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ROBUST SOLUTION OF MULTI-MODEL STOCHASTIC SINGULAR LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM: REGULARIZATION APPROACH

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Dedicated to the memory of Professor Josef Shinar

Abstract. A finite horizon multi-model stochastic linear-quadratic optimal control problem is considered. For this problem, we treat the case where the problem's functional does not contain a control function. This means that the problem under consideration is a singular optimal control problem. The solution to this problem is defined. To solve the considered problem, it is associated with a new optimal control problem for the same multi-model system. The functional in the new problem is the sum of the original functional and an integral of the square of the Euclidean norm of the vector-valued control with a small positive weighting coefficient. Thus, the new problem is regular. Moreover, it is a multi-model stochastic cheap control problem. Using the solvability conditions, the solution of this cheap control problem is reduced to solution of the following two problems: (i) a terminal-value problem for an extended matrix Riccati type differential equation; (ii) a nonlinear optimization (mathematical programming) problem. Asymptotic behaviour of solutions to these problems is analyzed. Using this asymptotic analysis, the optimal value of the functional of the original multi-model stochastic singular optimal control problem is obtained and the solution to this problem is derived. An illustrative academic example is presented.

Keywords. Asymptotic analysis; Cheap control problem; Multi-model stochastic optimal control problem; Regularization; Singular optimal control problem.

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1. Introduction

Multi-model differential systems represent the class of uncertain systems depending on an unknown numerical parameter belonging to some given set, finite or infinite and compact. Thus, a multi-model differential system represents a set of single-model differential systems, each of which is associated with one of the aforementioned parameters. Optimal control problem of a multi-model differential system can be either of Min-Max or of Max-Min type optimization

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problem where the maximum is searched with respect to the parameter, while the minimum is searched with respect to the control of a properly chosen functional.

Singular optimal control problem is such an optimal control problem that the first-order optimality conditions Maximum Principle ([57]), Robust Maximum Principle ([7]), Hamilton-Jacobi-Bellman equation ([5]) are not applicable for obtaining its solution. Single-model singular optimal control problems are extensively studied in the literature. Several approaches to the analysis and solution of such problems are widely used. Thus, higher order necessary/sufficient control optimality conditions can be useful in solving the singular optimal control problems (see, e.g., [4, 14, 38, 43, 48] and references therein). However, the higher order optimality conditions fail to yield a candidate optimal control/optimal control for the problem, which does not have an optimal control in the class of regular (non generalized) functions, even if the problem's functional has a finite infimum/supremum in this class of functions. The second approach is based on the design of a singular optimal control as a sequence of regular open-loop controls, i.e., a sequence of regular control functions of time, along which the functional tends to its infimum/supremum (see, e.g., [30, 31, 43] and references therein). A generalization of this approach is the extension approach (see [32, 33, 34]). The third approach combines geometric and analytic methods. Namely, this approach is based on a decomposition of the state space into the "singular" and "regular" subspaces, and a design of an optimal open loop control as a sum of impulsive (in the singular subspace) and regular (in the regular subspace) functions (see, e.g., [17, 18, 36, 65] and references therein). The fourth approach proposes to look for a solution of a singular optimal control problem in a properly defined class of generalized functions (see, e.g., [67]). Finally, the fifth approach is based on a regularization of the original singular problem by a "small" correction of its "singular" functional (see, e.g., [23, 24, 25, 26] and references therein). Such a regularization is a kind of the Tikhonov's regularization of ill-posed problems [63]. This approach yields the solution of the original problem in the form of a sequence of state-feedback controls.

However, to the best of our knowledge, the multi-model singular optimal control problem was considered in the open literature only in the work [26]. In this paper, the finite horizon multi-model deterministic singular linear-quadratic optimal control problem was studied with respect to Min-Max of its functional. In the present paper, we consider the finite horizon multi-model stochastic singular linear-quadratic optimal control problem with respect to Max-Min of its properly designed functional. We solve this problem by application of the regularization approach, which yields a new regular optimal control problem. The latter problem is a multi-model stochastic cheap control problem. To the best of our knowledge, a multi-model cheap control problem was considered only in the work [26] where the deterministic version of the problem was treated with respect to Min-Max of its functional. Asymptotic analysis of the multi-model stochastic cheap control problem with respect to Max-Min of its functional, obtained in the present paper, is carried out. Based on this analysis, a properly defined solution to the original multi-model stochastic singular control problem is obtained and the optimal value of the functional of this problem is derived.

The paper is organized as follows. In the next section (Section 2), the rigorous formulation of the considered problem is presented. Also, we present the main definitions. In Section 3, we regularize the original singular problem. This regularization yields a new problem - multi-model stochastic cheap control problem. The solvability conditions of this new problem are derived. In Section 4, these solvability conditions are analyzed asymptotically. In Section 5, an asymptotically suboptimal solution to the multi-model stochastic cheap control problem is designed. Based on the results of Sections 4 and 5, in Section 6, the solution to the original multi-model stochastic singular optimal control problem is derived and the optimal value of the functional of this problem is obtained. In Section 7, an illustrative academic example is presented. Section 8 is devoted to concluding remarks.

The following main notations are applied in the paper.

1. E^n denotes the n-dimensional real Euclidean space.
2. $\|\cdot\|$ denotes the Euclidean norm either of a vector or of a matrix.
3. The superscript "T" denotes the transposition of a matrix A (A^T), or of a vector x (x^T).
4. $L^2[a,b;E^n]$ denotes the linear space of n-dimensional vector-valued real functions, square-integrable in the finite interval $[a,b]$.
5. $O_{n_1\times n_2}$ is used for the zero matrix of the dimension $n_1\times n_2$, excepting the cases where the dimension of the zero matrix is obvious. In such cases, the notation 0 is used for the zero matrix.
6. I_n is the n-dimensional identity matrix.
7. $\mathrm{col}(x,y)$, where $x\in E^n$, $y\in E^m$, denotes the column block-vector of the dimension $n+m$ with the upper block x and the lower block y.
8. The inequality $A\le B$, where A and B are quadratic symmetric matrices of the same dimension, means that the matrix $B-A$ is positive semi-definite.

2. Problem Formulation and Main Definitions

Consider the following multi-model Ito differential stochastic system:

$$dw_k(t)=\big[\mathscr{A}_k(t)w_k(t)+\mathscr{B}_k(t)u(t)\big]dt+\sum_{i=1}^{m}\sigma_{i,k}(t)d\eta_i(t),\ \ w_k(0)=\widetilde{w}_0,\ \ t\in[0,t_f],$$
$$k\in\{1,2,...,K\},\quad K>1,\tag{2.1}$$

where $w_k(t)$, $(k\in\{1,2,...,K\})$ is a state in the multi-model system and $w_k(t)\in E^n$, $(k=1,2,...,K)$; $u(t)$ is a control in the multi-model system and $u(t)\in E^r$, $(r\le n)$; $t_f>0$ is a given time instant; $\eta(t)=\mathrm{col}\big(\eta_1(t),...,\eta_m(t)\big)$, $t\ge 0$, $(\eta(0)=0)$ is the m-dimensional standard Wiener process defined on the filtered probability space $\{\Omega,\mathscr{F},P\}$ [2]; $\mathscr{A}_k(t)$ and $\mathscr{B}_k(t)$, $t\in[0,t_f]$, $(k=1,2,...,K)$ are given matrix-valued continuous functions of corresponding dimensions; $\sigma_{i,k}(t)$, $t\in[0,t_f]$, $(i=1,...,m;\ k=1,2,...,K)$ are given vector-valued continuous functions of corresponding dimensions; $\widetilde{w}_0\in E^n$ is a given vector.

Let us consider the following functional:

$$\mathscr{F}(u,k)\stackrel{\triangle}{=}\mathrm{E}\left[w_k^T(t_f)\widetilde{\mathscr{H}}w_k(t_f)+\int_0^{t_f}w_k^T(t)\widetilde{\mathscr{D}}(t)w_k(t)dt\right],\quad k\in\{1,2,...,K\},\tag{2.2}$$

where $\widetilde{\mathscr{H}}$ is a constant symmetric positive semi-definite $n\times n$-matrix; for any $t\in[0,t_f]$, $\widetilde{\mathscr{D}}(t)$ is a symmetric positive semi-definite $n\times n$-matrix.

We also consider the vectors

$$w\stackrel{\triangle}{=}\mathrm{col}(w_1,w_2,...,w_K)\in E^{Kn},\quad w^0\stackrel{\triangle}{=}\mathrm{col}(\widetilde{w}_0,\widetilde{w}_0,...,\widetilde{w}_0)\in E^{Kn}.\tag{2.3}$$

Based on [40], (Chapter 5, Sections 5.1-5.2), let us consider the set Γ of all vector-valued functionals $\gamma\big(\chi(t),t\big) = \mathrm{col}\Big(\gamma_1\big(\chi(t),t\big),...,\gamma_r\big(\chi(t),t\big)\Big)$, each entry $\gamma_i\big(\chi(t),t\big)$, $(i=1,...,r)$ of which is a functional given and measurable with respect to the Borel σ-algebra on the set $\Delta = \{\big(\chi(t),t\big) : t \in [0,t_f],\ \chi(t) \in C[0,t_f;E^{Kn}]\}$.

Definition 2.1. The control $u = \gamma(w(t),t)$, $t \in [0,t_f]$ is called admissible in the multi-model system (2.1) if:
(i) $\gamma(\cdot,\cdot) \in \Gamma$;
(ii) for any $w^0 \in E^{Kn}$ of the form in (2.3), the initial-value problem (2.1) with $k = 1,2,...,K$ and $u(t) = \gamma\big(w(t),t\big)$ has the unique (with the probability 1) continuous solution $w(t)$, $t \in [0,t_f]$;
(iii) for the solution $w(t) = \mathrm{col}\big(w_1(t),...,w_K(t)\big)$ of (2.1) and any $k \in \{1,...,K\}$, the functional $\mathscr{F}(u,k)$ exists, i.e.,

$$0 \le \mathrm{E}\left[w_k^T(t_f)\widetilde{\mathscr{H}}w_k(t_f) + \int_0^{t_f} w_k^T(t)\widetilde{\mathscr{D}}(t)w_k(t)dt\right] < +\infty;$$

(iv)

$$0 \le \mathrm{E}\left[\int_0^T \gamma^T\big(t,w(t)\big)\gamma\big(w(t),t\big)dt\right] < +\infty.$$

In what follows, the set of all admissible controls in the multi-model system (2.1) is denoted by U.

Remark 2.2. Due to [40], (Chapter 5, Sections 5.1-5.2), a linear vector-valued functional of the form $u(t,w(t)) = \mathscr{P}(t)w(t)$ with the matrix-valued gain $\mathscr{P}(t)$, continuous for $t \in [0,t_f]$, belongs to the set U. In what follows, we call such a gain $\mathscr{P}(t)$ admissible in a linear control of the aforementioned form (or simply admissible).

Consider the following set (the simplex) in the space E^K:

$$\Omega_\lambda \stackrel{\triangle}{=} \left\{\lambda = \mathrm{col}(\lambda_1,\lambda_2,...,\lambda_K) \in E^K :\ \lambda_1 \ge 0,\ \lambda_2 \ge 0,...,\lambda_K \ge 0,\ \sum_{k=1}^{K}\lambda_k = 1\right\}. \tag{2.4}$$

Based on the functional $\mathscr{F}(u,k)$ and the set Ω_λ, we construct the performance index evaluating the control process of the multi-model system (2.1)

$$\mathscr{J}(u,\lambda) \stackrel{\triangle}{=} \lambda^T\mathscr{F}(u) \to \max_{\lambda\in\Omega_\lambda}\inf_{u\in U}, \tag{2.5}$$

where

$$\mathscr{F}(u) \stackrel{\triangle}{=} \mathrm{col}\big(\mathscr{F}(u,1),\mathscr{F}(u,2),...,\mathscr{F}(u,K)\big). \tag{2.6}$$

Remark 2.3. Since the control $u(\cdot)$ is not present explicitly in the functional $\mathscr{F}(u,k)$ (and, therefore, in the functional $\mathscr{J}(u,\lambda)$), the first-order optimality conditions fail to yield a solution (an optimal control $u(\cdot)$ and the corresponding vector λ) to the problem (2.1),(2.5). Thus, this problem is a singular optimal control problem.

Remark 2.4. Since for any $k \in \{1,2,...,K\}$, any $u\big(w(t),t\big) \in U$ and any $w^0 \in E^{Kn}$ of the form in (2.3), the value of the functional $\mathscr{F}(u,k)$ with $u(t) = u\big(w(t),t\big)$ is non-negative, then for any aforementioned $w^0 \in E^{Kn}$ and any $\lambda \in \Omega_\lambda$, there exists a finite infimum $\mathscr{J}_\lambda^*(w^0)$ of the functional $\mathscr{J}(u,\lambda)$ with respect to $u(t) = u\big(w(t),t\big) \in U$ in the problem (2.1),(2.5). Moreover, since for any $u\big(w(t),t\big)\in U$ and any $w^0\in E^{Kn}$ of the form in (2.3), $\mathscr{J}(u,\lambda)$ is continuous with respect to $\lambda\in\Omega_\lambda$ and the set Ω_λ is bounded and closed, then there exists finite value

$$\mathscr{J}^*(w^0)\stackrel{\triangle}{=}\max_{\lambda\in\Omega_\lambda}\mathscr{J}^*_\lambda(w^0). \tag{2.7}$$

Consider a sequence of the functions $u^*_q\big(w(t),t\big)\in U$ and a sequence of the vectors $\lambda^*_q\in\Omega_\lambda$, $(q=1,2,...)$.

Definition 2.5. The sequence of the pairs $\Big\{\Big(u^*_q\big(w(t),t\big),\lambda^*_q\Big)\Big\}_{q=1}^{+\infty}$ is called a solution of the optimal control problem (2.1),(2.5) if for any $w^0\in E^{Kn}$ of the form in (2.3):
(a) there exist finite $\lim_{q\to+\infty}\mathscr{J}\Big(u^*_q\big(w(t),t\big),\lambda^*_q\Big)$;
(b) the following equality is valid:

$$\lim_{q\to+\infty}\mathscr{J}\Big(u^*_q\big(w(t),t\big),\lambda^*_q\Big)=\mathscr{J}^*(w^0).$$

In this case, the value $\mathscr{J}^*(w^0)$ is called an optimal value of the functional in the problem (2.1),(2.5).

The objective of the paper is to obtain the solution of the optimal control problem (2.1),(2.5) and to derive the expression for the optimal value of the functional in this problem.

3. Regularization of the Optimal Control Problem (2.1),(2.5)

3.1. **Multi-Model Stochastic Cheap Control Problem.** We are going to derive the solution of the singular problem (2.1),(2.5) by its regularization. Namely, we approximate the original singular problem (2.1),(2.5) with a parameter dependent regular optimal control problem. This new problem has the same multi-model dynamics (2.1) as the original singular problem has. However, the functional in the new problem has the regular form. Namely, the functional in the new problem has the form

$$\mathscr{J}_\varepsilon(u,\lambda)=\lambda^T\mathscr{F}_\varepsilon(u), \tag{3.1}$$

where

$$\begin{aligned}\mathscr{F}_\varepsilon(u)&\stackrel{\triangle}{=}\mathrm{col}\big(\mathscr{F}_\varepsilon(u,1),\mathscr{F}_\varepsilon(u,2),...,\mathscr{F}_\varepsilon(u,K)\big),\\ \mathscr{F}_\varepsilon(u,k)&\stackrel{\triangle}{=}\mathrm{E}\left[w_k^T(t_f)\widetilde{\mathscr{H}}w_k(t_f)+\int_0^{t_f}\big[w_k^T(t)\widetilde{\mathscr{D}}(t)w_k(t)+\varepsilon^2u^T(t)u(t)\big]dt\right],\\ &\qquad k\in\{1,2,...,K\},\end{aligned} \tag{3.2}$$

$\varepsilon>0$ is a small parameter.

Remark 3.1. For the new problem (2.1),(3.1)-(3.2), we choose the same set of all admissible state-feedback controls as it was done for the original problem (2.1),(2.5), i.e., the set U (see Definition 2.1). Moreover, similarly to the original optimal control problem (2.1),(2.5), the functional (3.1) in the new problem is minimized by a proper choice of $u=u\big(w(t),t\big)\in U$ and maximized by a proper choice of $\lambda\in\Omega_\lambda$.

Remark 3.2. Since the parameter $\varepsilon > 0$ is small, the problem (2.1),(3.1)-(3.2) is a cheap control problem, i.e., an optimal control problem with a control cost much smaller than a state cost in the functional. Single-model cheap control problems (deterministic and stochastic) have been studied extensively in the literature (see, e.g., [6, 10, 19, 20, 22, 23, 24, 25, 27, 28, 37, 41, 45, 47, 51, 52, 53, 54, 55, 59, 61, 62] and references therein). However, to the best of our knowledge, a multi-model cheap control problem was studied only in the work [26] where a deterministic finite horizon linear-quadratic optimal control problem was considered. It is important to note that, due to the smallness of the control cost, a cheap control problem can be transformed to an optimal control problem for a singularly perturbed system. Various results in the topic of optimal control problems for singularly perturbed single-model systems can be found, for instance, in [3, 8, 9, 11, 13, 15, 21, 41, 42, 44, 46, 49, 50, 51, 52, 56, 58, 66] and references therein. However, to the best of our knowledge, an optimal control problem for a singularly perturbed multi-model system was studied only in the work [26].

3.2. **Solvability Conditions of the Optimal Control Problem (2.1),(3.1)-(3.2).** Based on the results of the book [7] (Section 16.3), let us introduce into the consideration the following block-diagonal $Kn \times Kn$-matrices:

$$\mathscr{A}(t) \stackrel{\triangle}{=} \begin{pmatrix} \mathscr{A}_1(t) & O_{n\times n} & \ldots & O_{n\times n} \\ O_{n\times n} & \mathscr{A}_2(t) & \ldots & O_{n\times n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ O_{n\times n} & O_{n\times n} & \ldots & \mathscr{A}_K(t) \end{pmatrix}, \quad \mathscr{H} \stackrel{\triangle}{=} \begin{pmatrix} \widetilde{\mathscr{H}} & O_{n\times n} & \ldots & O_{n\times n} \\ O_{n\times n} & \widetilde{\mathscr{H}} & \ldots & O_{n\times n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ O_{n\times n} & O_{n\times n} & \ldots & \widetilde{\mathscr{H}} \end{pmatrix},$$

$$\mathscr{D}(t) \stackrel{\triangle}{=} \begin{pmatrix} \widetilde{\mathscr{D}}(t) & O_{n\times n} & \ldots & O_{n\times n} \\ O_{n\times n} & \widetilde{\mathscr{D}}(t) & \ldots & O_{n\times n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ O_{n\times n} & O_{n\times n} & \ldots & \widetilde{\mathscr{D}}(t) \end{pmatrix}, \quad \Lambda(\lambda) \stackrel{\triangle}{=} \begin{pmatrix} \lambda_1 I_n & O_{n\times n} & \ldots & O_{n\times n} \\ O_{n\times n} & \lambda_2 I_n & \ldots & O_{n\times n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ O_{n\times n} & O_{n\times n} & \ldots & \lambda_K I_n \end{pmatrix}, \tag{3.3}$$

where λ_k, $(k = 1,2,...,K)$ are scalar nonnegative parameters satisfying the condition $\sum_{k=1}^{K} \lambda_k = 1$, i.e., the vector $\lambda \stackrel{\triangle}{=} \mathrm{col}(\lambda_1, \lambda_2, ..., \lambda_K)$ belongs to the set Ω_λ.

Along with the above introduced block-diagonal matrices, let us introduce the following block-form matrix and block-form vectors:

$$\mathscr{B}(t) \stackrel{\triangle}{=} \begin{pmatrix} \mathscr{B}_1(t) \\ \mathscr{B}_2(t) \\ \ldots \\ \mathscr{B}_K(t) \end{pmatrix}, \qquad \Theta_i(t) \stackrel{\triangle}{=} \begin{pmatrix} \sigma_{i,1}(t) \\ \sigma_{i,2}(t) \\ \ldots \\ \sigma_{i,K}(t) \end{pmatrix}, \quad i = 1,...,m. \tag{3.4}$$

Based on the matrices in (3.3) and (3.4), we consider the following terminal-value problem for the matrix Riccati differential equation:

$$\frac{d\mathscr{M}(t)}{dt} = -\mathscr{M}(t)\mathscr{A}(t) - \mathscr{A}^T(t)\mathscr{M}(t) + \mathscr{M}(t)\mathscr{S}(t,\varepsilon)\mathscr{M}(t) - \Lambda(\lambda)\mathscr{D}(t), \; t \in [0,t_f],$$
$$\mathscr{M}(t_f) = \Lambda(\lambda)\mathscr{H}, \tag{3.5}$$

where

$$\mathscr{S}(t,\varepsilon) = \frac{1}{\varepsilon^2}\mathscr{B}(t)\mathscr{B}^T(t). \tag{3.6}$$

Remark 3.3. For any $\lambda \in \Omega_\lambda$ and any $\varepsilon > 0$, the terminal-value problem (3.5) has the unique solution $\mathscr{M}(t) = \mathscr{M}(t,\lambda,\varepsilon)$ in the entire interval $[0,t_f]$, and $\mathscr{M}^T(t,\lambda,\varepsilon) = \mathscr{M}(t,\lambda,\varepsilon)$.

Theorem 3.4. *For a given $\varepsilon > 0$, the solution of the multi-model cheap control problem (2.1),(3.1)-(3.2) is $\Big(u = u^*_\varepsilon\big(w(t),t\big), \lambda = \lambda^*(\varepsilon)\Big)$, where*

$$u^*_\varepsilon\big(w(t),t\big) = -\frac{1}{\varepsilon^2}\mathscr{B}^T(t)\mathscr{M}\big(t,\lambda^*(\varepsilon),\varepsilon\big)w(t), \quad w(t) \in E^{Kn}, \quad t \in [0,t_f], \tag{3.7}$$

$$\lambda^*(\varepsilon) = \text{argmax}_{\lambda\in\Omega_\lambda}\mathscr{I}(\lambda,\varepsilon), \tag{3.8}$$

$$\mathscr{I}(\lambda,\varepsilon) = \big(w^0\big)^T\mathscr{M}(0,\lambda,\varepsilon)w^0 + \sum_{i=1}^{m}\int_0^{t_f}\Theta_i^T(t)\mathscr{M}(t,\lambda,\varepsilon)\Theta_i(t)dt. \tag{3.9}$$

The optimal value $\mathscr{I}^*_\varepsilon$ of the functional in the problem (2.1),(3.1)-(3.2) is

$$\mathscr{I}^*_\varepsilon = \mathscr{I}\big(\lambda^*(\varepsilon),\varepsilon\big). \tag{3.10}$$

Proof. First of all, let us note the following. Using the equation (3.3), the functional (3.1)-(3.2) can be rewritten as:

$$\mathscr{J}_\varepsilon(u,\lambda) = \mathrm{E}\left[w^T(t_f)\Lambda(\lambda)\mathscr{H}w(t_f) + \int_0^{t_f}\big[w^T(t)\Lambda(\lambda)\mathscr{D}(t)w(t) + \varepsilon^2u^T(t)u(t)\big]dt\right]. \tag{3.11}$$

Now, for any given $\lambda \in \Omega_\lambda$ and $\varepsilon > 0$, let us consider the optimal control problem, consisting of the set of equations (2.1) for all $k = 1,...,K$ and of the functional (3.11) to be minimized by a proper choice of the control $u(t) = u\big(w(t),t\big) \in U$. Due to the particular (undelayed) case of the results of [39],

$$u_\varepsilon\big(w(t),t,\lambda\big) = -\frac{1}{\varepsilon^2}\mathscr{B}^T(t)\mathscr{M}\big(t,\lambda,\varepsilon\big)w(t), \quad w(t) \in E^{Kn}, \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \quad \varepsilon > 0 \tag{3.12}$$

is the optimal control and $\mathscr{I}(\lambda,\varepsilon)$ is the optimal value of the functional in the aforementioned problem. Hence

$$\mathscr{J}_\varepsilon\Big(u_\varepsilon\big(w(t),t,\lambda\big),\lambda\Big) = \mathscr{I}(\lambda,\varepsilon), \quad \lambda \in \Omega_\lambda, \quad \varepsilon > 0, \tag{3.13}$$

and

$$\lambda^*(\varepsilon) = \text{argmax}_{\lambda\in\Omega_\lambda}\mathscr{I}(\lambda,\varepsilon) = \text{argmax}_{\lambda\in\Omega_\lambda}\mathscr{J}_\varepsilon\Big(u_\varepsilon\big(w(t),t,\lambda\big),\lambda\Big), \quad \varepsilon > 0. \tag{3.14}$$

The equations (3.7) and (3.12) mean that

$$u^*_\varepsilon\big(w(t),t\big) = u_\varepsilon\big(w(t),t,\lambda^*(\varepsilon)\big). \tag{3.15}$$

Thus, the equations (3.13),(3.14) and (3.15) directly imply the statement of the theorem. □

Consider the following single-model controlled Ito differential stochastic system:

$$dw(t) = \big[\mathscr{A}(t)w(t) + \mathscr{B}(t)u(t)\big]dt + \sum_{i=1}^{m}\Theta_i(t)d\eta_i, \quad w(0) = w^0, \quad t \in [0,t_f]. \tag{3.16}$$

Let, for any given $\varepsilon > 0$ and $k \in \{1,...,K\}$, $\mathscr{F}_\varepsilon^*(k)$ be the value of the functional $\mathscr{F}_\varepsilon(u,k)$ (see the equation (3.2)) calculated along the trajectory $w(t) = \mathrm{col}(w_1(t), w_2(t), ..., w_K(t))$ ($w_k(t) \in E^n$) of the system (3.16) with $u(t) = u_\varepsilon^*\big(w(t),t\big)$ (see the equation (3.7)).

As an immediate consequence of Theorem 3.4, we have the following assertion.

Corollary 3.5. *For a given $\varepsilon > 0$, the following equality is valid:*

$$\sum_{k=1}^{K} \lambda_k^*(\varepsilon)\mathscr{F}_\varepsilon^*(k) = \mathscr{I}_\varepsilon^*, \tag{3.17}$$

where $\mathrm{col}\big(\lambda_1^*(\varepsilon), \lambda_2^*(\varepsilon), ..., \lambda_K^*(\varepsilon)\big) = \lambda^*(\varepsilon)$ given by (3.8); $\mathscr{I}_\varepsilon^*$ (see the equation (3.10)) is the optimal value of the functional $\mathscr{J}_\varepsilon(u,\lambda)$ in the optimal control problem (2.1),(3.1)-(3.2).

4. Asymptotic Analysis of the Solvability Conditions to the Problem (2.1),(3.1)-(3.2)

4.1. **Transformation of the Terminal-Value Problem (3.5) and the Optimization Problem (3.8)-(3.9).** In what follows, we assume that:

AI. For any $k \in \{1,2,...,K\}$ and any $t \in [0,t_f]$, the matrix $\mathscr{B}_k(t)$ has the column rank r.

AII. For any $k \in \{1,2,...,K\}$ and any $t \in [0,t_f]$, $\det\Big(\mathscr{B}_k^T(t)\widetilde{\mathscr{D}}(t)\mathscr{B}_k(t)\Big) \neq 0$.

AIII. For any $k \in \{1,2,...,K\}$, $\widetilde{\mathscr{H}}\mathscr{B}_k(t_f) = 0$.

AIV. The matrix-valued functions $\mathscr{A}_k(t)$, $(k = 1,2,...,K)$ are continuously differentiable in the interval $[0,t_f]$.

AV. The matrix-valued functions $\mathscr{B}_k(t)$, $(k = 1,2,...,K)$ and $\widetilde{\mathscr{D}}(t)$ are twice continuously differentiable in the interval $[0,t_f]$.

By $\mathscr{B}_c(t)$, $t \in [0,t_f]$, we denote a complement matrix-valued function to the matrix-valued function $\mathscr{B}(t)$, $t \in [0,t_f]$ defined in (3.4). Thus, for any $t \in [0,t_f]$, the dimension of the matrix $\mathscr{B}_c(t)$ is $Kn \times (Kn-r)$, and the block-form matrix $\big(\mathscr{B}_c(t), \mathscr{B}(t)\big)$ is invertible. Due to the definition of the matrix-valued function $\mathscr{B}(t)$, as well as the assumption AV and the results of the book [29] (Section 3.3), the matrix-valued function $\mathscr{B}_c(t)$ can be chosen twice continuously differentiable in the interval $[0,t_f]$.

By virtue of the results of the work [26] (Lemma 1), we have the following assertion.

Proposition 4.1. *Let the assumptions AII and AV be satisfied. Then, there exist numbers $0 < \nu_{\min} \le \nu_{\max}$ such that, for all $t \in [0,t_f]$ and all $\lambda \in \Omega_\lambda$, the following relation is valid:*

$$\nu_{\min} I_r \le \mathscr{B}^T(t)\Lambda\mathscr{D}(t)\mathscr{B}(t) \le \nu_{\max} I_r. \tag{4.1}$$

Thus, for all $t \in [0,t_f]$ and all $\lambda \in \Omega_\lambda$, the matrix $\mathscr{B}^T(t)\Lambda\mathscr{D}(t)\mathscr{B}(t)$ is invertible and

$$\frac{1}{\nu_{\max}} I_r \le \big(\mathscr{B}^T(t)\Lambda\mathscr{D}(t)\mathscr{B}(t)\big)^{-1} \le \frac{1}{\nu_{\min}} I_r. \tag{4.2}$$

Consider the following matrix-valued functions of $(t,\lambda) \in [0,t_f] \times \Omega_\lambda$:

$$\mathscr{L}(t,\lambda) = \mathscr{B}_c(t) - \mathscr{B}(t)\big(\mathscr{B}^T(t)\Lambda\mathscr{D}(t)\mathscr{B}(t)\big)^{-1}\mathscr{B}^T(t)\Lambda\mathscr{D}(t)\mathscr{B}_c(t), \quad \mathscr{R}(t,\lambda) = \big(\mathscr{L}(t,\lambda), \mathscr{B}(t)\big). \tag{4.3}$$

Remark 4.2. Due to Proposition 4.1 and the results of [29] (Section 3.3), we have the following. For all $(t,\lambda) \in [0,t_f] \times \Omega_\lambda$, the matrix $\mathscr{R}(t,\lambda)$ is invertible and $\|\mathscr{R}(t,\lambda)\|$, $\|\mathscr{R}^{-1}(t,\lambda)\|$ are bounded. Moreover, the matrix-valued function $\mathscr{R}(t,\lambda)$ is twice continuously differentiable with respect to $t \in [0,t_f]$ uniformly in $\lambda \in \Omega_\lambda$, and this function is continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$.

Using the matrix-valued function $\mathscr{R}(t,\lambda)$ and its properties, we transform the unknown $\mathscr{M}(t)$ in the terminal-value problem (3.5) as follows:

$$\mathscr{M}(t) = \left(\mathscr{R}^T(t,\lambda)\right)^{-1} M(t)\mathscr{R}^{-1}(t,\lambda), \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \tag{4.4}$$

where $M(t)$ is a new unknown matrix-valued function.

By virtue of the results of [29] (Section 3.3) and by use of the equation (3.6), Proposition 4.1 and Remark 4.2, we directly have the following assertion.

Proposition 4.3. *Let the assumptions AI-AV be valid. Then, for any $\varepsilon > 0$ and any $\lambda \in \Omega_\lambda$, the transformation (4.4) converts the terminal-value problem (3.5) to the new terminal-value problem*

$$\begin{aligned} \frac{dM(t)}{dt} = -A(t,\lambda)M(t) - M(t)A^T(t,\lambda) + M(t)S(\varepsilon)M(t) - D(t,\lambda), \quad t \in [0,t_f], \\ M(t_f) = H(\lambda), \end{aligned} \tag{4.5}$$

where

$$A(t,\lambda) = \mathscr{R}^{-1}(t,\lambda)\left[\mathscr{A}(t)\mathscr{R}(t,\lambda) - d\mathscr{R}(t,\lambda)/dt\right], \tag{4.6}$$

$$B(t) = \mathscr{R}^{-1}(t,\lambda)\mathscr{B}(t) = \begin{pmatrix} O_{(Kn-r)\times r} \\ I_r \end{pmatrix} \stackrel{\triangle}{=} B, \tag{4.7}$$

$$S(\varepsilon) = \frac{1}{\varepsilon^2}BB^T = \begin{pmatrix} O_{(Kn-r)\times(Kn-r)} & O_{(Kn-r)\times r} \\ O_{r\times(Kn-r)} & (1/\varepsilon^2)I_r \end{pmatrix}, \tag{4.8}$$

$$D(t,\lambda) = \mathscr{R}^T(t,\lambda)\Lambda\mathscr{D}(t)\mathscr{R}(t,\lambda) = \begin{pmatrix} D_1(t,\lambda) & O_{(Kn-r)\times r} \\ O_{r\times(Kn-r)} & D_2(t,\lambda) \end{pmatrix}, \tag{4.9}$$

$$H(\lambda) = \mathscr{R}^T(t_f,\lambda)\Lambda\mathscr{H}\mathscr{R}(t_f,\lambda) = \begin{pmatrix} H_1(\lambda) & O_{(Kn-r)\times r} \\ O_{r\times(Kn-r)} & O_{r\times r} \end{pmatrix}, \tag{4.10}$$

$$D_1(t,\lambda) = \mathscr{L}^T(t,\lambda)\Lambda\mathscr{D}(t)\mathscr{L}(t,\lambda), \quad D_2(t,\lambda) = \mathscr{B}^T(t)\Lambda\mathscr{D}(t)\mathscr{B}(t), \tag{4.11}$$

$$H_1(\lambda) = \mathscr{L}^T(t_f,\lambda)\Lambda\mathscr{H}\mathscr{L}(t_f,\lambda). \tag{4.12}$$

For all $t \in [0,t_f]$ and $\lambda \in \Omega_\lambda$, the matrix $D_1(t,\lambda)$ is positive semi-definite, while the matrix $D_2(t,\lambda)$ is positive definite. For all $\lambda \in \Omega_\lambda$, the matrix $H_1(\lambda)$ is positive semi-definite, and the matrix-valued function $H_1(\lambda)$ is continuous. The matrix-valued functions $A(t,\lambda)$, $D(t,\lambda)$ are continuously differentiable with respect to $t \in [0,t_f]$ uniformly in $\lambda \in \Omega_\lambda$, and these functions are continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$.

Remark 4.4. Similarly to Remark 3.3, for any $\lambda \in \Omega_\lambda$ and any $\varepsilon > 0$, the terminal-value problem (4.5) has the unique solution $M(t) = M(t,\lambda,\varepsilon)$ in the entire interval $[0,t_f]$, and $M^T(t,\lambda,\varepsilon) = M(t,\lambda,\varepsilon)$.

Corollary 4.5. *Let the assumptions AI-AV be valid. Then, for any $\varepsilon > 0$, the transformation (4.4) convert the optimization problem (3.8)-(3.9) to the equivalent optimization problem*

$$\lambda^* = \lambda^*(\varepsilon) = \text{argmax}_{\lambda\in\Omega_\lambda} J(\lambda,\varepsilon), \tag{4.13}$$

$$J(\lambda,\varepsilon) = \big(z^0(\lambda)\big)^T M(0,\lambda,\varepsilon) z^0(\lambda) + \sum_{i=1}^{m}\int_0^{t_f}\big(\Phi_i(t,\lambda)\big)^T M(t,\lambda,\varepsilon)\Phi_i(t,\lambda)dt, \tag{4.14}$$

where $M(t,\lambda,\varepsilon)$ is the solution of the terminal-value problem (4.5);

$$z^0(\lambda) = \mathscr{R}^{-1}(0,\lambda)w^0, \quad \Phi_i(t,\lambda) = \mathscr{R}^{-1}(t,\lambda)\Theta_i(t), \;\; t\in[0,t_f], \;\; \lambda\in\Omega_\lambda, \;\; i=1,...,m. \tag{4.15}$$

Moreover,

$$J\big(\lambda^*(\varepsilon),\varepsilon\big) = \mathscr{I}\big(\lambda^*(\varepsilon),\varepsilon\big), \quad \varepsilon \geq 0. \tag{4.16}$$

Proof. The statements of the corollary follow immediately from Theorem 3.4 and Proposition 4.3. □

4.2. **Asymptotic Solution of the Terminal-Value Problem (4.5).** Due to the block form of the matrix $S(\varepsilon)$ (see the equation (4.8)), the right-hand side of the differential equation in (4.5) has a singularity with respect to ε at $\varepsilon = 0$. To remove this singularity, we look for the solution $M(t) = M(t,\lambda,\varepsilon)$ of the problem (4.5) in the following block form:

$$M(t,\lambda,\varepsilon) = \begin{pmatrix} M_1(t,\lambda,\varepsilon) & \varepsilon M_2(t,\lambda,\varepsilon) \\ \\ \varepsilon M_2^T(t,\lambda,\varepsilon) & \varepsilon M_3(t,\lambda,\varepsilon) \end{pmatrix}, \tag{4.17}$$

where the matrices $M_1(t,\lambda,\varepsilon)$, $M_2(t,\lambda,\varepsilon)$ and $M_3(t,\lambda,\varepsilon)$ are of the dimensions $(Kn-r)\times(Kn-r)$, $(Kn-r)\times r$ and $r\times r$, respectively; $M_1^T(t,\lambda,\varepsilon) = M_1(t,\lambda,\varepsilon)$, $M_3^T(t,\lambda,\varepsilon) = M_3(t,\lambda,\varepsilon)$.

Similarly to the block form of the matrix $M(t,\lambda,\varepsilon)$, we also partition the matrix $A(t,\lambda)$ into blocks as:

$$A(t,\lambda) = \begin{pmatrix} A_1(t,\lambda) & A_2(t,\lambda) \\ A_3(t,\lambda) & A_4(t,\lambda) \end{pmatrix}, \tag{4.18}$$

where the matrices $A_1(t,\lambda)$, $A_2(t,\lambda)$, $A_3(t,\lambda)$ and $A_4(t,\lambda)$ are of the dimensions $(Kn-r)\times(Kn-r)$, $(Kn-r)\times r$, $r\times(Kn-r)$ and $r\times r$, respectively.

Substitution of the block forms of the matrices $S(\varepsilon)$, $D(t,\lambda)$, $H(\lambda)$, $M(t,\lambda,\varepsilon)$, $A(t,\lambda,\varepsilon)$ (see the equations (4.8), (4.9), (4.10), (4.17), (4.18)) into the problem (4.5) yields after a routine matrix algebra the following equivalent terminal-value problem in the time interval $[0,t_f]$:

$$\frac{dM_1(t,\lambda,\varepsilon)}{dt} = -M_1(t,\lambda,\varepsilon)A_1(t,\lambda) - \varepsilon M_2(t,\lambda,\varepsilon)A_3(t,\lambda) - A_1^T(t,\lambda)M_1(t,\lambda,\varepsilon) - \\ \varepsilon A_3^T(t,\lambda)M_2^T(t,\lambda,\varepsilon) + M_2(t,\lambda,\varepsilon)M_2^T(t,\lambda,\varepsilon) - D_1(t,\lambda), \quad M_1(t_f,\lambda,\varepsilon) = H_1(\lambda), \tag{4.19}$$

$$\varepsilon\frac{dM_2(t,\lambda,\varepsilon)}{dt} = -M_1(t,\lambda,\varepsilon)A_2(t,\lambda) - \varepsilon M_2(t,\lambda,\varepsilon)A_4(t,\lambda) - \varepsilon A_1^T(t,\lambda)M_2(t,\lambda,\varepsilon) \\ -\varepsilon A_3^T(t,\lambda)M_3(t,\lambda,\varepsilon) + M_2(t,\lambda,\varepsilon)M_3(t,\lambda,\varepsilon), \quad M_2(t_f,\lambda,\varepsilon) = 0, \tag{4.20}$$

$$\varepsilon\frac{dM_3(t,\lambda,\varepsilon)}{dt} = -\varepsilon M_2^T(t,\lambda,\varepsilon)A_2(t,\lambda) - \varepsilon M_3(t,\lambda,\varepsilon)A_4(t,\lambda) - \varepsilon A_2^T(t,\lambda)M_2(t,\lambda,\varepsilon) \\ -\varepsilon A_4^T(t,\lambda)M_3(t,\lambda,\varepsilon) + \Big(M_3(t,\lambda,\varepsilon)\Big)^2 - D_2(t,\lambda), \quad M_3(t_f,\lambda,\varepsilon) = 0. \tag{4.21}$$

Remark 4.6. Since the terminal-value problem (4.19)-(4.21) is equivalent to the terminal-value problem (4.5), then (due to Remark 4.4), for any $\lambda \in \Omega_\lambda$ and any $\varepsilon > 0$, the problem (4.19)-(4.21) has the unique solution $\{M_1(t,\lambda,\varepsilon), M_2(t,\lambda,\varepsilon), M_3(t,\lambda,\varepsilon)\}$ in the entire interval $[0,t_f]$. Also, it should be noted that, for any $\lambda \in \Omega_\lambda$, the terminal-value problem (4.19)-(4.21) is a singularly perturbed one for a set of Riccati-type matrix differential equations. In what follows of this subsection, based on the Boundary Function Method (see, e.g., [64]), we construct and justify the zero-order asymptotic solution of this problem. We seek this asymptotic solution in the form

$$M_{j0}(t,\lambda,\varepsilon) = M^o_{j0}(t,\lambda) + M^b_{j0}(\tau,\lambda), \quad j = 1,2,3, \quad \tau = (t-t_f)/\varepsilon, \tag{4.22}$$

where the terms with the upper index "o" constitute the so called outer solution, while the terms with the upper index "b" are the boundary correction terms in a left-hand neighbourhood of $t = t_f$; $\tau \le 0$ is a new independent variable, called the stretched time. For any $t \in [0,t_f)$, $\tau \to -\infty$ as $\varepsilon \to +0$. Equations and conditions for obtaining the outer solution and the boundary correction terms are derived by substituting the representation (4.22) into the terminal-value problem (4.19)-(4.21) instead of $M_j(t,\lambda,\varepsilon)$, $(j = 1,2,3)$, and equating the coefficients for the same power of ε on both sides of the resulting equations, separately the coefficients depending on t and on τ.

4.2.1. *Obtaining the Boundary Correction* $M^b_{10}(\tau)$. For this boundary layer correction, we have the equation

$$\frac{dM^b_{10}(\tau,\lambda)}{d\tau} = 0, \quad \tau \le 0, \quad \lambda \in \Omega_\lambda. \tag{4.23}$$

By virtue of the Boundary Function Method, we require that the boundary layer correction terms tend to zero for τ tending to $-\infty$. Thus, with respect to $M^b_{10}(\tau,\lambda)$, we require that

$$\lim_{\tau\to-\infty} M^b_{10}(\tau,\lambda) = 0. \tag{4.24}$$

Moreover, we require that the limit (4.24) is uniform with respect to $\lambda \in \Omega_\lambda$.

From the equations (4.23)-(4.24), we directly have

$$M^b_{10}(\tau,\lambda) \equiv 0, \quad \tau \le 0, \quad \lambda \in \Omega_\lambda. \tag{4.25}$$

4.2.2. *Obtaining the Outer Solution Terms.* The equations and conditions for these terms are the following for all $t \in [0,t_f]$ and $\lambda \in \Omega_\lambda$:

$$\frac{dM^o_{10}(t,\lambda)}{dt} = -M^o_{10}(t,\lambda)A_1(t,\lambda) - A_1^T(t,\lambda)M^o_{10}(t,\lambda)$$
$$+M^o_{20}(t,\lambda)\big(M^o_{20}(t,\lambda)\big)^T - D_1(t,\lambda), \quad M^o_{10}(t_f,\lambda) = H_1(\lambda), \tag{4.26}$$

$$-M^o_{10}(t,\lambda)A_2(t,\lambda) + M^o_{20}(t,\lambda)M^o_{30}(t,\lambda) = 0, \tag{4.27}$$

$$\big(M^o_{30}(t,\lambda)\big)^2 - D_2(t,\lambda) = 0, \tag{4.28}$$

Remark 4.7. Let us observe that in the system (4.26)-(4.28), the unknown matrix-valued functions $M^o_{20}(t,\lambda)$ and $M^o_{30}(t,\lambda)$ are not subject to any terminal conditions. This occurs because in (4.26)-(4.28) these unknowns are subject to the algebraic (but not differential) equations.

Solving the algebraic equation (4.28) and taking into account the positive definiteness of the matrix $D_2(t,\lambda)$, we obtain

$$M_{30}^o(t,\lambda) = \big(D_2(t,\lambda)\big)^{1/2}, \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \tag{4.29}$$

where the superscript "1/2" denotes the unique symmetric positive definite square root of corresponding symmetric positive definite matrix.

Remark 4.8. Due to Proposition 4.3, $\|M_{30}^o(t,\lambda)\|$ is bounded for all $(t,\lambda) \in [0,t_f] \times \Omega_\lambda$. Moreover, due to Proposition 4.3 and the Implicit Function Theorem [60], the matrix-valued function $M_{30}^o(t,\lambda)$ is continuously differentiable with respect to $t \in [0,t_f]$ uniformly in $\lambda \in \Omega_\lambda$, and $\|dM_{30}^o(t,\lambda)/dt\|$ is bounded for all $(t,\lambda) \in [0,t_f] \times \Omega_\lambda$. In addition, since $D_2(t,\lambda)$ is continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$, then $M_{30}^o(t,\lambda)$ also is continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$.

Solving the equation (4.27) with respect to $M_{20}^o(t,\lambda)$ and using (4.29), we have

$$M_{20}^o(t,\lambda) = M_{10}^o(t,\lambda)A_2(t,\lambda)\big(D_2(t,\lambda)\big)^{-1/2}, \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \tag{4.30}$$

where the superscript "$-1/2$" denotes the inverse matrix for the unique symmetric positive definite square root of corresponding symmetric positive definite matrix.

Substituting (4.30) into (4.26), we obtain after a routine rearrangement the following terminal-value problem with respect to $M_{10}^o(t,\lambda)$ for all $\lambda \in \Omega_\lambda$:

$$\frac{dM_{10}^o(t,\lambda)}{dt} = -M_{10}^o(t,\lambda)A_1(t,\lambda) - A_1^T(t,\lambda)M_{10}^o(t,\lambda)$$
$$+M_{10}^o(t,\lambda)S_1^o(t,\lambda)M_{10}^o(t,\lambda) - D_1(t,\lambda), \quad t \in [0,t_f], \quad M_{10}^o(t_f,\lambda) = H_1(\lambda), \tag{4.31}$$

where

$$S_1^o(t,\lambda) = A_2(t,\lambda)D_2^{-1}(t,\lambda)A_2^T(t,\lambda). \tag{4.32}$$

Remark 4.9. Remember that, for all $t \in [0,t_f]$ and all $\lambda \in \Omega_\lambda$, the matrices $D_1(t,\lambda)$, $H_1(\lambda)$ are positive semi-definite and the matrix $D_2(t,\lambda)$ is positive definite (see Proposition 4.3). Therefore, for all $\lambda \in \Omega_\lambda$, the terminal-value problem (4.31) has the unique solution $M_{10}^o(t,\lambda)$ in the entire interval $[0,t_f]$. Moreover, due to Proposition 4.3, $\|M_{10}^o(t,\lambda)\|$ and $\|dM_{10}^o(t,\lambda)/dt\|$ are bounded for all $(t,\lambda) \in [0,t_f] \times \Omega_\lambda$. Therefore, due to Remark 4.8 and the equations (4.29),(4.30), $\|M_{20}^o(t,\lambda)\|$ and $\|dM_{20}^o(t,\lambda)/dt\|$ are bounded for all $(t,\lambda) \in [0,t_f] \times \Omega_\lambda$. In addition, since $A_1(t,\lambda)$, $S_1^o(t,\lambda)$, $D_1(t,\lambda)$ are continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$ and $H_1(\lambda)$ is continuous with respect to $\lambda \in \Omega_\lambda$ then, by virtue of the results of [35] (Chapter 5), $M_{10}^o(t,\lambda)$ also is continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$. Therefore, due to the equations (4.29),(4.30), Remark 4.8 and the continuity of $A_2(t,\lambda)$ with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$, $M_{20}^o(t,\lambda)$ also is continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$.

4.2.3. *Obtaining the Boundary Correction Terms $M_{20}^b(\tau,\lambda)$ and $M_{30}^b(\tau,\lambda)$.* These terms are obtained as the solution of the terminal-value problem

$$\frac{dM_{20}^b(\tau,\lambda)}{d\tau} = M_{20}^o(t_f,\lambda)M_{30}^b(\tau,\lambda) + M_{20}^b(\tau,\lambda)M_{30}^o(t_f,\lambda) + M_{20}^b(\tau,\lambda)M_{30}^b(\tau,\lambda),$$
$$\frac{dM_{30}^b(\tau,\lambda)}{d\tau} = M_{30}^o(t_f,\lambda)M_{30}^b(\tau,\lambda) + M_{30}^b(\tau,\lambda)M_{30}^o(t_f,\lambda) + \big(M_{30}^b(\tau,\lambda)\big)^2,$$
$$M_{20}^b(0,\lambda) = -M_{20}^o(t_f,\lambda), \quad M_{30}^b(0,\lambda) = -M_{30}^o(t_f,\lambda), \tag{4.33}$$

where $\tau \le 0$, $\lambda \in \Omega_\lambda$.

Substitution of the expressions for $M_{30}^o(t,\lambda)$ and $M_{20}^o(t,\lambda)$ (see the equations (4.29) and (4.30)) into the terminal-value problem (4.33) and use of the terminal condition for $M_{10}^o(t,\lambda)$ (see the equation (4.31)) transform the problem (4.33) into the following terminal-value problem:

$$\frac{dM_{20}^b(\tau,\lambda)}{d\tau} = M_{20}^b(\tau,\lambda)\big[\big(D_2(t_f,\lambda)\big)^{1/2} + M_{30}^b(\tau,\lambda)\big]$$
$$+H_1(\lambda)A_2(t_f,\lambda)\big(D_2(t_f,\lambda)\big)^{-1/2}M_{30}^b(\tau,\lambda),$$
$$M_{20}^b(0,\lambda) = -H_1(\lambda)A_2(t_f,\lambda)\big(D_2(t_f,\lambda)\big)^{-1/2}, \quad \tau \le 0, \quad \lambda \in \Omega_\lambda, \tag{4.34}$$

$$\frac{dM_{30}^b(\tau,\lambda)}{d\tau} = \big(D_2(t_f,\lambda)\big)^{1/2}M_{30}^b(\tau,\lambda) + M_{30}^b(\tau,\lambda)\big(D_2(t_f,\lambda)\big)^{1/2} + \big(M_{30}^b(\tau,\lambda)\big)^2,$$
$$M_{30}^b(0,\lambda) = -\big(D_2(t_f,\lambda)\big)^{1/2}, \quad \tau \le 0, \quad \lambda \in \Omega_\lambda. \tag{4.35}$$

Based on the results of [29] (Section 4.5), we obtain the solution of the terminal-value problem (4.34)-(4.35) in the form

$$M_{20}^b(\tau,\lambda) = -2H_1(\lambda)A_2(t_f,\lambda)\big(D_2(t_f,\lambda)\big)^{-1/2}\exp\Big(2\big(D_2(t_f,\lambda)\big)^{1/2}\tau\Big)\Big[I_r$$
$$+\exp\Big(2\big(D_2(t_f,\lambda)\big)^{1/2}\tau\Big)\Big]^{-1}, \quad \tau \le 0, \quad \lambda \in \Omega_\lambda, \tag{4.36}$$

$$M_{30}^b(\tau,\lambda) = -2\big(D_2(t_f,\lambda)\big)^{1/2}\exp\Big(2\big(D_2(t_f,\lambda)\big)^{1/2}\tau\Big)\Big[I_r$$
$$+\exp\Big(2\big(D_2(t_f,\lambda)\big)^{1/2}\tau\Big)\Big]^{-1}, \quad \tau \le 0, \quad \lambda \in \Omega_\lambda. \tag{4.37}$$

Due to Proposition 4.1 (see the inequalities in (4.1)) and Proposition 4.3 (see the expression for $D_2(t,\lambda)$ in (4.11)), the matrix-valued functions $M_{20}^b(\tau,\lambda)$ and $M_{30}^b(\tau,\lambda)$ are exponentially decaying for $\tau \to -\infty$ uniformly with respect to $\lambda \in \Omega_\lambda$, i.e., they satisfy the inequalities

$$\big\|M_{20}^b(\tau,\lambda)\big\| \le a\exp(\beta\tau), \quad \big\|M_{30}^b(\tau,\lambda)\big\| \le a\exp(\beta\tau), \quad \tau \le 0, \quad \lambda \in \Omega_\lambda, \tag{4.38}$$

where $a > 0$ and $\beta > 0$ are some constants independent of $\lambda \in \Omega_\lambda$.

4.2.4. *Justification of the Asymptotic Solution to the Terminal-Value Problem (4.19)-(4.21).* Quite similarly to the results of [26] (Theorem 1), we have the following assertion.

Lemma 4.10. *Let the assumptions AI-AV be fulfilled. Then, there exists a number $\varepsilon_0 > 0$ independent of $\lambda \in \Omega_\lambda$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, the entries of the solution to the terminal-value problem (4.19)-(4.21) $\{M_1(t,\lambda,\varepsilon), M_2(t,\lambda,\varepsilon), M_3(t,\lambda,\varepsilon)\}$ satisfy the inequalities*

$$\left\|M_1(t,\lambda,\varepsilon) - M_{10}^o(t,\lambda)\right\| \le c\varepsilon, \quad \left\|M_j(t,\lambda,\varepsilon) - M_{j0}(t,\lambda,\varepsilon)\right\| \le c\varepsilon,$$
$$j = 2,3, \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \tag{4.39}$$

where $M_{j0}(t,\lambda,\varepsilon)$, $(j = 2,3)$ are given in (4.22); $c > 0$ is some constant independent of ε and $\lambda \in \Omega_\lambda$.

4.3. **Asymptotic Behaviour of the Solution to the Optimization Problem (4.13)-(4.14).** Along with the optimization problem (4.13)-(4.14), we consider the following optimization problem:

$$\lambda_0^* = \mathrm{argmax}_{\lambda \in \Omega_\lambda} J_0(\lambda), \tag{4.40}$$

$$J_0(\lambda) = \left(x^0(\lambda)\right)^T M_{10}^o(0,\lambda) x^0(\lambda) + \sum_{i=1}^{m} \int_0^{t_f} \Phi_{i,x}^T(t,\lambda) M_{10}^o(t,\lambda) \Phi_{i,x}(t,\lambda) dt, \tag{4.41}$$

where $x^0(\lambda)$ and $\Phi_{i,x}(t,\lambda)$ are the upper blocks of the same dimension $Kn - r$ of the vectors $z^0(\lambda)$ and $\Phi_i(t,\lambda)$, respectively.

In contrast with the optimization problem (4.13)-(4.14), the optimization problem (4.40)-(4.41) is independent of ε.

In what follows, we assume that:

AVI. The optimization problem (4.40)-(4.41) has the unique solution λ_0^*.

Using Remarks 4.2, 4.8, 4.9, as well as the inequalities in (4.38) and Lemma 4.10, we obtain (similarly to Lemma 2 and Theorem 3 of the work [26]) the following assertion.

Lemma 4.11. *Let the assumptions AI-AVI be fulfilled. Then, the function $J_0(\lambda)$ is continuous with respect to $\lambda \in \Omega_\lambda$ and, for any $\varepsilon \in (0, \varepsilon_0]$, the function $J(\lambda,\varepsilon)$ also is continuous with respect to $\lambda \in \Omega_\lambda$. Moreover, the following two limit equalities are valid:*

$$\lim_{\varepsilon \to +0} J(\lambda,\varepsilon) = J_0(\lambda) \ \text{ uniformly in } \lambda \in \Omega_\lambda, \tag{4.42}$$

$$\lim_{\varepsilon \to +0} \lambda^*(\varepsilon) = \lambda_0^*, \tag{4.43}$$

where $\lambda^*(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is the solution of the optimization problem (4.13)-(4.14).

As a direct consequence of Lemma 4.11, we have the following assertion.

Corollary 4.12. *Let the assumptions AI-AVI be fulfilled. Then, for the solution $\lambda^*(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ of the optimization problem (4.13)-(4.14), there exists a function $g^*(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon_0]$, such that* $\lim_{\varepsilon \to +0} g^*(\varepsilon) = 0$ *and*

$$\left|J\big(\lambda^*(\varepsilon),\varepsilon\big) - J_0\big(\lambda_0^*\big)\right| \le g^*(\varepsilon), \quad \varepsilon \in (0, \varepsilon_0]. \tag{4.44}$$

4.4. **Asymptotic Behaviour of the Value $\mathcal{F}_\varepsilon^*(k)$.** Remember that $\mathcal{F}_\varepsilon^*(k)$, $(\varepsilon > 0, k \in \{1,...,K\})$ is the value of the functional $\mathcal{F}_\varepsilon(u,k)$ (see the equation (3.2)) calculated along the trajectory $w(t) = \mathrm{col}(w_1(t), w_2(t), ..., w_K(t))$ $(w_k(t) \in E^n)$ of the system (3.16) with $u(t) = u_\varepsilon^*\big(w(t),t\big)$ (see the equations (3.7) and (3.8)-(3.9)).

Let us rewrite the functional (3.2) in the terms of the vector $w(t)$, i.e.,

$$\mathcal{F}_\varepsilon(u,k) \stackrel{\triangle}{=} \mathrm{E}\left[w^T(t_f)\widetilde{\mathcal{H}_k}w(t_f) + \int_0^{t_f}\left[w^T(t)\widetilde{\mathcal{D}}_k(t)w(t) + \varepsilon^2 u^T(t)u(t)\right]dt\right], \quad k \in \{1,2,...,K\}, \tag{4.45}$$

where $\widetilde{\mathcal{H}_k}$ and $\widetilde{\mathcal{D}}_k(t)$ are block-diagonal matrices of the dimension $Kn \times Kn$ with the k-th blocks $\mathcal{H}$ and $\mathcal{D}(t)$, respectively, on the main diagonal, while the other blocks of this diagonal are zero matrices of the dimension $n \times n$. Thus,

$$\widetilde{\mathcal{H}_k} = \mathrm{diag}\big(\underbrace{O_{n\times n}, ..., O_{n\times n}}_{k-1}, \mathcal{H}, \underbrace{O_{n\times n}, ..., O_{n\times n}}_{K-k}\big), \quad k \in \{1,...,K\}, \tag{4.46}$$

$$\widetilde{\mathcal{D}}_k(t) = \mathrm{diag}\big(\underbrace{O_{n\times n}, ..., O_{n\times n}}_{k-1}, \widetilde{\mathcal{D}}(t), \underbrace{O_{n\times n}, ..., O_{n\times n}}_{K-k}\big), \quad k \in \{1,...,K\}. \tag{4.47}$$

We study the asymptotic behaviour of the value $\mathcal{F}_\varepsilon^*(k)$ by two stages. At the first stage, we analyze the asymptotic behaviour of the value $\widetilde{\mathcal{F}_\varepsilon}(k,\lambda)$ of the functional (4.45) calculated along the trajectory $w(t)$ of the system (3.16) with $u(t) = u_\varepsilon\big(w(t),t,\lambda\big)$ (see the equation (3.12)). Remember that $u_\varepsilon\big(w(t),t,\lambda^*(\varepsilon)\big) = u_\varepsilon^*\big(w(t),t\big)$ (see the equation (3.15)).

At the second stage, based on this asymptotic analysis and on Lemma 4.11, we obtain the asymptotic behaviour of the value $\mathcal{F}_\varepsilon^*(k)$. Let us start with the first stage of this study. Based on the particular (undelayed) case of the work [39], we obtain that

$$\widetilde{\mathcal{F}_\varepsilon}(k,\lambda) = \big(w^0\big)^T\widetilde{\mathcal{M}_k}(0,\lambda,\varepsilon)w^0 + \sum_{i=1}^{m}\int_0^{t_f}\Theta_i^T(t)\widetilde{\mathcal{M}_k}(t,\lambda,\varepsilon)\Theta_i(t)dt, \quad k = 1,...,K, \tag{4.48}$$

where, for all $\lambda \in \Omega_\lambda$, $\varepsilon > 0$, and $k \in \{1,...,K\}$, $\widetilde{\mathcal{M}_k}(t,\lambda,\varepsilon)$, $t \in [0,t_f]$ is the unique solution of the following terminal-value problem for linear differential equation with respect to the unknown $Kn \times Kn$-matrix-valued function $\widetilde{\mathcal{M}_k}(t)$:

$$\frac{d\widetilde{\mathcal{M}_k}(t)}{dt} = -\widetilde{\mathcal{M}_k}(t)\widetilde{\mathcal{A}}(t,\lambda,\varepsilon) - \widetilde{\mathcal{A}}^T(t,\lambda,\varepsilon)\widetilde{\mathcal{M}_k}(t) - \widetilde{\mathcal{D}}_k(t) - \mathcal{M}(t,\lambda,\varepsilon)\mathcal{S}(t,\varepsilon)\mathcal{M}(t,\lambda,\varepsilon), \quad t \in [0,t_f], \quad \widetilde{\mathcal{M}_k}(t_f) = \widetilde{\mathcal{H}_k}, \tag{4.49}$$

$$\widetilde{\mathcal{A}}(t,\lambda,\varepsilon) = \mathcal{A}(t) - \mathcal{S}(t,\varepsilon)\mathcal{M}(t,\lambda,\varepsilon), \tag{4.50}$$

$\mathcal{M}(t,\lambda,\varepsilon)$ is the solution of the terminal-value problem (3.5) mentioned in Remark 3.3.

Let us partition the $Kn \times Kn$-matrix-valued function $\mathscr{R}(t,\lambda)$ into K blocks as follows:

$$\mathscr{R}(t,\lambda) = \begin{pmatrix} \mathscr{R}_1(t,\lambda) \\ \mathscr{R}_2(t,\lambda) \\ \dots \\ \mathscr{R}_K(t,\lambda) \end{pmatrix}, \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \tag{4.51}$$

where each of the blocks is of the dimension $n \times Kn$.

Using the equations (3.4),(4.3),(4.51), we can represent $\mathscr{R}_k(t,\lambda)$ in the form

$$\mathscr{R}_k(t,\lambda) = \Big(\mathscr{L}_k(t,\lambda), \mathscr{B}_k(t)\Big), \quad k = 1,2,...,K, \quad t \in [0,t_f], \tag{4.52}$$

where $\mathscr{L}_k(t,\lambda)$ is the k-th block from the above of the dimension $n \times (Kn - r)$ in the matrix $\mathscr{L}(t,\lambda)$, i.e., this block is obtained from the following block-form representation of the matrix $\mathscr{L}(t,\lambda)$:

$$\mathscr{L}(t,\lambda) = \begin{pmatrix} \mathscr{L}_1(t,\lambda) \\ \mathscr{L}_2(t,\lambda) \\ \dots \\ \mathscr{L}_K(t,\lambda) \end{pmatrix}, \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \tag{4.53}$$

and each of the blocks is of the dimension $n \times (Kn - r)$.

Using the matrix-valued function $\mathscr{R}(t,\lambda)$ and its block-form representation (4.52), as well as the block-form representation (4.53) of the matrix-valued function $\mathscr{L}(t,\lambda)$, we transform (similarly to (4.4)) the unknown $\widetilde{\mathscr{M}_k}(t)$ in the terminal-value problem (4.49) as follows:

$$\widetilde{\mathscr{M}_k}(t) = \big(\mathscr{R}^T(t,\lambda)\big)^{-1}\widetilde{M}_k(t)\mathscr{R}^{-1}(t,\lambda), \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \tag{4.54}$$

where $\widetilde{M}_k(t)$ is a new unknown matrix-valued function.

Based on Proposition 4.3, the transformation (4.54), along with the equations (4.4),(4.51)-(4.53) and the assumption AIII, converts the terminal-value problem (4.49) to the new terminal-value problem

$$\frac{d\widetilde{M}_k(t)}{dt} = -\widetilde{M}_k(t)\widetilde{A}(t,\lambda,\varepsilon) - \widetilde{A}^T(t,\lambda,\varepsilon)\widetilde{M}_k(t) - \widetilde{D}_k(t,\lambda) - M(t,\lambda,\varepsilon)S(\varepsilon)M(t,\lambda,\varepsilon),$$
$$t \in [0,t_f], \quad \widetilde{M}_k(t_f) = \widetilde{H}_k(\lambda), \tag{4.55}$$

where $M(t,\lambda,\varepsilon)$, $t \in [0,t_f]$ is the solution of the terminal-value problem (4.5),

$$\begin{aligned}\widetilde{A}(t,\lambda,\varepsilon) &= \mathscr{R}^{-1}(t,\lambda)\big[\widetilde{\mathscr{A}}(t,\lambda,\varepsilon)\mathscr{R}(t,\lambda) - d\mathscr{R}(t,\lambda)/dt\big] \\ &= \mathscr{R}^{-1}(t,\lambda)\big[\big(\mathscr{A}(t) - \mathscr{S}(t,\varepsilon)\mathscr{M}(t,\lambda,\varepsilon)\big)\mathscr{R}(t,\lambda) - d\mathscr{R}(t,\lambda)/dt\big] \\ &= \mathscr{R}^{-1}(t,\lambda)\big[\mathscr{A}(t)\mathscr{R}(t,\lambda) - d\mathscr{R}(t,\lambda)/dt\big] - \mathscr{R}^{-1}(t,\lambda)\mathscr{S}(t,\varepsilon)\mathscr{M}(t,\lambda,\varepsilon)\mathscr{R}(t,\lambda) \\ &= A(t,\lambda) - \mathscr{R}^{-1}(t,\lambda)\mathscr{S}(t,\varepsilon)\big(\mathscr{R}^T(t,\lambda)\big)^{-1}M(t,\lambda,\varepsilon) = A(t,\lambda) - S(\varepsilon)M(t,\lambda,\varepsilon),\end{aligned} \tag{4.56}$$

$$\widetilde{D}_k(t,\lambda) = \mathscr{R}_k^T(t,\lambda)\widetilde{\mathscr{D}}(t)\mathscr{R}_k(t,\lambda), \tag{4.57}$$

$$\widetilde{H}_k(\lambda)=\mathscr{R}_k^T(t_f,\lambda)\widetilde{\mathscr{H}}\mathscr{R}_k(t_f,\lambda)=\begin{pmatrix}\mathscr{L}_k^T(t_f,\lambda)\\ \mathscr{B}_k^T(t_f)\end{pmatrix}\widetilde{\mathscr{H}}\Big(\mathscr{L}_k(t_f,\lambda)\ ,\ \mathscr{B}_k(t_f)\Big)$$

$$=\begin{pmatrix}\mathscr{L}_k^T(t_f,\lambda)\widetilde{\mathscr{H}}\mathscr{L}_k(t_f,\lambda) & \mathscr{L}_k^T(t_f,\lambda)\widetilde{\mathscr{H}}\mathscr{B}_k(t_f)\\ \mathscr{B}_k^T(t_f)\widetilde{\mathscr{H}}\mathscr{L}_k(t_f,\lambda) & \mathscr{B}_k^T(t_f)\widetilde{\mathscr{H}}\mathscr{B}_k(t_f)\end{pmatrix}=\begin{pmatrix}\widetilde{H}_{k,1}(\lambda) & O_{(Kn-r)\times r}\\ O_{r\times(Kn-r)} & O_{r\times r}\end{pmatrix},$$

$$\widetilde{H}_{k,1}(\lambda)=\mathscr{L}_k^T(t_f,\lambda)\widetilde{\mathscr{H}}\mathscr{L}_k(t_f,\lambda). \tag{4.58}$$

Using the transformation (4.54) and the equation (4.15), we can rewrite the equation (4.48) as follows:

$$\widetilde{\mathscr{F}_\varepsilon}(k,\lambda)=\big(z^0(\lambda)\big)^T\widetilde{M}_k(0,\lambda,\varepsilon)z^0(\lambda)+\sum_{i=1}^{m}\int_0^{t_f}\Phi_i^T(t,\lambda)\widetilde{M}_k(t,\lambda,\varepsilon)\Phi_i(t,\lambda)dt,\ k=1,...,K, \tag{4.59}$$

where, for all $\lambda\in\Omega_\lambda$ and $\varepsilon>0$, $\widetilde{M}_k(t,\lambda,\varepsilon)$, $t\in[0,t_f]$ is the unique solution of the terminal-value problem (4.55).

Let us analyze the asymptotic behaviour (for $\varepsilon\to+0$) of the value $\widetilde{\mathscr{F}_\varepsilon}(k,\lambda)$ given by the equation (4.59). For this purpose, first, we are going to construct the zero-order asymptotic solution to the terminal-value problem (4.55).

4.4.1. *Zero-Order Asymptotic Solution to the Terminal-Value Problem (4.55).* Similarly to the block form (4.17) of the solution to the terminal-value problem (4.5), we look for the solution to the problem (4.55) in the following block form:

$$\widetilde{M}_k(t,\lambda,\varepsilon)=\begin{pmatrix}\widetilde{M}_{k,1}(t,\lambda,\varepsilon) & \varepsilon\widetilde{M}_{k,2}(t,\lambda,\varepsilon)\\ \varepsilon\widetilde{M}_{k,2}^T(t,\lambda,\varepsilon) & \varepsilon\widetilde{M}_{k,3}(t,\lambda,\varepsilon)\end{pmatrix}, \tag{4.60}$$

where the matrices $\widetilde{M}_{k,1}(t,\lambda,\varepsilon)$, $\widetilde{M}_{k,2}(t,\lambda,\varepsilon)$ and $\widetilde{M}_{k,3}(t,\lambda,\varepsilon)$ are of the dimensions $(Kn-r)\times(Kn-r)$, $(Kn-r)\times r$ and $r\times r$, respectively; $\widetilde{M}_{k,1}^T(t,\lambda,\varepsilon)=\widetilde{M}_{k,1}(t,\lambda,\varepsilon)$, $\widetilde{M}_{k,3}^T(t,\lambda,\varepsilon)=\widetilde{M}_{k,3}(t,\lambda,\varepsilon)$.

Taking into account the symmetry of the matrix $\widetilde{D}_k(t,\lambda)$, let us partition it into blocks as:

$$\widetilde{D}_k(t,\lambda)=\begin{pmatrix}\widetilde{D}_{k,1}(t,\lambda) & \widetilde{D}_{k,2}(t,\lambda)\\ \widetilde{D}_{k,2}^T(t,\lambda) & \widetilde{D}_{k,3}(t,\lambda)\end{pmatrix}, \tag{4.61}$$

where the matrices $\widetilde{D}_{k,1}(t,\lambda)$, $\widetilde{D}_{k,2}(t,\lambda)$ and $\widetilde{D}_{k,3}(t,\lambda)$ are of the dimensions $(Kn-r)\times(Kn-r)$, $(Kn-r)\times r$ and $r\times r$, respectively; $\widetilde{D}_{k,1}^T(t,\lambda)=\widetilde{D}_{k,1}(t,\lambda)$, $\widetilde{D}_{k,3}^T(t,\lambda)=\widetilde{D}_{k,3}(t,\lambda)$.

Substitution of the block forms of the matrices $S(\varepsilon)$, $M(t,\lambda,\varepsilon)$, $A(t,\lambda)$, $\widetilde{H}_k(\lambda)$, $\widetilde{M}(t,\lambda,\varepsilon)$, $\widetilde{D}_k(t,\lambda)$ (see the equations (4.8),(4.17),(4.18),(4.58),(4.60),(4.61)) into the problem (4.55) yields after a routine matrix algebra the following equivalent terminal-value problem in the time interval $[0,t_f]$:

$$\frac{d\widetilde{M}_{k,1}(t,\lambda,\varepsilon)}{dt} = -\widetilde{M}_{k,1}(t,\lambda,\varepsilon)A_1(t,\lambda) - \varepsilon\widetilde{M}_{k,2}(t,\lambda,\varepsilon)A_3(t,\lambda) - A_1^T(t,\lambda)\widetilde{M}_{k,1}(t,\lambda,\varepsilon) - \varepsilon A_3^T(t,\lambda)\widetilde{M}_{k,2}^T(t,\lambda,\varepsilon) + \widetilde{M}_{k,2}(t,\lambda,\varepsilon)M_2^T(t,\lambda,\varepsilon) + M_2(t,\lambda,\varepsilon)\widetilde{M}_{k,2}^T(t,\lambda,\varepsilon) - \widetilde{D}_{k,1}(t,\lambda) - M_2(t,\lambda,\varepsilon)M_2^T(t,\lambda,\varepsilon), \quad \widetilde{M}_{k,1}(t_f,\lambda,\varepsilon) = \widetilde{H}_{k,1}(\lambda), \tag{4.62}$$

$$\varepsilon\frac{d\widetilde{M}_{k,2}(t,\lambda,\varepsilon)}{dt} = -\widetilde{M}_{k,1}(t,\lambda,\varepsilon)A_2(t,\lambda) - \varepsilon\widetilde{M}_{k,2}(t,\lambda,\varepsilon)A_4(t,\lambda) - \varepsilon A_1^T(t,\lambda)\widetilde{M}_{k,2}(t,\lambda,\varepsilon) - \varepsilon A_3^T(t,\lambda)\widetilde{M}_{k,3}(t,\lambda,\varepsilon) + \widetilde{M}_{k,2}(t,\lambda,\varepsilon)M_3(t,\lambda,\varepsilon) + M_2(t,\lambda,\varepsilon)\widetilde{M}_{k,3}(t,\lambda,\varepsilon) - \widetilde{D}_{k,2}(t,\lambda) - M_2(t,\lambda,\varepsilon)M_3(t,\lambda,\varepsilon), \quad \widetilde{M}_{k,2}(t_f,\lambda,\varepsilon) = 0, \tag{4.63}$$

$$\varepsilon\frac{d\widetilde{M}_{k,3}(t,\lambda,\varepsilon)}{dt} = -\varepsilon\widetilde{M}_{k,2}^T(t,\lambda,\varepsilon)A_2(t,\lambda) - \varepsilon\widetilde{M}_{k,3}(t,\lambda,\varepsilon)A_4(t,\lambda) - \varepsilon A_2^T(t,\lambda)\widetilde{M}_{k,2}(t,\lambda,\varepsilon) - \varepsilon A_4^T(t,\lambda)\widetilde{M}_{k,3}(t,\lambda,\varepsilon) + \widetilde{M}_{k,3}(t,\lambda,\varepsilon)M_3(t,\lambda,\varepsilon) + M_3(t,\lambda,\varepsilon)\widetilde{M}_{k,3}(t,\lambda,\varepsilon) - \widetilde{D}_{k,3}(t,\lambda) - \big(M_3(t,\lambda,\varepsilon)\big)^2, \quad \widetilde{M}_{k,3}(t_f,\lambda,\varepsilon) = 0. \tag{4.64}$$

The problem (4.62)-(4.64) is a singularly perturbed linear terminal-value problem. In what follows of this subsection, based on the Boundary Function Method (see, e.g., [64]), we construct and justify the zero-order asymptotic solution of this problem. We seek this asymptotic solution in the form

$$\widetilde{M}_{k,j0}(t,\lambda,\varepsilon) = \widetilde{M}_{k,j0}^o(t,\lambda) + \widetilde{M}_{k,j0}^b(\tau,\lambda), \quad j=1,2,3, \quad \tau = (t-t_f)/\varepsilon, \tag{4.65}$$

where (like in (4.22)) the terms with the upper index "o" constitute the outer solution, while the terms with the upper index "b" are the boundary correction terms in a left-hand neighbourhood of $t = t_f$; $\tau \le 0$ is a new independent variable, called the stretched time. For any $t \in [0,t_f)$, $\tau \to -\infty$ as $\varepsilon \to +0$. Similarly to Section 4.2, equations and conditions for obtaining the outer solution and the boundary correction terms are derived by substituting the representations (4.65) and (4.22) into the terminal-value problem (4.62)-(4.64) instead of $\widetilde{M}_{k,j}(t,\lambda,\varepsilon)$ and $M_j(t,\lambda,\varepsilon)$, $(j=1,2,3)$, respectively, and equating the coefficients for the same power of ε on both sides of the resulting equations, separately the coefficients depending on t and on τ.

We start with the obtaining the boundary correction $\widetilde{M}_{k,10}^b(\tau)$. For this boundary correction, we have the equation

$$\frac{d\widetilde{M}_{k,10}^b(\tau,\lambda)}{d\tau} = 0, \quad \tau \le 0, \quad \lambda \in \Omega_\lambda. \tag{4.66}$$

By virtue of the Boundary Function Method, we require (like in Section 4.2) that the boundary layer correction terms tend to zero for τ tending to $-\infty$, i.e., for $M_{k,10}^b(\tau,\lambda)$ we require that

$$\lim_{\tau\to-\infty}\widetilde{M}_{k,10}^b(\tau,\lambda) = 0. \tag{4.67}$$

Moreover, we require that the limit (4.67) is uniform with respect to $\lambda \in \Omega_\lambda$.

From the equations (4.66)-(4.67), we directly have

$$\widetilde{M}_{k,10}^b(\tau,\lambda) \equiv 0, \quad \tau \le 0, \quad \lambda \in \Omega_\lambda. \tag{4.68}$$

Proceed to obtaining the outer solution terms. The equations and conditions for these terms are the following for all $t \in [0, t_f]$ and $\lambda \in \Omega_\lambda$:

$$\begin{aligned}\frac{d\widetilde{M}^o_{k,10}(t,\lambda)}{dt} &= -\widetilde{M}^o_{k,10}(t,\lambda)A_1(t,\lambda) - A_1^T(t,\lambda)\widetilde{M}^o_{k,10}(t,\lambda) \\ &+ \widetilde{M}^o_{k,20}(t,\lambda)\big(M^o_{20}(t,\lambda)\big)^T + M^o_{20}(t,\lambda)\big(\widetilde{M}^o_{k,20}(t,\lambda)\big)^T \\ -\widetilde{D}_{k,1}(t,\lambda) - M^o_{20}(t,\lambda)\big(M^o_{20}(t,\lambda)\big)^T, &\quad \widetilde{M}^o_{k,10}(t_f,\lambda) = \widetilde{H}_{k,1}(\lambda),\end{aligned} \tag{4.69}$$

$$\begin{aligned}-\widetilde{M}^o_{k,10}(t,\lambda)A_2(t,\lambda) + \widetilde{M}^o_{k,20}(t,\lambda)M^o_{30}(t,\lambda) + M^o_{20}(t,\lambda)\widetilde{M}^o_{k,30}(t,\lambda) \\ -\widetilde{D}_{k,2}(t,\lambda) - M^o_{20}(t,\lambda)M^o_{30}(t,\lambda) = 0,\end{aligned} \tag{4.70}$$

$$\widetilde{M}^o_{k,30}(t,\lambda)M^o_{30}(t,\lambda) + M^o_{30}(t,\lambda)\widetilde{M}^o_{k,30}(t,\lambda) - \widetilde{D}_{k,3}(t,\lambda) - \big(M^o_{30}(t,\lambda)\big)^2 = 0. \tag{4.71}$$

Remark 4.13. Similarly to Remark 4.7, the unknown matrix-valued functions $\widetilde{M}^o_{k,20}(t,\lambda)$ and $\widetilde{M}^o_{k,30}(t,\lambda)$ are not subject to any terminal conditions in the system (4.69)-(4.71) because in this system $\widetilde{M}^o_{k,20}(t,\lambda)$ and $\widetilde{M}^o_{k,30}(t,\lambda)$ are subject to the algebraic (but not differential) equations.

Solving the Lyapunov algebraic equation (4.71) and taking into account the symmetry and positive definiteness of the matrix $M^o_{30}(t,\lambda) = \big(D_2(t,\lambda)\big)^{1/2}$, we obtain by virtue of [16]

$$\widetilde{M}^o_{k,30}(t,\lambda) = \int_0^{+\infty} \exp\Big(-\big(D_2(t,\lambda)\big)^{1/2}\sigma\Big)\big[\widetilde{D}_{k,3}(t,\lambda) + D_2(t,\lambda)\big]\exp\Big(-\big(D_2(t,\lambda)\big)^{1/2}\sigma\Big)d\sigma, \tag{4.72}$$

where $t \in [0, t_f]$, $\lambda \in \Omega_\lambda$.

Remark 4.14. Due to Proposition 4.3, Remark 4.8 and the theorem on continuity of an improper integral with respect to a parameter [12, 60], the matrix-valued function $\widetilde{M}^o_{k,30}(t,\lambda)$ is continuously differentiable with respect to $t \in [0, t_f]$ uniformly in $\lambda \in \Omega_\lambda$, and $\big\|dM^o_{k,30}(t,\lambda)/dt\big\|$ is bounded for all $(t,\lambda) \in [0,t_f] \times \Omega_\lambda$. Moreover, since $D_2(t,\lambda)$ is continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$, then $\widetilde{M}^o_{k,30}(t,\lambda)$ also is continuous with respect to $\lambda \in \Omega_\lambda$ uniformly in $t \in [0,t_f]$. In addition, since the matrix $\widetilde{D}_{k,3}(t,\lambda)$ is positive semi-definite and the matrix $D_2(t,\lambda)$ is positive definite for all $(t,\lambda) \in [0,t_f] \times \Omega_\lambda$, then the matrix $\widetilde{M}^o_{k,30}(t,\lambda)$ is positive definite for all these (t,λ).

Solving the equation (4.70) with respect to $\widetilde{M}^o_{k,20}(t,\lambda)$, we obtain

$$\begin{aligned}\widetilde{M}^o_{k,20}(t,\lambda) = \big[\widetilde{M}^o_{k,10}(t,\lambda)A_2(t,\lambda) - M^o_{20}(t,\lambda)\widetilde{M}^o_{k,30}(t,\lambda) + \widetilde{D}_{k,2}(t,\lambda)\big]\big(M^o_{30}(t,\lambda)\big)^{-1} \\ + M^o_{20}(t,\lambda), \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda.\end{aligned} \tag{4.73}$$

Substituting (4.73) into (4.69) and taking into account the equations (4.29),(4.30),(4.32),(4.71), we obtain after a routine rearrangement the following terminal-value problem for the linear differential equation with respect to $\widetilde{M}^o_{k,10}(t,\lambda)$:

$$\frac{d\widetilde{M}^o_{k,10}(t,\lambda)}{dt} = -\widetilde{M}^o_{k,10}(t,\lambda)\big(A_1(t,\lambda) - S^o_1(t,\lambda)M^o_{10}(t,\lambda)\big)$$
$$-\big(A_1(t,\lambda) - S^o_1(t,\lambda)M^o_{10}(t,\lambda)\big)^T\widetilde{M}^o_{k,10}(t,\lambda)$$
$$-\Big(I_{Kn-r}\ ,\ -M^o_{10}(t,\lambda)A_2(t,\lambda)D_2^{-1}(t,\lambda)\Big)\widetilde{D}_k(t,\lambda)\begin{pmatrix} I_{Kn-r} \\ -D_2^{-1}(t,\lambda)A_2^T(t,\lambda)M^o_{10}(t,\lambda)\end{pmatrix},$$
$$t\in[0,t_f],\quad \widetilde{M}^o_{k,10}(t_f,\lambda) = \widetilde{H}_{k,1}(\lambda), \tag{4.74}$$

where $M^o_{10}(t,\lambda)$ is the solution of the terminal-value problem (4.31); $\lambda\in\Omega_\lambda$.

Remark 4.15. Since the differential equation in (4.74) is linear and, for any $\lambda\in\Omega_\lambda$, the coefficients of this equation are continuous functions with respect to $t\in[0,t_f]$, then the terminal-value problem (4.74) has the unique solution $\widetilde{M}^o_{k,10}(t,\lambda)$ in the entire interval $[0,t_f]$ for any $\lambda\in\Omega_\lambda$. Moreover, $\|\widetilde{M}^o_{k,10}(t,\lambda)\|$ and $\|d\widetilde{M}^o_{k,10}(t,\lambda)/dt\|$ are bounded for all $(t,\lambda)\in[0,t_f]\times\Omega_\lambda$. Therefore, due to Remark 4.14 and the equation (4.73), $\|\widetilde{M}^o_{k,20}(t,\lambda)\|$ and $\|d\widetilde{M}^o_{k,20}(t,\lambda)/dt\|$ are bounded for all $(t,\lambda)\in[0,t_f]\times\Omega_\lambda$. In addition, since the coefficients of the differential equation in (4.74) are continuous with respect to $\lambda\in\Omega_\lambda$ uniformly in $t\in[0,t_f]$ and $\widetilde{H}_{k,1}(\lambda)$ is continuous with respect to $\lambda\in\Omega_\lambda$ then, by virtue of the results of [35] (Chapter 5), $\widetilde{M}^o_{k,10}(t,\lambda)$ also is continuous with respect to $\lambda\in\Omega_\lambda$ uniformly in $t\in[0,t_f]$. Therefore, due to Remark 4.14 and the equation (4.73), $\widetilde{M}^o_{k,20}(t,\lambda)$ also is continuous with respect to $\lambda\in\Omega_\lambda$ uniformly in $t\in[0,t_f]$.

Now let us proceed to the obtaining the boundary correction terms $\widetilde{M}^b_{k,20}(\tau,\lambda)$ and $\widetilde{M}^b_{k,30}(\tau,\lambda)$. Using the system of the differential equations in (4.33), we obtain the terminal-value problem for $\widetilde{M}^b_{k,20}(\tau,\lambda)$ and $\widetilde{M}^b_{k,30}(\tau,\lambda)$

$$\frac{d\widetilde{M}^b_{k,20}(\tau,\lambda)}{d\tau} = \widetilde{M}^b_{k,20}(\tau,\lambda)\big(M^o_{30}(t_f,\lambda)+M^b_{30}(\tau,\lambda)\big)$$
$$+\big(M^o_{20}(t_f,\lambda)+M^b_{20}(\tau,\lambda)\big)\widetilde{M}^b_{k,30}(\tau,\lambda)+\widetilde{M}^o_{k,20}(t_f,\lambda)M^b_{30}(\tau,\lambda)+M^b_{20}(\tau,\lambda)\widetilde{M}^o_{k,30}(t_f,\lambda)$$
$$-\frac{dM^b_{20}(\tau,\lambda)}{d\tau},\quad \tau\le 0,\quad \widetilde{M}^b_{k,20}(0,\lambda) = -\widetilde{M}^o_{k,20}(t_f,\lambda), \tag{4.75}$$

$$\frac{d\widetilde{M}^b_{k,30}(\tau,\lambda)}{d\tau} = \widetilde{M}^b_{k,30}(\tau,\lambda)\big(M^o_{30}(t_f,\lambda)+M^b_{30}(\tau,\lambda)\big)$$
$$+\big(M^o_{30}(t_f,\lambda)+M^b_{30}(\tau,\lambda)\big)\widetilde{M}^b_{k,30}(\tau,\lambda)+\widetilde{M}^o_{k,30}(t_f,\lambda)M^b_{30}(\tau,\lambda)+M^b_{30}(\tau,\lambda)\widetilde{M}^o_{k,30}(t_f,\lambda)$$
$$-\frac{dM^b_{30}(\tau,\lambda)}{d\tau},\quad \tau\le 0,\quad \widetilde{M}^b_{k,30}(0,\lambda) = -\widetilde{M}^o_{k,30}(t_f,\lambda). \tag{4.76}$$

Since the differential equations in (4.75) and (4.76) are linear with continuous in $\tau \in (-\infty, 0]$ coefficients for all $\lambda \in \Omega_\lambda$, then the terminal-value problem (4.75)-(4.76) has the unique solution $\{\widetilde{M}_{k,20}^b(\tau,\lambda), \widetilde{M}_{k,30}^b(\tau,\lambda)\}$ in the entire interval $(-\infty, 0]$ for all $\lambda \in \Omega_\lambda$. In what follows, we are going to estimate this solution.

First of all let us observe that, due to the equation (4.33), the inequalities in (4.38) and Remarks 4.8, 4.9, the following inequalities are valid:

$$\left\| \frac{dM_{20}^b(\tau,\lambda)}{d\tau} \right\| \le a_1 \exp(\beta\tau), \quad \left\| \frac{dM_{30}^b(\tau,\lambda)}{d\tau} \right\| \le a_1 \exp(\beta\tau), \quad \tau \le 0, \quad \lambda \in \Omega_\lambda, \quad (4.77)$$

where $a_1 > 0$ is some constant independent of $\lambda \in \Omega_\lambda$; the positive constant β is introduced in (4.38).

Let us estimate, first, the component $\widetilde{M}_{k,30}^b(\tau,\lambda)$ of the solution to the problem (4.75)-(4.76).

By virtue of the results of [1], the solution to the problem (4.76) can be represented in the form

$$\begin{aligned} \widetilde{M}_{k,30}^b(\tau,\lambda) &= -\Psi^T(0,\tau,\lambda)\widetilde{M}_{k,30}^o(t_f,\lambda)\Psi(0,\tau,\lambda) \\ &+ \int_0^\tau \Psi^T(\sigma,\tau,\lambda)\Big[\widetilde{M}_{k,30}^o(t_f,\lambda)M_{30}^b(\sigma,\lambda) + M_{30}^b(\sigma,\lambda)\widetilde{M}_{k,30}^o(t_f,\lambda) \\ &- \frac{dM_{30}^b(\sigma,\lambda)}{d\sigma}\Big]\Psi(\sigma,\tau,\lambda)d\sigma, \quad \tau \in (-\infty,0], \quad \lambda \in \Omega_\lambda, \end{aligned} \quad (4.78)$$

where, for any given $\tau \in (-\infty, 0]$ and $\lambda \in \Omega_\lambda$, the $r \times r$-matrix-valued function $\Psi(\sigma,\tau,\lambda)$ is the unique solution of the problem

$$\frac{d\Psi(\sigma,\tau,\lambda)}{d\sigma} = -\big(M_{30}^o(t_f,\lambda) + M_{30}^b(\tau,\lambda)\big)\Psi(\sigma,\tau,\lambda), \quad \sigma \in [\tau,0], \quad \Phi(\tau,\tau,\lambda) = I_r. \quad (4.79)$$

Based on the results of [19] (Lemma 4.2) and taking into account Proposition 4.3, the equation (4.29), the inequalities in (4.38) and Remarks 4.8, 4.9, we have immediately the following estimate of $\Psi(\sigma,\tau,\lambda)$ for all $-\infty \le \tau \le \sigma \le 0$ and all $\lambda \in \Omega_\lambda$:

$$\big\|\Psi(\sigma,\tau,\lambda)\big\| \le a_2 \exp\big(-0.5\beta(\sigma-\tau)\big), \quad (4.80)$$

where $a_2 > 0$ is some constant independent of $\lambda \in \Omega_\lambda$; the positive constant β is introduced in (4.38).

Now, using the second inequalities in (4.38) and in (4.77), as well as the inequality (4.80), we directly obtain the estimate of $\widetilde{M}_{k,30}^b(\tau,\lambda)$ given in (4.89)

$$\big\|\widetilde{M}_{k,30}^b(\tau,\lambda)\big\| \le b_1 \exp(0.5\beta\tau), \quad \tau \in (-\infty,0], \quad \lambda \in \Omega_\lambda, \quad (4.81)$$

where $b_1 > 0$ is some constant independent of $\lambda \in \Omega_\lambda$.

Similarly to (4.81), taking into account this inequality, we estimate the component $\widetilde{M}_{k,20}^b(\tau,\lambda)$ of the solution to the terminal-value problem (4.75)-(4.76)

$$\big\|\widetilde{M}_{k,20}^b(\tau,\lambda)\big\| \le b_2 \exp(0.25\beta\tau), \quad \tau \in (-\infty,0], \quad \lambda \in \Omega_\lambda, \quad (4.82)$$

where $b_2 > 0$ is some constant independent of $\lambda \in \Omega_\lambda$.

Lemma 4.16. *Let the assumptions AI-AV be fulfilled. Then, there exists a number* $\widetilde{\varepsilon}_0 > 0$ *independent of* $\lambda \in \Omega_\lambda$ *such that, for all* $\varepsilon \in (0, \widetilde{\varepsilon}_0]$*, the entries of the solution to the terminal-value problem (4.62)-(4.64)* $\{\widetilde{M}_{k,1}(t,\lambda,\varepsilon), \widetilde{M}_{k,2}(t,\lambda,\varepsilon), \widetilde{M}_{k,3}(t,\lambda,\varepsilon)\}$ *satisfy the inequalities*

$$\|\widetilde{M}_{k,1}(t,\lambda,\varepsilon) - \widetilde{M}^o_{k,10}(t,\lambda)\| \le \widetilde{c}\varepsilon, \quad \|\widetilde{M}_{k,j}(t,\lambda,\varepsilon) - \widetilde{M}_{k,j0}(t,\lambda,\varepsilon)\| \le \widetilde{c}\varepsilon,$$
$$j = 2,3, \quad t \in [0,t_f], \quad \lambda \in \Omega_\lambda, \tag{4.83}$$

where $\widetilde{M}_{k,j0}(t,\lambda,\varepsilon)$, $(j = 2,3)$ *are given in (4.65);* $\widetilde{c} > 0$ *is some constant independent of* ε *and* $\lambda \in \Omega_\lambda$.

Proof. Let us make the following transformation of the variables in the problem (4.62)-(4.64):

$$\widetilde{M}_{k,1}(t,\lambda,\varepsilon) = \widetilde{M}^o_{k,10}(t,\lambda) + \widetilde{\Delta}_1(t,\lambda,\varepsilon),\ \widetilde{M}_{k,j}(t,\lambda,\varepsilon) = \widetilde{M}_{k,j0}(t,\lambda,\varepsilon) + \widetilde{\Delta}_j(t,\lambda,\varepsilon),\ j = 2,3, \tag{4.84}$$

where $\widetilde{\Delta}_j(t,\lambda,\varepsilon)$, $(j = 1,2,3)$ are new unknown matrix-valued functions; $\widetilde{\Delta}_1^T(t,\lambda,\varepsilon) = \widetilde{\Delta}_1(t,\lambda,\varepsilon)$, $\widetilde{\Delta}_3^T(t,\lambda,\varepsilon) = \widetilde{\Delta}_3(t,\lambda,\varepsilon)$.

Using the above introduced new unknown matrix-valued functions, we construct the following block-form matrix-valued function:

$$\widetilde{\Delta}(t,\lambda,\varepsilon) \stackrel{\triangle}{=} \begin{pmatrix} \widetilde{\Delta}_1(t,\lambda,\varepsilon) & \varepsilon\widetilde{\Delta}_2(t,\lambda,\varepsilon) \\ \varepsilon\widetilde{\Delta}_2^T(t,\lambda,\varepsilon) & \varepsilon\widetilde{\Delta}_3(t,\lambda,\varepsilon) \end{pmatrix}. \tag{4.85}$$

Now, let us substitute the representation (4.84) into the problem (4.62)-(4.64). Due to this substitution and the use of the equations (4.68),(4.69)-(4.71),(4.75)-(4.76), as well as the block representations of the matrices $S(\varepsilon)$, $M(t,\lambda,\varepsilon)$, $A(t,\lambda,\varepsilon)$, $\widetilde{H}_k(\lambda)$, $\widetilde{M}(t,\lambda,\varepsilon)$, $\widetilde{D}_k(t,\lambda)$ (see the equations (4.8),(4.17),(4.18),(4.58),(4.60),(4.61)), we obtain after a routine matrix algebra the terminal-value problem for $\widetilde{\Delta}(t,\lambda,\varepsilon)$

$$\frac{d\widetilde{\Delta}(t,\lambda,\varepsilon)}{dt} = -\widetilde{\Delta}(t,\lambda,\varepsilon)\widetilde{A}(t,\lambda,\varepsilon) - \widetilde{A}^T(t,\lambda,\varepsilon)\widetilde{\Delta}(t,\lambda,\varepsilon)$$
$$+\widetilde{\Gamma}(t,\lambda,\varepsilon), \quad t \in [0,t_f], \quad \widetilde{\Delta}(t_f,\lambda,\varepsilon) = 0, \tag{4.86}$$

where $\lambda \in \Omega_\lambda$; the matrix-valued function $\widetilde{\Gamma}(t,\lambda,\varepsilon)$ has the block form

$$\widetilde{\Gamma}(t,\lambda,\varepsilon) = \begin{pmatrix} \widetilde{\Gamma}_1(t,\lambda,\varepsilon) & \widetilde{\Gamma}_2(t,\lambda,\varepsilon) \\ \widetilde{\Gamma}_2^T(t,\lambda,\varepsilon) & \widetilde{\Gamma}_3(t,\lambda,\varepsilon) \end{pmatrix}. \tag{4.87}$$

The blocks $\widetilde{\Gamma}_j(t,\lambda,\varepsilon)$, $(j = 1,2,3)$ are expressed in a known form by the matrix-valued functions $M_{j0}(t,\lambda,\varepsilon)$, $\widetilde{M}_{k,j0}(t,\lambda,\varepsilon)$, $(j = 1,2,3)$. Moreover, using Lemma 4.10 and the estimates (4.38), (4.81)-(4.82), the following inequalities are obtained for $\widetilde{\Gamma}_j(t,\lambda,\varepsilon)$, $(j = 1,2,3)$:

$$\|\widetilde{\Gamma}_1(t,\lambda,\varepsilon)\| \le \widetilde{b}_1[\varepsilon + \exp(\widetilde{\beta}_1\tau)], \quad \|\widetilde{\Gamma}_l(t,\lambda,\varepsilon)\| \le \widetilde{b}_1\varepsilon\big(1 + \exp(\widetilde{\beta}_1\tau)\big),\ l = 2,3,$$
$$\tau = (t - t_f)/\varepsilon, \quad t \in [0,t_f], \quad \varepsilon \in (0,\varepsilon_0], \quad \lambda \in \Omega_\lambda, \tag{4.88}$$

where $\widetilde{b}_1 > 0$ and $\widetilde{\beta}_1 > 0$ are some constants independent of ε and $\lambda \in \Omega_\lambda$.

By virtue of the results of [1], the solution of the problem (4.86) can be represented in the form

$$\widetilde{\Delta}(t,\lambda,\varepsilon)=\int_{t_f}^{t}\widetilde{\Upsilon}^T(\sigma,t,\lambda,\varepsilon)\widetilde{\Gamma}(\sigma,\lambda,\varepsilon)\widetilde{\Upsilon}(\sigma,t,\lambda,\varepsilon)d\sigma,\ t\in[0,t_f],\ \lambda\in\Omega_\lambda,\ \varepsilon\in(0,\varepsilon_0],\quad(4.89)$$

where, for any given $t\in[0,t_f]$, $\lambda\in\Omega_\lambda$ and $\varepsilon\in(0,\varepsilon_0]$, the $Kn\times Kn$-matrix-valued function $\widetilde{\Upsilon}(\sigma,t,\lambda,\varepsilon)$ is the unique solution of the problem

$$\frac{d\widetilde{\Upsilon}(\sigma,t,\lambda,\varepsilon)}{d\sigma}=\widetilde{A}(\sigma,\lambda,\varepsilon)\widetilde{\Upsilon}(\sigma,t,\lambda,\varepsilon),\quad \widetilde{\Upsilon}(t,t,\lambda,\varepsilon)=I_{Kn},\quad \sigma\in[t,t_f].$$

By $\widetilde{\Upsilon}_1(\sigma,t,\lambda,\varepsilon)$, $\widetilde{\Upsilon}_2(\sigma,t,\lambda,\varepsilon)$, $\widetilde{\Upsilon}_3(\sigma,t,\lambda,\varepsilon)$ and $\widetilde{\Upsilon}_4(\sigma,t,\lambda,\varepsilon)$, we denote the upper left-hand, upper right-hand, lower left-hand and lower right-hand blocks of the matrix $widetilde\Upsilon(\sigma,t,\lambda,\varepsilon)$ of the dimensions $(Kn-r)\times(Kn-r)$, $(Kn-r)\times r$, $r\times(Kn-r)$ and $r\times r$, respectively, i.e.,

$$\widetilde{\Upsilon}(\sigma,t,\lambda,\varepsilon)=\begin{pmatrix}\widetilde{\Upsilon}_1(\sigma,t,\lambda,\varepsilon) & \widetilde{\Upsilon}_2(\sigma,t,\lambda,\varepsilon)\\ \widetilde{\Upsilon}_3(\sigma,t,\lambda,\varepsilon) & \widetilde{\Upsilon}_4(\sigma,t,\lambda,\varepsilon)\end{pmatrix}.\quad(4.90)$$

Based on the results of [19] (Lemma 4.2) and taking into account Proposition 4.3, Lemma 4.10, the equation (4.29), the inequalities in (4.38) and Remarks 4.8, 4.9, we have immediately the following estimates of these blocks for all $0\le t\le\sigma\le t_f$ and all $\lambda\in\Omega_\lambda$:

$$\begin{gathered}\big\|\widetilde{\Upsilon}_l(\sigma,t,\lambda,\varepsilon)\big\|\le\widetilde{b}_2,\quad l=1,3,\quad \big\|\widetilde{\Upsilon}_2(\sigma,t,\lambda,\varepsilon)\big\|\le\widetilde{b}_2\varepsilon,\\ \big\|\widetilde{\Upsilon}_4(\sigma,t,\lambda,\varepsilon)\big\|\le\widetilde{b}_2\big[\varepsilon+\exp\big(-0.5\beta(\sigma-t)/\varepsilon\big)\big],\quad \varepsilon\in(0,\varepsilon_1],\end{gathered}\quad(4.91)$$

where $\varepsilon_1>0$ is some sufficiently small number; $\widetilde{b}_2>0$ is some constant independent of ε and $\lambda\in\Omega_\lambda$; the positive constant β was introduced in (4.38).

Using the block representations of the matrices $\widetilde{\Delta}(t,\lambda,\varepsilon)$, $\widetilde{\Gamma}(t,\lambda,\varepsilon)$, $\Upsilon(\sigma,t,\lambda,\varepsilon)$ (see the equations (4.85),(4.87),(4.90)), as well as using the inequalities (4.88),(4.91) and their uniformity with respect to $\lambda\in\Omega_\lambda$ and $\varepsilon\in\big(0,\min\{\varepsilon_0,\varepsilon_1\}\big]$, we obtain the following inequalities:

$$\|\widetilde{\Delta}_j(t,\lambda,\varepsilon)\|\le\widetilde{c}\varepsilon,\quad j=1,2,3,\quad t\in[0,t_f],\quad \lambda\in\Omega_\lambda,\quad \varepsilon\in(0,\widetilde{\varepsilon}_0],\quad(4.92)$$

where $\widetilde{\varepsilon}_0=\min\{\varepsilon_0,\varepsilon_1\}$; $\widetilde{c}>0$ is some constant independent of λ, ε and j.

Using the equation (4.84) and the inequalities in (4.92), we directly obtain the inequalities in (4.83). This completes the proof of the lemma. □

4.4.2. *Asymptotic Behaviour of the value* $\widetilde{\mathscr{F}_\varepsilon}(k,\lambda)$. Let us partition vectors $z^0(\lambda)$ and $\Phi_i(t,\lambda)$, $(i=1,...,m)$ into blocks as:

$$\begin{gathered}z^0(\lambda)=\mathrm{col}(x^0(\lambda),y^0(\lambda)),\\ \Phi_i(t,\lambda)=\mathrm{col}\big(\Phi_{i,x}(t,\lambda),\Phi_{i,y}(t,\lambda)\big),\quad i=1,...,m,\quad t\in[0,t_f],\quad \lambda\in\Omega_\lambda,\end{gathered}\quad(4.93)$$

where $x^0(\lambda)\in E^{Kn-r}$, $y^0(\lambda)\in E^r$, $\Phi_{i,x}(t,\lambda)\in E^{Kn-r}$, $\Phi_{i,y}(t,\lambda)\in E^r$.

Substituting the block representations of the matrix $\widetilde{M}_k(t,\lambda,\varepsilon)$ and of the vectors $z^0(\lambda)$ and $\Phi_i(t,\lambda)$, $(i=1,...,m)$ (see the equations (4.60) and (4.93)) into the expression for $\widetilde{\mathscr{F}_\varepsilon}(k,\lambda)$ (see the equation (4.59)), we obtain

$$\widetilde{\mathscr{F}_\varepsilon}(k,\lambda)=\big((x^0(\lambda))^T,(y^0(\lambda))^T\big)\begin{pmatrix}\widetilde{M}_{k,1}(0,\lambda,\varepsilon) & \varepsilon\widetilde{M}_{k,2}(0,\lambda,\varepsilon)\\ \\ \varepsilon\widetilde{M}^T_{k,2}(0,\lambda,\varepsilon) & \varepsilon\widetilde{M}_{k,3}(0,\lambda,\varepsilon)\end{pmatrix}\begin{pmatrix}x^0(\lambda)\\ y^0(\lambda)\end{pmatrix}$$
$$+\sum_{i=1}^{m}\int_0^{t_f}\big(\Phi^T_{i,x}(t,\lambda),\Phi^T_{i,y}(t,\lambda)\big)\begin{pmatrix}\widetilde{M}_{k,1}(t,\lambda,\varepsilon) & \varepsilon\widetilde{M}_{k,2}(t,\lambda,\varepsilon)\\ \\ \varepsilon\widetilde{M}^T_{k,2}(t,\lambda,\varepsilon) & \varepsilon\widetilde{M}_{k,3}(t,\lambda,\varepsilon)\end{pmatrix}\begin{pmatrix}\Phi_{i,x}(t,\lambda)\\ \Phi_{i,y}(t,\lambda)\end{pmatrix}dt, \tag{4.94}$$

where $\lambda\in\Omega_\lambda$, $\varepsilon>0$.

Let us introduce into the consideration the value

$$\widetilde{\mathscr{F}_0}(k,\lambda)\stackrel{\triangle}{=}\big(x^0(\lambda)\big)^T\widetilde{M}^o_{k,10}(0,\lambda)x^0(\lambda)+\sum_{i=1}^{m}\int_0^{t_f}\Phi^T_{i,x}(t,\lambda)\widetilde{M}^o_{k,10}(t,\lambda)\Phi_{i,x}(t,\lambda)dt,\ \ \lambda\in\Omega_\lambda. \tag{4.95}$$

Now, using the equations (4.94) and (4.95), as well as Lemma 4.16, the inequalities (4.81)-(4.82) and the boundedness of the vectors z^0, $\Phi_i(t,\lambda)$ for all $\lambda\in\Omega_\lambda$, $t\in[0,t_f]$, $(i=1,...m)$, we directly obtain the inequality

$$\big|\widetilde{\mathscr{F}_\varepsilon}(k,\lambda)-\widetilde{\mathscr{F}_0}(k,\lambda)\big|\le\widetilde{c}_1\varepsilon,\quad k=1,...,K,\quad \lambda\in\Omega_\lambda,\quad \varepsilon\in(0,\widetilde{\varepsilon}_0], \tag{4.96}$$

where $\widetilde{c}_1>0$ is some constant independent of λ and ε.

4.4.3. *Calculation of* $\lim_{\varepsilon\to+0}\mathscr{F}^*_\varepsilon(k)$. First of all let us remember the following. The equation (4.59) presents the expression for the value $\widetilde{\mathscr{F}_\varepsilon}(k,\lambda)$ of the functional (4.45), calculated along the trajectory $w(t)=\mathrm{col}(w_1(t),w_2(t),...,w_K(t))$, $(w_k(t)\in E^n)$ of the system (3.16) with $u(t)=u_\varepsilon\big(w(t),t,\lambda\big)$ given by the equations (3.12). From the other hand, $\mathscr{F}^*_\varepsilon(k)$ is the value of the functional (3.2) calculated along the trajectory $w(t)$ of the system (3.16) with $u(t)=u^*_\varepsilon\big(w(t),t\big)$ (see the equations (3.7) and (3.8)). Also, let us note that, due to the equations (4.46) and (4.47), the functionals (4.45) and (3.2) coincide with each other. Therefore, we have that

$$\mathscr{F}^*_\varepsilon(k)=\widetilde{\mathscr{F}_\varepsilon}\big(k,\lambda^*(\varepsilon)\big),\quad k=1,...,K,\quad \varepsilon>0, \tag{4.97}$$

where $\lambda^*(\varepsilon)$ is the solution of the optimization problem (3.8)-(3.9) and, due to Corollary 4.5, of the optimization problem (4.13)-(4.14).

Using the equality (4.97), the limit equality (4.43), the inequality (4.96), and taking into account the continuity of $\widetilde{\mathscr{F}_0}(k,\lambda)$, $(k=1,...,K)$ with respect to $\lambda\in\Omega_\lambda$, we have immediately the following limit equality

$$\lim_{\varepsilon\to+0}\mathscr{F}^*_\varepsilon(k)=\widetilde{\mathscr{F}_0}(k,\lambda^*_0),\quad k=1,...,K. \tag{4.98}$$

As a direct consequence of Corollary 3.5 and the equation (4.98), we obtain the limit equality

$$\lim_{\varepsilon\to+0}\mathscr{I}^*_\varepsilon=\sum_{k=1}^{K}\lambda^*_{0,k}\widetilde{\mathscr{F}_0}(k,\lambda^*_0),\quad \mathrm{col}(\lambda^*_{0,1},\lambda^*_{0,2},...,\lambda^*_{0,K})=\lambda^*_0. \tag{4.99}$$

Remember that $\mathscr{I}_\varepsilon^*$ (see the equation (3.10)) is the optimal value of the functional $\mathscr{J}_\varepsilon(u,\lambda)$ in the optimal control problem (2.1),(3.1)-(3.2).

5. Asymptotically Suboptimal Solution of the Problem (2.1),(3.1)-(3.2)

5.1. **Formal Construction of the Control Component in the Suboptimal Solution.** Due to the equation (4.4), the solution $\mathscr{M}\big(t,\lambda^*(\varepsilon),\varepsilon\big)$, $t\in[0,t_f]$ of the terminal-value problem (3.5) with $\lambda=\lambda^*(\varepsilon)$ has the form

$$\mathscr{M}\big(t,\lambda^*(\varepsilon),\varepsilon\big)=\Big(\mathscr{R}^T\big(t,\lambda^*(\varepsilon)\big)\Big)^{-1}M\big(t,\lambda^*(\varepsilon),\varepsilon\big)\mathscr{R}^{-1}\big(t,\lambda^*(\varepsilon)\big),\quad t\in[0,t_f],\quad \varepsilon>0, \tag{5.1}$$

where $M\big(t,\lambda^*(\varepsilon),\varepsilon\big)$ is the solution of the terminal-value problem (4.5) with $\lambda=\lambda^*(\varepsilon)$.

Substituting (5.1) into the expression (3.7) for the control component $u_\varepsilon^*\big(w(t),t\big)$ of the solution to the problem (2.1),(3.1)-(3.2) and taking into account the equations (4.7),(4.17), we can rewrite this control in the form

$$u_\varepsilon^*\big(w(t),t\big)=-\frac{1}{\varepsilon}\Big(M_2^T\big(t,\lambda^*(\varepsilon),\varepsilon\big)\ ,\ M_3\big(t,\lambda^*(\varepsilon),\varepsilon\big)\Big)\mathscr{R}^{-1}\big(t,\lambda^*(\varepsilon)\big)w(t),$$
$$w(t)\in E^{Kn},\quad t\in[0,t_f],\quad \varepsilon>0, \tag{5.2}$$

where $M_2\big(t,\lambda^*(\varepsilon),\varepsilon\big)$ and $M_3\big(t,\lambda^*(\varepsilon),\varepsilon\big)$ are the corresponding components of the solution to the terminal-value problem (4.19)-(4.21) with $\lambda=\lambda^*(\varepsilon)$.

Replacing in the right-hand side of (5.2) $\lambda^*(\varepsilon)$ with λ_0^*, as well as $M_2\big(t,\lambda^*(\varepsilon),\varepsilon\big)$ with $M_{20}^o(t,\lambda_0^*)$ and $M_3\big(t,\lambda^*(\varepsilon),\varepsilon\big)$ with $M_{30}^o(t,\lambda_0^*)$, we obtain the following state-feedback control:

$$\widehat{u}_\varepsilon\big(w(t),t\big)\stackrel{\triangle}{=}-\frac{1}{\varepsilon}\Big(\big(M_{20}^o(t,\lambda_0^*)\big)^T\ ,\ M_{30}^o(t,\lambda_0^*)\Big)\mathscr{R}^{-1}(t,\lambda_0^*)w(t),$$
$$w(t)\in E^{Kn},\quad t\in[0,t_f],\quad \varepsilon>0. \tag{5.3}$$

By virtue of Remark 2.2, for all $\varepsilon>0$, $\widehat{u}_\varepsilon\big(w(t),t\big)\in U$, i.e., this control is admissible in the problem (2.1),(3.1)-(3.2). In what follows of this section, we are going to show that the pair $\Big(\widehat{u}_\varepsilon\big(w(t),t\big),\lambda_0^*\Big)$ provides the value $\widehat{\mathscr{J}_\varepsilon}$ of the functional in the problem (2.1),(3.1)-(3.2) to be arbitrary close to the optimal value of this functional for all sufficiently small $\varepsilon>0$.

Let $\widehat{\mathscr{F}_\varepsilon}(k)$, $(k=1,...,K)$ be the value of the functional $\mathscr{F}_\varepsilon(u,k)$ (see the equation (3.2)), calculated along the trajectory $w(t)=\mathrm{col}(w_1(t),w_2(t),...,w_K(t))$, $(w_k(t)\in E^n)$ of the system (3.16) with $u(t)=\widehat{u}_\varepsilon\big(w(t),t\big)$ given by the equation (5.3). Then,

$$\widehat{\mathscr{J}_\varepsilon}=\sum_{k=1}^{K}\lambda_{0,k}^*\widehat{\mathscr{F}_\varepsilon}(k),\quad \varepsilon>0, \tag{5.4}$$

where $\lambda_{0,k}^*$, $(k=1,2,...,K)$ are the same as in (4.99).

5.2. **Asymptotic Analysis of the Value $\widehat{\mathscr{F}_{\varepsilon}}(k)$.** Consider the matrix-valued functions

$$M_0^o(t,\varepsilon) \stackrel{\triangle}{=} \begin{pmatrix} M_{10}^o(t,\lambda_0^*) & \varepsilon M_{20}^o(t,\lambda_0^*) \\ \varepsilon\big(M_{20}^o(t,\lambda_0^*)\big)^T & \varepsilon M_{30}^o(t,\lambda_0^*) \end{pmatrix}, \quad t\in[0,t_f], \quad \varepsilon>0,$$
$$\mathscr{M}_0^o(t,\varepsilon) \stackrel{\triangle}{=} \big(\mathscr{R}^T(t,\lambda_0^*)\big)^{-1} M_0^o(t,\varepsilon)\mathscr{R}^{-1}(t,\lambda_0^*), \quad t\in[0,t_f], \quad \varepsilon>0. \tag{5.5}$$

Using these matrix-valued functions, we can rewrite the control (5.3) in the form

$$\widehat{u}_{\varepsilon}\big(w(t),t,\lambda_0^*\big) = -\frac{1}{\varepsilon^2}\mathscr{B}^T(t)\mathscr{M}_0^o(t,\varepsilon)w(t), \quad w(t)\in E^{Kn}, \quad t\in[0,t_f], \quad \varepsilon>0. \tag{5.6}$$

Now, similarly to the equations (4.48)-(4.50), we obtain that

$$\widehat{\mathscr{F}_{\varepsilon}}(k) = \big(w^0\big)^T\widehat{\mathscr{M}_k}(0,\varepsilon)w^0 + \sum_{i=1}^{m}\int_0^{t_f}\Theta_i^T(t)\widehat{\mathscr{M}_k}(t,\varepsilon)\Theta_i(t)dt, \quad k=1,...,K, \tag{5.7}$$

where, for all $\varepsilon>0$ and $k\in\{1,...,K\}$, $\widehat{\mathscr{M}_k}(t,\varepsilon)$, $t\in[0,t_f]$ is the unique solution of the following terminal-value problem for linear differential equation with respect to the unknown $Kn\times Kn$-matrix-valued function $\widehat{\mathscr{M}_k}(t)$:

$$\frac{d\widehat{\mathscr{M}_k}(t)}{dt} = -\widehat{\mathscr{M}_k}(t)\widehat{\mathscr{A}}(t,\varepsilon) - \widehat{\mathscr{A}}^T(t,\varepsilon)\widehat{\mathscr{M}_k}(t) - \widetilde{\mathscr{D}}_k(t) - \mathscr{M}_0^o(t,\varepsilon)\mathscr{S}(t,\varepsilon)\mathscr{M}_0^o(t,\varepsilon),$$
$$t\in[0,t_f], \quad \widehat{\mathscr{M}_k}(t_f) = \widetilde{\mathscr{H}_k}, \tag{5.8}$$

$$\widehat{\mathscr{A}}(t,\varepsilon) = \mathscr{A}(t) - \mathscr{S}(t,\varepsilon)\mathscr{M}_0^o(t,\varepsilon), \tag{5.9}$$

$\widetilde{\mathscr{H}_k}$ and $\widetilde{\mathscr{D}}_k(t)$ are given by (4.46) and (4.47), respectively.

Let us transform (similarly to (4.54)) the unknown $\widehat{\mathscr{M}_k}(t)$ in the terminal-value problem (5.8) as follows:

$$\widehat{\mathscr{M}_k}(t) = \big(\mathscr{R}^T(t,\lambda_0^*)\big)^{-1}\widehat{M}_k(t)\mathscr{R}^{-1}(t,\lambda_0^*), \quad t\in[0,t_f], \tag{5.10}$$

where $\widehat{M}_k(t)$ is a new unknown matrix-valued function.

Due to this transformation, we obtain (similarly to (4.55)-(4.58)) the terminal-value problem for the unknown matrix-valued function $\widehat{M}_k(t)$

$$\frac{d\widehat{M}_k(t)}{dt} = -\widehat{M}_k(t)\widehat{A}(t,\varepsilon) - \widehat{A}^T(t,\varepsilon)\widehat{M}_k(t) - \widehat{D}_k(t) - M_0^o(t,\varepsilon)S(\varepsilon)M_0^o(t,\varepsilon),$$
$$t\in[0,t_f], \quad \widehat{M}_k(t_f) = \widehat{H}_k, \tag{5.11}$$

where $M_0^o(t,\varepsilon)$ is given in (5.5),

$$\widehat{A}(t,\varepsilon) = A(t,\lambda_0^*) - S(\varepsilon)M_0^o(t,\varepsilon), \tag{5.12}$$

$$\widehat{D}_k(t) = \mathscr{R}_k^T(t,\lambda_0^*)\widetilde{\mathscr{D}}(t)\mathscr{R}_k(t,\lambda_0^*), \tag{5.13}$$

$$\widehat{H}_k(\lambda) = \begin{pmatrix} \widehat{H}_{k,1} & O_{(Kn-r)\times r} \\ O_{r\times(Kn-r)} & O_{r\times r} \end{pmatrix}, \quad \widehat{H}_{k,1} = \mathscr{L}_k^T(t_f,\lambda_0^*)\widetilde{\mathscr{H}}\mathscr{L}_k(t_f,\lambda_0^*). \tag{5.14}$$

Using the transformation (5.10) and the equation (4.15), we can rewrite the equation (5.7) as follows:

$$\widehat{\mathscr{F}_\varepsilon}(k) = \big(z^0(\lambda_0^*)\big)^T \widehat{M}_k(0,\varepsilon) z^0(\lambda_0^*) + \sum_{i=1}^{m}\int_0^{t_f} \Phi_i^T(t,\lambda_0^*)\widehat{M}_k(t,\varepsilon)\Phi_i(t,\lambda_0^*)dt,\ k=1,...,K, \quad (5.15)$$

where, for all $\varepsilon > 0$, $\widehat{M}_k(t,\varepsilon)$, $t \in [0,t_f]$ is the unique solution of the terminal-value problem (5.11).

Further asymptotic analysis of the value $\widehat{\mathscr{F}_\varepsilon}(k)$ is based on the zero-order asymptotic solution to the problem (5.11).

5.2.1. *Zero-Order Asymptotic Solution to the Terminal-Value Problem (5.11).* Similarly to the block form (4.60) of the solution to the terminal-value problem (4.55), we look for the solution to the problem (5.11) in the following block form:

$$\widehat{M}_k(t,\varepsilon) = \begin{pmatrix} \widehat{M}_{k,1}(t,\varepsilon) & \varepsilon\widehat{M}_{k,2}(t,\varepsilon) \\ \varepsilon\widehat{M}_{k,2}^T(t,\varepsilon) & \varepsilon\widehat{M}_{k,3}(t,\varepsilon) \end{pmatrix}, \quad (5.16)$$

where the matrices $\widehat{M}_{k,1}(t,\varepsilon)$, $\widehat{M}_{k,2}(t,\varepsilon)$ and $\widehat{M}_{k,3}(t,\varepsilon)$ are of the dimensions $(Kn-r)\times(Kn-r)$, $(Kn-r)\times r$ and $r\times r$, respectively; $\widehat{M}_{k,1}^T(t,\varepsilon) = \widehat{M}_{k,1}(t,\varepsilon)$, $\widehat{M}_{k,3}^T(t,\varepsilon) = \widehat{M}_{k,3}(t,\varepsilon)$.

Taking into account the symmetry of the matrix $\widehat{D}_k(t)$, we partition it into blocks as:

$$\widehat{D}_k(t) = \begin{pmatrix} \widehat{D}_{k,1}(t) & \widehat{D}_{k,2}(t) \\ \widehat{D}_{k,2}^T(t) & \widehat{D}_{k,3}(t) \end{pmatrix}, \quad (5.17)$$

where the matrices $\widehat{D}_{k,1}(t)$, $\widehat{D}_{k,2}(t)$ and $\widehat{D}_{k,3}(t)$ are of the dimensions $(Kn-r)\times(Kn-r)$, $(Kn-r)\times r$ and $r\times r$, respectively; $\widehat{D}_{k,1}^T(t) = \widehat{D}_{k,1}(t)$, $\widehat{D}_{k,3}^T(t) = \widehat{D}_{k,3}(t)$.

Substitution of the block forms of the matrices $S(\varepsilon)$, $A(t,\lambda)$, $M_0^o(t,\varepsilon)$, $\widehat{H}_k$, $\widehat{M}_k(t,\varepsilon)$, $\widehat{D}_k(t)$ (see the equations (4.8),(4.18),(5.5),(5.14),(5.16),(5.17)) into the problem (5.11) yields after a routine matrix algebra the following equivalent terminal-value problem in the time interval $[0,t_f]$:

$$\begin{aligned}\frac{d\widehat{M}_{k,1}(t,\varepsilon)}{dt} &= -\widehat{M}_{k,1}(t,\varepsilon)A_1(t,\lambda_0^*) - \varepsilon\widehat{M}_{k,2}(t,\varepsilon)A_3(t,\lambda_0^*) - A_1^T(t,\lambda_0^*)\widehat{M}_{k,1}(t,\varepsilon) \\ &\quad -\varepsilon A_3^T(t,\lambda_0^*)\widehat{M}_{k,2}^T(t,\varepsilon) + \widehat{M}_{k,2}(t,\varepsilon)\big(M_{20}^o(t,\lambda_0^*)\big)^T + M_{20}^o(t,\lambda_0^*)\widehat{M}_{k,2}^T(t,\varepsilon) \\ &\quad -\widehat{D}_{k,1}(t) - M_{20}^o(t,\lambda_0^*)\big(M_{20}^o(t,\lambda_0^*)\big)^T, \quad \widehat{M}_{k,1}(t_f,\varepsilon) = \widehat{H}_{k,1}, \end{aligned} \quad (5.18)$$

$$\begin{aligned}\varepsilon\frac{d\widehat{M}_{k,2}(t,\varepsilon)}{dt} &= -\widehat{M}_{k,1}(t,\varepsilon)A_2(t,\lambda_0^o) - \varepsilon\widehat{M}_{k,2}(t,\varepsilon)A_4(t,\lambda_0^o) - \varepsilon A_1^T(t,\lambda_0^o)\widehat{M}_{k,2}(t,\varepsilon) \\ &\quad -\varepsilon A_3^T(t,\lambda_0^o)\widehat{M}_{k,3}(t,\varepsilon) + \widehat{M}_{k,2}(t,\varepsilon)M_{30}^o(t,\lambda_0^o) + M_{20}^o(t,\lambda_0^o)\widehat{M}_{k,3}(t,\varepsilon) \\ &\quad -\widehat{D}_{k,2}(t) - M_{20}^o(t,\lambda_0^o)M_{30}^o(t,\varepsilon), \quad \widehat{M}_{k,2}(t_f,\varepsilon) = 0, \end{aligned} \quad (5.19)$$

$$\varepsilon\frac{d\widehat{M}_{k,3}(t,\varepsilon)}{dt} = -\varepsilon\widehat{M}_{k,2}^{T}(t,\varepsilon)A_2(t,\lambda_0^*) - \varepsilon\widehat{M}_{k,3}(t,\varepsilon)A_4(t,\lambda_0^*) - \varepsilon A_2^T(t,\lambda_0^*)\widehat{M}_{k,2}(t,\varepsilon)$$
$$-\varepsilon A_4^T(t,\lambda_0^*)\widehat{M}_{k,3}(t,\varepsilon) + \widehat{M}_{k,3}(t,\varepsilon)M_{30}^o(t,\lambda_0^*) + M_{30}^o(t,\lambda_0^*)\widehat{M}_{k,3}(t,\varepsilon)$$
$$-\widehat{D}_{k,3}(t) - \big(M_{30}^o(t,\lambda_0^*)\big)^2, \quad \widehat{M}_{k,3}(t_f,\varepsilon) = 0. \quad (5.20)$$

Like the problem (4.62)-(4.64), the problem (5.18)-(5.20) also is a singularly perturbed linear terminal-value problem. In what follows of this subsection, similarly to Subsection 4.4.1, we construct and justify the zero-order asymptotic solution of this problem. We seek this asymptotic solution in the form

$$\widehat{M}_{k,j0}(t,\varepsilon) = \widehat{M}_{k,j0}^o(t) + \widehat{M}_{k,j0}^b(\tau), \quad j = 1,2,3, \quad \tau = (t-t_f)/\varepsilon, \quad (5.21)$$

where (like in (4.65)) the terms with the upper index "o" constitute the outer solution, while the terms with the upper index "b" are the boundary correction terms in a left-hand neighbourhood of $t = t_f$; $\tau \le 0$ is a new independent variable, called the stretched time. For any $t \in [0,t_f)$, $\tau \to -\infty$ as $\varepsilon \to +0$. Similarly to Subsection 4.4.1, equations and conditions for obtaining the outer solution and the boundary correction terms are derived by substituting the representation (5.21) into the terminal-value problem (5.18)-(5.20) instead of $\widehat{M}_{k,j}(t,\varepsilon)$, $(j = 1,2,3)$ and equating the coefficients for the same power of ε on both sides of the resulting equations, separately the coefficients depending on t and on τ.

Thus, similarly to (4.68), we have

$$\widehat{M}_{k,10}^b(\tau) \equiv 0, \quad \tau \le 0. \quad (5.22)$$

Furthermore, similarly to (4.69)-(4.71), we obtain the equations and conditions for the outer solution terms for all $t \in [0,t_f]$

$$\frac{d\widehat{M}_{k,10}^o(t)}{dt} = -\widehat{M}_{k,10}^o(t)A_1(t,\lambda_0^*) - A_1^T(t,\lambda_0^*)\widehat{M}_{k,10}^o(t)$$
$$+\widehat{M}_{k,20}^o(t)\big(M_{20}^o(t,\lambda_0^*)\big)^T + M_{20}^o(t,\lambda_0^*)\big(\widehat{M}_{k,20}^o(t,\lambda_0^*)\big)^T$$
$$-\widehat{D}_{k,1}(t) - M_{20}^o(t,\lambda_0^*)\big(M_{20}^o(t,\lambda_0^*)\big)^T, \quad \widehat{M}_{k,10}^o(t_f) = \widehat{H}_{k,1}, \quad (5.23)$$

$$-\widehat{M}_{k,10}^o(t)A_2(t,\lambda_0^*) + \widehat{M}_{k,20}^o(t)M_{30}^o(t,\lambda_0^*) + M_{20}^o(t,\lambda_0^*)\widehat{M}_{k,30}^o(t)$$
$$-\widehat{D}_{k,2}(t) - M_{20}^o(t,\lambda_0^*)M_{30}^o(t,\lambda_0^*) = 0, \quad (5.24)$$

$$\widehat{M}_{k,30}^o(t)M_{30}^o(t,\lambda_0^*) + M_{30}^o(t,\lambda_0^*)\widehat{M}_{k,30}^o(t) - \widehat{D}_{k,3}(t) - \big(M_{30}^o(t,\lambda_0^*)\big)^2 = 0. \quad (5.25)$$

Comparing the system (5.23)-(5.25) with the system (4.69)-(4.71) and taking into account that $\widehat{D}_k(t) = \widetilde{D}_k(t,\lambda_0^*)$, $\widehat{H}_{k,1} = \widetilde{H}_{k,1}(\lambda_0^*)$, we directly obtain the following equalities:

$$\widehat{M}_{k,j0}^o(t) = \widetilde{M}_{k,j0}^o(t,\lambda_0^*), \quad j = 1,2,3, \quad t \in [0,t_f]. \quad (5.26)$$

Now proceed to the obtaining the boundary correction terms $\widehat{M}^b_{k,20}(\tau)$ and $\widehat{M}^b_{k,30}(\tau)$. Similarly to (4.75)-(4.76), we obtain the terminal-value problem for $\widehat{M}^b_{k,20}(\tau)$ and $\widehat{M}^b_{k,30}(\tau)$

$$\frac{d\widehat{M}^b_{k,20}(\tau)}{d\tau} = \widehat{M}^b_{k,20}(\tau)M^o_{30}(t_f,\lambda^*_0) + M^o_{20}(t_f,\lambda^*_0)\widehat{M}^b_{k,30}(\tau), \quad \tau \le 0,$$
$$\widehat{M}^b_{k,20}(0) = -\widehat{M}^o_{k,20}(t_f), \tag{5.27}$$

$$\frac{d\widehat{M}^b_{k,30}(\tau)}{d\tau} = \widehat{M}^b_{k,30}(\tau)M^o_{30}(t_f,\lambda^*_0) + M^o_{30}(t_f,\lambda^*_0)\widehat{M}^b_{k,30}(\tau), \quad \tau \le 0,$$
$$\widehat{M}^b_{k,30}(0) = -\widehat{M}^o_{k,30}(t_f). \tag{5.28}$$

Since, due to the equation (4.29), the matrix $M^o_{30}(t_f,\lambda^*_0)$ is positive definite, then (similarly to (4.81)-(4.82)) we have the estimates of $\widehat{M}^b_{k,30}(\tau)$ and $\widehat{M}^b_{k,20}(\tau)$

$$\left\|\widehat{M}^b_{k,j0}(\tau)\right\| \le \widehat{b}\exp(\widehat{\beta}\tau), \quad j=2,3, \quad \tau \le 0, \tag{5.29}$$

where $\widehat{b} > 0$ and $\widehat{\beta} > 0$ are some constants.

Thus, we have completed the formal construction of the zero-order asymptotic solution to the terminal-value problem (5.18)-(5.20). The following assertion justifies this asymptotic solution.

Lemma 5.1. *Let the assumptions AI-AV be fulfilled. Then, there exists a number* $\widehat{\varepsilon}_0 > 0$ *such that, for all* $\varepsilon \in (0,\widehat{\varepsilon}_0]$*, the entries of the solution* $\left\{\widehat{M}_{k,1}(t,\varepsilon),\widehat{M}_{k,2}(t,\varepsilon),\widehat{M}_{k,3}(t,\varepsilon)\right\}$ *to the terminal-value problem (5.18)-(5.20) satisfy the inequalities*

$$\left\|\widehat{M}_{k,1}(t,\varepsilon) - \widehat{M}^o_{k,10}(t)\right\| \le \widehat{c}\varepsilon, \quad \left\|\widehat{M}_{k,j}(t,\varepsilon) - \widehat{M}_{k,j0}(t,\varepsilon)\right\| \le \widehat{c}\varepsilon,$$
$$j=2,3, \quad t \in [0,t_f], \tag{5.30}$$

where $\widehat{M}_{k,j0}(t,\varepsilon)$*,* $(j=2,3)$ *are given in (5.21);* $\widehat{c} > 0$ *is some constant independent of* ε.

Proof. The lemma is proven quite similarly to Lemma 4.16. □

5.2.2. *Asymptotic Estimate of the Value* $\widehat{\mathscr{F}_\varepsilon}(k)$. Substituting (4.93),(5.16) into the equation (5.15), and using Lemma 5.1, the equation (5.26) and the inequalities in (5.29), we directly obtain the following estimate of the value $\widehat{\mathscr{F}_\varepsilon}(k)$

$$\left|\widehat{\mathscr{F}_\varepsilon}(k) - \widetilde{\mathscr{F}_0}(k,\lambda^*_0)\right| \le \widehat{c}_1\varepsilon, \quad k=1,...,K, \quad \varepsilon \in (0,\widehat{\varepsilon}_0], \tag{5.31}$$

where $\widetilde{\mathscr{F}_0}(k,\lambda)$ is given by the equation (4.95); $\widehat{c}_1 > 0$ is some constant independent of ε.

5.2.3. *Calculation of* $\lim_{\varepsilon\to+0}\widehat{\mathscr{J}_\varepsilon}$. Using the equations (4.99),(5.4) and the inequalities in (5.31), we obtain immediately the limit equality

$$\lim_{\varepsilon\to+0}\widehat{\mathscr{J}_\varepsilon} = \sum_{k=1}^{K}\lambda^*_{0,k}\widetilde{\mathscr{F}_0}(k,\lambda^*_0) = \lim_{\varepsilon\to+0}\mathscr{I}^*_\varepsilon. \tag{5.32}$$

Remember that $\mathscr{I}^*_\varepsilon$ (see the equation (3.10)) is the optimal value of the functional $\mathscr{J}_\varepsilon(u,\lambda)$ in the optimal control problem (2.1),(3.1)-(3.2), while $\widehat{\mathscr{J}_\varepsilon}$ is the value of the functional in the problem (2.1),(3.1)-(3.2) generated by the pair $\left(\widehat{u}_\varepsilon\big(w(t),t\big),\lambda_0^*\right)$, where $\widehat{u}_\varepsilon\big(w(t),t\big)$ is given by (5.3). Thus, the equality (5.32) means that the pair $\left(\widehat{u}_\varepsilon\big(w(t),t\big),\lambda_0^*\right)$ indeed provides the value $\widehat{\mathscr{J}_\varepsilon}$ of the functional in the problem (2.1),(3.1)-(3.2) to be arbitrary close to the optimal value of this functional for all sufficiently small $\varepsilon > 0$.

Remark 5.2. In addition to the equality (5.32), let us note the following. Using this equality, as well as the equations (3.10),(4.16),(4.41) and the inequality (4.44), we directly obtain

$$\sum_{k=1}^{K}\lambda_{0,k}^*\widetilde{\mathscr{F}_0}(k,\lambda_0^*) = J_0(\lambda_0^*). \tag{5.33}$$

6. Solution of the Optimal Control Problem (2.1),(2.5)

Theorem 6.1. *Let the assumptions AI-AVI be fulfilled. Then, the following equality is valid:*

$$\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} = J_0\big(\lambda_0^*\big), \tag{6.1}$$

where $\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*}$ is the infimum of the functional $\mathscr{J}\left(u,\lambda_0^*\right)$ with respect to $u(t) = u\big(w(t),t\big) \in U$ in the problem (2.1),(2.5) (see Remark 2.4); the function $J_0(\lambda)$ is defined in the equation (4.41); the vector λ_0^* is defined by the equation (4.40).

Proof (by contradiction). Assume that the equality (6.1) is wrong, i.e., we assume that $\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} \neq J_0\big(\lambda_0^*\big)$. Let us show that this assumption implies the inequality

$$\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} < J_0\big(\lambda_0^*\big). \tag{6.2}$$

Using the equations (2.2)-(2.5),(3.1)-(3.2),(5.3)-(5.4) and Remark 2.4, we directly obtain the following chain of the inequalities and the equality:

$$\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} \le \mathscr{J}\left(\widehat{u}_\varepsilon(\cdot),\lambda_0^*\right) \le \mathscr{J}_\varepsilon\big(\widehat{u}_\varepsilon(\cdot),\lambda_0^*\big) = \widehat{\mathscr{J}_\varepsilon}, \quad \varepsilon > 0. \tag{6.3}$$

Thus, for all $\varepsilon > 0$, $\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} \le \widehat{\mathscr{J}_\varepsilon}$. The latter, along with the equations (5.32)-(5.33), directly yields that $\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} \le J_0\big(\lambda_0^*\big)$. This inequality and the above assumed inequality $\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} \neq J_0\big(\lambda_0^*\big)$ mean the fulfillment of (6.2).

Since (6.2) is valid and $\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*}$ is the infimum of the functional $\mathscr{J}\left(u,\lambda_0^*\right)$ with respect to $u(t) = u\big(w(t),t\big) \in U$ in the problem (2.1),(2.5), then there exists a control $\widetilde{u}(\cdot) \in U$ such that

$$\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} < \mathscr{J}\left(\widetilde{u}(\cdot),\lambda_0^*\right) < J_0\big(\lambda_0^*\big). \tag{6.4}$$

Taking into account that $u_\varepsilon^*(\cdot)$ given by (3.7) is the control component of the solution $\big(u_\varepsilon^*(\cdot),\lambda^*(\varepsilon)\big)$ to the problem (2.1),(3.1)-(3.2), we directly have

$$\mathscr{I}_\varepsilon^* = \mathscr{I}\big(\lambda^*(\varepsilon),\varepsilon\big) = \mathscr{J}_\varepsilon\big(u_\varepsilon^*(\cdot),\lambda^*(\varepsilon)\big) \le \mathscr{J}_\varepsilon\big(\widetilde{u}(\cdot),\lambda^*(\varepsilon)\big) = \mathscr{J}\left(\widetilde{u}(\cdot),\lambda^*(\varepsilon)\right) + \widetilde{a}\varepsilon^2, \quad \varepsilon > 0, \tag{6.5}$$

where

$$0 \le \widetilde{a} = \mathrm{E}\left[\int_0^{t_f}\widetilde{u}^T\left(\widetilde{w}(t),t\right)\widetilde{u}\big(\widetilde{w}(t),t\big)dt\right] < +\infty,$$

$\widetilde{w}(t) \stackrel{\triangle}{=} \mathrm{col}\big(\widetilde{w}_1(t), \widetilde{w}_2(t), ..., \widetilde{w}_K(t)\big)$, $t \in [0, t_f]$ is the trajectory of the initial-value problem (2.1) with $k = 1, 2, ..., K$ generated by $u(t) = \widetilde{u}(\cdot) \in U$.

Thus,

$$\mathscr{I}_\varepsilon^* \le \mathscr{J}\big(\widetilde{u}(\cdot), \lambda^*(\varepsilon)\big) + \widetilde{a}\varepsilon^2, \quad \varepsilon > 0. \tag{6.6}$$

Furthermore, using the equations (2.5)-(2.6),(4.43), we obtain the limit equality

$$\lim_{\varepsilon \to +0} \mathscr{J}\big(\widetilde{u}(\cdot), \lambda^*(\varepsilon)\big) = \mathscr{J}\big(\widetilde{u}(\cdot), \lambda_0^*\big). \tag{6.7}$$

Now, the inequality (6.6), along with the equalities (5.32)-(5.33),(6.7), directly yields the following inequality: $J_0\big(\lambda_0^*\big) \le \mathscr{J}\big(\widetilde{u}(\cdot), \lambda_0^*\big)$, which contradicts the right-hand side inequality in (6.4). This contradiction means that the above assumed inequality $\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} \ne J_0\big(\lambda_0^*\big)$ is wrong. Therefore, the equality (6.1) is correct. Thus, the theorem is proven. □

Corollary 6.2. *Let the assumptions AI-AVI be fulfilled. Then, the following equality is valid:*

$$\mathscr{J}^*(w^0) = J_0\big(\lambda_0^*\big), \tag{6.8}$$

where $\mathscr{J}^*(w^0)$ *is given by the equation (2.7).*

Proof First of all, let us note that

$$\mathscr{J}^*(w^0) \ge \mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*} = J_0\big(\lambda_0^*\big). \tag{6.9}$$

Using the equations (2.2)-(2.5),(3.1)-(3.2),(3.12), Remark 2.4, Theorem 3.4 and its proof, we directly obtain the following chain of the inequalities and the equality:

$$\mathscr{J}_\lambda^*(w^0) \le \mathscr{J}\Big(u_\varepsilon\big(w(t), t, \lambda\big), \lambda\Big) \le \mathscr{J}_\varepsilon\Big(u_\varepsilon\big(w(t), t, \lambda\big), \lambda\Big) = \mathscr{I}(\lambda, \varepsilon), \quad \lambda \in \Omega_\lambda, \quad \varepsilon > 0, \tag{6.10}$$

yielding

$$\mathscr{J}^*(w^0) = \max_{\lambda \in \Omega_\lambda} \mathscr{J}_\lambda^*(w^0) \le \max_{\lambda \in \Omega_\lambda} \mathscr{I}(\lambda, \varepsilon) = \mathscr{I}\big(\lambda^*(\varepsilon), \varepsilon\big) = \mathscr{I}_\varepsilon^*, \quad \varepsilon > 0. \tag{6.11}$$

Using (6.11) and the equations (5.32)-(5.33), we directly have

$$\mathscr{J}^*(w^0) \le \lim_{\varepsilon \to +0} \mathscr{I}_\varepsilon^* = J_0\big(\lambda_0^*\big). \tag{6.12}$$

Now, the equations (6.9) and (6.12) yield immediately the equality (6.8), which completes the proof of the corollary. □

Consider the sequence of numbers $\{\varepsilon_q\}_{q=1}^{+\infty}$ satisfying the conditions

$$0 < \varepsilon_q \le \min\{\widetilde{\varepsilon}_0, \widehat{\varepsilon}_0\}, \quad q = 1, 2, ...; \qquad \lim_{q \to +\infty} \varepsilon_q = 0. \tag{6.13}$$

Using this sequence, consider the sequence of state-feedback controls in the optimal control problem (2.1),(2.5)

$$\big\{\widehat{u}_q\big(w(t), t\big)\big\}_{q=1}^{+\infty} \stackrel{\triangle}{=} \big\{\widehat{u}_{\varepsilon_q}\big(w(t), t, \lambda_0^*\big)\big\}_{q=1}^{+\infty}, \tag{6.14}$$

where $\widehat{u}_\varepsilon\big(w(t), t, \lambda_0^*\big)$ is defined in (5.3).

Theorem 6.3. *Let the assumptions AI-AVI be fulfilled. Then, the sequence of the pairs* $\left\{\left(\widehat{u}_q\big(w(t),t\big),\lambda_0^*\right)\right\}_{q=1}^{+\infty}$ *is the solution the optimal control problem (2.1),(2.5), i.e.,*

$$\lim_{q\to+\infty}\mathscr{J}\left(\widehat{u}_q\big(w(t),t\big),\lambda_0^*\right)=\mathscr{J}^*(w^0). \tag{6.15}$$

Proof. Due to the equations (2.2)-(2.5),(3.1)-(3.2),(5.4), Remark 2.4, Theorem 6.1 and Corollary 6.2, we have the following chain of the equalities and the inequalities:

$$\mathscr{J}^*(w^0)=\mathscr{J}_\lambda^*(w^0)\big|_{\lambda=\lambda_0^*}\le\mathscr{J}\left(\widehat{u}_q\big(w(t),t\big),\lambda_0^*\right)\le\mathscr{J}_{\varepsilon_q}\left(\widehat{u}_q\big(w(t),t\big),\lambda_0^*\right)=\widehat{\mathscr{J}_{\varepsilon_q}}. \tag{6.16}$$

Using the equations (5.32)-(5.33),(6.13) and Corollary 6.2, we obtain the following:

$$\mathscr{J}^*(w^0)=J_0\big(\lambda_0^*\big)=\lim_{q\to+\infty}\widehat{\mathscr{J}_{\varepsilon_q}}.$$

The latter, along with (6.16) yields immediately the equality (6.15). Thus, the theorem is proven. □

7. Illustrative Example

Consider the following two-model Ito differential stochastic system:

$$\begin{aligned} dw_{1,k}(t)&=\rho_k w_{2,k}(t)dt+d\eta(t), \quad w_{1,k}(0)=1, \quad t\in[0,4], \quad k\in\{1,2\},\\ dw_{2,k}(t)&=u(t)dt+2d\eta, \quad w_{2,k}(0)=2, \quad t\in[0,4], \quad k\in\{1,2\}, \end{aligned} \tag{7.1}$$

where $w_{1,k}(t)$, $w_{2,k}(t)$, $u(t)$ are scalar functions; $\eta(t)$, $t\ge 0$, $(\eta(0)=0)$ is the scalar standard Wiener process defined on the filtered probability space $\{\Omega,\mathscr{F},P\}$ [2]; $\rho_1=2$, $\rho_2=1$.

Comparing the system (7.1) with the system (2.1), one can conclude that (7.1) is a particular case of (2.1) where $n=2$, $r=1$, $m=1$, $t_f=4$, $K=2$,

$$\mathscr{A}_1=\begin{pmatrix}0&2\\0&0\end{pmatrix},\quad \mathscr{A}_2=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \mathscr{B}_1=\mathscr{B}_2=\begin{pmatrix}0\\1\end{pmatrix},$$
$$\sigma_{1,1}=\sigma_{1,2}=\begin{pmatrix}1\\2\end{pmatrix},\quad \widetilde{w}_0=\begin{pmatrix}1\\2\end{pmatrix}. \tag{7.2}$$

In this example, we choose the functional $\mathscr{F}(u,k)$ as:

$$\mathscr{F}(u,k)=\mathrm{E}\left[w_{1,k}^2(4)+\int_0^4 w_{2,k}^2(t)dt\right],\quad k\in\{1,2\}. \tag{7.3}$$

Comparison of the functional (7.3) and the functional (2.2) yields that (7.3) is a particular case of (2.2) where

$$\widetilde{\mathscr{H}}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \widetilde{\mathscr{D}}=\begin{pmatrix}0&0\\0&1\end{pmatrix}. \tag{7.4}$$

Using the functional (7.3) and the set

$$\Omega_\lambda\stackrel{\triangle}{=}\big\{\lambda=\mathrm{col}(\lambda_1,\lambda_2)\in E^2:\lambda_1\ge 0,\ \lambda_2\ge 0,\ \lambda_1+\lambda_2=1\big\}, \tag{7.5}$$

we construct the performance index evaluating the control process of the two-model system (7.1)

$$\mathscr{J}(u,\lambda) \stackrel{\triangle}{=} \lambda_1 \mathscr{F}(u,1) + \lambda_2 \mathscr{F}(u,2) \to \max_{\lambda \in \Omega_\lambda} \inf_{u \in U}, \tag{7.6}$$

where the set U is a particular case of such a set presented in Definition 2.1.

Remark 7.1. The two-model singular optimal control problem (7.1),(7.6) is a particular case of the multi-model singular optimal control problem (2.1),(2.5). The solution of the problem (7.1),(7.6) will allow us to clearly illustrate the theoretical results of the previous sections, while avoiding too complicated analytical/numerical calculations. Such an illustration allows not to overload the paper and, therefore, to keep its readability. Also, let us note that the two-model system (7.1) and the functional (7.3) are stochastic versions of the two-model deterministic system and the corresponding functional considered in the illustrative example of the work [26].

Proceed to the construction of the solution to the optimal control problem (7.1),(7.6). Due to Theorem 6.3, first, we should check up the fulfilment of the assumptions AI-AVI in this problem. Based on the equations (7.2),(7.4), we have immediately that the assumptions AI-AV are fulfilled. The fulfilment of the assumption AVI will be verified in the sequel of this section. Based on Theorem 6.3 and the equations (5.3),(6.14), one can conclude the following. To construct the aforementioned solution, the matrix-valued functions $\mathscr{R}(t,\lambda)$, $P^o_{20}(t,\lambda)$, $P^o_{30}(t,\lambda)$ should be obtained. We start with the obtaining $\mathscr{R}(t,\lambda)$. Due to the equations (3.4),(4.3),(7.2), this matrix depends on the complement matrix $\mathscr{B}_c$ to the matrix $\mathscr{B} = \mathrm{col}(0,1,0,1)$. Similarly to [26], we choose the matrix $\mathscr{B}_c$ as follows:

$$\mathscr{B}_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, using this matrix, as well as the equations (3.3),(3.4),(4.3) and the data of the example (7.2),(7.4), we obtain the following matrices:

$$\mathscr{L}(t,\lambda) \equiv \mathscr{L}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \\ 0 & -\lambda_1 & 0 \end{pmatrix}, \quad \mathscr{R}(t,\lambda) \equiv \mathscr{R}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -\lambda_1 & 0 & 1 \end{pmatrix}, \tag{7.7}$$

where λ_1 and λ_2 are the respective entries of the vector $\lambda \in \Omega_\lambda$.

Due to the results of Subsection 4.2.2, to obtain the matrices $M^o_{20}(t,\lambda)$, $M^o_{30}(t,\lambda)$, first, we should obtain the matrices $A_1(t,\lambda)$, $A_2(t,\lambda)$, $D_1(t,\lambda)$, $D_2(t,\lambda)$, $H_1(\lambda)$, $S^o_1(\lambda)$. Using the equations (3.3),(4.6),(4.11), (4.12),(4.18),(4.32), as well as the data of the example (7.2),(7.4) and the above calculated matrices $\mathscr{L}(t,\lambda)$, $\mathscr{R}(t,\lambda)$, we obtain after a routine matrix algebra

$$
\begin{gathered}
A_1(t,\lambda)\equiv A_1(\lambda)=\begin{pmatrix} 0 & 2\lambda_2 & 0\\ 0 & 0 & 0\\ 0 & -\lambda_1 & 0\end{pmatrix},\ A_2(t,\lambda)\equiv A_2(\lambda)=\begin{pmatrix}2\\0\\1\end{pmatrix},\\
D_1(t,\lambda)\equiv D_1(\lambda)=\begin{pmatrix} 0 & 0 & 0\\ 0 & \lambda_1\lambda_2 & 0\\ 0 & 0 & 0\end{pmatrix},\quad D_2(t,\lambda)\equiv D_2=1,\\
H_1(\lambda)=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \lambda_2\end{pmatrix},\quad S_1^o(\lambda)\equiv S_1^o=\begin{pmatrix}4 & 0 & 2\\ 0 & 0 & 0\\ 2 & 0 & 1\end{pmatrix}.
\end{gathered}
\tag{7.8}
$$

Using the equations (4.29),(4.30),(7.8) and the symmetry of the matrix $M_{10}^o(t,\lambda)$, we directly have

$$
\begin{gathered}
M_{30}^o(t,\lambda)\equiv M_{30}^o=1,\\
M_{20}^o(t,\lambda)=\begin{pmatrix}2M_{10,11}^o(t,\lambda)+M_{10,13}^o(t,\lambda)\\ 2M_{10,12}^o(t,\lambda)+M_{10,23}^o(t,\lambda)\\ 2M_{10,13}^o(t,\lambda)+M_{10,33}^o(t,\lambda)\end{pmatrix},
\end{gathered}
\tag{7.9}
$$

where $M_{10,ij}^o(t,\lambda)$, $(i=1,2,3;\ j=1,2,3)$ is the entry of the matrix $M_{10}^o(t,\lambda)$ placed in its i-th row and j-th column.

Solving the terminal-value problem (4.31) with $t_f=4$ and the data from (7.8), we obtain

$$
\begin{aligned}
2M_{10,11}^o(t,\lambda)+M_{10,13}^o(t,\lambda)&=\frac{2\lambda_1}{(4\lambda_1+\lambda_2)(4-t)+1},\\
2M_{10,12}^o(t,\lambda)+M_{10,23}^o(t,\lambda)&=\frac{3\lambda_1\lambda_2(4-t)}{(4\lambda_1+\lambda_2)(4-t)+1},\\
2M_{10,13}^o(t,\lambda)+M_{10,33}^o(t,\lambda)&=\frac{\lambda_2}{(4\lambda_1+\lambda_2)(4-t)+1}.
\end{aligned}
\tag{7.10}
$$

Furthermore, for the sake of further calculations, we need the entries $M_{10,11}^o(t,\lambda)$, $M_{10,13}^o(t,\lambda)$, $M_{10,33}^o(t,\lambda)$, which are the following:

$$
\begin{aligned}
M_{10,11}^o(t,\lambda)&=\frac{\lambda_1\lambda_2(4-t)+\lambda_1}{(4\lambda_1+\lambda_2)(4-t)+1},\\
M_{10,13}^o(t,\lambda)&=-\frac{2\lambda_1\lambda_2(4-t)}{(4\lambda_1+\lambda_2)(4-t)+1},\\
M_{10,33}^o(t,\lambda)&=\frac{4\lambda_1\lambda_2(4-t)+\lambda_2}{(4\lambda_1+\lambda_2)(4-t)+1}.
\end{aligned}
\tag{7.11}
$$

Now, we should obtain the solution λ_0^* of the optimization problem (4.40)-(4.41). The maximized function $J_0(\lambda)$ of this problem depends on the vector $x^0(\lambda)$ and on the vector-valued function $\Phi_{1,x}(t,\lambda)$. From the equations (2.3),(3.4),(4.15),(4.93), as well as the data of the example (7.2) and the equation (7.7), we obtain the vectors $x^0(\lambda)$ and $\Phi_{1,x}(t,\lambda)$

$$x^0(\lambda) \equiv x^0 = \mathrm{col}(1,0,1), \quad \Phi_{1,x}(t,\lambda) \equiv \Phi_{1,x} = \mathrm{col}(1,0,1). \tag{7.12}$$

Based on the equation (4.41) and using the equation (7.12), we obtain (after a routine calculations) the function $J_0(\lambda)$ in the form

$$J_0(\lambda) = \frac{4\lambda_1\lambda_2+1}{4(4\lambda_1+\lambda_2)+1} + \frac{4\lambda_1\lambda_2}{4\lambda_1+\lambda_2} + \frac{4\lambda_1+\lambda_2-\lambda_1\lambda_2}{(4\lambda_1+\lambda_2)^2}\ln\big(4(4\lambda_1+\lambda_2)+1\big), \tag{7.13}$$

where the vector $\lambda \in \Omega_\lambda$ (see the equation (7.5)).

This function has the unique maximum point $\lambda_0^* \in \Omega_\lambda$, meaning the fulfilment of the assumption AVI in this example. This maximum point is $\lambda_0^* = \mathrm{col}(\lambda_{0,1}^*, \lambda_{0,2}^*) = (0.037, 0.963)$. The corresponding maximal value $J_0\big(\lambda_0^*\big)$ of the function $J_0(\lambda)$ is 1.814. By virtue of Definition 2.5 and Corollary 6.2, the optimal value of the functional in the two-model singular optimal control problem (7.1),(7.6) is $\mathscr{J}^*(w^0) = 1.814$.

Using Theorem 6.3, as well as the equations (5.3),(6.14),(7.9) and the vector $\lambda_0^* = \mathrm{col}(0.037, 0.963)$, we obtain the control entry in the solution to the two-model stochastic singular optimal control problem (7.1),(7.6)

$$\begin{aligned}\{\widehat{u}_q\big(w(t),t\big)\}_{q=1}^{+\infty} = \Bigg\{ -\frac{1}{\varepsilon_q}\Bigg[\frac{0.074w_{1,1}(t)}{1.111(4-t)+1} + \left(\frac{0.107(4-t)}{1.111(4-t)+1} + 0.037\right) w_{2,1}(t) \\ + \frac{0.963w_{1,2}(t)}{1.111(4-t)+1} + \left(\frac{0.107(t-4)}{1.111(4-t)+1} + 0.963\right) w_{2.2}(t)\Bigg]\Bigg\}_{q=1}^{+\infty},\end{aligned}$$

where the sequence of numbers $\{\varepsilon_q\}_{q=1}^{+\infty}$ is given by (6.13).

Thus, the sequence of the pairs $\left\{\left(\widehat{u}_q\big(w(t),t\big), \lambda_0^*\right)\right\}_{q=1}^{+\infty}$ is the solution of the two-model stochastic singular optimal control problem (7.1),(7.6), and

$$\max_{\lambda\in\Omega_\lambda} \inf_{u\in U} \Big(\lambda_1\mathscr{F}(u,1) + \lambda_2\mathscr{F}(u,2)\Big) = 1.814.$$

8. Concluding Remarks

CRI. In this paper, the finite horizon multi-model stochastic linear-quadratic optimal control problem was considered. The functional of this problem does not contain the control function which means that the considered optimal control problem is singular. The definition of the solution to this problem was proposed.

CRII. The original control problem is solved by the regularization approach. Namely, this problem is transformed approximately to an auxiliary regular optimal control problem. The latter has the same multi-model system of dynamics and a similar functional augmented by a finite horizon integral of the square of the Euclidean norm of the vector-valued control with a small positive weight (a small parameter). Thus, the auxiliary problem is a finite horizon multi-model stochastic linear-quadratic optimal control problem with a cheap control.

CRIII. The solution of this multi-model stochastic cheap control problem was reduced to the consecutive solution of the following two problems. The first problem is the terminal-value problem for the extended matrix Riccati differential equation. This problem depends not only on the aforementioned small parameter, but also on an auxiliary vector-valued parameter. The dimension of the latter equals to the number of the models in the multi-model system, and this vector-valued parameter belongs to the proper bounded and closed set in the corresponding Euclidean space. The second problem is the nonlinear optimization problem. The cost function of this problem depends on the small parameter, and this cost function is maximized with respect to the vector-valued parameter.

CRIV. An asymptotic analysis of each of the aforementioned two problems was carried out. Namely, for the first problem, zero-order asymptotic solutions is formally constructed and justified. It was shown that this asymptotic solution is valid uniformly with respect to the vector-valued parameter. For the second problem, the continuity of its solution with respect to the small parameter as the latter tends to zero was shown.

CRV. Based on this asymptotic analysis, the expression of the optimal value of the functional in the original singular optimal control problem was derived. The solution to the original problem also was obtained.

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