \sigma_p(H_n, E_n^*E_n);$$
$$\sigma_p(A_1, A_3A_3^*) = \sigma_p(H_n, F_nF_n^*);\ \sigma_p(A_1, A_2A_3) = \sigma_p(H_n, E_nF_n),$$

or

$$\sigma\Big(A_1, A_2^*A_2, A_2A_2^*, A_3A_3^*, A_2A_3\Big) = \sigma\Big(H_n, E_n^*E_n, E_nE_n^*, F_nF_n^*, E_nF_n\Big).$$

Then (A_1, A_2, A_3) is unitary equivalent to (H_n, E_n, F_n).

2. Background

2.1. Twisted $S_\nu U(2)$ pseudogroups and infinitesimal generators of their representations.

As mentioned above, $S_\nu U(2)$ were introduced by Woronowicz in [24] as deformations of $SU(2)$. Here we recall necessary definitions and basic results regarding representations of $S_\nu U(2)$. The details might be found in [24]. Please, note that in order to adjust the material to our presentation, we slightly changed the notation adopted in [24].

Let $\nu \in [-1, 1]$, and A be the C^*-algebra that is generated by 2 elements, α and γ, satisfying the relations:

$$\begin{gathered}\alpha^*\alpha + \gamma^*\gamma = I,\ \alpha\alpha^* + \nu^2\gamma^*\gamma = I \\ \gamma^*\gamma = \gamma\gamma^*,\ \alpha\gamma = \nu\gamma\alpha,\ \alpha\gamma^* = \nu\gamma^*\alpha.\end{gathered} \tag{2.1}$$

A is considered to be the non-commutative algebra of continuous functions on $S_\nu U(2)$. Let

$$u_\nu = \begin{bmatrix} \alpha & -\nu\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \in M_2 \otimes A. \tag{2.2}$$

Then

1) The $*$-algebra $\mathscr{A}$ generated by elements of u is dense in A.
2) u_ν is an invertible element of $M_2 \otimes A$.
3) If $\nu \neq 0$, the structure of a Hopf algebra is introduced on A by

$$\Phi : A \to A \otimes A,$$
$$(id \otimes \Phi)u_\nu = \begin{bmatrix} \alpha \otimes \alpha - \nu\gamma^* \otimes \gamma & -\nu\big(\alpha \otimes \gamma^* + \gamma^* \otimes \alpha^*\big) \\ \gamma \otimes \alpha + \alpha^*\gamma & -\nu\gamma \otimes \gamma^* + \alpha^* \otimes \alpha^* \end{bmatrix}$$

and

$$k : \mathscr{A} \to \mathscr{A}$$
$$\forall a \in \mathscr{A},\ \ k(k(a^*)^*) = a;\ \ (id \otimes k)u_\nu = u_\nu^{-1}.$$

We use the notation $u \odot v$ for the product of $N \times N$ matrices with entrees in $*$-algebras B and B' where the usual product of matrix elements is replaced by the tensor product, that is

$$u = \left[u_{ij}\right]_{i,j=1}^{N}, \; v = \left[v_{ij}\right]_{i,j=1}^{N} \Longrightarrow u \odot v = \left[w_{ij}\right]_{i,j=1}^{N}, \; w_{ij} = \sum_{k=1}^{N} a_{ik} \otimes b_{kj}.$$

Let V be a finite dimensional vector space. We say that v is a representation of $S_\nu U(2)$ acting on V, if v is an invertible element of the algebra $B(V) \otimes A$ and

$$(id \otimes \Phi)v = v \odot v \in B(V) \otimes A \otimes A,$$

where

$$(m_1 \otimes a) \odot (m_2 \otimes b) = (m_1 m_2) \otimes a \otimes b.$$

In particular, the matrix u_ν given by (2.2) determines a representation acting on $\mathbb{C}^2$ which is called *the fundamental representation of* $S_\nu U(2)$.

Further, let $\mathscr{M}$ be the subalgebra with unity in M_4 consisting of all complex-valued 4x4 matrices having non-zero elements only in the first row and on the main diagonal. The following elements of $\mathscr{M}$

$$\alpha_m(\nu) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \nu^{-1} & 0 & 0 \\ 0 & 0 & \nu^{-2} & 0 \\ 0 & 0 & 0 & \nu^{-1} \end{bmatrix}, \; \gamma_m(\nu) = \begin{bmatrix} 0 & 0 & 0 & -\nu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_m^*(\nu) = \begin{bmatrix} 1 & 0 & -\nu^2 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & \nu^2 & 0 \\ 0 & 0 & 0 & \nu \end{bmatrix}, \; \gamma_m^*(\nu) = \begin{bmatrix} 0 & \nu^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfy the commutation relations (2.1). Therefore, $\exists \; \phi : \mathscr{A} \to \mathscr{M}$ such that

$$\phi(\alpha) = \alpha_m(\nu), \; \phi(\gamma) = \gamma_m(\nu), \; \phi(\alpha^*) = \alpha_m^*(\nu), \; \phi(\gamma^*) = \gamma_m^*(\nu).$$

Denote by $e_0^\nu(a), \chi_0^\nu(a), \chi_1^\nu(a), \chi_2^\nu(a), f_0^\nu(a) f_1^\nu(a), f_2^\nu(a)$ the linear fuctionals on $\mathscr{A}$ determed by the entrees of ϕ, that is

$$\phi(a) = \begin{bmatrix} e_0^\nu(a) & \chi_0^\nu(a) & \chi_1^\nu(a) & \chi_2^\nu(a) \\ 0 & f_0^\nu(a) & 0 & 0 \\ 0 & 0 & f_1^\nu(a) & 0 \\ 0 & 0 & 0 & f_2^\nu(a) \end{bmatrix}, \; a \in \mathscr{A}.$$

For a representation v of $S_\nu U(2)$ acting on V, $v \in B(\mathscr{V}) \otimes \mathscr{A}$. Therefore, for a linear functional ψ on A we may introduce the operator $(id \otimes \psi)v \in B(V)$.

Definition 2.1. The operators:

$$E_\nu(v) = (id \otimes \chi_0^\nu)v, \; H_\nu(v) = (id \otimes \chi_1^\nu)v, \; F_\nu(v) = (id \otimes \chi_2^\nu)v$$

are called *infinitesimal generators* of v.

For the fundamental representation u_ν of $S_\nu U(2)$ these generators are:

$$E(u_\nu) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; H(u_\nu) = \begin{bmatrix} 1 & 0 \\ 0 & -\nu^2 \end{bmatrix}, \; F(u_\nu) = \begin{bmatrix} 0 & 0 \\ -\nu & 0 \end{bmatrix}.$$

When $\nu = 1$ these matrices generate $\mathfrak{sl}(2)$, as they coincide with (1.3) except for $F(u_1) = -f$.

Theorem 2.2 (Woronowicz [24])**.** *Let v and v' be finite dimensional representations of $S_\nu U(2)$. Then*

- *Infinitesimal generators determine a finite dimensional representation of $S_\nu U(2)$. For equivalent representations v and v' the sets of infinitesimal generators are equivalent.*
- *The infinitesimal genrators of v satisfy the following commutation relations:*

$$\begin{aligned} \nu F_\nu(v)E_\nu(v)) - \frac{1}{\nu}E_\nu(v))F_\nu(v) &= H_\nu(v) \\ \nu^2 H_\nu(v)E_\nu(v) - \frac{1}{\nu^2}E_\nu(v)H_\nu(v) &= (1+\nu^2)E_\nu(v) \\ \nu^2 F_\nu(v)H_\nu(v) - \frac{1}{\nu^2}H_\nu(v)F_\nu(v) &= (1+\nu^2)F_\nu(v) \end{aligned} \tag{2.3}$$

If V is Hilbert, and v is self-adjoint, then

$$-\nu E_\nu(v)^* = F_\nu(v), \;\; H_\nu^*(v) = H_\nu(v).$$

Any tuple (A_0,A_1,A_2) satisfying relations (2.3) is called *an infinitesimal representation* of $S_\nu U(2)$. The following result claims that for each $n \in \mathbb{N}$ there exists only 1, up to an equivalence, irreducible n-dimensional infinitesimal representation of $S_\nu U(2)$ and specifically describes it.

Theorem 2.3 (Wornowicz [24])**.** *Let (A_0,A_1,A_2) be a finite dimensional irreducible infinitesimal representation of $S_\nu U(2)$. Then, the eigenvalues of A_1 are real and, denoting λ_{max} the maximal one, we have the following possibilities:*

1) $\lambda_{max} = -\frac{\nu^2}{1-\nu^2}$. *Then $dim(V) = 1$, and $\exists\, c \in \mathbb{C}$ such that*

$$A_0 = c\frac{\nu}{1-\nu^2}I,\; A_1 = -\frac{\nu^2}{1-\nu^2}I,\; A_2 = \frac{1}{c}\frac{\nu^2}{1-\nu^2}I.$$

2) $\lambda_{max} = \frac{\nu^2}{1-\nu^2}(\nu^{-4n}-1)$, *where $n = 0, 1/2, 1, 3/2, ..$ is a non-negative integer or half-integer. Then $dim(V) = 2n+1$ and $\exists\, \eta_{-n}, ..., \eta_n$ - a basis of V such that*

$$A_0\eta_k = -c_{k+1}\eta_{k+1},\; A_1\eta_k = \frac{\nu^2}{1-\nu^2}(\nu^{-4k}-1)\eta_k,\; A_2\eta_k = \nu c_k \eta_{k-1}, \tag{2.4}$$

where $c_k = \frac{\nu}{1-\nu^2}\left[(\nu^{-2k}-\nu^{2n})(\nu^{-2n}-\nu^{2(k-1)})\right]^{1/2}$ and $\eta_{n+1} = 0$.

3) *If V is Hilbert, and the representation is self-adjoint, the case 1) cannot happen, and the basis $\eta_{-n}, ..., \eta_n$ is orthogonal.*

2.2. **Joint spectra of commuting tuples.** The following results could be found in [3].

Theorem 2.4 (Theorem A, [3])**.** *Suppose that $(A_1,\cdots,A_n)$ is a tuple of compact, self-adjoint operators acting on a Hilbert space H. Then the operators in $(A_1,...,A_n)$ pairwise commute if and only if $\sigma_p(A_1,...,A_n)$ consists of locally finite union of complex hyperplanes in $\mathbb{C}^n$.*

Theorem 2.5 (Theorem B, [3])**.** *Suppose $(A_1,\cdots,A_n)$ is a tuple of $N\times N$ normal matrices. Then the following conditions are equivalent:*

(a) The matrices in $(A_1,...,A_n)$ pairwise commute.

(b) *$\sigma_p(A_1,...,A_n)$ is the union of finitely many complex hyperplanes in $\mathbb{C}^n$.*
(c) *The complex polynomial $p(z_1,\cdots,z_N)=\det(z_1A_1+\cdots+z_nA_n-I)$ is completely reducible.*

2.3. **Spectral compressions.** In the proofs of Theorems 1.2 and 1.4, we need the following result, which is an adjusted to our needs and simplified version of the first statement in Theorem 1.15 in [20].

Let A_1 be a normal matrix with eigenvalues $\lambda_1,\dots,\lambda_m$ and spectral resolution $A_1=\sum_{j=1}^m \lambda_j P_{\lambda_j}$. Spectral projections P_{λ_j} are given by $P_{\lambda_j}=\frac{1}{2\pi i}\int_{\gamma_j}(w-A_1)^{-1}dw$, where γ_j is a conotur that separates λ_j from the rest of the spectrum of A_1, and the rank of P_{λ_j} is equal to the multiplicity of λ_j.

Theorem 2.6. *Let A_1,A_2 be $n\times n$ matrices with A_1 being normal. Suppose that $\lambda x_1+\mu x_2=1$ is in $\sigma_p(A_1,A_2)$. Then*

- *λ is an eigenvalue of A_1.*
- *If the multiplicity of this eigenvalue (and, therefore, of this line as well) is equal to 1, then*

$$P_\lambda A_2 P_\lambda=\mu P_\lambda. \tag{2.5}$$

3. Proof of Theorem 1.2

Fix an orthonormal eigenbasis $\zeta=(\zeta_0,...,\zeta_{n-1})$ for A_1, so that A_1 is diagonal in this basis and write A_2 and A_3 in this basis.

First we observe that condition (b) of Theorem 1.2 implies that

$$\sigma(A_1)=\sigma(H_{n,\nu})=\left\{\tfrac{\nu^2}{1-\nu^2}\left(\nu^{2(n-2k-1)}-1\right):\ k=0,...,n-1\right\}, \tag{3.1}$$

$$\begin{aligned}\sigma(A_2A_2^*)=\sigma(A_2^*A_2)&=\sigma(E_{n,\nu}E_{n,\nu}^*)\\ &=\left\{\nu^2c_k(\nu)^2:\ k=0,\dots,n-1\right\}.\end{aligned} \tag{3.2}$$

In particular, (3.1) and the condition (a) of Theorem 1.2 imply that A_1 is self-adjoint and its spectrum is simple. Further, it also follows from condition (b) of Theorem 1.2 that

$$\begin{aligned}&\sigma_p(A_1,A_2A_2^*)=\sigma_p(H_{n,\nu},E_{n,\nu}E_{n,\nu}^*)\\ &=\prod_{j=0}^{n-1}\left(\frac{\nu^2}{1-\nu^2}\left(\nu^{2(n-2j-1)}-1\right)x_1-\nu^2c_{j+1}(\nu)^2x_2-1\right),\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\sigma_p(A_1,A_2^*A_2)=\sigma_p(H_{n,\nu},E_{n,\nu}^*E_{n,\nu})\\ &=\prod_{j=0}^{n-1}\left(\frac{\nu^2}{1-\nu^2}\left(\nu^{2(n-2j-1)}-1\right)x_1-\nu^2c_j(\nu)^2x_2-1\right),\end{aligned} \tag{3.4}$$

where in (3.3) - (3.4) we have $c_0(\nu)=c_n(\nu)=0$, which agrees with (1.7). Thus, the projective joint spectra (3.3) and (3.4) consist of projective lines. Since all three matrices A_1, $A_2A_2^*$, and $A_2^*A_2$ are self-adjoint, Theorems 2.4 and 2.5 imply that A_1 commutes with both $A_2A_2^*$ and $A_2^*A_2$.

Since A_1 has a simple spectrum, we conclude that ζ is an eigenbasis for $A_2A_2^*$ and for $A_2^*A_2$, and, therefore, both $A_2A_2^*$ and $A_2^*A_2$ are diagonal matrices with

$$A_2A_2^*\zeta_j = \nu^2 c_{j+1}(\nu)^2\zeta_j,\ A_2^*A_2\zeta_j = \nu^2 c_j(\nu)^2\zeta_j,\ j = 0,...,n-1. \tag{3.5}$$

Considering that for each j, the j-th diagonal entry of $A_2A_2^*$ is equal to the sum of squares of absolute values of the entrees of the j-th row of A_2 and that the j-th diagonal entry of $A_2^*A_2$ is the sum of squares of absolute values of the entries of the j-th column of A_2, we conclude that the first column and the last row of A_2 are 0.

Write

$$A_2 = \begin{bmatrix} 0 & b_{11} & \cdots & b_{1\ n-1} \\ 0 & b_{21} & \cdots & b_{2\ n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & b_{n-1\ 1} & \cdots & b_{n-1\ n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since $A_2A_2^*$ is diagonal, the rows of the matrix $B = \left[b_{ij}\right]_{i,j=1}^{n-1}$ and the columns of B^* are orthogonal. Also, (3.5) shows that for every $1 \le j \le n-1$

$$\sum_{m=1}^{n-1} |b_{jm}|^2 = \nu^2 c_{j+1}(\nu)^2,$$

and, therefore, $B = DC$, where D is the diagonal $(n-1)\times(n-1)$ matrix with $\nu c_{j+1}(\nu),\ j = 0,...,n-2$, as the j-th entry on the main diagonal and $C = [c_{ij}]_{i,j=1}^{n-1}$ is a unitary $(n-1)\times(n-1)$ matrix. Of course, this yields the following representation of A_2: $A_2 = \tilde{D}\tilde{C}$, where $\tilde{D}$ is the diagonal $n\times n$ matrix obtained from D by adding the n-th 0 column and row, and $\tilde{C}$ is the $n\times n$ matrix obtained from C by also adding the first 0 column and the n-th 0 row. Since $\tilde{D}\tilde{D}^* = \tilde{D}^*\tilde{D} = A_2A_2^*$, we obtain $A_2^*A_2 = \tilde{C}^*A_2A_2^*\tilde{C}$. Let us write $U = \tilde{C} + W$, where W is the following $n\times n$ matrix:

$$W = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Then U is a unitary $n\times n$ matrix, and

$$A_2 = \tilde{D}(U-W), \tag{3.6}$$

$$A_2^*A_2 = (U^* - W^*)A_2A_2^*(U-W). \tag{3.7}$$

Since $\tilde{D}W = W^*A_2A_2^* = A_2A_2^*W = 0$, (3.6) and (3.7) yield

$$A_2 = \tilde{D}U, \tag{3.8}$$

$$A_2^*A_2 = U^*A_2A_2^*U. \tag{3.9}$$

By (3.5) the .entrees of $A_2^*A_2$ on the main diagonal are obtained from the main diagonal entrees of $A_2A_2^*$ by the circular permutation

$$\mathscr{P} = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & 1 & \cdots & n-1 \end{pmatrix}.$$

We denote the corresponding $n \times n$ matrix by the same letter $\mathscr{P}$, so that

$$\mathscr{P} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$A_2^* A_2 = \mathscr{P}^* A_2 A_2^* \mathscr{P}. \tag{3.10}$$

Relations (3.9) and (3.10) and the fact that U and $\mathscr{P}$ are unitary imply

$$U^* A_2 A_2^* U = \mathscr{P}^* A_2 A_2^* \mathscr{P} \Longrightarrow A_2 A_2^* U \mathscr{P}^* = U \mathscr{P}^* A_2 A_2^*. \tag{3.11}$$

Next, we show that the following result holds.

Lemma 3.1. *For every $\nu \in [-1,1] \setminus \{0\}$ except for a finite number of points in this punctured interval, $A_2 A_2^*$ and $A_2^* A_2$ have simple spectra, and for each of these exceptional values of ν the multiplicity of every spectral point of these matrices does not exceed 2.*

Proof. 1. First, let $|\nu| < 1$. By (3.2), $A_2 A_2^*$ and $A_2^* A_2$ have eigenvalues with multiplicities, if and only if, for some $0 \le i \ne j \le n-1$, we have $c_i(\nu)^2 = c_j(\nu)^2$, which by (1.7) happens when

$$\left(\nu^{n-2i-1} - \nu^{n-1}\right)\left(\nu^{1-n} - \nu^{n-2i+1}\right) = \left(\nu^{n-2j-1} - \nu^{n-1}\right)\left(\nu^{1-n} - \nu^{n-2j+1}\right).$$

This gives us

$$\begin{gathered} \tfrac{1}{\nu^{2i}} - 1 - \nu^{2(n-2i)} + \nu^{2(n-i)} = \tfrac{1}{\nu^{2j}} - 1 - \nu^{2(n-2j)} + \nu^{2(n-j)}, \\ \left(1 - \nu^{2(n-i)} + \nu^{2n}\right)\nu^{2j} = \left(1 - \nu^{2(n-j)} + \nu^{2n}\right)\nu^{2i}, \\ \left(\nu^{2j} - \nu^{2i}\right) - \nu^{2n}\left(\tfrac{\nu^{2j}}{\nu^{2i}} - \tfrac{\nu^{2i}}{\nu^{2j}}\right) + \nu^{2n}\left(\nu^{2j} - \nu^{2i}\right) = 0, \\ \left(\nu^{2j} - \nu^{2i}\right)\left(1 + \nu^{2n} - \nu^{2(n-j)} - \nu^{2(n-i)}\right) = 0. \end{gathered}$$

Write $\nu^2 = z$. Since $i \ne j$ the last equation implies

$$1 + z^n - z^{n-j} - z^{n-i} = 0. \tag{3.12}$$

If $i + j \le n,\ n - j \ge i$, then $1 + z^n - z^{n-j} - z^{n-i} \ge 1 + z^n - z^i - z^{n-i} = \left(1 - z^i\right)\left(1 - z^{n-i}\right) > 0$. Thus, if $c_i(\nu)^2 = c_j(\nu)^2$, we have $i + j > n$. Further, suppose that $j > i$, so that $n - j < n - i$. Write (3.12) as $\left(1 - z\right)\left(1 + z + \cdots + z^{n-j-1} - z^{n-i} - \cdots - z^{n-1}\right) = 0$, which implies

$$1 + z + \cdots + z^{n-j-1} - z^{n-i} - \cdots - z^{n-1} = 0. \tag{3.13}$$

Since $i + j > n$, the number of negative coefficients of the polynomial in the left-hand side of (3.13) is bigger than the number of positive ones, so when z runs from 0 to 1 this polynomial changes sign, and, therefore, has a root between 0 and 1. On the other hand, Descartes' rule of signs shows that this polynomial can have at most 1 positive root, and we conclude that for every pair $(i,j),\ i < j$ with $i + j > n$ there exists exactly 1 root $0 < z_{ij} = \nu_{ij}^2 < 1$ such that $\left[c_i(\pm\nu_{ij})\right]^2 = \left[c_j(\pm\nu_{ij})\right]^2$. Let us denote by $\tilde{S}$ the following finite subset of (-1,1):

$$\tilde{S} = \left\{ \pm\nu_{ij} = \pm\sqrt{z_{ij}} :\ i + j > n,\ i < j \right\}. \tag{3.14}$$

For every $\nu \in (-1,1)\setminus \left(\tilde{S}\cup 0\right)$ the matrices $A_2A_2^*$ and $A_2^*A_2$ have simple spectra. Also, suppose that $i<j<k,\ i+j>n,\ j+k>n$ and $z_{ij}=z_{ik}$, which means that $\nu_{ij}=\nu_{ik}$ and $c_i(\nu_{ij})^2=c_j(\nu_{ij})^2=c_k(\nu_{ij})^2$ is an eigenvalue of $A_2A_2^*$ of multiplicity greater than or equal to 3. Then it follows from (3.13) that

$$1+z_{ij}+\cdots+z_{ij}^{n-k-1}-z_{ij}^{n-i}-\cdots-z_{ij}^{n-1}=0,$$
$$1+z_{ij}+\cdots+z_{ij}^{n-k-1}+z_{ij}^{n-k}+\cdots+z_{ij}^{n-j-1}-z_{ij}^{n-i}-\cdots-z_{ij}^{n-1}=0,$$

a contradiction due to $z_{ij}>0$. This shows that for $\nu\in\tilde{S}$ no eigenvalue of $A_2A_2^*$ has multiplicity higher than 2.

2. Now, let $|\nu|=1$. By (1.8), $c_i(\pm1)^2=i(n-i)$, so it is easily seen that $c_i(\pm1)^2=c_j(\pm1)^2$ if and only if $i+j=n$.

This finishes the proof of Lemma 3.1 □

The following is a corollary to the proof of Lemma 3.1.

Corollary 3.2. *Suppose that (i_1,j_1) and (i_2,j_2) are such that $z_{i_1j_1}=z_{i_2j_2}$ and $i_1<i_2$. Then $j_1>j_2$ and $i_2-i_1>j_1-j_2$.*

Proof. We have by (3.13)

$$1+z_{i_1j_1}+\cdots+z_{i_1j_1}^{n-j_1-1}-z_{i_1j_1}^{n-i_1}-\cdots-z_{i_1j_1}^{n-1}=0 \tag{3.15}$$
$$1+z_{i_1j_1}+\cdots+z_{i_1j_1}^{n-j_2-1}-z_{i_1j_1}^{n-i_2}-\cdots-z_{i_1j_1}^{n-1}=0. \tag{3.16}$$

Since $i_2>i_1,\ n-i_2<n-i_1 \implies z_{i_1j_1}^{n-i_2}+\cdots+z_{i_1j_1}^{n-1}>z_{i_1j_1}^{n-i_1}+\cdots+z_{i_1j_1}^{n-1}\implies j_2<j_1$. Also, (3.15) and (3.16) imply $z_{i_1j_1}^{n-j_1}+\cdots+z_{i_1j_1}^{n-j_2-1}=z_{i_1j_1}^{n-i_2}+\cdots+z_{i_1j_1}^{n-i_1-1}$. Since $0<z_{i_1j_1}<1$ and $n-j_2-1<n-i_2$, we see that

$$(n-j_2-1)-(n-j_1-1)=j_1-j_2<(n-i_1-1)-(n-i_2-1)=i_2-i_1.$$

□

Write $S=\tilde{S}\cup\{\pm1\}$, where $\tilde{S}$ is given by (3.14). Then, for every $\nu\notin S$, $A_2A_2^*$ and $A_2^*A_2$ have simple spectra.

We now return to the proof of Theorem 1.2. There are 3 different cases.

1). $\nu\notin S$.

Since in this case $A_2A_2^*$ is a diagonal matrix with distinct entries on the main diagonal, relation (3.11) shows that $U\mathscr{P}^*$ is a diagonal unitary matrix, so that $U\mathscr{P}^*=\Lambda=diag(e^{i\theta_1},\dots,e^{i\theta_n})$. Hence,

$$U=\Lambda\mathscr{P}. \tag{3.17}$$

Now, (3.8) yields

$$A_2=\Lambda E_{n,\nu}. \tag{3.18}$$

A similar argument that uses

$$\sigma_p(A_1,A_3A_3^*)=\sigma_p(H_{n,\nu},F_{n,\nu}F_{n,\nu}^*);\ \sigma_p(A_1,A_3^*A_3)=\sigma_p(H_{n,\nu},F_{n,\nu}^*F_{n,\nu})$$

shows that, for $\nu\notin S$,

$$A_3=F_{n,\nu}\Sigma^*, \tag{3.19}$$

where Σ is a diagonal matrix with unimodular entries on the main diagonal.

Further, (3.18), (3.19) and relation $\sigma_p(A_1,A_2A_3)=\sigma_p(H_{n,\nu},E_{n,\nu}F_{n,\nu})$ from item b) in Theorem 1.2 imply that $\Lambda=\Sigma$. Thus

$$A_2=\tilde{\Lambda}E_{n,\nu}\tilde{\Lambda}^*,\ A_3=\tilde{\Lambda}E_{n,\nu}\tilde{\Lambda}^*,$$

where $\tilde{\Lambda}=diag\big(1,e^{-i\theta_1},e^{-i(\theta_1+\theta_2)},...,e^{-i(\theta_1+..\theta_{n-1})}\big)$. Since $\tilde{\Lambda}$ is diagonal, it commutes with A_1, which shows that $(A_1,A_2,A_3)=\tilde{\Lambda}(H_{n,\nu},E_{n,.\nu},F_{n,\nu})\tilde{\Lambda}^*$. This finishes the proof in the case $\nu\notin S$.

2). $\nu\in S,\ |\nu|<1$.

Let $\mathscr{M}(\nu)$ be the set of all pairs (ij) such that $\nu^2=z_{ij}$.

In this case, (3.11) does not imply that $U\mathscr{P}^*$ is necessary a diagonal matrix, but rather the following.

Let us denote by $Q_{ij},\ i<j$ the $n\times n$ matrix corresponding to the exchange of the i-th and the j-th rows by the left multiplication. Then either $U\mathscr{P}^*$ is diagonal, or $\exists\mathscr{K}\subset\mathscr{M}(\nu)$ such that

$$U=\Lambda\Bigg(\prod_{(i,j)\in\mathscr{K}}Q_{i-1\ j-1}\Bigg)\mathscr{P}. \tag{3.20}$$

Observe that $det(x_1H_{n,\nu}+x_2E_{n,\nu}-I)$ is a polynomial independent of x_2.

Claim 3.21. *If $\mathscr{K}$ is not empty, then* (3.20) *implies that the polynomial $det\big(x_1A_1+x_2A_2-I\big)$ contains non-trivial monomials with positive powers of x_2.*

Of course, this contradicts $\sigma_p(A_1,A_2)=\sigma_p(H_{n,\nu},E_{n,\nu})$, so once the claim is proved, the present case is reduced to the one considered above. Suppose that $\mathscr{K}\neq\emptyset$. Write

$$x_1A_1+x_2A_2-I=\big[d_{ij}(x_1,x_2)\big]_{i,j=0}^{n-1}, \tag{3.22}$$

$$det\big(x_1A_1+x_2A_2-I\big)=\sum_{\mathscr{P}}\prod_{l=0}^{n-1}(-1)^{sign(\mathscr{P})}d_{l\,j_l}(x_1,x_2), \tag{3.23}$$

where the sum is taken over all substitutions $\mathscr{P}=\begin{pmatrix}0&\cdots&n-1\\ j_0&\cdots&j_{n-1}\end{pmatrix}$. We also remark that each non-diagonal entry $d_{ij}(x_1,x_2),\ i\neq j$ is either 0, or a constant times x_2.

Let $\mathscr{K}=\big\{(i_1,j_1),\dots,(i_k,j_k)\big\},\ i_1<i_2<\cdots<i_k$. Corollary 3.2 implies $i_1<\cdots<i_k<j_k<\cdots<j_1$, and, as we saw above, $i_s+j_s>n$.

First, let us consider the case when $j_k-i_k=1$. In this situation the entries on the main diagonal of $x_1A_1+x_2A_2-I$ are

$$d_{ii}(x_1,x_2)=\tfrac{\nu^2}{1-\nu^2}\big(\nu^{2(n-2i-1)}-1\big)x_1-1,\ i\neq i_k,$$
$$d_{i_ki_k}(x_1,x_2)=\tfrac{\nu^2}{1-\nu^2}\big(\nu^{2(n-2i-1)}-1\big)x_1+e^{i\theta_{i_k-1}}\nu c_{i_k}(\nu)x_2-1,$$

and it follows that $det\big(x_1A_1+x_2A_2-I\big)$ contains the following terms with the first power of x_2

$$e^{i\theta_{i_k-1}}\nu c_{i_ki_k}(\nu)x_2\prod_{i\neq i_k}\Big[\frac{\nu^2}{1-\nu^2}\big(\nu^{2(n-2i-1)}-1\big)x_1-1\Big].$$

Now, suppose that $j_k-i_k>1$. In this case all entries on the main diagonal of $x_1A_1+x_2A_2-I$ are linear functions in x_1 and do not contain terms with x_2. Also, all non-trivial entries $d_{ij}(x_1,x_2)$

outside of the main diagonal are:

$$\begin{aligned} d_{ii+1}(x_1,x_2) = e^{i\theta_i}\nu c_i(\nu)x_2,\ i=0,...,n-2,\ i\neq i_s-1, j_s-1,\\ s=1,...,k \\ d_{i_s-1\ j_s}(x_1,x_2) = e^{i\theta_{j_s-1}}\ \nu c_{j_s}x_2,\ s=1,...,k \\ d_{j_s-1\ i_s}(x_1,x_2) = e^{i\theta_{i_s-1}}\nu c_{i_s}x_2,\ s=1,...,k. \end{aligned} \tag{3.24}$$

Since we have only $n-1$ non-zero entries outside of the main diagonal, each non-zero product in the rigfht-hand side of 3.23 contains an entry from the main diagonal.
Let us call a *cycle* a set of pairs $\gamma = \{(m_1,m_2),(m_2,m_3),...,(m_r,m_{r+1})\}$ if $m_l \neq m_{l+1}$ and $m_{r+1}=m_1$. We say that r is *the length of the cycle* and denote it by $\ell(\gamma)$. We say that a cycle is *simple*, if it does not contain proper subcycles, and we call a cycle γ *non-trivial*, if all entries $d_{m_i,m_{i+1}}(x_1,x_2)$ do not vanish.

Given a substitution $\mathscr{P} = \begin{pmatrix} 0 & \cdots & n-1 \\ j_0 & \cdots & j_{n-1} \end{pmatrix}$, the set of pairs $(0,j_0),(1,j_1),\dots,(n-1,j_{n-1})$ is comprised of fixied pairs (m,m) and simple cycles. It is easily seen from (3.24) that each product in the right-hand side of (3.23) is a non-trivial polynomial in x_1 times x_2 raised to the power equal to the sum of the lengths of simple cycles in $\mathscr{P}$.

Every entry of the first column and the last row of the matrix $x_1A_1+x_2A_2-I$ outside of the main diagonal is 0. Every other column and row contains only 1 non-zero entry outside of the main diagonal. Thus, every pair (ij), $i\neq j$ corresponding to a non-zero entry of $x_1A_1+x_2A_2-I$ may belong to only one non-trivial simple cycle.

It is easily seen that all non-trivial simple cycles are:

$$\begin{aligned} \gamma_s = \Big\{(i_s-1,j_s),(j_s,j_s+1),\dots,(j_{s-1}-2,j_{s-1}-1),(j_{s-1}-1,i_{s-1})\Big\},\\ (i_{s-1},i_{s-1}+1),\dots,(i_s-2,i_s-1). \\ \delta_s = \Big\{(j_s-1,i_s),(i_s,i_s+1),\dots,(i_{s+1}-2,i_{s+1}-1),(i_{s+1}-1,j_{s+1})\Big\},\\ (j_{s+1},j_{s+1}+1),\dots,(j_s-2,j_s-1), \end{aligned} \tag{3.25}$$

The corresponding lengths are:

$$\begin{aligned} \ell(\gamma_s) &= j_{s-1}-j_s+i_s-i_{s-1},\\ \ell(\delta_s) &= j_s-j_{s+1}+i_{s+1}-i_s,\\ \sum_s \Big(\ell(\gamma_s)+\ell(\delta_s)\Big) &= j_1-i_1. \end{aligned}$$

Index pairs (i_1-1,j_1) and $(i,i+1)$, for $0\le i\le i_1-2$ and $j_1\le i\le n-2$ do not belong to any non-trivial cycle. Thus, there is only one substitution $\mathscr{P}$ that contains all non-trivial cycles γ_s and δ_s listed in (3.25) and fixing pairs (i_1-1,i_1-1) and (i,i) with $0\le i\le i_1-2$ and $j_1\le i\le$

$n-1$. This shows that the polynomial $det(x_1A_1+x_2A_2-I)$ contains all monomials from

$$\pm x_2^{j_1-i_1}e^{i(\theta_0+\cdots+\theta_{n-1})}\Big(\prod_{m=i_1}^{j_1}\nu c_m(\nu)\Big)\prod_{m=0}^{i_1-1}\Big(\frac{\nu^2}{1-\nu^2}\big(\nu^{2(n-2m-1)}-1\big)x_1-1\Big)$$
$$\times\prod_{m=j_1}^{n-1}\Big(\frac{\nu^2}{1-\nu^2}\big(\nu^{2(n-2m-1)}-1\big)x_1-1\Big),$$

which proves Claim 3.21.
3). Finally, let $|\nu|=1$.

In this case $\mathscr{M}(1)=\mathscr{M}(-1)=\Big\{(k,n-k):\ k=1,...,n-1\Big\}$. The proof in this case is similar to the one in the case $\nu\in S,\ |\nu|<1$: we prove that the similar claim $\mathscr{K}\neq\emptyset$ implies the existence of non-zero monomials with positive degrees of x_2, and this proof goes along practically the same lines.

Theorem 1.2 is completely proved.

4. Proof of Theorem 1.4

Since $\sigma_p(A_1,A_2A_2^*)=\sigma_p(H_n,E_nE_n^*)$ and $\sigma_p(A_1,A_2^*A_2)=\sigma_p(H_n,E_n^*E_n)$, and H_n has a simple spectrum, $\sigma_p(A_1,A_2A_2^*)$, and $\sigma_p(A_1,A_2^*A_2)$ consist of complex lines, and the spectral points of $A_2A_2^*$ and $A_2^*A_2$ have multiplicity 2 or 1. A proof similar to the one of Theorem 1.2 shows that a relation similar to (3.18) holds. Namely,

$$A_2=\Lambda E_n=\begin{bmatrix} 0 & e^{i\theta_0}(n-1) & 0 & \cdots & 0\\ 0 & 0 & 2e^{i\theta_1}(n-2) & \cdots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & e^{i\theta_{n-2}}(n-1)\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}, \tag{4.1}$$

where, again, $\Lambda=diag(e^{i\theta_0},\dots,e^{i\theta_{n-1}})$ is a diagonal matrix.

In our case

$$F_nF_n^*=diag\big(0,\underbrace{1,\dots,1}_{n-1}\big), \tag{4.2}$$

so that, as above, by Theorem 2.4, A_1 and $A_3A_3^*$ commute, and, since the spectrum of A_1 is simple, $A_3A_3^*$ is a diagonal matrix, and relation (4.2) implies that the rows of A_3 from 1 to $n-1$ form an orthonormal system. In particular, if $A_3=\Big[a_{ij}^3\Big]_{i,j=0}^{n-1}$, the Hilbert-Schmidt norm of A_3 satisfies

$$\| A_3\|_{HS}^2=trace(A_3A_3^*)=\sum_{i,j=0}^{n-1}|a_{ij}^3|^2=n-1. \tag{4.3}$$

Further,

$$\sigma_p(A_1,A_2A_3)=\sigma_p(H_n,E_nF_n)=\Big\{(x_1,x_2)\in\mathbb{C}^2:$$
$$\Big(-(n-1)x_1-1\Big)\prod_{j=0}^{n-2}\Big((n-1-2j)x_1+(j+1)(n-j-1)x_2-1\Big)\Big\}. \tag{4.4}$$

We now apply Theorem 2.6. Denote by $P_j,\ j=0,...,n-1$ the projection (2.5) corresponding to the eigenvalue $n-1-2j$ of A_1. By (4.1), $P_jA_2A_3P_j = e^{i\theta_j}(j+1)(n-j-1)a^3_{j+1\,j}P_j$, $j=0,...,n-2$. Theorem 2.6 implies $e^{i\theta_j}(j+1)(n-1-j)a^3_{j+1\ j} = (j+1)(n-1-j)$, so that $a^3_{j+1\ j} = e^{-i\theta_j}$. Now, (4.3) yields $A_3 = F_n\Lambda^*$, which finishes the proof of Theorem 1.4.

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