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CONVERGENCE OF ALGORITHMS BASED ON UNIONS OF QUASI-NONEXPANSIVE MAPS

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In Memory of Professor Petr Zabreiko

Abstract. Recently, M. K. Tam considered a framework for the analysis of iterative algorithms which can be described in terms of a structured set-valued operator in 2018. At each point in the ambient space, the value of the operator is expressed as a finite union of values of single-valued paracontracting operators. Tam proved that the associated fixed point iteration is locally convergent around strong fixed points. We generalize Tam's result and investigate the global convergence of his algorithm for an arbitrary starting point. In this paper, this result is generalized for an operator expressed as a finite union of values of single-valued quasi-nonexpansive operators.
Keywords. Convergence analysis; Fixed point; Nonexpansive mapping; Paracontracting operator.
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1. Introduction

In the past six decades, fixed point theory of nonlinear mappings is under the spotlight; see, e.g., [3, 8, 11, 12, 14, 15, 16, 17, 19, 21, 22, 29] and the references cited therein. The most important fixed point theorem is from Banach [1], which is the existence of a unique fixed point of strict contractions. In addition, it also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Various feasibility problems, which find real applications in medical science, computer science, management science, and others, receive much attention; see, e.g., [4, 5, 6, 10, 13, 18, 20, 23, 24, 26, 27, 28]. In particular, in [7, 25], it was considered a framework for the convergence analysis which is described in terms of a structured set-valued operator. For each point in the ambient space, it is assumed that the operator value is expressed as a finite union of values of single-valued paracontracting operators. For such algorithms, a convergence result was proved which generalizes the pertinent result in [2]. The result in [25] involves an application of its main result to sparsity constrained minimisation. Indeed, it was demonstrated in [25] that the associated fixed point iterations is locally convergent around strong fixed points. In [30], we proved the global convergence of the

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algorithm for an arbitrary starting point. Here, this result is further generalized for an operator expressed as a finite union of values of single-valued quasi-nonexpansive operators.

Let $C \subset X$ be its nonempty, closed set of a metric space (X,ρ). Put $B(x,r) = \{y \in X : \rho(x,y) \leq r\}$ for each $x \in X$ and each $r > 0$. Define $\mathrm{Fix}(S) = \{x \in C : S(x) = x\}$ for each mapping S on C. Fix $\theta \in C$ and suppose that the following assumption holds: For each $M > 0$, $B(\theta,M) \cap C$ is compact.

Assume that m is a natural number, $T_i : C \to C$, $i = 1,\dots,m$ are continuous operators and that the following assumption holds:

For each $i \in \{1,\dots,m\}$, each $z \in \mathrm{Fix}(T_i)$, each $x \in C$ and each $y \in C \setminus \mathrm{Fix}(T_i)$, we have $\rho(z,T_i(x)) \leq \rho(z,x)$ and $\rho(z,T_i(y)) < \rho(z,y)$. Note that the operators satisfying (A2) are called paracontractions [9]. Let, for every point $x \in X$, $\phi(x) \subset \{1,\dots,m\}$ be a given nonempty set, that is, $\phi : X \to 2^{\{1,\dots,m\}} \setminus \{\emptyset\}$. Suppose that the following assumption holds:

There exists $\delta > 0$ such that, for each $y \in B(x,\delta) \cap C$, $\phi(y) \subset \phi(x)$ for each $x \in C$. Define $T(x) = \{T_i(x) : i \in \phi(x)\}$ for each $x \in C$, $\bar{F}(T) = \{z \in C : T_i(z) = z, i = 1,\dots,m\}$ and $F(T) = \{z \in C : z \in T(z)\}$. Set $\bar{F}(T) \neq \emptyset$ and denote by $\mathrm{Card}(D)$ the cardinality of a set D. For each $z \in R^1$, set

$$\lfloor z \rfloor = \max\{i : i \text{ is an integer and } i \leq z\}.$$

In the sequel, we suppose that the sum over empty set is zero. We studied the asymptotic behavior of sequences of iterates $x_{t+1} \in F(x_t)$, $t = 0,1,\dots$ in [30]. In particular, we were interested in their convergence to a fixed point of T. This iterative algorithm was introduced in [25] which also contains its application to sparsity constrained minimization.

The following result of [30] shows that almost all iterates of our set-valued mappings are approximated solutions of the corresponding fixed point problem. Many results of this type are collected in [27, 28].

Theorem 1.1. *Let $M > 0$, $\varepsilon \in (0,1)$, and $\bar{F}(T) \cap B(\theta,M) \neq \emptyset$. Then there exists an integer $Q \geq 1$ such that for each sequence $\{x_i\}_{i=0}^{\infty} \subset C$ which satisfy $\rho(x_0,\theta) \leq M$ and $x_{t+1} \in T(x_t)$ for each integer $t \geq 0$ the inequality $\rho(x_t,\theta) \leq 3M$ holds for all integers $t \geq 0$,*

$$Card(\{t \in \{0,1,\dots,\} : \rho(x_t,x_{t+1}) > \varepsilon\}) \leq Q$$

and $\lim_{t\to\infty} \rho(x_t,x_{t+1}) = 0$.

The following global result is due to [30].

Theorem 1.2. *Assume that a sequence $\{x_t\}_{t=0}^{\infty} \subset C$ and that for each integer $t \geq 0$, $x_{t+1} \in T(x_t)$. Then there exist $x_* = \lim_{t\to\infty} x_t$ and a natural number t_0 such that for each integer $t \geq t_0$ $\phi(x_t) \subset \phi(x_*)$ and if an integer $i \in \phi(x_t)$ satisfies $x_{t+1} = T_i(x_t)$, then $T_i(x_*) = x_*$.*

Theorem 1.2 generalizes the main result of [25] which establishes a local convergence of the iterative algorithm for iterates starting from a point belonging to a neighborhood of a strong fixed point belonging to the set $\bar{F}(T)$. In the present paper we generalize these two theorem for an essential larger class of operators.

2. Preliminaries and the first result

Let $C \subset X$ be a nonempty and closed subset of (X,ρ), a metric space. Recall that, for each $x \in X$ and each $r > 0$, $B(x,r) = \{y \in X : \rho(x,y) \leq r\}$. For each $x \in X$ and each nonempty set

$D \subset X$, set $\rho(x,D) = \inf\{\rho(x,y) :\ y \in D\}$. Recall that for each mapping $S : C \to C$, $\mathrm{Fix}(S) = \{x \in C :\ S(x) = x\}$. Fix $\theta \in C$. Suppose that the following assumption holds:

(A1) For each $M > 0$, the set $B(\theta,M) \cap C$ is compact.

Assume that $\psi : C \to [0,\infty)$ is a continuous function, m is a natural number, $T_i : C \to C$, $i = 1,\dots,m$ are continuous operators, and that the following assumptions hold:

(A2) For each $M > 0$, $\{z \in C :\ \psi(z) \leq M\}$ is bounded;

(A3) For each $i \in \{1,\dots,m\}$, each $x \in C$ and each $y \in C \setminus \mathrm{Fix}(T_i)$, we have $\psi(T_i(x)) \leq \psi(x)$ and $\psi(T_i(y)) < \psi(y)$.

Note that, in Section 1, we discuss of a particular case of the problem studied here with $\psi(x) = \rho(z_*,x)$, $x \in X$, where z_* is a common joint fixed point of mappings T_i, $i = 1,\dots,m$.. Let, for every point $x \in C$,

$$\phi(x) \subset \{1,\dots,m\} \tag{1}$$

be a nonempty set. In other words, $\phi : X \to 2^{\{1,\dots,m\}} \setminus \{\emptyset\}$. Suppose that the following assumption holds:

(A4) For each $x \in C$, there exists $\delta > 0$ such that, for each $y \in B(x,\delta) \cap C$, $\phi(y) \subset \phi(x)$. Define $T(x) = \{T_i(x) :\ i \in \phi(x)\}$ for each $x \in C$ $F(T) = \{z \in C :\ z \in T(z)\}$. Denote by $\mathrm{Card}(D)$ the cardinality of a set D. For each $z \in R^1$, set

$$\lfloor z \rfloor = \max\{i :\ i \text{ is an integer and } i \leq z\}.$$

In the sequel, we suppose that the sum over empty set is zero.

We study the asymptotic behavior of sequences of iterates $x_{t+1} \in F(x_t)$, $t = 0,1,\dots$ and prove in Section 4 the following result which shows that almost all iterates of our set-valued mappings are approximated solutions of the corresponding fixed point problem.

Theorem 2.1. *Let $M > 0$ and $\varepsilon \in (0,1)$. Then there exists an integer $Q \geq 1$ such that, for each sequence $\{x_i\}_{i=0}^{\infty} \subset C$ which satisfy $\psi(x_0) \leq M$ and $x_{t+1} \in T(x_t)$ for each integer $t \geq 0$ the inequality $\psi(x_t) \leq M$ holds for all integers $t \geq 0$,*

$$Card(\{t \in \{0,1,\dots,\} :\ \rho(x_t,x_{t+1}) > \varepsilon\}) \leq Q$$

and $\lim_{t\to\infty} \rho(x_t,x_{t+1}) = 0$.

3. An auxiliary result

We use all the notation, definitions, and assumptions introduced in Section 2.

Lemma 3.1. *Let $M,\varepsilon > 0$. Then there exists $\delta > 0$ such that for each $s \in \{1,\dots,m\}$ and each $x \in C \cap B(\theta,M)$ satisfying*

$$\rho(x,T_s(x)) > \varepsilon \tag{2}$$

the inequality

$$\psi(T_s(x)) \leq \psi(x) - \delta \tag{3}$$

is true.

Proof. Let $s \in \{1,\dots,m\}$. It is sufficient to prove that there exists $\delta > 0$ such that, for each $x \in C \cap B(\theta,M)$ satisfying (2), inequality (3) is true. Then for each integer $k \geq 1$, one assumes that there exists

$$x_k \in C \cap B(\theta,M) \tag{4}$$

such that

$$\rho(x_k, T_s(x_k)) > \varepsilon \tag{5}$$

and

$$\psi(T_s(x_k)) > \psi(x_k) - k^{-1}. \tag{6}$$

In view of (A1) and (4), extracting a subsequence and re-indexing, we may assume without loss of generality that there exists

$$x_* = \lim_{k\to\infty} x_k. \tag{7}$$

By (4)-(7) and the continuity of T_s, $\rho(x_*, \theta) \le M$,

$$\rho(x_*, T_s(x_*)) = \lim_{k\to\infty} \rho(x_k, T_s(x_k)) \ge \varepsilon$$

and $\psi(T_s(x_*)) \ge \psi(x_*)$, which contradicts (A3). The contradiction that we have reached proves Lemma 3.1. □

4. Proof of Theorem 2.1

(A2) implies that there exists $M_1 > 0$ such that

$$\{x \in X : \ \psi(x) \le M\} \subset B(\theta, M_1). \tag{8}$$

Lemma 3.1 implies that there exists $\delta \in (0, \varepsilon)$ such that the following property holds:

(a) for each $s \in \{1, \dots, m\}$ and each $x \in C \cap B(\theta, M_1)$ satisfying $\rho(x, T_s(x)) > \varepsilon$, we have $\psi(T_s(x)) < \psi(x) - \delta$. Choose a natural number

$$Q \ge M\delta^{-1}. \tag{9}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset C$,

$$\psi(x_0) \le M \tag{10}$$

and that for each integer $t \ge 0$,

$$x_{t+1} \in F(x_t). \tag{11}$$

Let $t \ge 0$ be an integer. In view of (8), (10) and (A3),

$$\psi(x_{t+1}) \le \phi(x_t) \le \phi(x_0) \le M, \tag{12}$$

$$\rho(x_t, \theta) \le M_1, \ t = 0, 1, \dots. \tag{13}$$

Let $t \ge 0$ be an integer and

$$\rho(x_t, x_{t+1}) > \varepsilon. \tag{14}$$

There exists $s \in \{1, \dots, m\}$ such that

$$x_{t+1} = T_s(x_t). \tag{15}$$

Property (a) and equations (13)-(15) imply that $\psi(x_{t+1}) = \psi(T_s(x_t)) \le \psi(x_t) - \delta$. Thus the following property holds:

(b) if an integer $t \ge 0$ and $\rho(x_t, x_{t+1}) > \varepsilon$, then $\psi(x_{t+1}) \le \psi(x_t) - \delta$. Assume that $n \ge 1$ is an integer. Property (b) and equations (10), (12) imply that

$$\begin{aligned} M &\ge \psi(x_0) \ge \psi(x_0) - \psi(x_{n+1}) \\ &= \sum_{t=0}^{n} (\psi(x_t) - \psi(x_{t+1})) \\ &\ge \sum \{\psi(x_t) - \psi(x_{t+1}) : \ t \in \{0, \dots, n\}, \ \rho(x_t, x_{t+1}) > \varepsilon\} \end{aligned}$$

$$\geq \delta \mathrm{Card}(\{t \in \{0,\dots,n\} :\ \rho(x_t,x_{t+1}) > \varepsilon\})$$

and in view of (9), $\mathrm{Card}(\{t \in \{0,\dots,n\} :\ \rho(x_t,x_{t+1}) > \varepsilon\}) \leq M\delta^{-1} \leq Q$. Since n is an arbitrary natural number, we conclude that

$$\mathrm{Card}(\{t \in \{0,1,\dots\} :\ \rho(x_t,x_{t+1}) > \varepsilon\}) \leq Q.$$

Since ε is any element of $(0,1)$, Theorem 2.1 is proved.

5. The second main result

We assume that assumptions (A1), (A2) and (A4) hold and there exists

$$z_* \in \cap_{j=1}^m \mathrm{Fix}(T_j), \tag{16}$$

that there exists a function $D : C \times C \to [0,\infty)$ such that

$$\psi(x) = D(z_*,x),\ x \in C \tag{17}$$

and that the following assumptions hold:

(B1) For each $i \in \{1,\dots,m\}$, each $x \in C$ and each $y \in C \setminus \mathrm{Fix}(T_i)$, we have $D(z_*,T_i(x)) \leq D(z_*,x)$ and $D(z_*,T_i(y)) < D(z_*,y)$.

(B2) For each $i \in \{1,\dots,m\}$, each $z \in \mathrm{Fix}(T_i)$ and each $y \in C$, we have $D(z,T_i(y)) \leq D(z,y)$.

(B3) For each $z \in F(T)$, the function $D(z,\cdot) : C \to [0,\infty)$ is continuous and for each $M,\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in B(\theta,M) \cap C$ and $D(z,x) \leq \delta$, then $\rho(z,x) \leq \varepsilon$.

Clearly, (B1) implies (A3). Here our setting is less general then in Section 2 but it is still contain problems studied in [30] as well as problems in spaces with a generalized metric including well-known Bregman distance [22, 27].

Theorem 5.1. *Assume that a sequence $\{x_t\}_{t=0}^\infty \subset C$ and that for each integer $t \geq 0$, $x_{t+1} \in T(x_t)$. Then there exist $x_* = \lim_{t\to\infty} x_t$ and a natural number t_0 such that for each integer $t \geq t_0$ $\phi(x_t) \subset \phi(x_*)$ and if an integer $i \in \phi(x_t)$ satisfies $x_{t+1} = T_i(x_t)$, then $T_i(x_*) = x_*$.*

Proof. Choose $M_0 > \psi(x_0)$. By (A2), there exists $M_1 > M_0$ such that

$$\{y \in C :\ \psi(y) \leq M_0\} \subset B(\theta,M_1). \tag{18}$$

In view of Theorem 2.1 and (A2), the sequence $\{x_t\}_{t=0}^\infty$ is bounded. In view of (A1), it has a limit point $x_* \in C$ and a subsequence $\{x_{t_k}\}_{k=0}^\infty$ such that

$$x_* = \lim_{k\to\infty} x_{t_k}. \tag{19}$$

In view of (A4) and (19), we may assume without loss of generality that

$$\phi(x_{t_k}) \subset \phi(x_*),\ k = 1,2,\dots \tag{20}$$

and that there exists

$$\widehat{p} \in \phi(x_*) \tag{21}$$

such that

$$x_{t_k+1} = T_{\widehat{p}}(x_{t_k}),\ k = 1,2,\dots. \tag{22}$$

It follows from Theorem 2.1, the continuity of $T_{\widehat{p}}$ and equation (19)

$$T_{\widehat{p}}(x_*) = \lim_{k\to\infty} T_{\widehat{p}}(x_{t_k}) = \lim_{k\to\infty} x_{t_k+1} = \lim_{k\to\infty} x_{t_k} = x_*. \tag{23}$$

Set $I_1 = \{i \in \phi(x_*) :\ T_i(x_*) = x_*\}$, $I_2 = \phi(x_*) \setminus I_1$. In view of (23), one has

$$\widehat{p} \in I_1. \tag{24}$$

Fix $\delta_0 \in (0,1)$ such that

$$\rho(x_*, T_i(x_*)) > 2\delta_0,\ i \in I_2. \tag{25}$$

Assumption (A4), the continuity of $T_i,\ i = 1,\dots,m$ and (25) imply that there exists $\delta_1 \in (0,\delta_0)$ such that for each $x \in B(x_*,\delta_1) \cap C$,

$$\phi(x) \subset \phi(x_*), \tag{26}$$

$$\rho(x, T_i(x)) > \delta_0,\ i \in I_2. \tag{27}$$

Fix

$$\varepsilon_1 \in (0,\delta_1). \tag{28}$$

Theorem 2.1 and (B3) imply that there exists an integer $q_1 \geq 1$ and $\varepsilon_0 \in (0,\varepsilon_1)$ such that the following property holds:

(a) $D(x_*, x_{q_1}) \leq \varepsilon_0$, $\rho(x_t, x_{t+1}) \leq \varepsilon_1$ for each integer $t \geq q_1$ and if $x \in B(\theta, M_1) \cap C$ satisfies $D(x_*, x) \leq \varepsilon_0$, then $\rho(x_*, x) \leq \varepsilon_1$. Let $t \geq q_1$ be an integer and that

$$D(x_*, x_t) \leq \varepsilon_0. \tag{29}$$

Property (a) and equations (17), (18) and (29) imply that $\rho(x_t, x_*) \leq \varepsilon_1$. It follows from (26)-(28) that

$$\phi(x_t) \subset \phi(x_*) \tag{30}$$

and

$$\rho(x_t, T_i(x_t)) > \delta_0,\ i \in I_2. \tag{31}$$

In view of (30), there exists $s \in \phi(x_*)$ such that $x_{t+1} = T_s(x_t)$. Property (a) implies that

$$\rho(x_t, T_s(x_t)) = \rho(x_t, x_{t+1}) \leq \varepsilon_1. \tag{32}$$

It follows from (24), (28), (31), and (32) that $s \in I_1,\ T_s(x_*) = x_*$. Assumption (B2) imply that

$$D(x_*, x_{t+1}) = D(x_*, T_s(x_t)) \leq D(x_*, x_t) \leq \varepsilon_0.$$

Thus we have shown that if an integer $t \geq q_1$ satisfies $D(x_*, x_t) \leq \varepsilon_0$, then $D(x_*, x_{t+1}) \leq \varepsilon_0$ and $\rho(x_t, x_{t+1}) \leq \varepsilon_1$, which implies that, for each integer $t \geq q_1$, $D(x_*, x_t) \leq \varepsilon_0$ and $\rho(x_t, x_{t+1}) \leq \varepsilon_1$. Since ε i an arbitrary element of $(0,\delta_1)$, we conclude that $\lim_{t\to\infty} x_t = x_*$ and Theorem 5.1 is proved. □

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