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SOME RECENT RESULTS ON THE LAVRENTIEV PHENOMENON

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To Francis Clarke, on the occasion of his seventy-fifth birthday, with deep admiration and gratitude

Abstract. The paper is a survey on some of our recent results on the Lavrentiev phenomenon, namely the fact that the infimum of an integral functional on the set of Sobolev functions might be strictly less than the one taken on Lipschitz functions. Francis Clarke has not only inspired many of our methods and techniques, but has also given us the opportunity to meet and work together.

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1. Introduction

Consider the three following problems in the Calculus of Variations:

$$\text{Minimize } E_1 : u \in W^{1,1}([0,1]) \mapsto \int_0^1 (u(x)^3 - x)^2 (u'(x))^6 \, dx,$$
$$u(0) = 0, u(1) = 1.$$

$$\text{Minimize } E_2 : u \in W^{1,1}(B^2, \mathbb{R}) \mapsto \int_{B^2} |\nabla u(x)|^p + a(x)|\nabla u(x)|^q \, dx,$$
$$u(x) = tw(x) \text{ on } \partial B^2,$$

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where B^2 is the unit ball in $\mathbb{R}^2$, $1 \le p < 2$, $q > 3$,

$$a(x) = \begin{cases} \frac{x_1 x_2}{|(x_1,x_2)|} & \text{if } x_1x_2 > 0, \\ 0 & \text{if } x_1x_2 \le 0, \end{cases} \quad , \quad w(x) = \begin{cases} 1 & \text{if } x_1 > 0, x_2 > 0, \\ x_2 & \text{if } x_1 < 0, x_2 > 0, \\ 0 & \text{if } x_1 < 0, x_2 < 0, \\ x_1 & \text{if } x_1 > 0, x_2 < 0, \end{cases}$$

and t is any constant which is chosen sufficiently large in terms of p, q.

$$\text{Minimize } E_3 : u \in W^{1,1}(B^2, \mathbb{R}^2) \mapsto \int_{B^2} (|u(x)|^2 - 1)^2 |Du(x)|^2 \, dx,$$
$$u(x) = x \text{ on } \partial B^2.$$

Here, $Du(x)$ is a matrix with two rows and two columns, and $|Du(x)|^2$ is simply the sum of the squares of the coefficients of $Du(x)$.

The above problems can be considered as special cases of the basic problem in the Calculus of Variations:

$$\text{Minimize } E : u \in W^{1,1}_\varphi(\Omega, \mathbb{R}^k) \mapsto \int_\Omega f(x, u(x), Du(x)) \, dx$$

where $\Omega \subset \mathbb{R}^N$ and $W^{1,1}_\varphi(\Omega, \mathbb{R}^k)$ is the set of those $u \in W^{1,1}(\Omega, \mathbb{R}^k)$ which coincide with a given function φ on $\partial\Omega$. These three examples have a common feature: the Lavrentiev phenomenon. More specifically, they all show that the infimum of the energy on the set of Sobolev maps $W^{1,1}_\varphi(\Omega)$ may be strictly less than the infimum of the energy on the set of Lipschitz maps, i.e.:

$$\inf\{E(u) : u \in W^{1,1}_\varphi(\Omega, \mathbb{R}^k)\} < \inf\{E(u) : u \in W^{1,\infty}_\varphi(\Omega, \mathbb{R}^k)\}.$$

This gap between the $W^{1,1}(\Omega)$ infimum and the Lipschitz one for certain variational problems has been discovered by Lavrentiev in 1927 ([32]). In the first example, due to Manià [34], $N = 1$, the domain Ω is the interval $(0,1)$, $k = 1$ and φ is the identity on $\partial(0,1) = \{0,1\}$. In the second one, constructed by Zhikov [42], $N = 2$, Ω is the unit ball and $k = 1$. The third example has been considered more recently by Alberti and Majer [1] and in this case $N = 2$, Ω is the unit ball B^2, $k = 2$ and φ is the identity again. From now on, we will simplify the notation by writing $W^{1,1}(\Omega)$ instead of $W^{1,1}(\Omega, \mathbb{R}^k)$. The fact that the admissible maps are vector valued will be clear from the context. We also remark that, in all three examples, the Lagrangians are continuous, nonnegative and convex with respect to the last variable, which implies that the corresponding functionals are weakly lower semicontinuous on $W^{1,1}_\varphi(\Omega)$.

The Lavrentiev phenomenon may be seen as a serious flaw in the mainstream strategy to prove the existence of *classical* solutions. This strategy is based on two main steps: first, the Direct Method in the Calculus of Variations provides a weak minimum in a Sobolev space, for which the derivatives must be understood in the distributional sense. Once the existence of such a Sobolev minimum is known, one tries to prove that this is a classical minimum, namely that it is at least Lipschitz continuous (and thus differentiable almost everywhere), or C^1, or even analytic. This path was studied by Tonelli himself [41]. For $N = 1$, new results appeared in the last decades: we quote Clarke and Vinter milestone [26] for coercive Lagrangians, the seminal

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We can modify E_3 with an additional term $\varepsilon|Du(x) - I|^p$ to obtain a coercive functional, for which the Direct Method applies. The Lavrentiev phenomenon still occurs when $1 < p < 2$ and ε is sufficiently small.

paper [24] of Clarke for problems of slow growth and the more recent works that weaken the regularity assumptions on the Lagrangian (e.g. [4, 5, 6, 18, 27, 39]).

When the Lavrentiev phenomenon occurs, the minimizer provided by the Direct Method cannot be a classical minimizer. For instance, in the third example above, the minimizer turns out to be $u_0(x) = x/|x|$; it is worth noticing that u_0 is not even continuous on B^2. We also emphasize the fact that the infimum of E over $W^{1,\infty}(\Omega)$ equals the infimum over the set $C^1(\bar{\Omega})$. Indeed, any Lipschitz function u can be approximated by smooth functions u_i in the sense that u_i and ∇u_i converge almost everywhere to u and ∇u respectively. Since u and its gradient are bounded, one can further require that the u_i's and their gradients are uniformly bounded. We can conclude with the help of the dominated convergence theorem that $\lim_{i\to+\infty} E(u_i) = E(u)$. As a consequence, there is no gap between the infimum over $W^{1,\infty}(\Omega)$ and the infimum over $C^1(\bar{\Omega})$. A similar argument *does not apply* between $W^{1,1}(\Omega)$ and $W^{1,\infty}(\Omega)$, even if the latter space is dense in the former and in spite of the fact that E is strongly lower semicontinuous on $W^{1,1}$. The Lavrentiev phenomenon dramatically emphasizes the lack of upper semicontintuity of the energy.

In order to better understand the discontinuity of E, one can rely on the relaxation approach, see e.g. [16], and define the relaxed functional as

$$E_{rel}(u) = \sup\{D(u) : D \text{ is weakly lower semicontinuous on } W^{1,1}_{\varphi}(\Omega), D \leq E \text{ on } W^{1,\infty}_{\varphi}(\Omega)\}. \tag{1.1}$$

Observe that E itself satisfies the properties required on D (here, we need the convexity of f with respect to the last variable, which in turn implies the weak lower semicontinuity of the functional E). As a consequence, the relaxed energy E_{rel} is not lower than E. Actually, it agrees with E on $W^{1,\infty}_{\varphi}$. The main motivation to introduce the relaxed functional is emphasized by the following identity:

$$\inf\{E_{rel}(u), u \in W^{1,1}_{\varphi}(\Omega)\} = \inf\{E(u), u \in W^{1,\infty}_{\varphi}(\Omega)\}.$$

In particular, if the relaxed energy E_{rel} coincide with the energy E on $W^{1,1}_{\varphi}(\Omega)$, then there is no Lavrentiev phenomenon; that is,

$$\inf\{E(u), u \in W^{1,1}_{\varphi}(\Omega)\} = \inf\{E(u), u \in W^{1,\infty}_{\varphi}(\Omega)\}.$$

In such a situation, applying the Direct Method in the setting of Sobolev maps does not introduce artificial solutions for the original problem. In contrast, we say that there is a Lavrentiev gap when $E_{rel} \not\equiv E$.

The main goal of this article is to present a survey on some of our recent results identifying general classes of variational problems which forestall the Lavrentiev gap. In section 2 we describe some new results for the one-dimensional case ($N = 1$). In particular we extend known results to the non-autonomous case; that is, when f depends on the three variables x, u and Du. We also discard the Lavrentiev phenomenon for Lagrangians that might take the value $+\infty$, a quite difficult case, rarely considered in the literature. Section 3 is devoted to the multidimensional scalar case ($N > 1, k = 1$) in the autonomous setting when f does not depend on x. In the final section 4, we consider the non autonomous case, still for multidimensional scalar problems.

Notation. Throughout the paper, given an integer $N \geq 1$, we denote by $B_r(x)$ the open ball of radius $r > 0$ and center x in $\mathbb{R}^N$. When $x = 0$, we simply write B_r.

2. The one dimensional case

In this section, $p \geq 1$, $\Omega =]a,b[$ is an interval and the Lagrangian

$$\Lambda : [a,b] \times \mathbb{R}^N \times \mathbb{R}^N \to [0,+\infty[\cup \{+\infty\},\ (t,y,u) \mapsto \Lambda(t,y,u)$$

is Lebesgue-Borel measurable in $(t,(y,u))$, i.e., measurable with respect to the σ-algebra generated by the products of Lebesgue measurable subsets of $[a,b]$ (for t) and Borel measurable subsets of $\mathbb{R}^N \times \mathbb{R}^N$ (for (y,u)): this guarantees that if $y,u : [a,b] \to \mathbb{R}^N$ are measurable then $t \mapsto \Lambda(t,y(t),u(t))$ is measurable (see [25, Proposition 6.34]). We point out that in Sections 2.1 and 2.2 the Lagrangian Λ is allowed to take the value $+\infty$. We consider the integral functional

$$E(y) = \int_a^b \Lambda(t,y(t),y'(t))\,dt, \quad y \in W^{1,p}([a,b],\mathbb{R}^N).$$

Basic assumption on the effective domain. In Sections 2.1 and 2.2 we assume that the *effective domain* $\mathrm{Dom}(\Lambda)$ of Λ, given by

$$\mathrm{Dom}(\Lambda) := \{(t,y,u) \in [a,b] \times \mathbb{R}^N \times \mathbb{R}^N : \Lambda(t,y,u) < +\infty\},$$

is of the form $\mathrm{Dom}(\Lambda) = [a,b] \times D_\Lambda$, with $D_\Lambda \subset \mathbb{R}^N \times \mathbb{R}^N$: thus, for all $t' \in [a,b]$ and all $(y,u) \in \mathbb{R}^N \times \mathbb{R}^N$, we have $\Lambda(t',y,u) < +\infty$ whenever $\Lambda(t,y,u) < +\infty$ for some $t \in [a,b]$.

In this section we consider Lagrangians that do not necessarily satisfy any kind of growth condition. Thus, one cannot expect neither existence, nor Lipschitz regularity.

2.1. The Lavrentiev gap and the boundary conditions.

We fix $A,B \in \mathbb{R}^N$ and consider the problems

$$\min E(y) : y \in W^{1,p}([a,b],\mathbb{R}^N),\ y(a) = A, y(b) = B \tag{$\mathrm{P}_{a,b}$}$$

and

$$\min E(y) : y \in W^{1,p}([a,b],\mathbb{R}^N),\ y(a) = A. \tag{P_{a}}$$

Absolutely continuous functions that satisfy the boundary condition(s) will be said to be *admissible*.

Definition 2.1. Let $y \in W^{1,p}([a,b],\mathbb{R}^N)$ be such that $E(y) < +\infty$. The Lavrentiev gap does not occur for (P_a) (resp. $(\mathrm{P}_{a,b})$) at y if there is a sequence $(y_k)_k$ of Lipschitz admissible functions for the problem such that:

(1) $\lim\limits_{k\to+\infty} E(y_k) = E(y)$;
(2) $y_k \to y$ in $W^{1,p}([a,b],\mathbb{R}^N)$.

Definition 2.2. The Lavrentiev phenomenon does not occur for (P_a) (resp. $(\mathrm{P}_{a,b})$) if there is a minimizing sequence $(y_k)_k$ for the problem, of Lipschitz functions, satisfying the boundary condition $y_k(a) = A$ (resp. $y_k(a) = A, y_k(b) = B$).

Unfortunately the phenomenon may happen even when the Lagrangian is a polynomial:

Example 2.3 (Manià, [34])**.** Consider the (non-autonomous) problem of minimizing

$$E(y) = \int_0^1 (y^3 - t)^2 (y')^6\,dt : y \in W^{1,1}([0,1]),\ y(0) = 0,\ y(1) = 1. \tag{P}$$

Then $y_*(t) := t^{1/3}$ is a minimizer and $E(y_*) = 0$. Not only y_* is not Lipschitz; it turns out with some computations (see [15, §4.3]) that:

$$0 = \min E = E(y_*) < \inf\{E(y) : y \in \mathrm{Lip}([0,1]),\, y(0) = 0, y(1) = 1\},$$

where by $\mathrm{Lip}([0,1])$ we denote the Lipschitz functions defined in $[0,1]$. This Lavrentiev phenomenon may still persist even if the Lagrangian is a polynomial satisfying Tonelli's existence conditions, see [3].

However, as noticed in [15], the situation changes drastically if the initial boundary condition is allowed to vary along the minimizing sequence $(y_k)_k$. Indeed consider the sequence $(y_k)_k$, where each y_k is obtained by truncating y_* at $1/k$, $k \in \mathbb{N}_{\geq 1}$ (see Fig. 1):

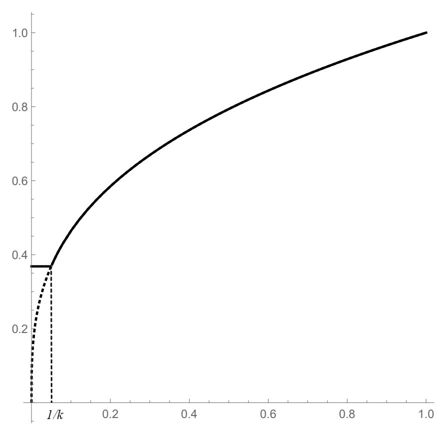

FIGURE 1. The function $y_k, k \geq 1$ in Example 2.3.

$$y_k(t) := \begin{cases} 1/k^{1/3} & \text{if } t \in [0, 1/k], \\ t^{1/3} & \text{otherwise.} \end{cases}$$

Then $(y_k)_k$ is a sequence of Lipschitz functions satisfying

$$y_k(1) = y(1) = 1, \quad E(y_k) \to E(y_*), \quad y_k \to y_* \text{ in } W^{1,1}(0,1).$$

Therefore, no Lavrentiev phenomenon occurs for the variational problem with (just) the end-point condition $y(1) = 1$:

$$\min E(y) = \int_0^1 (y^3 - t)^2 (y')^6 \, dt : y \in W^{1,1}([0,1]),\; y(1) = 1.$$

Though the occurrence of the phenomenon is often related to the non-autonomous case, there are cases of autonomous Lagrangians that exhibit it.

Example 2.4 ([21], [22])**.** Let

$$\Lambda(y,u) := \begin{cases} \left(u - \dfrac{1}{2y}\right)^2 & \text{if } y \neq 0, \\ +\infty & \text{if } y = 0. \end{cases}$$

Then:

(1) $y_*(t) := \sqrt{t}$ is a minimizer of (P_0) with the boundary conditions $y(0) = 0, y(1) = 1$ and $E(y_*) = 0$;
(2) If $y \in \mathrm{Lip}([0,1]), y(0) = 0$ then $E(y) = +\infty$;
(3) Λ violates Condition (B_y) in Theorem 2.5 below: for every $r > 0$, Λ is unbounded on $y_*([0,1]) \times B_r$.

In particular the Lavrentiev gap occurs at y_* for (P_0).

We present in § 2.2 some results for both problems (P_a) and ($\mathrm{P}_{a,b}$); in § 2.3 we introduce additional state and velocity constraints.

2.2. **Unconstrained problems.** In the autonomous case the non-occurrence of the Lavrentiev phenomenon for (P_a) was established by Alberti and Serra Cassano [2] for a wide range of Lagrangians, containing those that are bounded on bounded sets.

Theorem 2.5 (Non-occurrence of the Lavrentiev gap, Alberti & Serra Cassano (1994), [2])**.** *Suppose that Λ is autonomous and Borel measurable. Let y be such that $E(y) < +\infty$. Assume, moreover, that there is a neighbourhood $\mathcal{O}_y$ of $y([a,b])$ in $\mathbb{R}^N$ such that*

$$\exists r_y > 0 \quad \Lambda \text{ is bounded on } \mathcal{O}_y \times B_{r_y}. \tag{B_y}$$

Then the Lavrentiev gap does not occur at y for the problem with one prescribed initial condition (P_a).

A remark in [2] led to believe that the conclusion of Theorem 2.5 is valid also for the two endpoint constraint problem ($\mathrm{P}_{a,b}$); actually this is not so as it was recently pointed out in [37]: Condition (B_y) in Theorem 2.5 is not enough to ensure the non-occurrence of the gap when one wishes to preserve both endpoint constraints. The following counter-example, where the Lagrangian equals zero on its effective domain, simplifies an example of G. Alberti.

Example 2.6 (Occurrence of the Lavrentiev gap in an autonomous problem with both endpoint constraints)**.** Let, for $(y,u) \in \mathbb{R} \times \mathbb{R}$,

$$\Lambda(y,u) := \begin{cases} 0 & \text{if } y \le 0 \text{ or } y > 0,\, u \le \dfrac{1}{2y}, \\ +\infty & \text{otherwise.} \end{cases}$$

The effective domain of Λ is represented in Fig. 2. Note that:

- Λ is autonomous;
- Λ is lower semicontinuous on $\mathbb{R}^2$ and $\Lambda(y,\cdot)$ is convex for all $y \in \mathbb{R}$;
- For all $K > 0$, Λ is bounded on $]-K,K[\times[-\dfrac{1}{2K},\dfrac{1}{2K}]$.

Thus Λ satisfies all the assumptions of Theorem 2.5 at any y. In particular, there is no Lavrentiev gap at y for the problem with the initial endpoint condition $y(0) = 0$, and thus, there is no Lavrentiev phenomenon, a fact that could be immediately realized noticing that the Lipschitz function equal to 0 is a minimum of E. Now let $y_*(t) = \sqrt{t}, t \in [0,1]$. Since $y_*' = \dfrac{1}{2y_*}$ then y_* minimizes E among the absolutely continuous functions y satisfying $y(0) = 0, y(1) = 1$. The Lavrentiev gap occurs at y_* for the two endpoint problem (P):
Claim. $E(y) = +\infty$ *for every Lipschitz* $y : [0,1] \to \mathbb{R}$ *satisfying* $y(0) = 0, y(1) = 1$. Indeed,

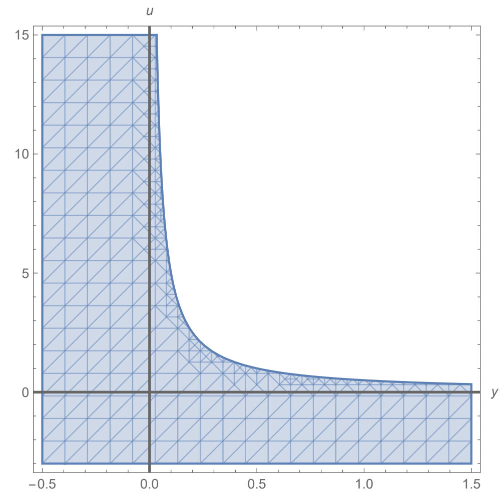

FIGURE 2. The domain of $\Lambda(\cdot,\cdot)$ in Example 2.6

assume the contrary: let y be such a function and suppose $E(y) < +\infty$. Let $0 \le t_1 < t_2 \le 1$ be such that $y(t_1) = 0, y(t_2) = 1$ and $y(]t_1,t_2[) =]0,1[$. Since $E(y) < +\infty$, then

$$y'(t)y(t) \le \frac{1}{2} \text{ a.e. on } [t_1,t_2]. \tag{2.1}$$

Note that, since $\lim_{t\to t_1} \frac{1}{2y(t)} = +\infty$ and y' is bounded, then necessarily (2.1) is strict on a non-negligible set. It follows that

$$\int_{t_1}^{t_2} y(t)y'(t)\,dt < \int_{t_1}^{t_2} \frac{1}{2}\,dt = \frac{t_2-t_1}{2} \le \frac{1}{2}. \tag{2.2}$$

However the change of variable $\zeta = y(t)$ (which is justified, for instance, by the chain rule [33, Theorem 1.74]), gives

$$\int_{t_1}^{t_2} y(t)y'(t)\,dt = \int_0^1 \zeta\,d\zeta = \frac{1}{2},$$

contradicting (2.2). This proves the claim.

Some natural questions arise:

(1) Is it true that a Lagrangian that is autonomous and bounded on bounded sets prevents the occurrence of the Lavrentiev phenomenon in the case of a two endpoint constrained problem?
(2) Can the phenomenon be prevented in the non-autonomous case? Some sufficient conditions aimed at its non-occurrence are well established (see [17]). However, they do usually involve regularity properties of the Lagrangian that are not present in the autonomous case.

Recently, it appeared that Condition (S) on the first (time) variable of the Lagrangian is the appropriate assumption to be added in order to study the non-autonomous case.

Condition (S). For every $K \ge 0$ there are $\kappa, \beta \ge 0, \gamma \in L^1([a,b])$, $\varepsilon_* > 0$ satisfying, for a.e. $t \in [a,b]$

$$|\Lambda(t_2,z,u) - \Lambda(t_1,z,u)| \le \big(\kappa\Lambda(t,z,u) + \beta|u|^p + \gamma(t)\big)\,|t_2-t_1|$$

whenever $t_1, t_2 \in [t-\varepsilon_*, t+\varepsilon_*] \cap [a,b]$, $z \in B_K$, $u \in \mathbb{R}^N$, $(t,z,u) \in \mathrm{Dom}(\Lambda)$.

Remark 2.7. Condition (S) is fulfilled if $\Lambda = \Lambda(y,u)$ is autonomous. In the smooth setting, Condition (S) ensures the fulfillment of the Erdmann–Du Bois-Reymond (EDBR) condition. In the nonsmooth framework, it plays a key role in the proof of the Lipschitz regularity under slow growth conditions in the pioneering work of Clarke [24] and turns out to ensure the validity of the EDBR (see [4, 6]). Actually the proof of [4] is based on Clarke's maximum principle.

Theorem 2.8 gives an answer to the questions formulated above.

Theorem 2.8 (Non-occurrence of the Lavrentiev gap [37])**.** *Suppose that Λ satisfies Condition (S) and let $y \in W^{1,p}([a,b],\mathbb{R}^N)$ with $y(a) = A, y(b) = B$ be such that $E(y) < +\infty$. Assume that there is a neighbourhood $\mathscr{O}_y$ of $y([a,b])$ in $\mathbb{R}^N$ such that:*

$$\exists r_y > 0 \quad \Lambda \text{ is bounded on } [a,b] \times \mathscr{O}_y \times B_{r_y}. \tag{B_y}$$

Then:

(1) There is no Lavrentiev gap for $(\mathscr{P}_a)$ at y.
(2) There is no Lavrentiev gap for $(\mathscr{P}_{a,b})$ at y assuming, instead of (B_y)*, that:*

$$\forall r > 0 \quad \Lambda \text{ is bounded on } [a,b] \times \mathscr{O}_y \times B_r. \tag{B_y^+}$$

Sketch of the proof of Theorem 2.8. The proof of the first claim in Theorem 2.8 follows narrowly that of [2, Theorem 2.4], obtained there for problem (P_a) in the case of autonomous Lagrangians.
We first use a Lusin type approximation of y, thus obtaining a Lipschitz sequence $(z_k)_k$ that converges to y in the $W^{1,p}$ norm and satisfies both boundary conditions $z_k(a) = y(a), z_k(b) = y(b)$. It may happen, however, that $E(z_k)$ does not converge to $E(y)$: we then reparametrize each z_k by setting $y_k = z_k \circ \psi_k$, where $\psi_k : [a,b] \to [a,b_k], b_k \le b$ is a suitable Lipschitz, bijective function in such a way that $E(y_k) \to E(y)$ as $k \to +\infty$. In general $b_k < b$ whence $y_k(b) \neq y(b)$: Condition (B_y^+) allows to build each ψ_k in such a way that $b_k = b$. □

Remark 2.9. Notice that the Lagrangian in Example 2.6 violates Condition $(\mathrm{B}_{y_*}^+)$; indeed Λ takes the value $+\infty$ on $y_*([0,1]) \times [-r,r]$ whenever $r > \dfrac{1}{2}$.

2.3. **Non-occurrence of the phenomenon with state and velocity constraints for extended valued Lagrangians.** The construction of the approximating Lipschitz sequence in the proof of Theorem 2.8, based on a Lusin's type approximation, may not preserve state or velocity constraints. Moreover, Condition (B_y^+) reduces drastically the possibility to apply the result to Lagrangians that take the value $+\infty$.
Here, we fix a subset Δ of $\mathbb{R}^N$ and a cone $\mathscr{U}$ in $\mathbb{R}^N$ and consider the constrained problems

$$\begin{cases} \min E(y) : y \in W^{1,p}([a,b];\mathbb{R}^N),\, y(a) = A, y(b) = B, \\ y(t) \in \Delta \ \text{ for all } t \in [a,b], \quad y'(t) \in \mathscr{U} \ \text{ a.e. } t \in (a,b) \end{cases} \tag{$\mathrm{P}_{a,b}^{\Delta,\mathscr{U}}$}$$

and

$$\begin{cases} \min E(y) : y \in W^{1,p}([a,b],\mathbb{R}^N),\, y(a) = A, \\ y(t) \in \Delta \ \text{ for all } t \in [a,b], \quad y'(t) \in \mathscr{U} \ \text{ a.e. } t \in [a,b]. \end{cases} \tag{$\mathrm{P}_{a}^{\Delta,\mathscr{U}}$}$$

In this context, the non-occurrence of the gap/phenomenon means, referring to Definition 2.1 and Definition 2.2, that the Lipschitz functions $(y_k)_k$ satisfy not only the given boundary conditions but also the given constraints.
In addition to the assumptions of Theorem 2.8, we require here the *radial convexity* of Λ with respect to the velocity variable and, in the case of the two endpoint constraints problem, that $\mathrm{Dom}(\Lambda)$ is open in $[a,b]\times\mathbb{R}^N\times\mathbb{R}^N$. When Λ takes only real values, Theorem 2.10 below is proved in [38] in a wider context of optimal control. The extended valued case is considered in [36] and proved by means of the methods introduced in [35]. In both cases the main source of inspiration is Clarke's formulation of a weak growth condition introduced in his paper [24] in 1993. As in Theorem 2.8 the conditions that ensure the non-occurrence of the gap depend on the number of boundary constraints. Let us denote by ∂B_r the sphere of center 0 and radius r in $\mathbb{R}^N$.

Theorem 2.10 (Non-occurrence of the Lavrentiev gap for problems with constraints [38, 36])**.** *Let $y\in W^{1,p}([a,b],\mathbb{R}^N)$ be such that $E(y)<+\infty$. Assume that Λ satisfies Condition (S) and, moreover, that:*

a) $0<r\mapsto\Lambda(t,y,ru)$ *is convex for all* $(t,y,u)\in\mathrm{Dom}(\Lambda)$ *and* $y\in\Delta$, $u\in\mathscr{U}$*;*
b) $\forall(t,y,u)\in\mathrm{Dom}(\Lambda)$, $u\in\mathscr{U}$ $\Lambda(t,y,ru)<+\infty\ \forall r\in]0,1]$. *Moreover* $0<r\mapsto\Lambda(t,y,ru)$ *has a non-empty subdifferential at* $r=1$*: there is* $q\in\mathbb{R}$ *such that*

$$\forall r>0 \qquad \Lambda(t,y,ru)-\Lambda(t,y,u)\geq q(r-1).$$

Then:

(1) There is no Lavrentiev gap for $(\mathscr{P}_a^{\Delta,\mathscr{U}})$ *at* y *provided that*

$$\exists r_y>0 \quad \Lambda \text{ is bounded on } \big([a,b]\times y([a,b])\times(\partial B_{r_y}\cap\mathscr{U})\big)\cap\mathrm{Dom}(\Lambda). \tag{B_y^σ}$$

(2) There is no Lavrentiev gap for $(\mathscr{P}_{a,b}^{\Delta,\mathscr{U}})$ *at* y *provided that, in addition to a) and b) and* (B_y^σ),

$$D_\Lambda\cap(\Delta\times\mathscr{U}) \text{ is interior to } D_\Lambda$$

and there is $\lambda_y>\frac{\|y'\|_1}{b-a}$ such that, for every compact subset A of $D_\Lambda\cap(\Delta\times\mathscr{U})$,

$$\Lambda \text{ is bounded on } [a,b]\times\big(A\cap(y([a,b])\times B_{\lambda_y})\big).$$

In both cases each element of the approximating sequence to y in energy and norm is a Lipschitz reparametrization of y.

In the autonomous, real valued case ($\Lambda(t,y,u)=L(y,u)$), the conclusion of Claim (2) of Theorem 2.10 with $\mathscr{U}=\mathbb{R}^N$ was obtained by Cellina, Ferriero and Marchini in [20] under the additional hypotheses that $L(y,u)$ is continuous and convex in u.

Remark 2.11. (1) In Condition (B_y^σ), one could use, instead of the sphere ∂B_{r_y}, any set that suitably encloses the origin (see [36]).
(2) The existence of a non-empty subdifferential in b) of Theorem 2.10 is fulfilled if $(t,y,r_0u)\in\mathrm{Dom}(\Lambda)$ for some $r_0>1$ or if Λ is equal to 0 on its effective domain: indeed in the first case $r=1$ belongs to the interior of the effective domain of $r\mapsto\Lambda(t,y,ru)$, in the second $0\in\partial_r(\Lambda(t,y,ru))_{r=1}$.

(3) Concerning Point (2) of Theorem 2.10, it is worth noticing that the requirement $\lambda_y > \frac{\|y'\|_1}{b-a}$ implies that $\lambda_y > \operatorname{essinf}|y'|$; this latter condition appears in Clarke's [24, Growth Hypothesis (H)] in the context of existence and regularity of minimizers.

Remark 2.12. Condition (B_y) in Theorem 2.8 implies the validity of (B_y^σ) in Theorem 2.10; differently from the first where one needs a bound of the values of $\Lambda(t,\cdot,u)$ in a *neighbourhood* of a product of the form $[a,b]\times y([a,b])\times B_r$, the latter condition involves just the behavior of Λ on the elements of $[a,b]\times y([a,b])\times(\partial B_r\cap\mathscr{U})$ *contained* in the effective domain of Λ, for a suitable $r>0$: notice in particular that we restrict here the attention to the values of u on ∂B_r, instead of B_r. Actually, the methods of [2, 37] need the boundedness of $\Lambda(t,y,v)$ for v in a *whole* ball (not just its boundary): weakening this requirement was the main aim in [21], carried on successfully just in dimension 1 by requiring the boundedness of Λ along the graph of two Lipschitz functions, one positive and the other negative. Similarly, Condition (B_y^+) of Theorem 2.8 implies the validity of Condition ($\mathrm{B}_y^\Subset$). Notice that the validity of (B_y^+) has a strong impact on the structure of the effective domain of Λ which is forced in that case to contain $[a,b]\times y([a,b])\times\mathbb{R}^N, r>0$. Instead, Condition ($\mathrm{B}_y^\Subset$) respects the integrity of $\mathrm{Dom}(\Lambda)$. The situation is illustrated in Fig. 3 for the autonomous case with $N=1$.

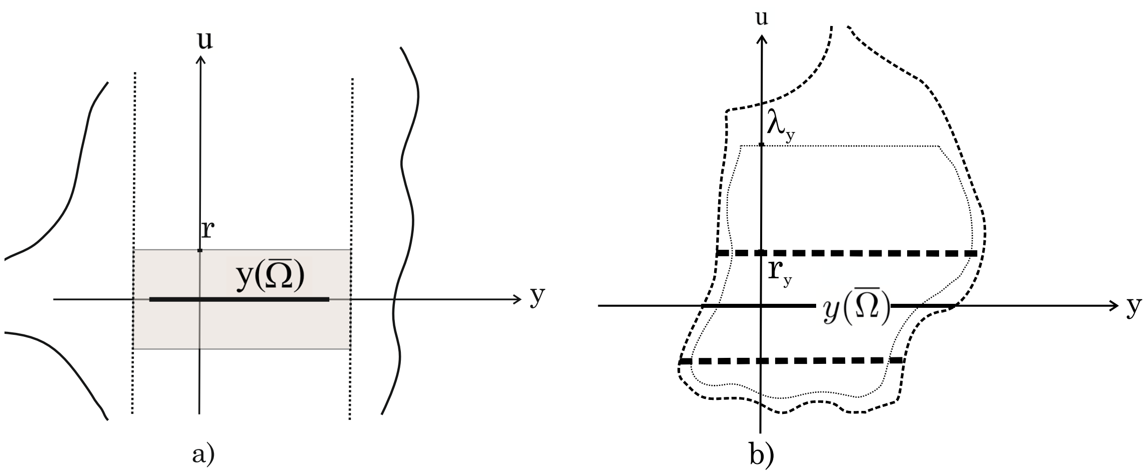

FIGURE 3. The effective domain of an autonomous Lagrangian $\Lambda=\Lambda(y,u)$ ($N=1$) and the validity of the assumptions of Theorem 2.8 (case a)) compared with those of Theorem 2.10 (case b)) for problem ($\mathrm{P}_{a,b}$).

Remark 2.13. Assumption b) in Theorem 2.10 is quite natural: indeed if a minimizer y_* exists for the problems considered here, even locally in a suitable sense, then for almost every $t\in[a,b]$, the map $0<r\mapsto L(t,y_*(t),ry_*'(t))$ is "convex" at $r=1$, in the sense that it has a non-empty convex subdifferential [4, 6].

Remark 2.14. Many of the assumptions of Theorem 2.10 are not there for a purely technical reason. Example 2.4 shows that Claim 1 may fail if (B_y^σ) does not hold. Example 2.6 shows that, in Claim 2 of Theorem 2.10 one cannot drop the assumption that D_Λ is open, the other assumptions being trivially satisfied (with $\mathscr{U}=\mathbb{R}$).

Example 2.15 [21, 36] shows that Claim 1 may fail even if just b) does not hold.

Example 2.15. Let

$$\Lambda(y,u)=\begin{cases}0 & \text{if } u=\dfrac{1}{2y}, y\neq 0\\ +\infty & \text{otherwise.}\end{cases}$$

Let $y_*(t)=\sqrt{y}$, $t\in[0,1]$. Since $y_*'=\dfrac{1}{2y_*}$ a.e. in $[0,1]$ we have $E(y_*)=0$ so that y_* minimizes E among the absolutely continuous functions y on $[0,1]$ satisfying $y(0)=0$. Now let y be Lipschitz in $[0,1]$ such that $y(0)=0$ and $E(y)<+\infty$. Then $\Lambda(y,y')<+\infty$ a.e. in $(0,1)$ so that

$$\begin{cases}y'=\dfrac{1}{2y} \text{ a.e. in } (0,1)\\ y(0)=0.\end{cases}$$

It follows that $y(t)=\sqrt{t}$, a contradiction. Thus $E(y)=+\infty$ whenever y is Lipschitz and $y(0)=0$. What fails here, among the assumptions of Theorem 2.10, is the fact that the projections of the effective domain onto the u variable are either empty or a non-zero singleton and thus are not in general star-shaped with respect to 0.

Remark 2.16 (Non-occurrence of the Lavrentiev phenomenon)**.** Theorems 2.5, 2.8 and 2.10 deal with the non-occurrence of the gap at a prescribed function y. Once the assumptions are satisfied for every admissible trajectory, one obtains as a byproduct the non-occurrence of the phenomenon. In particular, for the case of two prescribed endpoints, in the framework of Theorem 2.8 one has to impose that Λ is bounded on bounded sets, and a fortiori that Λ is real valued. Instead, in the spirit of Theorem 2.10, in place of (B_y^{σ}) and ($\mathrm{B}_y^{\Subset}$), one has to assume, respectively, that:

- for every compact subset $\mathscr{K}$ of Δ there exits $r_{\mathscr{K}}>0$ such that Λ is bounded on $([a,b]\times\mathscr{K}\times(\partial B_{r_{\mathscr{K}}}\cap\mathscr{U}))\cap\mathrm{Dom}(\Lambda)$;
- Λ is bounded on $[a,b]\times A$, for every compact subset A of $D_\Lambda\cap(\Delta\cap\mathscr{U})$. We are not aware of other results on the non-occurrence of the Lavrentiev phenomenon for extended valued Lagrangians in the presence of both prescribed endpoints.

Example 2.17. Theorem 2.10 can be appreciated with the following example, considered in [19]. Let $\Lambda(t,y,u)=\dfrac{1}{|y|}+\dfrac{1}{2}|u|^2$ for $t\in[0,1], y\in\Delta:=\mathbb{R}^3\setminus\{0\}, u\in\mathbb{R}^3$. The problem (P) of minimizing $E(y)=\displaystyle\int_0^1\Lambda(t,y(t),y'(t))\,dt$ among the absolutely continuous function that satisfy given endpoint conditions is related to the existence of Keplerian orbits: such a minimizer exists, as shown in [30]. A natural question is whether the Lavrentiev phenomenon occurs here. Now, Theorem 2.8 is of no help since, $\Lambda(t,y,u)$ being unbounded on *every* strip $[a,b]\times(B_K\setminus\{0\})\times B_r$ ($K,r>0$), one cannot ensure the validity of (B_y^{+}) for every admissible trajectory y (see Remark 2.16). Instead, the assumptions of Claim 2 of Theorem 2.10 are satisfied for every admissible trajectory y. Indeed

- $\Lambda(t,y,u)$ is convex in u and thus radially convex;
- If $K\subset\Delta=\mathbb{R}^3\setminus\{0\}$ is compact and $r>0$ then Λ is continuous and thus bounded on $[a,b]\times K\times B_r$, so that Λ fulfills ($\mathrm{B}_y^{\Subset}$) for every admissible trajectory y.

In particular, by applying Theorem 2.10 to a minimizing sequence, it follows that the Lavrentiev phenomenon does not occur.

Sketch of the proof of Theorem 2.10. The argument is inspired by [19], [24] and [35]. Instead of recurring, as in the proof of Theorem 2.8, to a Lusin's approximation of y, we just reparametrize y and take profit of the radial convexity assumption in order to estimate the energy of the reparametrized function in terms of $E(y)$. The construction is based on the following remark. Let φ be an absolutely continuous, increasing change of variable on $[a,b]$ with a Lipschitz inverse. Let y be an admissible trajectory, and set $\bar{y}(t) := y(\varphi^{-1}(t))$. Notice that, by taking high values of $\varphi'(\tau)$, one lowers the norm of the derivative of $\bar{y}(\varphi(\tau))$. The change of variable $t = \varphi(\tau)$ yields

$$E(\bar{y}) = \int_a^b \Lambda(t, \bar{y}(t), \bar{y}'(t))\, dt = \int_a^b \Lambda\Big(\varphi(\tau), y(\tau), \frac{y'(\tau)}{\varphi'(\tau)}\Big)\, \varphi'(\tau)\, d\tau.$$

Supposing that Λ smooth, the derivative of $\mu \mapsto \Lambda\Big(\varphi, y, \dfrac{u}{\mu}\Big)\, \mu$ at $\mu = 1$ is

$$\Lambda(\varphi, y, u) - u \cdot \nabla_u \Lambda(\varphi, y, u).$$

One then builds a suitable sequence of increasing and one-to-one changes of variable $\varphi_k : [a,b] \to [a,b]$ in such a way that $\varphi_k' \geq 1$ and:

- $y_k := y \circ \varphi_k$ is Lipschitz;
- $y_k \to y$ in the $W^{1,p}$ norm;
- $E(y_k)$ tends to $E(y)$ as $k \to +\infty$.

The convexity of $0 < \mu \mapsto \Phi(\mu) := \Lambda\big(t, y, \dfrac{u}{\mu}\big)\, \mu$ allows to compare $\Lambda(\varphi_k, y, y')$ with $\Lambda\big(\varphi_k, y, \dfrac{y'}{\varphi_k'}\big)\, \varphi_k'$; Condition ($\mathrm{B}_y^{\sigma}$) enables to preserve the estimate after applying the integral sign. Finally, Condition (S) yields an estimate of the integral $\displaystyle\int_a^b \Lambda(\varphi_k, y, y')\, d\tau$ in terms of $E(y)$. It may happen, however, that $\varphi_k(b) > b$: one has to introduce a suitable set where $0 < c \leq \varphi_k' < 1$, for a suitable constant c. Condition ($\mathrm{B}_y^{\Subset}$) is needed to give an upper bound of $\displaystyle\int_a^b \Lambda\big(\varphi_k, y, \frac{y'}{\varphi_k'}\big)\, \varphi_k'\, d\tau$ in terms of $\displaystyle\int_a^b \Lambda(\varphi_k, y, y')\, d\tau$ in the set where φ_k' is smaller than 1. □

Remark 2.18. It is worth mentioning that the proof of Theorem 2.10 is constructive (see [38]). Moreover, if in addition to the assumptions, the Lagrangian satisfies a suitable slow growth (e.g., Clarke's Condition (H) introduced in [24], satisfied for instance by $\sqrt{1 + |u|^2}$), then there exists a minimizing sequence of *equi-Lipschitz* functions. Without entering into the details (see [35, Corollary 5.5]), we point out how Clarke's *vision* in [24] inspired these recent results: they are all based on some relations between the convex subdifferential of the map Φ (introduced above) when u is large and when u is small enough. Whereas Condition (H) imposes a suitable inequality between the two quantities, we use in the proof of Theorem 2.10 some more general bounds of the convex subdifferential of Φ at $\mu = 1$, that are valid under the boundedness assumptions on Λ (namely conditions (B_y^{w}) and ($\mathrm{B}_y^{\Subset}$)).

3. The multidimensional autonomous case

3.1. **The framework.** In the one dimensional setting, there is no Lavrentiev gap when the integrand is continuous and does not depend on x. This result holds true even if the admissible maps are vector-valued.

In the third example considered in the Introduction, the integrand $f_3 : (x,s,\xi) \in B^2 \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \mapsto (s^2-1)^2|\xi|^2 \in [0,+\infty)$ is autonomous, and yet the Lavrentiev phenomenon occurs. This proves that the one dimensional result for autonomous integrands does not generalize to *vectorial* problems when the domain is higher dimensional. In this section, we thus focus on multidimensional *scalar* problems, namely when the admissible maps are defined on a subset of $\mathbb{R}^N$, $N \geq 2$, and take their values in $\mathbb{R}$.

We introduce a bounded Lipschitz open set Ω in $\mathbb{R}^N$, a Lipschitz function $\varphi : \mathbb{R}^N \to \mathbb{R}$ and a nonnegative continuous function $f : (s,\xi) \in \mathbb{R} \times \mathbb{R}^N \mapsto f(s,\xi) \in \mathbb{R}^+$. We point out that throughout this section, f does not depend on the x variable. We still denote by E the energy functional which is defined on the Sobolev set $W^{1,1}_\varphi(\Omega)$ of those $u \in W^{1,1}_{loc}(\mathbb{R}^N)$ which agree with φ on $\mathbb{R}^N \setminus \Omega$:

$$E : u \in W^{1,1}_\varphi(\Omega) \mapsto \int_\Omega f(u(x), \nabla u(x))\,dx.$$

In this setting, a classical result states that the Lavrentiev phenomenon does not occur provided that f is jointly convex with respect to both variables (s,ξ): for every $s,s' \in \mathbb{R}$ and $\xi,\xi' \in \mathbb{R}^N$,

$$f(\theta s + (1-\theta)s', \theta\xi + (1-\theta)\xi') \leq \theta f(s,\xi) + (1-\theta) f(s',\xi'), \qquad \forall \theta \in [0,1].$$

This statement has a long history and is the result of many contributions. To the best of our knowledge, the first proof is contained in the book by Ekeland and Temam [29, Proposition 2.6, p. 312], in the case when f only depends on the ξ variable and when φ is zero on the boundary. Then, De Arcangelis and Trombetti [28, Proposition 2.4] have extended this result to general boundary conditions φ, but still assuming that f does not depend on the s variable. We have adapted the proof to the case when f is jointly convex with respect to (s,ξ) and enlarged the class of domains that one can consider [13]. Let us also quote another approach by Cellina and Bonfanti [7] who have established the absence of the Lavrentiev phenomenon when f has a separate dependence on s and ξ; that is,

$$f(s,\xi) = g(|\xi|) + h(s),$$

where g and h are smooth convex functions satisfying some suitable growth assumptions (here, $|\xi|$ means the Euclidean norm of ξ).

However, the joint convexity with respect to (s,ξ) is not a natural assumption in our framework. Indeed, the weak lower semicontinuity of E, which is a key property for both the relaxation of the functional (compare (1.1)) and the Direct Method, only requires the convexity with respect to ξ. Moreover, we have already seen in Section 2 that in the one-dimensional case, there is no Lavrentiev gap for continuous autonomous integrands. In this section, we present a recent result giving a similar conclusion in the scalar multidimensional setting, when f is convex just with respect to ξ.

3.2. **Non occurrence of the Lavrentiev gap for autonomous problems.** Here is the main result of Section 3.

Theorem 3.1. [11] *Let Ω be a Lipschitz bounded open set in $\mathbb{R}^N$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz bounded function. Assume that the continuous map $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ is convex with respect to the last variable. Then for every $u \in W^{1,1}_\varphi(\Omega)$, there exists a sequence $(u_i)_{i\geq 1} \subset W^{1,\infty}_\varphi(\Omega)$ which converges to u in $W^{1,1}(\Omega)$ and such that $(E(u_i))_{i\geq 1}$ converges to $E(u)$.*

It follows from Theorem 3.1 that no Lavrentiev gap occurs between $W^{1,1}_{\varphi}(\Omega)$ and $W^{1,\infty}_{\varphi}(\Omega)$.

When φ is a smooth function on $\mathbb{R}^N$, it also follows from Theorem 3.1 and a standard regularisation argument that no Lavrentiev gap can occur between $W^{1,1}_{\varphi}(\Omega)$ and $C^{\infty}_{\varphi}(\overline{\Omega})$.

If $u \in W^{1,p}_{\varphi}(\Omega)$ for some $p \geq 1$, then the sequence $(u_i)_{i\geq 1}$ that we construct explicitly in the proof of Theorem 3.1 converges to u in $W^{1,p}_{\varphi}(\Omega)$.

3.3. **Ideas of the proof.** Let us explain the main arguments to prove a weaker result than the theorem stated above: Given $u \in W^{1,1}(\Omega)$, we want to construct a sequence $(u_i)_{i\geq 1} \subset W^{1,\infty}(\Omega)$ converging to u in $W^{1,1}(\Omega)$ and such that $\lim_{i\to+\infty} E(u_i) = E(u)$. Here, we ignore the boundary condition; that is, we do not require that each u_i agree with u on $\partial\Omega$. This rougher construction still allows to illustrate the main ideas of the proof.

In order to simplify the presentation, let us also assume that f is superlinear with respect to the last variable, uniformly with respect to the first one:

$$\limsup_{|\xi|\to+\infty} \inf_{s\in\mathbb{R}} \frac{f(s,\xi)}{|\xi|} = +\infty.$$

The convolution approach. One of the classical tools to rule out the Lavrentiev gap relies on the approximation by convolution: given $u \in W^{1,1}(\Omega)$ and a family of mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$, let us consider the maps:

$$u_\varepsilon = u * \rho_\varepsilon.$$

We recall that any $u \in W^{1,1}_{\varphi}(\Omega)$ is implicitly extended by φ outside Ω, so that the convolution u_ε is well-defined. The maps u_ε converge to u in $W^{1,1}(\Omega)$. It follows from the Fatou lemma that the functional E is lower semicontinuous on $W^{1,1}(\Omega)$. This implies that $\liminf_{\varepsilon\to 0} E(u_\varepsilon) \geq E(u)$. If $E(u) = +\infty$, then the equality holds and the proof is complete. In the rest of the proof, we thus assume that $E(u) < +\infty$. We have to establish the limsup inequality:

$$\limsup_{\varepsilon\to 0} E(u_\varepsilon) \leq E(u). \tag{3.1}$$

If f is *jointly convex* with respect to (s,ξ), then we deduce from the Jensen inequality that:

$$f(u*\rho_\varepsilon, \nabla u*\rho_\varepsilon) \leq f(u,\nabla u)*\rho_\varepsilon. \tag{3.2}$$

By integrating over Ω, this gives

$$\int_\Omega f(u*\rho_\varepsilon(x), \nabla u*\rho_\varepsilon(x))\,dx \leq \int_\Omega f(u,\nabla u)*\rho_\varepsilon(x)\,dx.$$

Since $E(u) < +\infty$, the function $x \mapsto f(u(x),\nabla u(x))$ is summable on Ω and thus the right hand side converges to $f(u,\nabla u)$ in $L^1(\Omega)$. Then, it remains to take the limsup in both sides to obtain (3.1).

Under our assumptions, f is not assumed to be convex with respect to s. As a consequence, the above argument involving this form of the Jensen inequality does not apply. As a matter of fact, we do not even know if (3.1) is true for this particular approximation of u. We are thus led to rely on a new strategy to approximate u.

The parametric problem. In a nutshell, the idea is to mollify the graph of u instead of u itself. In this step of the proof, our main source of inspiration is [10] where Bouchitté and Fragalà address a different question, namely a new duality theory in the Calculus of Variations. The common feature with our problem is the non convexity of the integrand with respect to the s variable. In order to overcome this obstacle, they introduce a *convexification recipe*: to the original energy E, they associate the parametric functional G which is convex with respect to all variables.

In order to better explain the parametric formulation of the problem, let us start from the classical example of the minimum area problem. The non parametric formulation amounts to the minimization of the energy:

$$E_a : u \in W^{1,1}(\Omega) \mapsto \int_\Omega \sqrt{1+|\nabla u|^2}\, dx.$$

Then $E_a(u)$ is the area of the graph of u over Ω, namely

$$E_a(u) = \mathscr{H}^N(\mathrm{Graph}_u),$$

where we have denoted $\mathrm{Graph}_u = \{(x,u(x)) : x \in \Omega\}$ while $\mathscr{H}^N$ is the N dimensional Hausdorff measure on $\mathbb{R}^N \times \mathbb{R}$. Another way to consider this quantity is to interpret it as the perimeter of the hypograph of u:

$$E_a(u) = \mathrm{Per}\,(\{(x,t) : t \leq u(x)\}).$$

Let us introduce the indicator function χ_u of the hypograph of u:

$$\chi_u : (x,t) \in \Omega \times \mathbb{R} \mapsto \begin{cases} 1 & \text{if } t \leq u(x), \\ 0 & \text{if } t > u(x). \end{cases}$$

If u is smooth on $\overline{\Omega}$, then the Stokes formula applied on the hypograph of u implies that for every $\varphi \in C_c^\infty(\Omega \times \mathbb{R}, \mathbb{R}^{N+1})$,

$$\int_{\Omega\times\mathbb{R}} \chi_u(x,t) \mathrm{div}\, \varphi(x,t)\, dx\, dt = -\int_{\mathrm{Graph}_u} \varphi n_u(x,t)\, d\mathscr{H}^N(x,t),$$

where n_u is the inner normal to the hypograph of u:

$$n_u = \frac{1}{\sqrt{1+|\nabla u|^2}}(\nabla u, -1).$$

This proves that the distributional derivative of χ_u is $n_u \mathscr{H}^N \llcorner \mathrm{Graph}_u$ which belongs to the set of finite $\mathbb{R}^{N+1}$-valued measures on $\Omega \times \mathbb{R}$. The total variation of this measure is

$$|D\chi_u| = \mathscr{H}^N \llcorner \mathrm{Graph}_u.$$

The density of $D\chi_u$ with respect to $|D\chi_u|$ is:

$$\frac{D\chi_u}{|D\chi_u|} = n_u.$$

The above calculation can be extended to $W^{1,1}$ functions provided that we give a suitable definition to the graph of u.

We have thus proved:

$$E_a(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\, dx = \int_{\Omega\times\mathbb{R}} d|D\chi_u| = \int_{\Omega\times\mathbb{R}} |n_u|\, d|D\chi_u|.$$

We can write the latter quantity as:

$$\int_{\Omega\times\mathbb{R}} h_a\left(\frac{D\chi_u}{|D\chi_u|}\right) d|D\chi_u|,$$

where $h_a(\xi,\lambda) := |(\xi,\lambda)|$ for every $(\xi,\lambda)\in\mathbb{R}^N\times(-\infty,0)$ (the notation $|\cdot|$ here refers to the Euclidean norm in $\mathbb{R}^{N+1}$). Let us observe that one can obtain the new integrand $|D\chi_u|$ from the old one $\sqrt{1+|\nabla u|^2}$ by a homogeneization technique. Indeed, denoting by f_a the Lagrangian $\xi\mapsto\sqrt{1+|\xi|^2}$, we have

$$\forall\lambda<0, \qquad h_a(\xi,\lambda) = -\lambda f_a\left(\frac{\xi}{-\lambda}\right) = |(\xi,\lambda)|.$$

The identity $E_a(u) = \int_{\Omega\times\mathbb{R}} h_a\left(\frac{D\chi_u}{|D\chi_u|}\right) d|D\chi_u|$ can be generalized to a large family of functionals in the calculus of variations, in particular in our framework where the energy $E(u) = \int_\Omega f(u,\nabla u)\,dx$ has a nonnegative continuous integrand $f:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^+$ which is convex and superlinear with respect to the last variable. Indeed, following the same arguments as for the minimum area problem, one can prove that

$$E(u) = G(\chi_u)$$

where

$$G(\chi_u) = \int_{\Omega\times\mathbb{R}} h\left(t,\frac{D\chi_u}{|D\chi_u|}\right) d|D\chi_u|,$$

and

$$h:(t,\xi,\lambda)\in\mathbb{R}\times\mathbb{R}^N\times(-\infty,0)\mapsto -\lambda f(t,\xi/(-\lambda)).$$

The next step is to extend the functional G as a convex functional on the vector space of those $v\in L^1_{loc}(\Omega\times\mathbb{R})$ such that Dv is an $\mathbb{R}^{N+1}$-valued Borel measure. We first need to extend h on the whole $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$. In view of the superlinearity of f with respect to the last variable, there is only one way to do so if one expects a lower semicontinuous function on the whole space: we define $h=+\infty$ on $\mathbb{R}\times\mathbb{R}^N\times[0,+\infty)$ except at the origin, where we set $h(t,0,0)=0$ for every $t\in\mathbb{R}$. The resulting h is indeed lower semicontinuous, and convex in the last variables (ξ,λ). We can now extend the parametric energy G:

$$G(v) = \int_{\Omega\times\mathbb{R}} h\left(t,\frac{Dv}{|Dv|}(x,t)\right) d|Dv|(x,t),$$

where v is any $L^1_{loc}(\Omega\times\mathbb{R})$ function such that Dv is an $\mathbb{R}^{N+1}$-valued Borel measure.

As before, the notation $|Dv|$ refers to the total variation of the measure Dv while $Dv/|Dv|$ is the density of Dv with respect to its total variation. Observe that the integrand of G does not depend on v itself but just on its distributional derivative. This stands in strong contrast with E for which the integrand f depends both on u and ∇u.

Since for every $t\in\mathbb{R}$, the map $(\xi,\lambda)\mapsto h(t,\xi,\lambda)$ is convex, the map G is convex as well, whereas E is not necessarily convex on $W^{1,1}(\Omega)$. In that sense, the minimization problem associated to G is a convexification of the original problem related to E. By exploiting the convexity of G, we are in a better position to establish a new Jensen type inequality. The price to pay is that we enlarge the set of admissible functions. Indeed, the new functional G is not only defined for the functions of the form χ_u with $u\in W^{1,1}_\varphi(\Omega)$, but on the set of those locally summable maps v which have bounded variations. Let us emphasize the fact that these maps v

depend on $N+1$ variables. In the sequel, we refer to the x variable as the horizontal one, while we call t the vertical variable.

The horizontal regularisation. As already mentioned, we do not approximate the admissible map u itself, but the indicator function χ_u of its hypograph. It is crucial to decouple the regularisation with respect to the horizontal variable x and the one related to the vertical variable t. Regarding the former regularisation, we simply take the convolution of χ_u with a smooth mollifier:

$$\chi_u *_x \rho_\varepsilon(x,t) = \int_{\mathbb{R}^N} \chi_u(x-y,t)\rho_\varepsilon(y)\,dy, \qquad \forall (x,t) \in \Omega \times \mathbb{R}.$$

As above, the function u is implicitly extended to $\mathbb{R}^N$ by setting $u = \varphi$ on $\mathbb{R}^N \setminus \Omega$. Observe that the convolution is only defined with respect to the horizontal variable x, even if χ_u also depends on the t variable.

This approximation satisfies the crucial property:

$$\lim_{\varepsilon \to 0} G(\chi_u *_x \rho_\varepsilon) = G(\chi_u). \tag{3.3}$$

In order to prove the *liminf* inequality: $\liminf_{\varepsilon\to 0} G(\chi_u *_x \rho_\varepsilon) \geq G(\chi_u)$, one relies on a classical theorem due to Reshetnyak. The *limsup* part instead is a consequence of the following inequality between positive measures:

$$h(t, D\chi_u(\cdot,t) *_x \rho_\varepsilon(x)) \leq h(t, D\chi_u(\cdot,t)) *_x \rho_\varepsilon(x). \tag{3.4}$$

Here, given an $\mathbb{R}^{N+1}$-valued Radon measure μ on $\Omega \times \mathbb{R}$, we have denoted by $h(t,\mu)$ the positive measure

$$h(t,\mu(x,t)) = h\left(t, \frac{\mu}{|\mu|}(x,t)\right) |\mu|(x,t).$$

We use this notation in (3.4) for $\mu = D\chi_u *_x \rho_\varepsilon$ and $\mu = D\chi_u$.

The inequality (3.4) reduces to the classical Jensen inequality when one replaces the measure $D\chi_u$ by a function $\ell \in L^1(\Omega \times \mathbb{R}, \mathbb{R}^N)$. More specifically, for every $t \in \mathbb{R}$,

$$\begin{aligned} h(t, \ell(\cdot,t) *_x \rho_\varepsilon(x)) &= h\left(t, \int_{\mathbb{R}^N} \ell(x-y,t)\rho_\varepsilon(y)\,dy\right) \\ &\leq \int_{\mathbb{R}^N} h(t, \ell(x-y,t))\rho_\varepsilon(y)\,dy = h(t,\ell(\cdot,t)) *_x \rho_\varepsilon(x). \end{aligned}$$

The above calculation just requires the convexity of h with respect to the last variables, since the convolution does not involve the t variable.

Proving this inequality for $D\chi_u$ instead of ℓ requires the disintegrations of the measures $D\chi_u$ and $D\chi_u *_x \rho_\varepsilon$ according to the product $\Omega \times \mathbb{R}$. This leads to (3.4). By integrating the latter over Ω, one can then derive the limsup inequality for G and finally obtains (3.3).

The vertical regularisation. We proceed to intertwine the *horizontal smoothing* described in the paragraph above with a *vertical* regularisation. The latter amounts to strengthening the monotonicity of $\chi_u *_x \rho_\varepsilon(x,t)$ with respect to the t variable. More specifically, given a new parameter $\delta > 0$, we define the maps

$$v_{\varepsilon,\delta}(x,t) = \chi_u *_x \rho_\varepsilon(x,t) + \delta\alpha(t).$$

Here, $\alpha : \mathbb{R} \to (0,1)$ is a smooth decreasing function with $\lim_{t\to-\infty} \alpha(t) = 1$, $\lim_{t\to+\infty} \alpha(t) = 0$. The role of α is to guarantee that the new approximation $v_{\varepsilon,\delta}$ decreases with respect to the vertical variable t uniformly with respect to x. Moreover, (3.3) becomes

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} G(\chi_u *_x \rho_\varepsilon + \delta\alpha) = G(\chi_u).$$

As we shall see below, the conjunction of the horizontal and the vertical regularisations yields the Lipschitz continuity of the approximating sequence that we now define.

Coming back to functions defined on Ω. Fix $\varepsilon, \delta > 0$. For every $x \in \Omega$, the map $t \mapsto v_{\varepsilon,\delta}(x,t)$ is decreasing, and moreover, using the definition of χ_u and the properties of α, one gets:

$$\lim_{t\to-\infty} v_{\varepsilon,\delta}(x,t) = 1+\delta, \qquad \lim_{t\to+\infty} v_{\varepsilon,\delta}(x,t) = 0.$$

As a consequence, for every $s \in (0,1)$ and every $x \in \Omega$, there is a unique $t \in \mathbb{R}$ such that $v_{\varepsilon,\delta}(x,t) = s$. We denote by $u^s_{\varepsilon,\delta}(x)$ this value of t.

Assume for an instant that $v_{\varepsilon,\delta}$ and $u^s_{\varepsilon,\delta}$ are smooth on $\Omega\times\mathbb{R}$. Starting from the identity $v_{\varepsilon,\delta}(x, u^s_{\varepsilon,\delta}(x)) = s$, the chain rule implies that

$$|\nabla u^s_{\varepsilon,\delta}(x)| = \frac{|\nabla_x v_{\varepsilon,\delta}(x, u^s_{\varepsilon,\delta}(x))|}{|\partial_t v_{\varepsilon,\delta}(x, u^s_{\varepsilon,\delta}(x))|}.$$

This expression gives an hint of the respective roles of ε and δ. The numerator $|\nabla_x v_{\varepsilon,\delta}(x, u^s_{\varepsilon,\delta}(x))|$ in the right-hand side is bounded uniformly in x thanks to the horizontal convolution with a mollifier. The denominator $|\partial_t v_{\varepsilon,\delta}(x, u^s_{\varepsilon,\delta}(x))| = -\partial_t v_{\varepsilon,\delta}(x, u^s_{\varepsilon,\delta}(x))$ is bounded from below since the $\delta\alpha$ term induces the uniform monotonicity in the t variable. We can thus conclude that $u^s_{\varepsilon,\delta}$ is Lipschitz continuous.

Actually, the above argument must be suitably modified, since $v_{\varepsilon,\delta}$ is not differentiable with respect to t in general and at this stage of the proof, no regularity property is known for $u^s_{\varepsilon,\delta}$. More specifically, we need to identify a condition on the function $v_{\varepsilon,\delta}$ which does not involve its partial derivatives and guarantees the Lipschitz continuity of $u^s_{\varepsilon,\delta}$. In order to guess this condition, let us remember that $v_{\varepsilon,\delta}$ is close to the indicator function of the hypograph of u. Now, a function $w : \Omega \to \mathbb{R}$ is Lipschitz continuous if there exists some $K > 0$ such that its graph lies below the family of cones of the form

$$C_{x'} := \{(x,t) : t \geq w(x') + K|x-x'|\}, \qquad x' \in \Omega.$$

Equivalently,

$$\chi_w(x, t+K|x-x'|) \leq \chi_w(x',t), \tag{3.5}$$

where χ_w is the indicator function of the hypograph of w. It turns out that replacing χ_w by $v_{\varepsilon,\delta}$ in (3.5) gives exactly the condition that we are looking for; that is, $u^s_{\varepsilon,\delta}$ is Lipschitz continuous for every $s \in (0,1)$ if and only if there exists $K > 0$ such that

$$v_{\varepsilon,\delta}(x, t+K|x-x'|) \leq v_{\varepsilon,\delta}(x',t), \qquad \forall x,x' \in \Omega, \forall t \in \mathbb{R}.$$

This characterization allows to establish the desired Lipschitz continuity of the approximating maps $u^s_{\varepsilon,\delta}$.

Letting (ε,δ) *go to* $(0,0)$. It follows from elementary arguments that for every $s\in(0,1)$, the family of maps $u^s_{\varepsilon,\delta}$ converges to u a.e. when ε and δ go to 0. The $W^{1,1}$ convergence of $u^s_{\varepsilon,\delta}$ to u and the convergence of the energies $E(u^s_{\varepsilon,\delta})$ to $E(u)$ are more delicate and rely on the coarea formula.

Let us point out that there is no reason for $u^s_{\varepsilon,\delta}$ to satisfy the boundary condition $u^s_{\varepsilon,\delta}=\varphi$ on $\partial\Omega$. We thus need to modify the values of the approximating maps near the boundary. We do not detail this part of the proof here. Let us just mention that it relies on a partition of unity argument and local approximations near the boundary. In order to overcome the lack of convexity with respect to u, one relies again on the parametric formulation of the problem.

4. THE MULTIDIMENSIONAL NON-AUTONOMOUS CASE

4.1. **The framework.** In this last section, we consider scalar problems in the multiple integrals Calculus of Variations, when the integrand depends on the three variables x,u and ∇u. We present here some of our very recent results contained in [12]. More specifically, we assume that the integrand f which now also depends on the space variable x, has the following properties:

Hypothesis (A). (A$_1$) The function $(x,s,\xi)\mapsto f(x,s,\xi)$ is a nonnegative Carathéodory function which is convex with respect to the variables (s,ξ), for a.e. $x\in\Omega$.

(A$_2$) For every $R>0$, there exists $\gamma_R\in L^1(\Omega)$ such that for every $|(s,\xi)|\le R$,

$$f(x,s,\xi)\le\gamma_R(x),\qquad \text{a.e. } x\in\Omega.$$

Given $p\ge 1$, a Lipschitz bounded open set $\Omega\subset\mathbb{R}^N$ and a Lipschitz function $\varphi\in W^{1,\infty}(\mathbb{R}^N)$, we consider the functional

$$E: u\in W^{1,p}_\varphi(\Omega)\mapsto\int_\Omega f(x,u(x),\nabla u(x))\,dx.$$

In order to state our main assumption, we need to introduce some notation:

Definition 4.1 (The functions $f^-_{x,\delta}$ and $(f^-_{x,\delta})^{**}$)**.** Let $x\in\Omega$ and $\delta\in(0,1)$. We denote by η the $(N+1)$-dimensional variable $\eta=(s,\xi)$.

(1) For every $\eta\in\mathbb{R}^{N+1}$, we define

$$f^-_{x,\delta}(\eta):=\operatorname{essinf}\{f(y,\eta): y\in B_\delta(x)\cap\Omega\}=\sup\{\rho\ge 0:\rho\le f(y,\eta)\text{ a.e. } y\in B_\delta(x)\cap\Omega\}.$$

(2) The function $f^-_{x,\delta}$ being nonnegative on $\mathbb{R}^{N+1}$, one can define $(f^-_{x,\delta})^{**}$ as the largest convex function on $\mathbb{R}^{N+1}$ lying below $f^-_{x,\delta}$.

Remark 4.2. Equivalently, $(f^-_{x,\delta})^{**}$ is the convex bidual of the nonnegative function $f^-_{x,\delta}$.

The convex function $(f^-_{x,\delta})^{**}$ may be far below $f(x,\cdot,\cdot)$.

Given a function $g:\mathbb{R}^{N+1}\to\mathbb{R}^+$ which is bounded from below, the Fenchel transform of g is defined as:

$$g^*(\ell):=\sup_{\eta\in\mathbb{R}^{N+1}}\left(\langle(\eta,\ell)\rangle-g(\eta)\right),$$

where we have denoted by $\langle\cdot,\cdot\rangle$ the standard scalar product on $\mathbb{R}^{N+1}$. One can thus define the Fenchel transform of g^* (which is bounded from below) and obtain in this way the convex bidual g^{**}, which is the largest convex function lying below g, see [29, Chapter 1].

Example 4.3. For instance, consider the double phase functional presented in the Introduction:

$$f(x,s,\xi)=|\xi|^p+a(x)|\xi|^q,$$

where $1\le p\le q$ and a is a nonnegative Lipschitz continuous function on $\mathbb{R}^N$. Then, if $x_0\in\Omega$ is such that $a(x_0)>0$ and $B_\delta(x_0)\cap\Omega\cap[a\equiv 0]\neq\emptyset$, for every $\xi\in\mathbb{R}^N$,

$$(f^-_{x_0,\delta})^{**}(s,\xi)=f^-_{x_0,\delta}(s,\xi)=|\xi|^p.$$

In particular, $f^-_{x_0,\delta}(s,\cdot)$ has a p-growth, while $f(x_0,s,\cdot)$ has a q-growth.

Our main assumption is related to the distance between $(f^-_{x,\delta})^{**}(s,\xi)$ and $f(x,s,\xi)$. For every $L>0$, we say that f satisfies the condition ($\mathrm{H}^{p,L}$) if the following holds:

Hypothesis ($\mathbf{H}^{p,L}$). There exists $A>0$, $b\in L^1(\Omega)$, $\delta_*\in(0,1)$ and a function $\theta:(0,\delta_*)\mapsto(0,1]$ with $\lim_{\delta\to 0}\theta(\delta)=1$ such that for every $\delta\in(0,\delta_*)$ and every $(x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^N$,

$$|(s,\xi)|^p+(f^-_{x,\delta})^{**}(s,\xi)\le L\delta^{-N}\Rightarrow f(x,\theta(\delta)s,\theta(\delta)\xi)\le A\left(|(s,\xi)|^p+(f^-_{x,\delta})^{**}(s,\xi)\right)+b(x). \tag{4.1}$$

Remark 4.4. Actually, the assumption ($\mathrm{H}^{p,L}$) unifies and substantially extends most of the classical conditions arising in the Lavrentiev literature; we refer to the classical reference [14] and the more recent paper [40] for a general overview, see also the introductions of the two papers [8, 9]. One of the main novelties in ($\mathrm{H}^{p,L}$) is that we require the estimate on the right-hand side of (4.1) to hold when both (s,ξ) *and* $(f^-_{x,\delta})^{**}(s,\xi)$ are not too large. This stands in strong contrast with the main condition in the seminal article [42] for instance, where Zhikov assumes that $f(x,s,\xi)$ is estimated by $(f^-_{x,\delta})^{**}(s,\xi)$ whenever $|(s,\xi)|^p\le L\delta^{-N}$.

4.2. Non-occurrence of the gap under ($\mathbf{H}^{p,L}$).

Here is the main result of Section 4.

Theorem 4.5. *Let φ be a bounded Lipschitz function on $\mathbb{R}^N$, Ω a Lipschitz bounded open set, $p\in[1,+\infty)$, and f satisfy the Basic Assumption* (A) *and Hypothesis* ($\mathrm{H}^{p,L}$) *for some $L>0$. Then there is no Lavrentiev gap between $W^{1,\infty}_\varphi(\Omega)$ and $W^{1,p}_\varphi(\Omega)$.*

Example 4.6. Let us first observe that the double phase functional $f:(x,s,\xi)\mapsto|\xi|^p+a(x)|\xi|^q$, with $a\in W^{1,\infty}(\mathbb{R}^N)$, satisfies ($\mathrm{H}^{p,L}$) for every $L>0$ provided that $q\le p\left(1+\frac{1}{N}\right)$. Indeed, for every $(x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^N$ and every $\delta\in(0,1)$,

$$f^-_{x,\delta}(s,\xi)=|\xi|^p+a^-_{x,\delta}|\xi|^q,$$

where $a^-_{x,\delta}=\inf_{B_\delta(x)\cap\Omega}a$. Since the latter is convex, we also have $f^-_{x,\delta}(s,\xi)=(f^-_{x,\delta})^{**}(s,\xi)$. We then use the Lipschitz continuity of a to get

$$\begin{aligned}f(x,s,\xi)&\le|\xi|^p+a^-_{x,\delta}|\xi|^q+\|\nabla a\|_{L^\infty(\mathbb{R}^N)}\delta|\xi|^q=(f^-_{x,\delta})^{**}(s,\xi)+\|\nabla a\|_{L^\infty(\mathbb{R}^N)}\delta|\xi|^{q-p}|\xi|^p\\&\le(f^-_{x,\delta})^{**}(s,\xi)(1+\|\nabla a\|_{L^\infty(\mathbb{R}^N)}\delta|\xi|^{q-p}),\end{aligned}$$

where the last line relies on the fact that $|\xi|^p\le(f^-_{x,\delta})^{**}(s,\xi)$.

If $|(s,\xi)|^p+(f^-_{x,\delta})^{**}(s,\xi)\le L\delta^{-N}$, then $|\xi|^p\le L\delta^{-N}$, so that $|\xi|^{q-p}\le(L\delta^{-N})^{\frac{q-p}{p}}$. This implies

$$f(x,s,\xi)\le(f^-_{x,\delta})^{**}(s,\xi)(1+L^{\frac{q-p}{p}}\|\nabla a\|_{L^\infty(\mathbb{R}^N)}\delta^{1-N\frac{q-p}{p}}).$$

If $q \leq p\left(1+\frac{1}{N}\right)$, then the exponent of δ is nonnegative and we can conclude that f satisfies ($\mathrm{H}^{p,L}$) for every $L>0$. In particular, we recover the classical restriction on p and q to discard the Lavrentiev gap for the double phase functional, see e.g. [42, Theorem 2.3].

We emphasize the fact that ($\mathrm{H}^{p,L}$) is not restricted to the $p-q$ growth condition, and does not involve the classical Δ_2 property either.

Example 4.7. For instance, the function

$$f(x,s,\xi) = \exp(|\xi|) + a(x)\exp(\alpha|\xi|), \tag{4.2}$$

satisfies ($\mathrm{H}^{p,L}$) provided that a is Lipschitz continuous and $\alpha \leq 1+1/N$. This specific integrand does not fall into the realm of any of the results that we know on the Lavrentiev phenomenon.

Example 4.8. Another interesting example of a functional to which Theorem 4.5 applies is given by the function

$$f(x,s,\xi) = g(|x|^\gamma|\xi|^p),$$

where $g:\mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing convex function and $\gamma>0$, $p\geq 1$. We further assume that there exists $\nu>1$ such that

$$\int_1^{+\infty} \frac{g(t)}{t^\nu}\,dt < +\infty.$$

Then f does satisfy ($\mathrm{H}^{p,L}$) whenever $\gamma > N\frac{\nu-2}{\nu-1}$. As a consequence of Theorem 4.5, there is no Lavrentiev gap for the corresponding energy. We are not aware of any comparable result in the literature for this integrand, without any further restriction on the growth of g.

4.3. **Ideas of the proof.** The role of the function $(f^-_{x,\delta})^{**}$ is now well established in the field and we can trace it back to the paper [42].

i) A regularizing sequence. Let us briefly sketch the standard approximation argument for a given map $u \in W^{1,p}_\varphi(\Omega)$ such that $E(u) < +\infty$. Let $(\rho_\varepsilon)_{\varepsilon>0}$ a smooth regularisation kernel such that each ρ_ε is compactly supported in B_ε. Then one defines the maps

$$u_\varepsilon(x) = u*\rho_\varepsilon(x) = \int_{B_\varepsilon} u(x-y)\rho_\varepsilon(y)\,dy.$$

Exactly as in (3.1), we deduce from the Fatou lemma the liminf inequality:

$$\liminf_{\varepsilon\to 0} E(u_\varepsilon) \geq E(u).$$

The x-dependence of f does not introduce any change at this point. In contrast, the limsup inequality does not readily follow from the Jensen inequality as in (3.2). Actually, one applies the latter to the function $(f^-_{x,\varepsilon})^{**}$ instead of f, for some given $x\in\Omega$ and $\varepsilon>0$ (observe that this is possible since $(f^-_{x,\varepsilon})^{**}$ is convex by construction):

$$(f^-_{x,\varepsilon})^{**}(u_\varepsilon(x),\nabla u_\varepsilon(x)) \leq (f^-_{x,\varepsilon})^{**}(u,\nabla u)*\rho_\varepsilon(x) \leq f(\cdot,u,\nabla u)*\rho_\varepsilon(x).$$

In the last inequality, we have used the fact that $(f^-_{x,\varepsilon})^{**} \leq f^-_{x,\varepsilon} \leq f$ everywhere on $B_\varepsilon(x)\times\mathbb{R}\times\mathbb{R}^N$.

ii) The role of ($\mathrm{H}^{p,L}$). We now rely on the assumption ($\mathrm{H}^{p,L}$) for some suitable $L>0$ (here, we admit that there exists $L>0$ for which the condition in the left-hand side of (4.1) is satisfied for

$(s,\xi)=(u_\varepsilon(x),\nabla u_\varepsilon(x))$; we also assume for simplicity that one can take $\theta\equiv 1$). Hence, there exists $A>0$ and $b\in L^1(\Omega)$ such that

$$f(x,u_\varepsilon(x),\nabla u_\varepsilon(x))\leq A\left(|(u_\varepsilon(x),\nabla u_\varepsilon(x))|^p+f(\cdot,u,\nabla u)*\rho_\varepsilon(x)\right)+b(x).$$

Since $E(u)<+\infty$, the right-hand side is summable. Integrating the above inequality on Ω, we can deduce from the Vitali theorem that $\limsup_{\varepsilon\to 0}E(u_\varepsilon)\leq E(u)$. Together with the liminf inequality, this allows to conclude that

$$\lim_{\varepsilon\to 0}E(u_\varepsilon)=E(u).$$

iii) Modification of u_ε. The main novelty in our approach lies in the way we modify the approximating maps u_ε to force them to coincide with φ on $\partial\Omega$. Here, we have been inspired by one *coup de génie* of Francis Clarke who introduced new seminal ideas in the Hilbert-Haar regularity theory for scalar multidimensional problems [23]. In this work, one of the most beautiful ideas is to compare a given minimum u with a *dilated* version of u, to obtain continuity properties for u. This stands in contrast with the usual approach where *translations* were used instead. The gain of relying on dilations is to substantially weaken the assumption on the boundary condition to obtain essentially the same conclusion on the minimum. Here, we go in the opposite direction; that is, we replace dilations by translations.

More specifically, from the classical references (see e.g. [29]) up to the most recent ones as [9], one first proves the non occurrence of Lavrentiev gaps on a star-shaped domain by using dilations as a preliminary step before taking the convolution with ρ_ε. This allows to work with maps which agree with the boundary condition φ on a neighborhood of $\partial\Omega$. In this situation, it becomes easy to modify the maps u_ε in such a way that they coincide with φ on $\partial\Omega$. In a second step, one extends the result to an arbitrary Lipschitz domain Ω, by relying on the fact that the latter can be written as the union of star-shaped subdomains. This extension is sometimes based on a partition of unity argument. The latter in turn involves problematic terms, which require suitable restrictions on the growth of f (when $p>N-1$ however, it is sometimes possible to avoid such restrictions, see [31, Theorem 2.3]). Using translations instead of dilations substantially simplify the argument. In particular, we do not appeal to this two steps strategy and directly work instead on any Lipschitz domain. Moreover, when working with translations, the calculations are exactly the same for u and ∇u, whereas performing dilations inevitably introduces a discrepancy between the function and its gradients. This discrepancy is another source of technical difficulties, which are generally overcome by assuming additionally that the function u is bounded. Our approach does not require such an assumption.

4.4. **On the assumption** ($\mathrm{H}^{p,L}$). In the examples presented above, as well as in the most classical ones that one can find in the literature, it is not difficult to check that ($\mathrm{H}^{p,L}$) is satisfied. To this end, we first observe that this condition is remarkably *stable*. For instance, the finite sum of nonnegative functions satisfying ($\mathrm{H}^{p,L}$), satisfies ($\mathrm{H}^{p,L}$) as well. If f and g are two integrands which are comparable, in the sense that $\frac{1}{A}f\leq g\leq Af$ for some $A>0$, then f satisfies ($\mathrm{H}^{p,L}$) if and only if g does.

We proceed to give a standard set of assumptions which make the verification of ($\mathrm{H}^{p,L}$) easier:

Whereas $\nabla(u(\cdot+\tau))=(\nabla u)(\cdot+\tau)$, one has $\nabla(u(\frac{\cdot}{\lambda}))=\frac{1}{\lambda}(\nabla u)(\frac{\cdot}{\lambda})$, with a nasty multiplying term $\frac{1}{\lambda}$ in the latter case.

Proposition 4.9 (A sufficient condition for the validity of ($\mathrm{H}^{p,L}$))**.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ satisfy the three following assumptions:*

(1) (*Structure assumption*) *There exist a continuous function $m : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ which is convex and nondecreasing in the last variable and a globally Lipschitz convex function $\kappa : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ such that*

$$\forall (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \qquad f(x,s,\xi) = m(x,\kappa(s,\xi));$$

(2) (*Growth assumption*) *There exists an increasing convex function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which is superlinear and such that*

$$\forall (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \qquad \psi \circ \kappa(s,\xi) \le f(x,s,\xi).$$

(3) (*Uniformity assumption for the x variable*) *There exist a non decreasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$, $\beta \in (0,1]$ and $C \ge 0$ such that:*

$$\forall x,y \in \Omega, \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N, \qquad |f(x,s,\xi) - f(y,s,\xi)| \le |x-y|^\beta h \circ \kappa(s,\xi) + C.$$

If

$$\sup_{\nu \ge \max(1,\psi(0))} \left(\frac{\nu^{-1}(h \circ \psi^{-1})(\nu)}{(\max(\nu, \psi^{-1}(\nu)^p))^{\frac{\beta}{N}}} \right) < +\infty \quad (\textit{Upper bound condition})$$

then for every $L > 0$ there is $A_L > 0$ such that

$$|(s,\xi)|^p + \psi \circ \kappa(s,\xi) \le L\delta^{-N} \Longrightarrow f(x,s,\xi) \le A_L((f^-_{x,\delta})^{**}(s,\xi) + 1).$$

In particular f satisfies ($\mathrm{H}^{p,L}$) *for every $L > 0$.*

As typical examples of functions κ, one can take $\kappa(\xi) = \|\xi\|$ where $\|\cdot\|$ stands for any norm on $\mathbb{R}^N$, or $\kappa(\xi) = |\xi_i|$. The latter allows to deal with *orthotropic* integrands, when f is the sum of functions f_i with similar structures and depending only on ξ_i, for $i = 1,\ldots,N$.

Example 4.10. For instance, consider

$$f(x,s,\xi) = \sum_{i=1}^N \exp(|\xi_i|) + a_i(x)\exp(\alpha_i|\xi_i|),$$

where each a_i is Lipschitz continuous on $\mathbb{R}^N$. This is the orthotropic version of the integrand $f(x,\xi) = \exp(|\xi|) + a(x)\exp(\alpha|\xi|)$ introduced in (4.2). For every $i = 1,\ldots,N$, the integrand $f_i(x,s,\xi_i) = \exp(|\xi_i|) + a_i(x)\exp(\alpha_i|\xi_i|)$ satisfies the assumptions of Proposition 4.9 with $\kappa(\xi) = |\xi_i|$, $\psi_i = \exp$ and $h_i = \|\nabla a\|_{L^\infty(\Omega)} \exp(\alpha_i \cdot)$, provided that $\alpha_i \le 1 + \frac{1}{N}$ for every $i = 1,\ldots,N$. It follows that each f_i satisfies ($\mathrm{H}^{p,L}$) for every $L > 0$, and the same is true for the sum f itself, thanks to the stability properties of the assumption ($\mathrm{H}^{p,L}$).

Remark 4.11. We acknowledge the fact however that the convex envelope $(f^-_{x,\varepsilon})^{**}$ is sometimes hard to calculate or estimate, and may be very far from f itself. It is an open problem whether ($\mathrm{H}^{p,L}$) could be replaced by the less demanding one:

There exists $A > 0$, $b \in L^1(\Omega)$, $\delta_* \in (0,1)$ and a function $\theta : (0,\delta_*) \mapsto (0,1]$ with $\lim_{\delta \to 0} \theta(\delta) = 1$ such that for every $\delta \in (0,\delta_*)$ and every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$,

$$|(s,\xi)|^p + f^-_{x,\delta}(s,\xi) \le L\delta^{-N} \Rightarrow f(x,\theta(\delta)s,\theta(\delta)\xi) \le A\Big(|(s,\xi)|^p + f^-_{x,\delta}(s,\xi) \Big) + b(x).$$

When f satisfies a Δ_2 type condition, the answer is positive and is essentially a consequence of the main results in [8].

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