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METHOD OF GUIDING FUNCTIONS AND BIRKHOFF-KELLOGG-ROTHE AND KAKUTANI FIXED POINT THEOREMS

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To the memory of Petr Petrovich Zabreiko, an outstanding scientist and a wonderful teacher

Abstract. We apply the method of guiding functions to prove an extension of the Gustafson-Schmitt theorem on a periodic solution to the case of differential inclusions. This result is used to obtain the Birkhoff-Kellogg-Rothe and Kakutani fixed point theorems for multivalued maps.

Keywords. Differential inclusion; Fixed point; Guiding function; Multivalued map; Periodic solution.

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1. INTRODUCTION

Professor Petr Zabreiko has made the significant contribution in the development of the method of guiding functions which demonstrated its high efficiency in the searching of periodic and bounded solutions of differential equations and inclusions, in the study of bifurcation phenomena and in some other problems; see, e.g., [4, 7, 8, 10, 11] and the references therein. In the present paper, we show how the modification of the method of guiding functions for differential inclusions can be applied to the proof of some fixed point theorems for multivalued maps whose justification is based usually on the topological degree theory (cf. [10, Theorem 2.2.21]).

To this end, we first prove the assertion on the existence of a periodic solution for a differential inclusion (Theorem 3.1) extending the theorem of G.B. Gustafson and K. Schmitt ([5], see also [3]). The application of this result yields the fixed point theorem for a multivalued map satisfying boundary condition of the Birkhoff-Kellogg-Rothe type (see [1], [12] and also [9]) which in turn implies the classical Kakutani fixed point theorem ([6], see also [10]).

2. PRELIMINARIES

Let us mention necessary facts from the theory of multivalued maps (see, e.g., [10]).

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Let $\mathscr{M} \subseteq \mathbb{R}^n$ be a nonempty closed subset. By the symbols $C(\mathscr{M})$, $K(\mathscr{M})$, $Cv(\mathscr{M})$, and $Kv(\mathscr{M})$ we will denote, respectively, the collections of all nonempty closed, compact, closed convex, and compact convex subsets of $\mathscr{M}$.

Let (X,ρ) be a metric space.

Definition 2.1. A multivalued map (multimap) $\mathscr{F}: X \to K(\mathbb{R}^n)$ is called upper semicontinuous at a point $x \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(x,x') < \delta$ implies $\mathscr{F}(x') \subset U_\varepsilon(\mathscr{F}(x))$, where U_ε denotes the ε-neighborhood of a set.

If $\mathscr{F}$ is upper semicontinuous at each point of X it is called upper semicontinuous (u.s.c.)

Definition 2.2. A multimap $\mathscr{F}: X \to C(\mathbb{R}^n)$ is called closed if its graph

$$Gr(F) = \{(x,y) \in X \times \mathbb{R}^n \colon y \in \mathscr{F}(x)\}$$

is a closed subset of $X \times \mathbb{R}^n$.

We will need the following property (see [10], Theorem 1.3.3).

Proposition 2.3. *Let a multimap $\mathscr{F}_0\colon X \to C(\mathbb{R}^n)$ be closed, a multimap $\mathscr{F}_1\colon X \to K(\mathbb{R}^n)$ u.s.c. and*

$$\mathscr{F}_0(x) \cap \mathscr{F}_1(x) \neq \emptyset, \quad \forall x \in X.$$

Then the intersection of multimaps $\mathscr{F} = \mathscr{F}_0 \cap \mathscr{F}_1\colon X \to K(\mathbb{R}^n)$,

$$\mathscr{F}(x) = \mathscr{F}_0(x) \cap \mathscr{F}_1(x), \quad \forall x \in X$$

is u.s.c.

Let us mention also the following property (cf. [10], Theorem 1.2.35).

Proposition 2.4. *Let $\mathscr{F}: X \to K(\mathbb{R}^n)$ be a u.s.c. multimap. Then, for each compact subset $Q \subset X$ the image $\mathscr{F}(Q) = \cup_{x\in Q}\mathscr{F}(x)$ is relatively compact, i.e., bounded.*

Now we collect some facts concerning the method of guiding functions for differential inclusions (see [4], [10], and [11]).

For $T > 0$, let $F\colon [0,T] \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$ be a u.s.c. multimap. By a solution of the differential inclusion

$$x'(t) \in F(t,x(t)) \tag{2.1}$$

we mean an absolutely continuous function $x\colon [0,T] \to \mathrm{R}^n$ satisfying (2.1) for a.e. $t \in [0,T]$.

A continuously differentiable function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is called the non-degenerate potential if there exists such $r_\varphi > 0$ that

$$grad\,\varphi(x) = \left\{\frac{\partial\varphi(x)}{\partial x_1}, \frac{\partial\varphi(x)}{\partial x_2}, \ldots, \frac{\partial\varphi(x)}{\partial x_n}\right\} \neq 0$$

for all $x \in \mathbb{R}^n$, $\|x\| \geq r_\varphi$.

Definition 2.5. A non-degenerate potential φ is called a guiding function of differential inclusion (2.1) if

$$\langle grad\,\varphi(x), y\rangle \leq 0$$

for all $y \in F(t,x)$, $0 \leq t \leq T$, and $\|x\| \geq r_\varphi$.

Proposition 2.6. *Let $F\colon [0,T]\times\mathbb{R}^n \to Kv(\mathbb{R}^n)$ be a u.s.c. globally bounded multimap. If differential inclusion (2.1) admits a guiding function φ satisfying the coercivity condition:*

$$\lim_{\|x\|\to\infty} |\varphi(x)| \to \infty,$$

then it has a solution $x(\cdot)$ satisfying the periodicity condition

$$x(0) = x(T).$$

3. Main results

3.1. **Existence of periodic solutions of differential inclusions.** Let $B\subset\mathbb{R}^n$ be a closed ball of radius $r>0$ centered at the origin; $S=\partial B$ its boundary. The following assertion is a version of the Gustafson-Schmitt theorem (see [3, 5]).

Theorem 3.1. *Let a multimap $F\colon [0,T]\times B \to Kv(\mathbb{R}^n)$ be u.s.c. and satisfy the following condition:*

($\star$) *for each $(t,x)\in[0,T]\times S$ there exists $y\in F(t,x)$ such that*

$$\langle x,y\rangle \le 0. \tag{3.1}$$

Then the differential inclusion

$$x'(t)\in F(t,x(t)) \quad a.e.\ t\in[0,T] \tag{3.2}$$

has a solution $x(\cdot)$ satisfying periodicity condition $x(0)=x(T)$ and such that

$$x(t)\in B, \forall t\in[0,T]. \tag{3.3}$$

Before proving this assertion, we consider its following weakened version.

Lemma 3.2. *Let a multimap $F\colon [0,T]\times B \to Kv(\mathbb{R}^n)$ be u.s.c. and satisfy the following condition:*

($\star\star$) *for each $(t,x)\in[0,T]\times S$ and all $y\in F(t,x)$, we have*

$$\langle x,y\rangle \le 0. \tag{3.4}$$

Then the conclusion of Theorem 3.1 holds true.

Proof. (i) Let $p\colon\mathbb{R}^n\to B$ be a radial projection:

$$p(x) = \begin{cases} x, & \|x\|\le r;, \\ r\frac{x}{\|x\|}, & \|x\|>r. \end{cases}$$

Let us extend the multimap F to $[0,T]\times\mathbb{R}^n$ by the formula

$$\widetilde{F}(t,x) = -x + p(x) + F(t,p(x)).$$

From the properties of multivalued maps (see, e.g., [10]) it follows that the multimap $\widetilde{F}\colon [0,T]\times\mathbb{R}^n\to Kv(\mathbb{R}^n)$ is u.s.c. Moreover, for each $(t,x)\in[0,T]\times\mathbb{R}^n$, $\|x\|\ge r$ and $\widetilde{y}\in\widetilde{F}(t,x)$ we have

$$\langle x,\widetilde{y}\rangle \le 0.$$

Indeed, if $\|x\|=r$, then $p(x)=x$ and the above relation follows from (3.4). In case $\|x\|>r$, for each $\widetilde{y}\in\widetilde{F}(t,x)$ we get

$$\widetilde{y} = -\mu x + y,$$

where $\mu > 0$ and $y \in F(t, p(x))$. But then from (3.4) it follows that

$$\langle x, \widetilde{y} \rangle = \langle x, -\mu x + y \rangle = -\mu \langle x, x \rangle + \langle x, y \rangle \leq -\mu \langle x, x \rangle < 0.$$

But this means that the function $\varphi(x) = \frac{1}{2}\|x\|^2$ is guiding for the differential inclusion

$$x'(t) \in \widetilde{F}(t, x(t)) \tag{3.5}$$

and hence according to Proposition 2.6 this inclusion possesses a solution $x(\cdot)$ satisfying condition $x(0) = x(T)$.

(ii) Let us show now that $x(t) \in B$, $\forall t \in [0, T]$ and hence $x(\cdot)$ is a solution of the inclusion

$$x'(t) \in F(t, x(t)), \quad \text{a.e. } t \in [0, T].$$

Consider the open subset m of $[0, T]$ given as

$$m = \{t \in [0, T] \colon x(t) \in \mathbb{R}^n \setminus B\},$$

which can be defined also as the set of such $t \in [0, T]$ for which $\|x(t) - p(x(t))\| > 0$.

Then, for $t \in m$, we have

$$\frac{d}{dt}\left[\frac{1}{2}\|x(t) - p(x(t))\|^2\right] = \langle x(t) - p(x(t)), x'(t) \rangle =$$

$$= \langle x(t) - p(x(t)), \widetilde{y}(t) \rangle = \langle x(t) - p(x(t)), -x(t) + p(x(t)) + y(t) \rangle,$$

where $y(t) \in F(t, p(x(t)))$.

Therefore, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|x(t) - p(x(t))\|^2\right] = -\|x(t) - p(x(t))\|^2 + \langle x(t) - p(x(t)), y(t) \rangle \leq$$

$$\leq -\|x(t) - p(x(t))\|^2 < 0, \tag{3.6}$$

since $x(t) - p(x(t)) = \lambda x(t)$, $\lambda > 0$ and we can apply relation (3.4).

Now, let $t_\star \in [0, T]$ be such that

$$\frac{1}{2}\|x(t_\star) - p(x(t_\star))\|^2 = \max_{t \in [0,T]} \frac{1}{2}\|x(t) - p(x(t))\|^2.$$

If $t_\star \in m \setminus \{0, T\}$, then

$$\frac{d}{dt}\left[\frac{1}{2}\|x(t_\star) - p(x(t_\star))\|^2\right] = 0,$$

in contradiction to (3.6). If $t_\star = 0$, then, by the periodicity condition, the maximum is achieved also at $t_\star = T$. Hence

$$\frac{d}{dt}\left[\frac{1}{2}\|x(T) - p(x(T))\|^2\right] \geq 0,$$

and in this case $t_\star$ also does not belong to m.

So, $t_\star \in [0, T] \setminus m$. For an arbitrary $t \in [0, T]$ we have

$$0 \leq \|x(t) - p(x(t))\|^2 \leq \|x(t_\star) - p(x(t_\star))\|^2 = 0,$$

i.e., $x(t) = p(x(t))$ and hence $x(t) \in B$. □

Proof of Theorm 3.1. Extend the multimap F to $[0,T]\times\mathbb{R}^n$ by the formula

$$\overline{F}(t,x)=F(t,p(x)).$$

It is easy to see that the multimap $\overline{F}$ possesses property $(\star)$ on the whole $[0,T]\times\mathbb{R}^n$. Consider the closed multimap $K\colon \mathbb{R}^n\to Cv(\mathbb{R}^n)$,

$$K(x)=\begin{cases}\mathbb{R}^n, & \|x\|\le r,\\ \{y\in\mathbb{R}^n\colon \langle x,y\rangle\le 0\}, & \|x\|>r.\end{cases}$$

Define the multimap $F_K\colon [0,T]\times\mathbb{R}^n\to Kv(\mathbb{R}^n)$,

$$F_K(t,x)=\overline{F}(t,x)\cap K(x).$$

Notice that this multimap coincides with F on $[0,T]\times B$ and satisfies condition $(\star\star)$ for each $(t,x)\in[0,T]\times(\mathbb{R}^n\setminus B)$. Moreover, from Proposition 2.3 it follows that F_K is u.s.c.

For an arbitrary $\varepsilon>0$, let $B_{r_\varepsilon}\subset\mathbb{R}^n$ be a closed ball of the radius $r_\varepsilon=r+\varepsilon$ centered at the origin. Then conditions of Lemma 3.2 are fulfilled for the multimap F_K on the set $[0,T]\times B_{r_\varepsilon}$ and hence there exists a solution $x_\varepsilon(\cdot)$ of the differential inclusion

$$x'(t)\in F_K(t,x(t)),\quad \text{a.e. } t\in[0,T], \tag{3.7}$$

satisfying condition $x_\varepsilon(0)=x_\varepsilon(T)$ and such that $x_\varepsilon(t)\in B_{r_\varepsilon}$, $t\in[0,T]$.

Now, let $\varepsilon_i>0$, $\varepsilon_i\to 0$, $i=1,2,\ldots$ be an arbitrary sequence and $\{x_{\varepsilon_i}(\cdot)\}$ the corresponding sequence of solutions to differential inclusion (3.7). Since there exists $\mathcal{N}>0$ such that

$$\|F_K(t,x)\|:=\sup\{\|y\|\colon y\in F_K(t,x)\}\le\mathcal{N},\quad \forall(t,x)\in[0,T]\times\mathbb{R}^n,$$

the sequence $\{x_{\varepsilon_i}(\cdot)\}$ satisfies the conditions of the Arzel$\grave{a}$ - Ascoli theorem and hence is relatively compact. So, we can assume, without loss of generality, that $x_{\varepsilon_i}(\cdot)\longrightarrow x(\cdot)\in C([0,T];\mathbb{R}^n)$. Since the integral multioperator generated by the multimap F_K is closed (see, e.g., [10], Corollary 1.5.34) the function $x(\cdot)$ is a solution to differential inclusion (3.7). It is clear that $x(t)\in B$, $\forall t\in B$, so $x(\cdot)$ is a desired solution of (3.2) satisfying the condition $x(0)=x(T)$.

3.2. **Fixed point theorems of Birkhoff-Kellogg-Rothe and Kakutani.** In this section, we show that Theorem 3.1 yields the following fixed point result for a multimap satisfying the boundary condition of Birkhoff-Kellog-Rothe type.

Theorem 3.3. *Let a multimap $\mathfrak{F}\colon B\to Kv(\mathbb{R}^n)$ be u.s.c. and satisfy the condition*

$$\mathfrak{F}(x)\cap B\neq\emptyset,\quad \forall x\in S.$$

Then $\mathfrak{F}$ has a fixed point $x_\star\in B$, $x_\star\in\mathfrak{F}(x_\star)$.

Proof. Consider the differential inclusion

$$x'(t)\in F(x(t)),\quad \text{a.e. } t\in[0,T], \tag{3.8}$$

where $F(x)=\mathfrak{F}(x)-x$. It is clear that the multimap $F\colon B\to Kv(\mathbb{R}^n)$ is u.s.c. Moreover, F satisfies conditions of Theorem 3.1. In fact, if for $x\in S$ an element $y\in\mathfrak{F}(x)$ is such that $y\in B$, then

$$\langle x,y-x\rangle\le 0.$$

Since differential inclusion (3.8) is autonomous, the application of Theorem 3.1 implies for each $\tau=\frac{T}{m}$, $m=1,2,\ldots$ the existence of its τ-periodic solution $x_\tau(\cdot)$ which is contained in B.

Consider a sequence $\{x_i\}_{i=1}^{\infty}$ consisting of $\frac{T}{2^i}$-periodic solutions of differential inclusion (3.8) which are contained in B. Again we can use the boundedness of the multimap F to apply the Arzelà - Ascoli theorem which yields the relative compactness of the sequence $\{x_i\}_{i=1}^{\infty}$. So, we can assume, without loss of generality, that $x_i(\cdot) \longrightarrow x(\cdot) \in C([0,T];\mathbb{R}^n)$. Applying, as earlier, the property of closedness of the integral multioperator generated by F we conclude that $x(\cdot)$ is a solution of differential inclusion (3.8).

Moreover, the function $x(\cdot)$ is $\frac{T}{2^i}$-periodic for each $i = 1,2,\ldots$ and hence it is a constant, $x(t) \equiv x_\star \in B$, $t \in [0,T]$.

So, $x(\cdot)$ is a constant solution of differential inclusion (3.8), i.e.,

$$0 \in F(x_\star) = \mathfrak{F}(x_\star) - x_\star,$$

and $x_\star$ is the desired fixed point of $\mathfrak{F}$. □

The known Kakutani fixed point theorem ([6], see also, e.g., [10]) may be regarded as a corollary of Theorem 3.3.

Theorem 3.4. *Let $M \subset \mathbb{R}^n$ be a convex closed and bounded subset. Then each u.s.c. multimap $\mathfrak{G}\colon M \to Kv(M)$ has a fixed point.*

Proof. Let $B \subset \mathbb{R}^n$ be a closed ball centered at the origin such that $M \subset B$. By the Tietze-Dugundji theorem (see, e.g. [2]) there exists a retraction $\theta\colon B \to M$, i.e., a continuous map such that $\theta(x) = x$, $\forall x \in M$. From properties of multivalued maps it follows that the multimap $\mathfrak{F}\colon B \to Kv(M)$, $\mathfrak{F}(x) = \mathfrak{G} \circ \theta(x)$ is u.s.c. and hence satisfies conditions of Theorem 3.3.

A fixed point $x_\star$ of a multimap $\mathfrak{F}$ obviously belongs to M and hence $\theta(x_\star) = x_\star$ and $x_\star$ is a fixed point of $\mathfrak{G}$. □

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