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DIRECTIONAL APPROACHES IN NONSMOOTH ANALYSIS

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Dedicated to Francis Clarke for the wonderful books, articles, and results that he offered to the readers

Abstract. We revisit some directional concepts used in nonsmooth analysis which can be used to complement or refine the fundamental approach of Clarke in his books and papers. We prove some results which remained unsolved or not studied and we present some questions.

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1. Introduction

The thesis [6] defended fifty years ago by Francis Clarke and his book "Optimization and Nonsmooth Analysis" [7] have been a breakthrough in optimization and analysis, even if some approaches of interest appeared earlier and later. They have been followed by books which are among the best books in mathematics.

The large success of the book [7] relied on several qualities: 1) it adresses a simple and useful class of functions, the class of locally Lipschitzian functions and the study can be extended to the class of lower semicontinuous functions; 2) the analysis of these functions could be related to a study of tangent and normal cones to subsets of normed (vector) spaces; 3) convexity of the generalized derivatives and tangent cones allowing a nice use of duality.

Slight variants of these notions appeared in [18], [19] and [25] bringing more accuracy. For instance, it was shown in [19] that the moderate subdifferential ∂_M introduced in [18] coincides with the singleton $\{f'(x)\}$ when f is differentiable at x. This difference with the Clarke subdifferential (which requires strict differentiability called here circa-differentiability in order to underline its main character and the absence of strict inequalities) was not enough to attract a large interest. Still, we believe that a directional approach may have some interest. Another directional approach has been developped more recently in a dual way. One may wonder whether

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there are links between the two approaches. Besides pointing out such a question, it is the purpose of the present study to reveal some properties of ∂_M not detected before and to put in full light the interest of a directional approach.

The notation we adopt is similar to the notation in [4], [8], [16] and [22]. In particular we use $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, $\mathbb{P} :=\{p \in \mathbb{R} : p > 0\}$.

2. Local versus directional notions

As it is well known, a property is local if it refers or applies to some neighborhood of the point of interest x of a metric or topological space. On the other hand, in a normed space (or topological vector space X, but we refrain to consider such a generalization) one can consider properties that are valid in a set of the form $V(x,u,\delta,\varepsilon)$, or $V(x,u,\varepsilon)$ when $\delta = \varepsilon$, where $x \in X$, $u \in X \backslash \{0\}$, δ, $\varepsilon \in \mathbb{P}$, the set of positive numbers and

$$V(x,u,\delta,\varepsilon) := \{x + tu' : u' \in B(u,\delta),\ t \in [0,\varepsilon[\}$$

or in the union over $i \in I$, a given index set, of such conical neighborhoods $V(x,u_i,\delta_i,\varepsilon_i)$. In the case of directional continuity of $g : X \to Y$, where Y is some topological space, one requires that for any neighborhood W of $g(x)$ in Y and $u \in X \backslash \{0\}$ there exist some δ, $\varepsilon \in \mathbb{P}$ such that for any $x' \in V(x,u,\delta,\varepsilon)$ one has $g(x') \in W$ or that $g^{-1}(W)$ contains a *sponge* around x in the terminology of Treiman, i.e. a union over $u \in X \backslash \{0\}$, or the unit sphere S_X of X, of conical neighborhoods $V(x,u,\delta_u,\varepsilon_u)$. That notion is natural but not much used, whereas *directional derivability* of g at x (which means that for all $u \in X \backslash \{0\}$ the quotient $t^{-1}(g(x+tu') - g(x))$ has a limit $g'(x,u)$ as $(t,u') \to (0_+,u)$) is of common use (and often confused with *radial derivability* which means that $t^{-1}(g(x+tu) - g(x))$ has a limit as $t \to 0_+$). When the limit $g'(x,u)$ is linear and continuous in its second variable, one usually says that g is *Hadamard or directionally differentiable at* x. The following easy lemma shows that several directional notions coincide with the usual notions when the source space X is finite dimensional.

Lemma 2.1. *If X is a finite dimensional normed space, any sponge around $x \in X$ is a neighborhood of x.*

In particular, as a well known consequence, Hadamard differentiability coincides with Fréchet differentiability when X is finite dimensional.

Proof. Let $S := \bigcup_{u \in S_X} V(x,u,\delta_u,\varepsilon_u)$ be a sponge around x, with $\delta_u, \varepsilon_u \in \mathbb{P}$ for all $u \in X \backslash \{0\}$. Since S_X is compact, one can find a finite subcovering $(B(u,\delta_u))_{u \in F}$ of the covering $(B(u,\delta_u))_{u \in S_X}$ of S_X. Taking the infimum ε of the finite family $(\varepsilon_u)_{u \in F}$ one gets a neighborhood V of x by setting $V := x + [0,\varepsilon]S_X$ contained in S. Alternatively, one can use sequences to prove the statement. □

In the sequel we shall say that a mapping $g : X \to Y$ is Lipschitzian at $x \in X$ in the direction $u \in X \backslash \{0\}$ if there exists $\varepsilon \in \mathbb{P}$ such that g is Lipschitzian on $V(x,u,\varepsilon)$ and that g is *directionally Lipschitzian at* x if it is Lipschitzian at $x \in X$ in any direction $u \in X \backslash \{0\}$. A weaker notion is *directional stability* (resp. *calmness when* $Y := \mathbb{R}$) at x which means that for any $u \in X \backslash \{0\}$ there exist $c \in \mathbb{R}_+$ and ε in $\mathbb{P}$ such that $\|f(x') - f(x)\| \leq c\,\|x' - x\|$ (resp. $f(x) - f(x') \leq c\,\|x' - x\|$) for all $x' \in V(x,u,\varepsilon)$. Such notions generalize stability (or Stepanov property) and calmness for real-valued functions respectively. One must be aware that the same terminology is used elsewhere with a different meaning.

A directional character can also be given to notions dealing with multifunctions or set limits. For instance, if $F : X \rightrightarrows Y$ is a multimap (or multifunction in a more usual terminology) from a normed space X to a topological space Y, the *directional inner limit* of F at $x \in X$ is the set of $z \in Y$ such that for any neighborhood W of z, any $u \in X \backslash \{0\}$, there exists some $\varepsilon \in \mathbb{P}$ such that for any $v \in V(x,u,\varepsilon)$ one has $F(v) \cap W \neq \varnothing$ or, equivalently, for any $u \in X \backslash \{0\}$, and any sequences $(t_n) \to 0_+$, $(u_n) \to u$ one has $F(x+t_n u_n) \cap W \neq \varnothing$ for n large enough. The multimap F is said to be directionally inner semicontinuous at x if $F(x)$ is contained in the directional inner limit of F at x. Also, below we use the obvious notion of sequential outer limit and directional closedness for a multimap.

Let us add that the notion of metric regularity has been weakened into directional metric regularity and studied by several authors: see [3], [5], [9], [11], [13], [14], [15], in particular for its connexions with directional normal cones and directional subdifferentials. Directional approaches have also been used for higher order notions [12], [24].

3. Tangent cones and normal cones

In the sequel X is a normed space, or a Banach space when completeness of X is required and a is a point in the closure $\mathrm{cl}E$ of a subset E of X.

Definition 3.1. The Clarke tangent cone (or circa-tangent cone) to a subset E of X at $a \in \mathrm{cl}E$ is the set $T^C(E,a)$ of $v \in X$ such that for any sequences $(t_n) \to 0_+$, $(e_n) \to a$ satisfying $e_n \in E$ for all n there exists a sequence $(v_n) \to v$ such that $e_n + t_n v_n \in E$ for all n.

The main weakness of this notion is the lack of accuracy: $T^C(E,a)$ does not give an idea of the shape of E around a as the following simple example shows.

Example. For $c \in \mathbb{P}$, let $E := \{(x,y) \in \mathbb{R}^2 : y \geq -c\,|x|\}$, a large cone when c is large. However for $a := (0,0)$ one has $T^C(E,a) = \{(x,y) \in \mathbb{R}^2 : y \geq c\,|x|\}$, a small cone.

A means to amend this weakness consists in restricting the class of sequences (e_n) occurring in the preceding definition. In [19], to define the *moderate tangent cone* $T^M(E,a)$ one takes sequences $(e_n) \to a$ in E such that $(t_n^{-1}(e_n - a))$ converges, whereas in [25], to define the *Treiman tangent cone* $T^B(E,a)$ one selects sequences, $(e_n) \to a$ in E such that $(t_n^{-1}(e_n - a))$ is bounded. More explicitely, we state the next definition.

Definition 3.2. The moderate (or Michel-Penot) tangent cone to E at $a \in \mathrm{cl}E$ is the set $T^M(E,a)$ of $v \in X$ such that for any $u \in X_0 := X \backslash \{0\}$ and any sequences $(t_n) \to 0_+$, $(u_n) \to u$ satisfying $e_n := a + t_n u_n \in E$ for all n there exists a sequence $(v_n) \to v$ such that $e_n + t_n v_n \in E$ for all n (or all n large enough, what is equivalent).

The moderate normal cone $N_M(E,a)$ to a subset E of X at $a \in \mathrm{cl}E$ is the polar cone to the moderate tangent cone to E at a.

This definition presents a reminiscence to the initial definition of the Clarke tangent cone $T^C(E,a)$. However, this cone is different from the Clarke tangent cone, even in finite dimensions, as the next simple example shows.

Example. Let $f : X \to Y$ be a map between two normed spaces which is differentiable at 0, with $f(0) = 0$. Then for its graph E, $T^M(E,(0,0))$ is the graph of its derivative $Df(0)$ at 0. But if f is not circa-differentiable (or strictly differentiable) at 0, $T^C(E,a)$ cannot be this subspace. $\square$

A directional version of the paratingent cone of Bouligand can also be given. We call it the *peritangent cone* or *directional paratingent cone* and denote it by $T^{dirP}(E,a)$. It is the set of $v \in X$ such that there exist $u \in X$ and sequences $(t_n) \to 0_+$, $(u_n) \to u$, $(v_n) \to v$ such that $e_n := a + t_n u_n \in E$ and $e_n + t_n v_n \in E$ for all n. However, the example of the graph of $x \mapsto |x|$ at $a := 0$, shows that it may be large, even if it is smaller than the paratingent cone.

Let us compare the moderate tangent cone to classical tangent cones. Among them are the *directional* (or Dini, or Bouligand or contingent) *tangent cone* $T^D(E,a)$, or just $T(E,a)$, the set of $v \in X$ such that for some sequences $(t_n) \to 0_+$, $(v_n) \to v$ one has $a + t_n v_n \in E$ for all $n \in \mathbb{N}$ and the *incident* (or adjacent ([2]) or classical) *tangent cone* $T^I(E,a)$ which is the set of $v \in X$ such that for any sequence $(t_n) \to 0_+$, there exists a sequence $(v_n) \to v$ such that $a + t_n v_n \in E$ for all $n \in \mathbb{N}$.

Introducing a bit of notation may be convenient. We denote by S_+ the set of sequences $s := (t_n) \to 0_+$, we write $(x_n) \overset{s}{\to} x$ if $(t_n^{-1}(x_n - x))$ converges to some $u \in X_0 := X \backslash \{0\}$, $(x_n) \overset{dir}{\to} x$ if $(x_n) \overset{s}{\to} x$ for some $s \in S_+$ and, for $E \subset X$, $a \in \mathrm{cl}(E)$, $s \in S_+$, we set

$$S(a,s,E) := \{(u_n) \in X^{\mathbb{N}} : a + t_n u_n \in E \ \forall n \in \mathbb{N}\},$$
$$T(E,a,s) := \{u \in X : \exists (u_n) \in S(a,s,E), (u_n) \to u\}.$$

Thus

$$T^D(E,a) = \bigcup_{s \in S_+} T(E,a,s), \qquad T^I(E,a) := \bigcap_{s \in S_+} T(E,a,s).$$

Since any converging sequence is bounded, denoting by $T^B(E,a)$ the Treiman tangent cone ([25]), one has $T^B(E,a) \subset T^M(E,a)$, and moreover

$$T^C(E,a) \subset T^B(E,a) \subset T^M(E,a) \subset T^I(E,a) \subset T^D(E,a).$$

Let us prove the third inclusion, which is new. Given $a \in \mathrm{cl}E$, $v \in T^M(E,a)$ and a sequence $(t_n) \to 0_+$, we can find a sequence $(e_n) \subset E$ such that $\|e_n - a\| \leq t_n/n$ for all $n \in \mathbb{N}$. Then $(u_n) := (t_n^{-1}(e_n - a)) \to 0$, so that, by definition of $T^M(E,a)$, there exists a sequence $(v_n) \to v$ such that $e_n + t_n v_n \in E$ for all $n \in \mathbb{N}$. Setting $v'_n = u_n + v_n$ one gets $(v'_n) \to v$ and $a + t_n v'_n = e_n + t_n v_n \in E$ for all $n \in \mathbb{N}$, so that $v \in T^I(E,a)$.

Taking dual cones, the preceding inclusions imply inclusions for the corresponding normal cones: with an obvious notation one has

$$N_D(E,a) \subset N_I(E,a) \subset N_M(E,a) \subset N_B(E,a) \subset N_C(E,a).$$

Let us note that when X is finite dimensional one has $T^B(E,a) = T^M(E,a)$: to see that, we take $v \in X \backslash T^B(E,a)$, so that there exist $(t_n) \to 0_+$, a bounded sequence (u_n) and $c > 0$ such that for $e_n := a + t_n u_n$ one has $d(v, t_n^{-1}(E - e_n)) \geq c$ for all $n \in \mathbb{N}$. Taking a converging subsequence $(u_k)_{k \in K}$ of (u_n), we see that $v \in X \backslash T^M(E,a)$.

Also, when E is *tangentable (or derivable) at* $a \in \mathrm{cl}E$ in the sense that $T^I(E,a) = T^D(E,a)$, a rather mild assumption, $T^M(E,a)$ is related to the *convex core* $T(E,a) \boxminus T(E,a)$ of $T(E,a) := T^D(E,a) = T^I(I,a)$, where for two subsets C, D of X one sets

$$C \boxminus D := \{x \in X : D + x \subset C\}.$$

More generally, one has:

Proposition 3.3. *(**tangent cones and sets differences**) The following inclusions hold:*

$$T^I(E,a) \boxminus T^D(E,a) \subset T^M(E,a) \subset T^I(E,a) \boxminus T^I(E,a).$$

In particular, when E is tangentable at a one has

$$T^M(E,a) = T^I(E,a) \boxminus T^I(E,a) = T^D(E,a) \boxminus T^D(E,a).$$

For the proof, which is elementary, we refer to [22, p. 395].

Example.(the pistil) Let $E := \{(r,s) \in \mathbb{R}^2 : s = \sqrt{|r|}\}$, $a := (0,0)$. Then $T^I(E,a) = T^D(E,a) = \mathbb{R}_+(0,1)$, so that $T^M(E,a) = \mathbb{R}_+(0,1)$. □

The next proposition ensues when E is tangentable at a, but this assumption is not necessary.

Proposition 3.4. *For any subset E of X and any $a \in \mathrm{cl}E$, the moderate tangent cone to E at a is convex: $T^M(E,a) + T^M(E,a) \subset T^M(E,a)$.*

Proof. Given $v \in T^M(E,a)$, $w \in T^M(E,a)$ and sequences $(t_n) \to 0_+, (u_n) \to u$ such that $e_n := a + t_n u_n \in E$ for all n, one can find sequences $(v_n) \to v$, $(w_n) \to w$ such that $e_n + t_n v_n \in E$, $e_n + t_n(v_n + w_n) = (e_n + t_n v_n) + t_n w_n \in E$ for all $n \in \mathbb{N}$, which shows that $v + w \in T^M(E,a)$. □

Proposition 3.5. *When E is convex, $T^M(E,a)$ coincides with the tangent cone of convex analysis: $T^M(E,a) = T(E,a) := T^D(E,a) = T^I(E,a) = \mathrm{cl}(\mathbb{R}_+(E-a))$.*

Proof. We already know that $T^M(E,a) \subset T^I(E,a) = T(E,a)$.

For the converse, since $T^M(E,a)$ is a closed cone and since $T(E,a) := \mathrm{cl}(\mathbb{R}_+(E-a))$, it suffices to prove that $E-a$ is contained in $T^M(E,a)$. Given $v \in E-a$, or $v := e-a$, with $e \in E$, taking sequences $(t_n) \to 0_+$ in $[0,1]$, $(u_n) \to u$ such that $e_n := a + t_n u_n \in E$ for all $n \in \mathbb{N}$, setting $v_n := v - t_n u_n$, $e'_n := t_n e + (1-t_n) e_n \in E$, one has $(v_n) \to v$, $e'_n = t_n(a+v) + (1-t_n)(a + t_n u_n) = a + t_n u_n + t_n v_n$, so that $v \in T^M(E,a)$. □

It would be of interest to examine whether the coincidence of the moderate tangent cone with the classical tangent cone remains valid for the many generalizations of convexity. We foresake this vast question here, but we note the following coincidence result.

Proposition 3.6. *When E is a submanifold of class C^1 or a C^1-submanifold with boundary, $T^M(E,a)$ coincides with the classical tangent cone $T(E,a)$ to E at a.*

Proof. That follows from the invariance of the moderate tangent cone under diffeomorphisms established below. □

Let us consider some rules. Similarly to the case of the Clarke tangent cone, the rule $T^M(E,a) \subset T^M(F,a)$ when $a \in E \subset F$ is not valid and this is an important weakness of both notions which is related to the lack of precision of these notions. Consequently, rules involving unions or intersections are not for free. Still, when $a \in E \subset F$ and for some remainder $o(\cdot)$ one has $d(y,E) \leq o(\|y-a\|)$ for $y \in F$ the inclusion $T^M(E,a) \subset T^M(F,a)$ holds.

Example. Let $X := \mathbb{R}^2$, $E := \mathbb{R} \times \{0\}$, $F = E \cup \{(x,y) : |y| = x^2\}$. Then $T^M(E,a) \subset T^M(F,a)$.

Let us give some calculus rules. The first one is a direct application of the definitions. It is valid for any norm on a product compatible with the product topology.

Proposition 3.7. *Let E and F be subsets of normed spaces X and Y respectively, let $a \in \mathrm{cl}E$, $b \in \mathrm{cl}F$. Then $T^M(E \times F,(a,b)) = T^M(A,a) \times T^M(B,b)$.*

As it is the case for the Clarke tangent cone, introducing a robust variant of the moderate tangent cone may be of interest for calculus rules. When E is a subset of X and $a \in \mathrm{cl}E$, it is convenient to write $(e_n) \to_E a$ if $(e_n) \to a$ and $e_n \in E$ for all $n \in \mathbb{N}$. Also, for a sequence $s := (t_n) \to 0_+$, we write $(e_n) \xrightarrow{s} a$ if $(t_n^{-1}(e_n - a))$ converges and we write $(e_n) \xrightarrow{s}_E a$ if $(e_n) \to_E a$ and $(e_n) \xrightarrow{s} a$.

The next corollary implies that the notion of moderate tangent cone to a subset of a manifold of class C^1 has a meaning.

Corollary 3.8. ***(Invariance of the moderate tangent cone)*** *Let E, U and F, V be subsets of normed spaces X and Y respectively, U and V being open, and containing $a \in \mathrm{cl}E$ and $b := g(a)$, let $g : E \cap U \to Y \cap V$ be an homeomorphism that is directionally differentiable at $a \in \mathrm{cl}E$, its inverse k being directionally differentiable at b. Then one has*

$$g'(a)(T^M(E,a)) = T^M(F,b).$$

Definition 3.9. Given a subset E of X and $a \in \mathrm{cl}E$, the moderate hypertangent cone to E at x is the set $H^M(E,a)$ of $v \in X$ such that for any sequences $s := (t_n) \to 0_+$, $(e_n) \xrightarrow{s}_E a$ and any sequence $(v_n) \to v$ one has $e_n + t_n v_n \in E$ for n large enough.

This cone is larger than the Clarke hypertangent cone $H^C(E,a)$ for which the convergence condition on the sequence $(t_n^{-1}(e_n - a))$ is not required, hence $H^M(E,a)$ is of larger use. It is easy to check the next relations.

$$H^C(E,a) \subset H^M(E,a) \subset T^M(E,a),$$
$$H^M(E,a) + T^M(E,a) = H^M(E,a).$$

Since $0 \in T^M(E,a)$, the last relation can be written

$$H^M(E,a) = H^M(E,a) \boxminus T^M(E,a).$$

Proposition 3.10. *If E has the moderate cone property at $a \in \mathrm{cl}E$ in the sense that $H^M(E,a)$ is nonempty, then one has the following inclusions:*

$$T^M(E,a) = \mathrm{cl}H^M(E,a) \qquad intT^M(E,a) \subset H^M(E,a).$$

Proof. The inclusion $\mathrm{cl}H^M(E,a) \subset T^M(E,a)$ is a consequence of the closedness of $T^M(E,a)$. For the reverse inclusion, let $v \in T^M(E,a)$ and let $w \in H^M(E,a)$, $(t_n) \to 0_+$. Since $H^M(E,a) + T^M(E,a) = H^M(E,a)$ we have $v + t_n w \in H^M(E,a)$ and $v = \lim_n (v + t_n w) \in \mathrm{cl}H^M(E,a)$.

Given $z \in \mathrm{int}T^M(E,a)$ and $w \in H^M(E,a)$, for $t > 0$ small enough we have $z = (z - tw) + tw \in T^M(E,a) + H^M(E,a) = H^M(E,a)$. □

Moderate hypertangent cones can be used for some rules in a way similar to the use of hypertangent cones. We just mention the next result about intersections which can be proved as in [22, Prop. 5.41]; in fact it is a consequence of Proposition 5.6 below when taking for g the identity map.

Proposition 3.11. *Let E, F be subsets of X and let $a \in \mathrm{cl}(E) \cap \mathrm{cl}(F)$ be such that $T^M(E,a) \cap H^M(F,a)$ is nonempty. Then $T^M(E,a) \cap T^M(F,a) \subset T^M(E \cap F,a)$.*

For the study of the moderate tangent cone to the inverse image $E := g^{-1}(F)$ of a subset F of Y, or the direct image $F := g(E)$ of a subset E of X by a map $g : X \to Y$ which is directionally derivable at $a \in \mathrm{cl}E$, we need the assumption that g is *directionally compatible with E and F*

at a in the sense that for any sequence $s := (t_n) \to 0_+$ and any $w \in T(F,b,s)$ with $b := g(a)$, there exists some $u \in T(E,a,s)$ such that $w = g'(a)u$. This assumption is satisfied if $g'(a)X = Y$ and if g is *directionally open at* a *with respect to* E and F in the sense that for any sequence $s := (t_n) \to 0_+$, $w \in T(F,b,s)$, $(w_n) \in S(b,s,F)$ satisfying $(w_n) \to w$ there exist $u \in T(E,a,s)$ and $(u_n) \to u$ such that $g(a+t_nu_n) = b+t_nw_n$ for all large $n \in \mathbb{N}$.

Theorem 3.12. *Let E and F be subsets of normed (vector) spaces X and Y respectively, let $g : X \to Y$ be directionally (or Hadamard) differentiable at $a \in E$ and directionally compatible with E and F, with $g(E) \subset F$, $b := g(a) \in F$. Then one has*

$$g'(a)(T^M(E,a)) \subset T^M(F,b),$$
$$g'(a)^\intercal(N_M(F,b)) \subset N_M(E,a).$$

Proof. Given $v \in T^M(E,a)$, let $z := g'(a)v$ and let $s := (t_n) \to 0_+$, $w \in T(F,b,s)$, $(w_n) \to w$ be such that $b+t_nw_n \in F$ for all $n \in \mathbb{N}$. We have to find a sequence $(z_n) \to z$ such that $b+t_nw_n+t_nz_n \in F$ for all n. By directional compatibility of g with E and F there exists $u \in T(E,a,s)$ such that $g'(a)u = w$. By definition of $T(E,a,s)$ there exists a sequence $(u_n) \to u$ such that $a+t_nu_n \in E$ for all n. Since $v \in T^M(E,a)$, there exists a sequence $(v_n) \to v$ such that $a+t_nu_n+t_nv_n \in E$ for all n. Thus $g(a+t_nu_n+t_nv_n) \in F$ and, since $\lim_n t_n^{-1}(g(a+t_nu_n)-(b+t_nw_n)) = g'(a)u-w = 0$, and since $g'(a)$ is linear and continuous, one has

$$\begin{aligned} g'(a)v = g'(a)(u+v) - g'(a)(u) &= \lim_n t_n^{-1}(g(a+t_nu_n+t_nv_n) - g(a+t_nu_n)) \\ &= \lim_n t_n^{-1}(g(a+t_nu_n+t_nv_n) - (b+t_nw_n)). \end{aligned}$$

Setting $z_n := t_n^{-1}(g(a+t_nu_n+t_nv_n)-(b+t_nw_n))$, we have $(z_n) \to z$ and $b+t_nw_n+t_nz_n = g(a+t_nu_n+t_nv_n) \in g(E) \subset F$, as required. Since s, $(w_n) \in S(b,s,F)$, $w = \lim_n w_n \in T(F,b,s)$ are arbitrary, that shows that $g'(a)v \in T^M(F,b)$.

The second inclusion follows: for any $y^* \in N_M(F,b)$, $v \in T^M(E,a)$ one has $\langle g'(a)^\intercal(y^*), v\rangle = \langle y^*, g'(a)(v)\rangle \le 0$ since $g'(a)(v) \in T^M(F,b)$, so that $g'(a)^\intercal(y^*) \in N_M(E,a)$. □

Corollary 3.13. *([22, Prop. 5.71]) Let E, F, a, b be as in the preceding theorem, let $g : E \to Y$ be directionally differentiable at a. Assume g is directionally open at a with respect to E and F. Then the conclusions of the theorem hold.*

This corollary has a similarity with a result for Clarke tangent cones ([22, Prop. 5.27]).

Proof. That follows from the fact that g is directionally compatible with E and F at a. □

Corollary 3.14. *Let E, F, a, b be as in the preceding theorem, let $g : E \to Y$ be directionally differentiable at a. Assume g has a local right inverse $k : W \to X$ on some neighborhood W of $b := g(a)$ (i.e. $g \circ k = I_W$, the identity map of W) that is directionally derivable at b and satisfies $k(W \cap F) \subset E$, $k(b) = a$. Then the conclusions of the theorem hold.*

Here we assume the derivative $k'(b)$ is an arbitrary positively homogeneous map, not necessarily a continuous linear map. Note that when g is circa-differentiable at a with $g'(a)(X) = Y$, $g'(a)^{-1}(0)$ being complemented, the existence of a differentiable right inverse k satisfying $k(b) = a$ is ensured, but here we do not make such an assumption.

Proof. It suffices to show that g is directionally open at a with respect to E and F. Given a sequence $s := (t_n) \to 0_+$, $w \in T(F,b,s)$, $(w_n) \in S(b,s,F)$ satisfying $(w_n) \to w$, setting $u_n :=$

$t_n^{-1}(k(b+t_nw_n)-a)$, we have $g(a+t_nu_n)=b+t_nw_n$ and $(u_n)\to u:=k'(b)w$, so that $u\in T(E,a,s)$ and g is directionally open at a with respect to E and F. $\square$

The existence of a required right inverse is ensured in the next proposition which relies on the following famous result.

Lemma 3.15. *(**Lyusternik-Graves Theorem**, [16, Thm 1.20], [22, Thm 2.67]) Let X and Y be Banach spaces, let W be an open subset of X, let $g:W\to Y$ be circa-differentiable at some $a\in W$ with a surjective derivative $g'(a)$ at a. Let $b:=g(a)$. Then g is open at a and there exist some ρ, σ, $\kappa\in\mathbb{P}:=]0,+\infty[$ and a right inverse $k:B(b,\sigma)\to W$ of g satisfying $k(b)=a$,*

$$\|x-k(y)\|\le\kappa\|g(x)-y\|\qquad\forall(x,y)\in B(a,\rho)\times B(b,\sigma).$$

Proposition 3.16. *Let X, Y, W, $F\subset Y$, a,b be as in the preceding lemma and let $g:W\to Y$ be circa-differentiable at a with a surjective derivative $g'(a)$ at a. Then g is directionally compatible with $E:=g^{-1}(F)$ and F.*

Proof. Given a sequence $s:=(t_n)\to0_+$ and $w\in T(F,b,s)$ with $b:=g(a)$, let sequences $s:=(t_n)\to0_+$ and $(w_n)\to w$ be such that $b+t_nw_n\in F$ for all n. We have to show that any $u\in X$ such that $g'(a)u=w$ belongs to $T(E,a,s)$. Let ρ, σ, $\kappa\in\mathbb{P}$ be as in the preceding lemma. For $t\in\mathbb{R}_+$, $\|u'-u\|$ small enough we have $x:=a+tu'\in B(a,\rho)$, $y:=b+tw\in B(b,\sigma)$ hence

$$\left\|a+tu'-k(b+tw)\right\|\le\kappa\left\|g(a+tu')-(b+tw)\right\|\le\kappa r(t+\left\|u'-u\right\|)$$

for a remainder r given by the differentiability of g at a. That shows that k is directionally derivable at a in the direction u with $k'(b)w=u$. In particular, for $u_n:=t_n^{-1}(k(b+t_nw_n)-a)$, we have $(u_n)\to u$ and $g(a+t_nu_n)=b+t_nw_n\in F$. Thus $a+t_nu_n\in g^{-1}(F)=E$ and $u\in T(E,a,s)$. $\square$

4. Moderate derivatives and moderate subdifferentials

It is easy to show that when $E\subset X\times\mathbb{R}$ is a *quasi-epigraph*, i.e. when $E+\{0\}\times\mathbb{R}_+\subset E$, for $e\in E$, the cone $T^M(E,e)$ is also a quasi-epigraph. In fact $T^M(E,e)$ is an epigraph as it is closed. When E is the epigraph of a function f and $e:=(x,f(x))$, it is sensible to describe the function $f^M(x,\cdot)$ associated to $T^M(E,e)$. That is the purpose of the next lemma whose proof is a direct consequence of the fact that $T^M(E,e)$ is a quasi-epigraph and a closed convex cone.

Lemma 4.1. *Let E be the epigraph of a function $f:X\to\mathbb{R}_\infty:=\mathbb{R}\cup\{+\infty\}$ finite at x, and let $e:=(x,f(x))$. Then, the moderate directional derivative of f at x $f^M(x,\cdot):X\to\overline{\mathbb{R}}$ defined by $f^M(x,v):=\inf\{s\in\mathbb{R}:(v,s)\in T^M(E,e)\}$ is such that*

$$T^M(E,e)=\operatorname{epi}f^M(x,\cdot)$$

and $f^M(x,\cdot)$ is lower semicontinuous and sublinear.

Definition 4.2. The moderate subdifferential of $f:X\to\mathbb{R}_\infty$ at $x\in\operatorname{dom}f$ is the set

$$\partial_Mf(x):=\{x^*\in X^*:x^*\le f^M(x,\cdot)\}=\{x^*\in X^*:(x^*,-1)\in N_M(E,e)\}.$$

The equality stems from the fact that $(x^*,-1)\in N_M(E,e)$ if, and only if,

$$\langle x^*,v\rangle-s\le0\qquad\forall(v,s)\in T^M(E,e).$$

The inclusion $T^C(E,a) \subset T^M(E,a)$ for $a \in \mathrm{cl}E$ ensures that $N_M(E,a) \subset N_C(E,a)$, the Clarke normal cone to E at a, and implies for a function f on X and $x \in \mathrm{dom}f$ the inclusion $\partial_M f(x) \subset \partial_C f(x)$.

Let us say that f is *directionally stable at x in the direction $u \in X\backslash\{0\}$* if

$$\limsup_{(t,u')\to(0_+,u)} \frac{1}{t}\left|f(x+tu') - f(x))\right| < +\infty$$

and that f is *directionally stable at x* if for any $u \in X_0 := X\backslash\{0\}$ the function f is directionally stable at x in the direction u. That is the case if f is *stable at x* in the sense that $\limsup_{x'\to x, x'\neq x} \frac{1}{\|x'-x\|}\left|f(x') - f(x))\right| < +\infty$. Assuming that f is directionally stable at x, one can give a simple expression to $f^M(x,\cdot)$, setting for $u,\ v \in X$

$$f^{\Diamond}(x,u,v) := \limsup_{(t,u',v')\to(0_+,u,v)} \frac{1}{t}(f(x+tu'+tv') - f(x+tu')),$$

$$f^{\Diamond}(x,v) := \sup_{u\in X\backslash\{0\}} f^{\Diamond}(x,u,v).$$

Let us note that when f is directionally Lipschitzian at x the expressions of $f^{\Diamond}(x,u,v)$ and $f^{\Diamond}(x,v)$ can be simplified into

$$f^{\Diamond}(x,u,v) = \limsup_{t\to 0_+} \frac{1}{t}(f(x+tu+tv) - f(x+tu)),$$

$$f^{\Diamond}(x,v) := \sup_{u\in X\backslash\{0\}} \limsup_{t\to 0_+} \frac{1}{t}(f(x+tu+tv) - f(x+tu)).$$

Moreover, when f is locally Lipschitzian around x, one can take the supremum over $u \in X$ rather than over $u \in X\backslash\{0\}$.

Proposition 4.3. *For any function $f : X \to \overline{\mathbb{R}}$ finite at $x \in X$ and any $v \in X$ one has $f^M(x,v) \leq f^{\Diamond}(x,v)$. If f is directionally stable at x equality holds for all $v \in X$.*

Proof. We first show that $f^M(x,\cdot) \leq f^{\Diamond}(x,\cdot)$ or $\mathrm{epi}f^{\Diamond}(x,\cdot) \subset \mathrm{epi}f^M(x,\cdot)$. Setting $E := \mathrm{epi}f$, $e := (x, f(x))$, we have to prove that $(v,s) \in T^M(E,e)$ for any $(v,s) \in \mathrm{epi}f^{\Diamond}(x,\cdot)$, or that for any $(u,r) \in X\times\mathbb{R}$ and any sequences $((u_n,r_n)) \to (u,r)$, $(t_n) \to 0_+$ such that $e_n := e + t_n(u_n,r_n) \in E$ for all $n \in \mathbb{N}$ there exists a sequence $((v_n,s_n)) \to (v,s)$ such that $e_n + t_n(v_n,s_n) \in E$ for all $n \in \mathbb{N}$. Since $(v,s) \in \mathrm{epi}f^{\Diamond}(x,u,\cdot)$, picking a sequence $((t_n,u_n,v_n)) \to (0_+,u,v)$ such that $(q_n) \to f^{\Diamond}(x,u,v)$ for

$$q_n := \frac{1}{t_n}(f(x+t_nu_n+t_nv_n) - f(x+t_nu_n)),$$

one has $\limsup_n q_n \leq s$ and there exists a sequence $(s_n) \to s$ such that $q_n \leq s_n$ for all n. Since $f(x+t_nu_n) \leq f(x) + t_nr_n$ for all n by definition of $E := \mathrm{epi}f$, we also have

$$\frac{1}{t_n}(f(x+t_nu_n+t_nv_n) - (f(x)+t_nr_n)) \leq s_n,$$

hence $(x+t_nu_n+t_nv_n, f(x)+t_nr_n+t_ns_n) \in E$ for all n. That shows that $(v,s) \in T^M(E,e)$.

Conversely, assume that f is directionally stable at x and let $(v,s) \in T^M(E,e)$. We have to prove that $(v,s) \in \mathrm{epi}f^{\Diamond}(x,\cdot)$. Given $u \in X$, $(u_n) \to u$, $(t_n) \to 0_+$, since f is directionally stable at x, taking subsequences if necessary, we may assume that $(r_n)_n := (t_n^{-1}(f(x+t_nu_n) - f(x)))_n$

converges to some $r \in \mathbb{R}$. Then $e_n := e + t_n(u_n, r_n) \in E$ for all $n \in \mathbb{N}$, and by definition of $T^M(E,e)$, one can find a sequence $((v_n, s_n)) \to (v,s)$ such that $e_n + t_n(v_n, s_n) \in E$ for all $n \in \mathbb{N}$. That means that $f(x + t_n u_n + t_n v_n) \le f(x) + t_n r_n + t_n s_n$ for all n. Since $f(x + t_n u_n) = f(x) + t_n r_n$ for all n, we have

$$f(x + t_n u_n + t_n v_n) - f(x + t_n u_n) \le t_n s_n,$$

hence $f^{\diamondsuit}(x,u,v) \le s$ since $(t_n) \to 0_+$, $(u_n) \to u$, $(v_n) \to v$ are arbitrary. Taking the supremum over u, we get $f^{\diamondsuit}(x,v) \le s$ or $(v,s) \in \operatorname{epi} f^{\diamondsuit}(x,\cdot)$. □

The following properties are easy consequences.

Corollary 4.4. *If f is locally Lipschitzian around $x \in X$ with rate k, then $\partial_M f(x)$ is contained in kB_{X^*} and $\partial_M(-f)(x) = -\partial_M f(x)$.*

We do not want to examine whether all the conditions in [23] are satisfied by ∂_M. That is dubious for its condition (S6) since it implies quasi-homotonicity. That leads us to revise the conditions listed in [23]. We do that in the next section by removing Condition (S6) and by adding another property not detected up to now, even for the Clarke subdifferential. Condition (S5) has already been obtained, but we present another proof under other assumptions.

Proposition 4.5. *(S5) Given Banach spaces X and Y, an open subset W of X, a map $g : W \to Y$ that is circa-differentiable at some $\overline{x} \in W$ with a surjective derivative $g'(\overline{x})$, if $f : X \to \overline{\mathbb{R}}$, $h : Y \to \overline{\mathbb{R}}$ are such that $f = h \circ g$ with h finite and directionally Lipschitzian at $\overline{y} := g(\overline{x})$, then $\partial_M h(\overline{y}) \circ g'(\overline{x}) \subset \partial_M f(\overline{x})$.*

Proof. Clearly, $f := h \circ g$ is directionally Lipschitzian at $\overline{x}$. Given $y^* \in \partial_M h(\overline{y})$, we have to prove that $x^* := y^* \circ g'(\overline{x}) \in \partial_M f(\overline{x})$, i.e. that for any u, $v \in X$, any sequences $(t_n) \to 0_+$, $(u_n) \to u$, $(x_n) := (\overline{x} + t_n u_n)$ there exists a sequence $(v_n) \to v$ such that

$$\langle y^* \circ g'(\overline{x}), v \rangle \le \limsup_n \frac{1}{t_n}(f(x_n + t_n v_n) - f(x_n)).$$

Since $y^* \in \partial_M h(\overline{y})$ and $(y_n) := (g(x_n))$ is such that $(t_n^{-1}(y_n - \overline{y})) \to g'(\overline{x})u$, we can find a sequence $(z_n) \to g'(\overline{x})v$ such that

$$\langle y^*, g'(\overline{x})v \rangle \le \limsup_n \frac{1}{t_n}(h(y_n + t_n z_n) - h(y_n)).$$

Since h is directionally Lipschitzian at $\overline{y}$, we have

$$\lim_n \frac{1}{t_n}(h(y_n + t_n z_n) - h(g(x_n) + t_n g'(\overline{x})v)) = 0,$$

hence

$$\langle y^*, g'(\overline{x})v \rangle \le \limsup_n \frac{1}{t_n}(f(x_n + t_n g'(\overline{x})v)) - f(x_n)).$$

Taking a local right inverse k of g that is differentiable at $\overline{y}$ and setting $x_n := k(y_n)$, $w_n := t_n^{-1}(k(y_n + t_n z_n) - k(y_n))$, so that $(t_n^{-1}(x_n - \overline{x})) \to k'(\overline{y})(g'(\overline{x})u)$,

$$\begin{aligned}(w_n) &= (t_n^{-1}(k(y_n + t_n z_n) - k(\overline{y})) - t_n^{-1}(k(y_n) - k(\overline{y}))) \\ &\to k'(\overline{y})(g'(\overline{x})u + z) - k'(\overline{y})(z) = k'(\overline{y})(g'(\overline{x})u)\end{aligned}$$

satisfying $g(x_n) = y_n$ and $f(x_n) \le h(y_n) + t_n^2$, $g(x_n) = y_n$ for all n in an infinite subset N of $\mathbb{N}$, It would be interesting to know whether the moderate subdifferential is *quasi-homotone* in the sense of [21, Def.1] that for any subset S of X and any function f on X such that $f \geq d_S$ with $f = 0$ on S, one has $\partial_M d_S(x) \subset \partial_M f(x)$ for all $x \in S$. Here d_S is the distance function to S given by $d_S(x) := \inf_{w\in S} \|w - x\|$ which plays a key role in nonsmooth analysis (see [7], [16], [23] for instance). Such a property has some interesting consequences. Among them is the coincidence of the normal cone to a set E at $a \in \mathrm{cl}E$ with the *metric normal cone* $N^m(E,a) := \mathbb{R}_+\partial d_E(a)$ and the relation $N^m(E,a) := [0,1]\partial d_E(a)$ (see [21, Lemma 3]). The answer seems to be negative. On the other hand, we note the following positive answer.

Proposition 4.6. *The Treiman subdifferential ∂_B is quasi-homotone.*

Proof. It is a simple adaptation of the proof for the Clarke subdifferential ([21, Prop. 1]). Let S be a subset of X, $x \in S$, $f : X \to \overline{\mathbb{R}}$ be such that $f \geq d_S$, $f = 0$ on S, and let E be the epigraph of f and $x_f := (x, f(x)) = (x,0)$. We have to prove that for any $x^* \in \partial_B d_S(x)$ and any $(v,s) \in T^B(E,x_f)$ we have $\langle x^*, v\rangle \leq s$. Given $(v,s) \in T^B(E,x_f)$, we take $(t_n) \to 0_+$, $((x_n,r_n)) \to_E (x,0)$ such that $((u_n, p_n)) := ((t_n^{-1}(x_n - x), t_n^{-1}r_n))$ is bounded. We pick $w_n \in S$ such that $\|w_n - x_n\| \leq d_S(x_n) + t_n^2$, hence $t_n^{-1}\|w_n - x\| \leq t_n^{-1}\|w_n - x_n\| + t_n^{-1}d_S(x_n) + t_n \leq \|u_n\| + t_n^{-1}r_n + t_n$ is bounded too. Since $(w_n,0) \in E$ for all $n \in \mathbb{N}$, by definition of $T^B(E,x_f)$ there exists a sequence $((v_n,s_n)) \to (v,s)$ such that $(w_n,0) + t_n(v_n,s_n) \in E$ for all $n \in \mathbb{N}$. Then

$$s_n \geq t_n^{-1} f(w_n + t_n v_n) \geq t_n^{-1} d_S(w_n + t_n v_n),$$
$$s \geq \limsup_n t_n^{-1} d_S(w_n + t_n v_n) = \lim_n t_n^{-1}(d_S(w_n + t_n v_n) - d_S(w_n)) \geq \langle x^*, v\rangle.$$

□

Note that the preceding proof cannot be adapted to the moderate subdifferential ∂_M since when $((t_n^{-1}(x_n - x), t_n^{-1}r_n))$ converges, we cannot conclude that $((t_n^{-1}(w_n - x), t_n^{-1}0))$ converges. Thus, we cannot assert that the moderate normal cone to a set E at $a \in \mathrm{cl}E$ coincides with the *metric normal cone* $N_M^m(E,a) := \mathbb{R}_+\partial_M d_E(a)$ and that the relation $N_M^m(E,a) = [0,1]\partial^m d_E(a)$ holds.

Now let us give a chain rule which is rather special, but which will be used later for a mean value theorem. We mimic the proof of [7, p. 395].

Lemma 4.7. *Let $f : X \to \overline{\mathbb{R}}$ be finite at $y \in X$ and directionally stable at $y := rz$, with $r \in \mathbb{R}$, $z \in X$ and let $h := f \circ g : \mathbb{R} \to \overline{\mathbb{R}}$, where $g(t) := tz$ for $t \in \mathbb{R}$. Then, identifying $\partial_M h(r)$ with a subset of $\mathbb{R}$ and the linear map $g : \mathbb{R} \to X$ with $z = g(1)$, one has*

$$\partial_M h(r) \subset \partial_M f(y) \circ g = \langle \partial_M f(y), z\rangle.$$

Proof. Given $r,\ s,\ t \in \mathbb{R}$ one has

$$\begin{aligned} h^\Diamond(r,s,t) &= \sup_{(r_n)} \limsup_n \frac{1}{r_n}(f(rz + r_n sz + r_n tz) - f(rz + r_n sz)) \\ &\leq f^\Diamond(rz, sz, tz) \leq f^\Diamond(rz, tz) = t f^M(rz, z) = (f^\Diamond(rz,\cdot) \circ g)(t) \end{aligned}$$

Thus, for any $r^* \in \partial_M h(r)$ one has

$$r^* t \leq \sup_s h^\Diamond(r,s,t) \leq t f^M(rz,z) = t \sup_{x^* \in \partial_M f(rz)} \langle x^*, z\rangle.$$

Taking $t = 1$ and then $t = -1$, and observing that $\langle \partial_M f(rz), z\rangle$ is an interval of $\mathbb{R}$, we get $r^* \in \langle \partial_M f(rz), z\rangle$. □

Theorem 4.8. *(**Mean Value Theorem**) Let $f : X \to \overline{\mathbb{R}}$ be finite and continuous on a segment $[x, x+z]$ of X. Then there exist $y \in [x, x+z]$ and $y^* \in \partial_M f(y)$ such that*

$$f(x+z) - f(x) = \langle y^*, z \rangle.$$

Proof. We first show that there is no loss of generality in assuming that $f(x+z) = f(x)$. In fact, taking $z^* \in X^*$ such that $\langle z^*, z \rangle = f(x) - f(x+z)$ we have $(f+z^*)(x+z) = (f+z^*)(z)$ and if $w \in [x, x+z]$ and $w^* \in \partial_M (f+z^*)(w)$ are such that $\langle w^*, z \rangle = 0$, for $y := w$ and $y^* = w^* - z^*$, we get $y^* \in \partial_M f(y)$ and

$$f(x+z) - f(x) = (f+z^*)(x+z) - (f+z^*)(x) - \langle z^*, z \rangle = \langle w^*, z \rangle - \langle z^*, z \rangle = \langle y^*, z \rangle$$

as expected.

Now, when $f(x+z) = f(x)$, let us set $g(t) := x + tz$ for $t \in \mathbb{R}$, $h := f \circ g$. Since f is lower semicontinuous on $[x, x+z]$, h attains its infimum on $[0,1]$ at some point r of $]0,1[$. Thus, for $y := g(r)$, one has $0 \in \partial_M h(r) = \langle \partial_M f(y), z \rangle$: there exists $y^* \in \partial_M f(y)$ such that $\langle y^*, z \rangle = 0$. $\square$

Another estimate of the moderate derivative can be given as follows. It uses the fact that when h, $k : X \to \overline{\mathbb{R}}$ are two positively homogeneous functions with epigraphs H and K respectively, their *deconvolution* $h \boxminus k$ given by

$$(h \boxminus k)(v) := \sup_{w \in \mathrm{dom}\, k} h(v+w) - k(w) \qquad v \in X,$$

has $H \boxminus K := \{z : z + K \subset H\}$ as its epigraph, as easily checked. In the case the epigraph H of h is the tangent cone $T^A(E, x_h)$ ($A := C$, D, I, M...) at $x_h := (x, h(x))$ to the epigraph E of a function f we denote by $f^A(x, \cdot)$ the functionn h. That is a familar way to associate a directional derivative f^A to a notion of tangent cone T^A.

Proposition 4.9. *The moderate derivative $f^M(x, \cdot)$ of f at $x \in \mathrm{dom}\, f$ satisfies*

$$f^D(x, \cdot) \boxminus f^I(x, \cdot) \le f^M(x, \cdot) \le f^I(x, \cdot) \boxminus f^D(x, \cdot).$$

Here $f^D(x, \cdot)$ (resp. $f^I(x, \cdot)$) denotes the function whose epigraph is $T^D(E, (x, f(x)))$ (resp. $T^I(E, (x, f(x)))$). In particular, if f is epiderivable at x in the sense that $f^D(x, \cdot) = f^I(x \cdot)$, then one has

$$f^M(x, \cdot) = f^D(x, \cdot) \boxminus f^D(x, \cdot) = f^I(x, \cdot) \boxminus f^I(x, \cdot).$$

Proof. Denoting by E the epigraph of f and setting $e := (x, f(x))$, these inequalities are consequences of the relations

$$T^I(E, e) \boxminus T^D(E, e) \subset T^M(E, e) \subset T^I(E, e) \boxminus T^I(E, e)$$

which can be proved as follows for any subset E of a normed space Z and any $e \in E$. Given $v \in T^I(E, e) \boxminus T^D(E, e)$, let $(t_n) \to 0_+$ and $(u_n) \to u$ be such that $e + t_n u_n \in E$ for all $n \in \mathbb{N}$. Then $u \in T^D(E, e)$, hence $u + v \in T^I(E, e)$, so that there exists a sequence $(v_n) \to v$ such that $e + t_n(u_n + v_n) \in E$ for all n. That proves that $v \in T^M(E, e)$. Now, given $v \in T^M(E, e)$, for any sequence $(t_n) \to 0_+$ and any $u \in T^I(E, e)$ we can find a sequence $(u_n) \to u$ such that $e + t_n u_n \in E$ for all n. By definition of $T^M(E, e)$, there exists a sequence $(v_n) \to v$ such that $e + t_n u_n + t_n v_n \in E$ for all n. Thus $u + v \in T^I(E, e)$, hence $v \in T^I(E, e) \boxminus T^I(E, e)$. $\square$

The following consequence is one of the main features of the moderate derivative.

Corollary 4.10. *If a function f is finite at $x \in X$ and directionally differentiable at $x \in X$, then $f^M(x, \cdot) = Df(x) := f'(x, \cdot)$ and $\partial_M f(x) = \{Df(x)\}$.*

5. Some properties of the moderate subdifferential

In this section we want to check whether the main properties considered in [16, p. 152] or [23] for instance are satisfied by the moderate subdifferential. Of course, it would be of interest to know whether the more complete list of [23] is satisfied. Since it does not seem to be the case, we modify the list of requirements characterizing subdifferentials that we considered in [23]. Hereafter we assume that the subdifferential ∂ is defined on a class $\mathscr{F}(X)$ of functions on X, with X in a class $\mathscr{X}$ of Banach spaces and we require the following conditions that ∂ should satisfy.

Definition 5.1. Whenever $f \in \mathscr{F}(X)$, $\partial f(x)$ with $x \in X$, X a member of $\mathscr{X}$ should satisfy the following conditions:

(C 0) (a) $\partial f(x) = \varnothing$ if $x \in X \backslash \mathrm{dom} f$;

(C 0) (b) if f,$g \in \mathscr{F}(X)$ coincide in a neighborhood of $\bar{x}$, then $\partial f(\bar{x}) = \partial g(\bar{x})$;

(C1) if f attains a local minimum at $\bar{x} \in X$, then $0 \in \partial f(\bar{x})$;

(C2) (a) if f is convex, then $\partial f(\bar{x}) = \{x^* \in X^* : f \geq x^* - x^*(\bar{x}) + f(\bar{x})\}$;

(C2) (b) if d is circa-differentiable at $\bar{x}$ then $\partial(f+d)(\bar{x}) = \partial f(\bar{x}) + d'(\bar{x})$;

(C3) (a) if $f = h \circ g$, with $g : X \to Y$ circa-differentiable at $\bar{x}$, with $g'(\bar{x})(X) = Y$ and $h \in \mathscr{F}(Y)$, then $\partial f(\bar{x}) = g'(\bar{x})^{\intercal}(\partial h(g(\bar{x}))) := \partial h(g(\bar{x})) \circ g'(\bar{x})$;

(C3) (b) if $f = j \circ g$, with $X := X_1 \times \ldots \times X_k$, $\bar{x} := (\bar{x}_1, \ldots, \bar{x}_k) \in X$, $g := (g_1, \ldots, g_k) : X \to \mathbb{R}^k$, $g_i \in \mathscr{F}(X_i)$, $j \in \mathscr{F}(\mathbb{R}^k)$ nondecreasing in each of its k arguments and circa-differentiable at $\bar{r} := g(\bar{x})$ with $D_i j(\bar{r}) (= \frac{\partial}{\partial r_i} j(\bar{r})) \neq 0$ for $i \in \mathbb{N}_k := \{1, \ldots, k\}$ then $\partial f(\bar{x}) \subset j'(g(\bar{x})) \circ (\partial g_1(\bar{x}_1) \times \ldots \times \partial g_k(\bar{x}_k))$.

The first two conditions are borrowed from [16, p. 152] and are very natural; on the other hand, we do not mention condition (e) of this reference requiring that $\partial f(x) \subset k B_{X^*}$ when f is k-Lipschitzian near x, as we consider it is a consequence of the other conditions. We note that here we do not retain condition (S4) of [23] which is not important for our aims and we discard its condition (S6) because this condition implies quasi-homotonicity. Condition (C1) is crucial for optimization; let us check it.

Proposition 5.2. *If f is finite at $\bar{x}$ and reaches a local minimum at $\bar{x}$, then $0 \in \partial_M f(\bar{x})$.*

Proof. Setting $e := (\bar{x}, f(\bar{x}))$, $E := \mathrm{epi} f$, we have to show that $-r = \langle (0,-1), (v,r) \rangle \leq 0$ for any $(v,r) \in T^M(E,e)$. Since $T^M(E,e) \subset T^D(E,e)$, we can find sequences $(t_n) \to 0_+$, $((v,r_n)) \to (v_n, r)$ such that $(\bar{x}, f(\bar{x})) + t_n(v_n, r_n) \in E$ for all $n \in \mathbb{N}$. Then $t_n r_n \geq f(\bar{x} + t_n v_n) - f(\bar{x}) \geq 0$ for $n \in \mathbb{N}$ large enough and we get $r \geq 0$. □

Condition (C2a) requiring that when f is convex $\partial_M f(\bar{x}) = \partial f(\bar{x})$, the Fenchel subdifferential of convex analysis, is a consequence of the property $N_M(E,e) = N(E,e)$ when E is a convex set and $e \in E$. Condition (C2b) is an easy consequence of the definitions.

Condition (C3a) can be decomposed into two assertions.

Proposition 5.3. *Let $g : X \to Y$ be a map between two Banach spaces which is circa-differentiable at $\bar{x}$, let $h : Y \to \overline{\mathbb{R}}$ be finite and directionally stable at $\bar{y} := g(\bar{x})$ and let $f := h \circ g$. Then $\partial_M f(\bar{x}) \subset \partial_M h(\bar{y}) \circ g'(\bar{x})$. If $g'(\bar{x})(X) = Y$ then $\partial_M f(\bar{x}) = \partial_M h(\bar{y}) \circ g'(\bar{x})$.*

That is a consequence of the rule about the expression of normal cones to inverse images, in view of the facts that $g \times I_{\mathbb{R}}$ is circa-differentiable at $(\bar{x}, g(\bar{x}))$ with a surjective derivative when $g'(\bar{x})$ is surjective and that $\mathrm{epi} f = (g \times I_{\mathbb{R}})^{-1}(\mathrm{epi} h)$. □

Let us prove that condition (C 3) (b) is satisfied by ∂_C and ∂_M if $\mathscr{F}(X)$ is the set of locally Lipschitzian functions or the set of directionally stable functions respectively. In order to ease the reading we just consider the assertion in the case $k=2$ which allows a simplified notation; the general case is similar. The case $k=1$ is a consequence of the invariance of the circa-tangent cone and of the moderate normal cone by homeomorphisms that are circa-differentiable at some point with invertible derivatives. We state it as it has its own interest.

Lemma 5.4. *Let $f := j \circ g$, where $g : X \to \mathbb{R}$ is stable (resp. locally Lipschitzian) at $x \in X$ and j is nondecreasing, circa-differentiable at $g(x)$ with $j'(g(x)) \neq 0$. Then $\partial_C f(x) = j'(g(x))\partial_C g(x)$ (resp. $\partial_M f(x) = j'(g(x))\partial_M g(x)$).*

Proposition 5.5. *Let $g : Y \to \mathbb{R}$, $h : Z \to \mathbb{R}$, be directionally stable at $y \in Y$ and $z \in Z$ respectively, let $j : \mathbb{R}^2 \to \mathbb{R}_\infty$ be nondecreasing in each of its variables, finite at $(g(y),h(z))$ and circa-differentiable there, with $(p,q) := j'(g(y),h(z))$, $p \neq 0$, $q \neq 0$. Then, for $f := j \circ (g \times h)$ one has*

$$\partial_M f(y,z) \subset j'(g(y),h(z))(\partial_M g(y) \times \partial_M h(z)).$$

If g and h are locally Lipschitzian around y and z respectively, then

$$\partial_C f(y,z) \subset j'(g(y),h(z))(\partial_C g(y) \times \partial_C h(z)).$$

Proof. We start with the Clarke subdifferential. By symmetry, we just prove that for $(y^*,z^*) \in \partial_C f(y,z)$ we have $y^*/p \in \partial_M g(y)$, or, in view of the lemma $y^* \in \partial_C f_z(y)$ with $f_z := f(\cdot,z)$. We use the fact that f is locally Lipschitzian. Given $v \in Y$, let $(t_n) \to 0_+$, $(y_n) \to y$, $(z_n) \to z$ be such that

$$\lim_n \frac{1}{t_n}[f(y_n + t_n v, z_n) - f(y_n, z_n)] = \limsup_{(t,y',z') \to (0_+,y,z)} \frac{1}{t}[f(y' + tv, z') - f(y', z')].$$

By circa-differentiability of j at $(g(y),h(z))$ there exists a sequence $(p_n) \to p$ such that

$$j(g(y_n + t_n v), h(z_n)) - f(g(y_n), h(z_n)) = p_n(g(y_n + t_n v)) - g(y_n)).$$

Since $(y^*,z^*) \in \partial_C f(y,z)$, for any $v \in X$ we have $\langle y^*, v\rangle = \langle (y^*,z^*),(v,0)\rangle$, hence

$$\langle y^*, v\rangle \leq \lim_n \frac{1}{t_n} p_n(g(y_n + t_n v)) - g(y_n)) \leq p \limsup_{(y',t) \to (y,0)} \frac{1}{t}(g(y' + tv) - g(y')).$$

That shows that $y^*/p \in \partial_C g(y)$.

In the case of the moderate subdifferential, it suffices to assume that g and h are directionally stable at $y \in Y$ and $z \in Z$ respectively, so that f is directionally stable at (y,z) and we can use $f^\diamondsuit$, the proof being similar. □

Remark. In the case of $f(y,z) := g(y) + h(z)$ and $\partial := \partial_C$, no assumption is required on g and h. That stems from the fact that when $(v,s) \in T^C(\text{epi} g, g(y))$, one easily sees that $(v,0,s) \in T^C(\text{epi}\, f, (y,z,f(y,z)))$. Thus for any $(y^*,z^*) \in \partial_C f(y,z)$ one has $\langle y^*, v\rangle = \langle (y^*,z^*),(v,0)\rangle \leq s$, hence $y^* \in \partial_C g(y)$. □

Under additional assumptions, one can get other properties. As in the case of the Clarke subdifferential ([22, Prop. 5.49]), the following result combines two rules. Here we say that a subset E of X is *moderately regular* or in short M-regular at $a \in E$ if $T^M(E,a) = T^D(E,a)$.

Proposition 5.6. *Let X, Y be two normed spaces, let F (resp. G) be a subset of X (resp. Y) and let $g : W \to Y$ be a mapping on an open subset W of X containing $a \in E := F \cap g^{-1}(G)$. Assume g is differentiable at a with derivative $A := g'(a)$ and that $A(T^M(F,a)) \cap H^M(G,b) \neq \varnothing$ for $b := g(a)$. Then*

$$T^M(F,a) \cap A^{-1}(T^M(G,b)) \subset T^M(E,a).$$

Equality holds when F is M-regular at a and G is M-regular at b and then E is M-regular at a.

Proof. We first show that $v \in T^M(E,a)$ whenever $v \in T^M(F,a) \cap A^{-1}(H^M(G,b))$. Let $s := (t_n) \to 0_+$ and let $(e_n) \xrightarrow{s}_E a$. Since $v \in T^M(F,a)$ there exists a sequence $(v_n) \to v$ such that $e_n + t_n v_n \in F$ for all n. Then

$$w_n := \frac{1}{t_n}(g(e_n + t_n v_n) - g(e_n))$$

converges to $w := A(v)$. Since $w \in H^M(G,b)$, we have $g(e_n) + t_n w_n \in G$ for n large enough, hence $e_n + t_n v_n \in F \cap g^{-1}(G)$ for these n. That shows that $v \in T^M(E,a)$.

Now given $v \in T^M(F,a) \cap A^{-1}(T^M(G,b))$, picking $v' \in T^M(F,a) \cap A^{-1}(H^M(G,b))$ and setting $v_n := v + 2^{-n} v'$, so that $v_n \in T^M(F,a) \cap A^{-1}(H^M(G,b))$ by convexity of $T^M(F,a)$ and the relation $T^M(G,b) + H^M(G,b) \subset H^M(G,b)$, the first part of the proof shows that $v_n \in T^M(E,a)$. This set is closed and since $(v_n) \to v$, we get $v \in T^M(E,a)$.

The last assertion is a consequence of the relation $T^D(E,a) \subset T^D(F,a) \cap A^{-1}(T^D(G,b))$ □

6. Relationships with directionally limiting normal cones and directionally limiting subdifferentials.

Let us recall that for $\varepsilon > 0$ the *ε-normal set* to a subset E of X at $a \in \mathrm{cl}E$ is the set $N^\varepsilon(E,a)$ of $x^* \in X^*$ such that for any $\varepsilon' > \varepsilon$ there exists some $\delta > 0$ for which $\langle x^*, e - a \rangle \leq \varepsilon' \|e - a\|$ for all $x \in E \cap B(a,\delta)$, or, in other terms,

$$x^* \in N^\varepsilon(E,a) \Longleftrightarrow \limsup_{e \to a,\ e \in E} \langle x^*, \frac{e-a}{\|e-a\|} \rangle \leq \varepsilon.$$

We set $N^\varepsilon(E,a) = \varnothing$ if $a \in X \backslash \mathrm{cl}E$. The *firm* or *Fréchet normal cone* to E at a is the cone

$$N_F(E,a) := \bigcap_{\varepsilon > 0} N^\varepsilon(E,a).$$

The following definition is a variant of a notion used in [3], [13], [17]. Note that here we do not select a particular direction.

Definition 6.1. The directionally limiting normal cone to E at $a \in E$ is the set $N_{dirL}(E,a)$ of weak*-limits of sequences $x_n^* \in N^{\varepsilon_n}(E,e_n)$ for some sequences $(\varepsilon_n) \to 0_+$, $s := (t_n) \to 0_+$, $(e_n) \xrightarrow{s}_E a$, where $(e_n) \xrightarrow{s}_E a$ means that $(e_n) \to a$ in E, and $(u_n) := (t_n^{-1}(e_n - a))$ converges to some $u \in X \backslash \{0\}$.

In Asplund spaces, the preceding definition can be simplified.

Proposition 6.2. *If X is an Asplund space, then $N_{dirL}(E,a)$ is the set of weak*-limits of sequences (a_n^*) such that $a_n^* \in N_F(E,a_n)$ for some sequence $(a_n) \xrightarrow{dir}_E a$.*

Proof. We note that $x_n^* \in N^{\varepsilon_n}(E,e_n)$ means that for any $\varepsilon > \varepsilon_n$, e_n is a local minimizer on E of $x_n^* - \varepsilon \left\| \cdot - e_n \right\|$. When X is an Asplund space that implies that there exist some $a_n \in E$ close to e_n and some $u_n^* \in B_{X^*}$ such that $a_n^* := x_n^* - 2\varepsilon_n u_n^* \in N_F(E,a_n)$. The result follows. □

The next result is close to [3, Lemma 2.1]. It relates $N_{dirL}(E,a)$ (and the Fréchet normal cone $N_F(E,a)$) to the limiting normal cone $N_L(E,a)$ to E at a, the sequential weak* outer limit of the firm (or Fréchet) normal cones $N_F(E,x)$ to E at nearby points x as $x \to_E a$.

Proposition 6.3. *For any subset E of a normed space X and any $a \in E$ one has*

$$N_{dirL}(E,a) \cup N_F(E,a) \subset N_L(E,a).$$

If X is finite dimensional this inclusion is an equality.

Proof. The first assertion stems from the fact that $(e_n) \xrightarrow{s}_E a$ implies that $(e_n) \to a$, $e_n \in E$ for all n. If X is finite dimensional, given $x^* \in N_L(E,a)$, $(e_n) \to_E a$ implies conversely that $(e_{k(n)}) \xrightarrow{s}_E a$ or $e_{k(n)} = a$ for some sequence $s := (k(n)) \to +\infty$. In the second case, we have $x^* \in N_F(E,a)$ since this cone is closed. □

Defining $\partial_{dirL} f(x)$ by

$$\partial_{dirL} f(x) := \{x^* \in X^* : (x^*, -1) \in N_{dirL}(E, x_f)\}$$

for $x_f := (x, f(x))$, we also have $\partial_{dirL} f(x) \subset \partial_L f(x) \subset \partial_C f(x)$.

Several authors have shown crucial properties for the notions of directional limiting normal cone and subdifferential. We refer to [3], [1], [11] for detailed properties and applications. Here we just point out the following closedness property and an important sum rule.

Proposition 6.4. *Let X be an Asplund space and let $f : X \to \mathbb{R}_\infty$, be directionally Lipschitzian at $x \in \operatorname{dom} f$. Then $\partial_{dirL} f(\cdot)$ is sequentially weak* directionally closed at x.*

Proof. Given $x^* = \mathrm{w}^* - \lim_n x_n^*$, where for some $u \in X$ with $\|u\| = 1$, $(u_n) \to u$, $s := (t_n) \to 0_+$, $x_n := x + t_n u_n$, one has $x_n^* \in \partial_{dirL} f(x_n)$, we have to prove that $x^* \in \partial_{dirL} f(x)$. By definition, for each $n \in \mathbb{N}$ we can find sequences $(u_{n,p})_p \to u_n'$, $(t_{n,p})_p \to 0_+$, $(x_{n,p}^*)$ with $x_{n,p}^* \in \partial_F f(x_n + t_{n,p} u_{n,p})$ with $\left\| u_{n,p} \right\| = 1$ for all n, p and $x_n^* = \mathrm{w}^* - \lim_{\mathrm{p}} x_{n,p}^*$. We may assume that $t_{n,p} \leq t_n^2$, so that $\left\| t_n^{-1}(x_n + t_{n,p} u_{n,p} - x) - u \right\| \leq t_n + \|u_n - u\|$ for all n, p. Thus for a map $k : \mathbb{N} \to \mathbb{N}$ satisfying $\lim_n k(n) = +\infty$ one has $x^* = \mathrm{w}^* - \lim_n x_{n,k(n)}^*$ and $(t_n^{-1}(x_n + t_{n,k(n)} u_{n,k(n)} - x)_n \to u$, so that $x^* \in \partial_{dirL} f(x)$. □

Theorem 6.5. *Let X be an Asplund space and let $f, g : X \to \mathbb{R}_\infty$, f being lower semicontinuous and g being directionally Lipschitzian at $x \in \operatorname{dom} f \cap \operatorname{dom} g$. Then*

$$\partial_{dirL}(f+g)(x) \subset \partial_{dirL} f(x) + \partial_{dirL} g(x).$$

Proof. Let $x^* \in \partial_{dirL}(f+g)(x)$, so that there exist $u \in X \backslash \{0\}$ and sequences $(t_n) \to 0_+$, $(u_n) \to u$, $(x_n^*) \xrightarrow{*} x^*$ such that $x_n^* \in \partial_F(f+g)(x_n)$ for all $n \in \mathbb{N}$, where $x_n := x + t_n u_n$. Since g is Lipschitzian around x_n, by the fuzzy sum rule for Fréchet subdifferentials, [16, Prop. 4.48] there exist (y_n), (z_n) in $B(x_n, t_n^2)$ with $(f(y_n)) \to f(x)$, $y_n^* \in \partial_F f(y_n)$, $z_n^* \in \partial_F g(z_n)$, $(\|y_n^* + z_n^* - x_n^*\|) \to 0$. Since (z_n^*) is bounded, taking a subsequence, we may assume that (z_n^*) weak* converges to some $z^* \in \partial_{dirL} g(x)$ and consequently (y_n^*) weak* converges to some $y^* \in \partial_{dirL} f(x)$ with $y^* + z^* = x^*$. □

Questions. 1) Can one compare $T^M(E,a)$ with the directional inner limit of $T^D(E,x)$ as $x \xrightarrow{E} a$? See [10], [20], [25, Lemma 2.1] for the links between $T^C(E,a)$ and the inner limit of $T^D(E,x)$ as $x \xrightarrow{E} a$.

2) When X is an Asplund space and $f : X \to \mathbb{R}_\infty$, is locally Lipschitzian at $x \in \operatorname{dom} f$, one has $\partial_C f(x) = \overline{\mathrm{co}}^* \partial_L f(x)$ (see [22, Thm 6.10] for instance). One may wonder whether there is a directional version of this result.

3) Is the subdifferential ∂_{dirL} quasi-homotone?

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