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EFFICIENCY CONDITIONS FOR NONCONVEX MATHEMATICAL PROGRAMMING PROBLEMS VIA WEAK SUBDIFFERENTIALS

TRAN VAN SU

Faculty of Mathematics, The University of Danang
University of Science and Education, Da Nang 550000, Vietnam

Dedicated to Professor Elijah Lucien Polak on the occasion of his 90th birthday

Abstract. In this paper, we study some new characterizations for the weak subdifferentials with lower Lipschitz functions in real normed space and its applications to nonconvex mathematical programming problems having set, inequality and equality constraints. First, some new properties of the weak subdifferential and the argumented normal cone are formulated. Second, the fuzzy sum rules, in general, in terms of weak subdifferentials are proposed. Third, we derive some necessary and sufficient optimality conditions for having the global minimum. Finally, some necessary and sufficient optimality conditions for the (weak) efficiency of such problems are obtained.
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1. Introduction

The current paper aims at contributing to the nonconvex mathematical programming theory and the necessary conditions for having the global minimum. Our work here can be viewed as the continuation of the scientists during the last twenty years concerning the weak subdifferentials and augmented normal cones; see [2, 4, 5, 6, 9, 12, 15, 16, 17, 18, 19, 20, 21, 22] and the references therein. As far as we know, Subgradients play a crucial role in the constructing the weak subdifferential notion and the augmented normal cone notion. Note that the weak subdifferential and the augmented normal cone have a compatible relationship involving indicator functions. The concept of weak subdifferential was first proposed by Kasimbeyli - Inceoglu - Mammadov in Refs. [2, 9], which is an overview of the classical subdifferential notion in real normed space. The concept of augmented weak subdifferential that we introduce for a vector-valued mapping in Definition 2.7 has not been reviewed before; see [18] and the cited references therein. We observe that the authors [18] provided only the concept of second-order

E-mail address: tvsu@ued.udn.vn
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classical subdifferential for a vector-valued mapping and then they obtained some properties of second-order scalar subdifferentials. Thus, the definition 2.7 in our paper is really useful and is a significant deterministic method to investigate nonsmooth nonconvex mathematical programming problems in infinite-dimensional space. For this goal, some the fuzzy sum rules for the augmented weak subdifferential and the weak subdifferential are formulated and then they are employed in deriving necessary and sufficient optimality conditions for the (weak) efficiency of a nonconvex mathematical programming problem having set, equality and inequality constraints. Also, some important properties of the weak subdifferential, the augmented weak subdifferential and the weak normal cone are studied in accordance with the class of lower Lipschitz functions together with the class of indicator functions.

In recent years, finding some key properties of the weak subdifferential and the (augmented) weak normal cone has been an important subject of study, as is shown in the works of Kasimbeyli and Mammadov [1, 2] and some other related authors. The author [8] introduced only the cononical generalized gradient notion and its applications to the nonsmooth mathematical programs. However, to the best of our knowledge, in most papers in the content of the reference, some important characterizations of the weak subdifferential/ and normal cone as well as the fuzzy sum rules are not discovered and, in our literature, in which those concepts are considered, the necessary and sufficient optimality conditions for efficiency in any nonconvex mathematical programming problem concerning lower Lipschitz functions are established. The equivalent formulation between the weak subdifferential and the augmented normal cone is also presented. We mention once again that the concept of weak subdifferential is provided based on the use of supporting cones instead of supporting hyperplanes, which was introduced by the authors in Refs. [3, 10, 11]. Such concept plays a crucial role in nonlinear analysis that allows establishing necessary/and sufficient optimality conditions for efficiency in any nonconvex mathematical programming problems (see [7, 8, 13, 20, 21, 22] for more details). This is the motivation for our work in the present literature.

The paper is organized as follows. The definition of weak subdifferentials, augmented normal cones and some preliminaries results are provided in the next section. Some characterizations of weak subdifferential and argumented normal cone along with some necessary and sufficient optimality conditions via weak subdifferentials and (augmented) weak normal cones for global minimum are presented in Section 3. Section 4 is devoted to establishing some necessary and sufficient optimality conditions in terms of weak subdifferentials for the (weak) efficiency of a nonconvex mathematical programming problem with set, inequality and equality constraints in real normed space. Finally, Section 5 presents some conclusions.

2. Preliminaries

Let X be a real normed space with a norm $\|\cdot\|$, and let X^* be the topological dual of X. By $\langle \xi, x\rangle$ we denote the value of the continous linear functional ξ defined on X at the vector $x \in X$. Let C be a nonempty subset of X and $\bar{x} \in C$. Let $\delta(\,.\,,C)$ be an indicator function of C, that is,

$$\delta(x,C) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

Let $\mathbb{R}^n$ be a $n-$dimensional Euclidean space, $\mathbb{R}^n_+$ be a non-negative orthant cone of $\mathbb{R}^n$, $\mathbb{R}^n_{++}$ be the topological interior of a non-negative orthant cone $\mathbb{R}^n_+$, $\mathscr{P}_n$ be the set of all continuous positively homogeneous subadditive convex functions on $\mathbb{R}^n$, $\mathbb{R}$ be the real numbers set, $\mathbb{N}$ be the positive integer numbers set and let $m,n,p,q \in \mathbb{N}$. Let a vector-valued mapping $k : X \to \mathbb{R}^n$. Then the domain, epigraph and hypograph of the mapping $k|_C$ are formulated respectively by

$$\begin{aligned}
&\text{dom}k|_C := \{x \in C | k(x) \neq \emptyset\},\\
&\text{epi}k|_C := \{(x,r) \in C \times \mathbb{R}^n | r \in k(x) + \mathbb{R}^n_+\},\\
&\text{hypo}k|_C := \{(x,r) \in C \times \mathbb{R}^n | r \in k(x) - \mathbb{R}^n_+\}.
\end{aligned}$$

Definition 2.1. ([2, 14]) Let l be a real-valued function defined on X and $\bar{x} \in X$.

(i) l is called lower locally Lipschitz at $\bar{x}$ if there exist a neighborhood of $\bar{x}$, say U, and a non-negative Lipschitz constant L such that $l(x) - l(\bar{x}) \geq -L\|x - \bar{x}\|$ for every $x \in U$. If the above inequality holds for every $x \in X$, then l is called lower Lipschitz at $\bar{x}$.

(ii) l is called calm at $\bar{x}$ if there exist a neighborhood of $\bar{x}$, say U, and a non-negative calm constant L such that $|l(x) - l(\bar{x})| \leq L\|x - \bar{x}\|$ for every $x \in U$.
It is said that l is Lipschitz around $\bar{x}$ if $|l(x) - l(y)| \leq L\|x - y\|$ for every $x, y \in U$. Especially l is called locally Lipschitz on X, if for each $\bar{x} \in X$, there exists a neighborhood U of $\bar{x}$ such that l is Lipschitz around $\bar{x}$. If l is Fréchet differentiable at $\bar{x}$, then its Fréchet derivative at $\bar{x}$ is denoted by $\nabla l(\bar{x})$.

We remark that the class of lower locally Lipschitz functions is wider than the class of calm functions and so is the class of locally Lipschitz functions.

Definition 2.2. ([1]) The augmented normal cone to C at $\bar{x}$ is the set

$$N^a_C(\bar{x}) = \{(\xi, r) \in X^* \times \mathbb{R}_+ \mid \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq 0 \ (\forall x \in C)\}.$$

Remark 2.3. Note that the augmented normal cone notion is extended from the usual normal cone notion in Convex Analysis. The set $N^a_C(\bar{x})$ is a nonempty closed convex cone. Observe that, if $\|\xi\| \leq r$, then

$$\sup_{x \in C \setminus \{\bar{x}\}} \left| \left\langle \xi, \frac{x - \bar{x}}{|x - \bar{x}|} \right\rangle \right| \leq r,$$

which ensures that $(\xi, r) \in N^a_C(\bar{x})$. An augmented normal cone consisting of only such elements is said to be trivial and formulated by $N^{triv}_C(\bar{x})$. It can be verified that $N^a_C(\bar{x}) \supset N^{triv}_C(\bar{x})$. In the case $C = X$, we have $N^{triv}_C(\bar{x}) \equiv N^a_C(\bar{x})$.

Definition 2.4. ([2]) Let $f : X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point where $f(\bar{x})$ is finite. A pair $(\xi, r) \in X^* \times \mathbb{R}_+$ is called the weak subgradient of f at $\bar{x}$ if

$$f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \ \text{ for all } x \in X. \tag{2.1}$$

The set

$$\partial^w f(\bar{x}) := \{(\xi, r) \in X^* \times \mathbb{R}_+ | f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \ (\forall x \in X)\}$$

is called the weak subdifferential of f at $\bar{x}$. If $\partial^w f(\bar{x}) \neq \emptyset$, then f is called weakly subdifferentiable at $\bar{x}$. If (2.1) is valid only $x \in C$, where $C \subset X$, then we say that f is weakly subdifferentiable at $\bar{x}$ on C. One uses the symbol $\partial^w_C f(\bar{x})$ to instead of the weak subdifferential of f at $\bar{x}$ on C. It is not difficult to verify that $\partial^w f(\bar{x}) \subset \partial^w_C f(\bar{x})$. Besides, if f is lower Lipschitz at $\bar{x}$, then there exists a non-negative Lipschitz constant L such that $(0, L) \in \partial^w f(\bar{x})$ and thus f is weakly subdifferentiable at $\bar{x}$.

Example 2.5. Let a real-valued function f be defined on X, given for all $x \in X$ by

$$f(x) = \begin{cases} 0 & \text{if } \|x\| \leq 1, \\ \|x\| & \text{if } \|x\| > 1. \end{cases}$$

For the illustration let us may take $\bar{x} \in C$ be arbitrary, where the set C is an unit ball of X, means that $C = \{x \in X | \|x\| \leq 1\} \subset X$. Then $\partial^w f(\bar{x}) = N_C^a(\bar{x})$. In fact, for every $(\xi, r) \in N_C^a(\bar{x})$ is equivalent to $\langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq 0$ $(\forall x \in C)$. Since $f(x) - f(\bar{x}) = \|x\| \geq 0$ for all $x \in X \setminus C$ and $f(x) - f(\bar{x}) = 0$, otherwise. Thus, $(\xi, r) \in N_C^a(\bar{x})$ is equivalent to $\langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \leq f(x) - f(\bar{x})$ $(\forall x \in X)$, that is $(\xi, r) \in \partial^w f(\bar{x})$, which completes the check.

Remark 2.6. ([2]) The nonempty set $\partial^w f(\bar{x})$ is closed convex in $X^* \times \mathbb{R}_+$. Further, it yields from Definition 2.4 that the pair $(\xi, r) \in X \times \mathbb{R}_+$ is a weak subgradient of f at $\bar{x} \in X$ if and only if there exists a continuous (super linear) concave function $k : X \to \mathbb{R}$ defind by

$$k(x) = f(\bar{x}) + \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \ (\forall x \in X),$$

satisfies $k(x) \leq f(x)$ for every $x \in X$ and $k(\bar{x}) = f(\bar{x})$, means that k supports f from below. Therefore, it entails that if f is weakly subdifferentiable at $\bar{x}$ and $(\xi, r) \in \partial^w f(\bar{x})$, then the graph of function k becomes a supporting surface to the epigraph of f on X at the point $(\bar{x}, f(\bar{x}))$. Especially, for the gradient of k at $\bar{x}$, that is $\nabla k(\bar{x})$, one can achieve that

$$\nabla k(\bar{x}) = \xi - rz, \ \ z = \frac{x - \bar{x}}{|x - \bar{x}|},$$

which guarantees the bounded of the norm of $\nabla k(\bar{x})$. This result will be useful in estimating the subgradients for finding the extremal points of a nonsmooth function.

Definition 2.7. Let $f : X \to \mathbb{R}^n$ be a vector-valued mapping and $\bar{x} \in X$ be a given point. A triple $(\xi, P, r) \in X^* \times (\mathscr{P}_n \setminus \{0\}) \times \mathbb{R}_+$ is called the augmented weak subgradient of f at $\bar{x}$ if

$$\langle P, f(x) - f(\bar{x}) \rangle \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \text{ for all } x \in X. \tag{2.2}$$

The set

$$\partial_a^w f(\bar{x}) := \Big\{ (\xi, P, r) \in X^* \times (\mathscr{P}_n \setminus \{0\}) \times \mathbb{R}_+ | \\ \langle P, f(x) - f(\bar{x}) \rangle \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\| \ (\forall x \in X) \Big\}$$

is called the augmented weak subdifferential of f at $\bar{x}$. If $\partial_a^w f(\bar{x}) \neq \emptyset$, then f is called augmented weakly subdifferentiable at $\bar{x}$. If (2.2) is valid only $x \in C$, where $C \subset X$, then we say that f is augmented weakly subdifferentiable at $\bar{x}$ on C. One uses the symbol $\partial_{a,C}^w f(\bar{x})$ to instead of the augmented weak subdifferential of f at $\bar{x}$ on C. It is not difficult to verify that $\partial_a^w f(\bar{x}) \subset \partial_{a,C}^w f(\bar{x})$.

Example 2.8. Let a $\mathbb{R}^3$-valued mapping f be defined on $\mathbb{R}$, given by $f(x) = (x, 2x, 3x)$ for all $x \in \mathbb{R}$. For the illustration let us take $\bar{x} := (0,0,0) \in \mathbb{R}^3$, $C = \mathbb{R}_+$ and $P := (P_1, P_2, P_3) \in \mathscr{P}_3 \setminus \{(0,0,0)\}$. Then

$$\partial_a^w f(\bar{x}) = \Big\{ (\xi, P, r) \in \mathbb{R} \times (\mathscr{P}_3 \setminus \{(0,0,0)\}) \times \mathbb{R}_+ \big| \, |\xi - \sum_{i=1}^3 iP_i| \leq r \Big\},$$

$$\partial_{a,C}^w f(\bar{x}) = \Big\{ (\xi, P, r) \in \mathbb{R} \times (\mathscr{P}_3 \setminus \{(0,0,0)\}) \times \mathbb{R}_+ \big| \, \xi - \sum_{i=1}^3 iP_i \leq r \Big\}.$$

Indeed, for any $(\xi,P,r)\in\mathbb{R}\times(\mathscr{P}_3\setminus\{(0,0,0)\})\times\mathbb{R}_+$, $(\xi,P,r)\in\partial^w_a f(\bar{x})$ is equivalent to

$$\langle\xi,x\rangle - r|x| \le (\sum_{i=1}^{3} iP_i)x\ (\forall x\in\mathbb{R}) \text{ i.e., } |\xi-\sum_{i=1}^{3} iP_i|\le r.$$

Similarly, we calculate that $(\xi,P,r)\in\partial^w_{a,C}f(\bar{x})$ is equivalent to the inequality $\xi-\sum_{i=1}^{3} iP_i\le r$, as it was checked.

Definition 2.9. ([5, 23]) Let $\emptyset\neq K\subset X$ and $\bar{x}\in X$.

(i) A function $l:X\to\mathbb{R}$ has a global minimum at $\bar{x}$ if $l(x)\ge l(\bar{x}),\ \forall x\in X$.
(i') A function $l:X\to\mathbb{R}$ has a global maximum at $\bar{x}$ if $l(x)\le l(\bar{x}),\ \forall x\in X$.
(ii) A function $l:X\to\mathbb{R}$ has a global minimum at $\bar{x}\in K$ on K if $l(x)\ge l(\bar{x})$ for all $x\in K$.
(ii') A function $l:X\to\mathbb{R}$ has a global maximum at $\bar{x}\in K$ on K if $l(x)\le l(\bar{x})$ for all $x\in K$.
(iii) A mapping $l:X\to\mathbb{R}^n$ has a weakly efficient solution at $\bar{x}\in K$ on K if $l(x)-l(\bar{x})\notin -\mathbb{R}^n_{++}$ for all $x\in K$.
(iv) A mapping $l:X\to\mathbb{R}^n$ has an efficient solution at $\bar{x}\in K$ on K if $l(x)-l(\bar{x})\notin -(\mathbb{R}^n_+\setminus\{0\})$ for all $x\in K$.

Remark 2.10. Obviously that condition (iii) implies condition (ii) and moreover this notions are coincide in the case $K=X$ and $n=1$. If l has a global minimum at $\bar{x}\in K$, then l also has a global minimum at $\bar{x}$ on K.

3. Some basic characterizations of weak subdifferential and argumented normal cone

In this section, we study some fundamental characterizations of the weak subdifferential and the augmented normal cone in real normed space.

Proposition 3.1. *Let $C\subset X$ and $f:X\to\mathbb{R}$ be weakly subdifferentiable at $\bar{x}\in C$. Then*

$$\partial^w f(\bar{x})+N^a_C(\bar{x})\subset\partial^w(f+\delta(\,.\,,C))(\bar{x})\subset\partial^w_C f(\bar{x}). \tag{3.1}$$

Proof. Let $(\xi,r)\in\partial^w f(\bar{x})$ and $(\eta,s)\in N^a_C(\bar{x})$ be arbitrary. For every $x\in X$, it follows from the definitions that

$$f(x)-f(\bar{x})\ge\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|+\langle\eta,x-\bar{x}\rangle-s\|x-\bar{x}\|,$$

which guarantees the following inequality

$$(f+\delta(\,.\,,C))(x)-(f+\delta(\,.\,,C))(\bar{x})\ge\langle\xi+\eta,x-\bar{x}\rangle-(r+s)\|x-\bar{x}\|.$$

Thus, $(\xi+\eta,r+s)\in\partial^w(f+\delta(\,.\,,C))(\bar{x})$, means that the first inclusion in (3.1) is valid.

For the last inclusion in (3.1), we always have

$$\partial^w(f+\delta(\,.\,,C))(\bar{x})\subset\partial^w_C(f+\delta(\,.\,,C))(\bar{x})=\partial^w_C f(\bar{x}),$$

which follows the claim. □

Proposition 3.2. *Let* $\dim X<+\infty$, *$f:X\to\mathbb{R}$ be a function that attains a global minimum on $C\subset X$ at $\bar{x}\in C$. If $-f$ is weakly subdifferentiable at $\bar{x}$, then we have*

$$\partial^w(-f)(\bar{x})+N^a_C(\bar{x})\subset N^{triv}_C(\bar{x})\subset N^a_C(\bar{x})\subset\partial^w_C f(\bar{x}). \tag{3.2}$$

Proof. By the initial hypotheses, $-f$ is weakly subdifferentiable at $\bar{x}$. Making use of Proposition 3.1, we conclude that $-f+\delta(\,.\,,C)$ is also weakly subdifferentiable at $\bar{x}$. By taking $(\xi,r)\in\partial^w(-f)(\bar{x})+N_C^a(\bar{x})$. Under (3.1) it entails that $(\xi,r)\in\partial^w(-f+\delta(\,.\,,C))(\bar{x})$, and thus, one can reach the following one

$$-f(x)+\delta(x,C)+f(\bar{x})-\delta(\bar{x},C)\geq\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|,\ (\forall x\in X).$$

Since $\bar{x}\in C$ and f attains a global minimum at $\bar{x}$, it yields that

$$0\geq(-f)(x)-(-f)(\bar{x})\geq\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|,\ (\forall x\in C),$$

which ensures that $\|\xi\|\leq r$. Thus, $(\xi,r)\in N_C^{triv}(\bar{x})\subset N_C^a(\bar{x})$. Because f attains a global minimum on C at $\bar{x}\in C$, it follows $f(x)-f(\bar{x})\geq 0$ for all $x\in C$. Hence, for any $(\xi,r)\in N_C^a(\bar{x})$ and for any $x\in C$, one can obtain that $\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|\leq f(x)-f(\bar{x})$ and so is (3.2). Therefore, we get the desired conclusions. □

Corollary 3.3. *Let* $\dim X<+\infty$, $f:X\to\mathbb{R}$ *be a function that attains a global maximum on* $C\subset X$ *at* $\bar{x}\in C$. *If* f *is weakly subdifferentiable at* $\bar{x}$, *then we have*

$$\partial^w f(\bar{x})+N_C^a(\bar{x})\subset N_C^{triv}(\bar{x})\subset N_C^a(\bar{x})\subset\partial_C^w(-f)(\bar{x}). \tag{3.3}$$

Proof. Since f attains a global maximum on C at $\bar{x}\in C$, so is $-f$ attains a global minimum on C at $\bar{x}\in C$. In view of Proposition 3.2 with observing $-f$ replacing f, we deduce that (3.3) is fulfilled, which completes the proof. □

Corollary 3.4. *Let* $\dim X<+\infty$, $f:X\to\mathbb{R}$ *be a function that attains a global maximum on* $C\subset X$ *at* $\bar{x}\in\operatorname{int}C$. *If* f *is weakly subdifferentiable at* $\bar{x}$, *then we have*

$$\partial^w f(\bar{x})+N_C^a(\bar{x})\subset N_C^{triv}(\bar{x})=N_C^a(\bar{x})\subset\partial_C^w(-f)(\bar{x}). \tag{3.4}$$

Proof. Taking into account Corollary 3.3, it suffices to show that $N_C^{triv}(\bar{x})\supset N_C^a(\bar{x})$. Since $\bar{x}\in\operatorname{int}C$, there exists $\delta>0$ such that the sphere $S_\delta(\bar{x})=\{x\in X\,|\,\|x-\bar{x}\|=\delta\}\subset C$. On the contrary, suppose that there is a pair $(\xi,r)\in N_C^a(\bar{x})$ but $(\xi,r)\notin N_C^{triv}(\bar{x})$, means that $\|\xi\|>r\geq 0$. One gets the right inequality $\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|\leq 0$ for every $x\in C$, which guarantees that

$$\langle\xi,x-\bar{x}\rangle-r\|x-\bar{x}\|\leq 0$$

for every $x\in S_\delta(\bar{x})$. Evidently,

$$\left\langle\xi,\frac{x-\bar{x}}{\|x-\bar{x}\|}\right\rangle-r\leq 0\ (\forall x\in S_\delta(\bar{x})),$$

or equivalently, $\langle\xi,x\rangle-r\leq 0$ for all $x\in S_1(0))$. Thus, $\|\xi\|\leq r$, this is a contradiction and completes the proof of Corollary 3.4. □

Proposition 3.5. *Let* $C\subset X$ *and* $f:X\to\mathbb{R}$ *be weakly subdifferentiable at* $\bar{x}\in C$. *Then*

(i) *If* $(\xi,r)\in\partial_C^w f(\bar{x})$, *then* $((\xi,-1),r)\in N^a_{\operatorname{epi}f|_C}(\bar{x},f(\bar{x}))$.

(ii) *If* $((\xi,\lambda),r)\in N^a_{\operatorname{epi}f|_C}(\bar{x},\mu)$ *with* $(\bar{x},\mu)\in\operatorname{epi}f|_C$, *then* $|\lambda|\leq r$.

Proof. Since $C \subset X$ and f is weakly subdifferentiable at $\bar{x} \in C$, it yields that f is weakly subdifferentiable on C at $\bar{x} \in C$, that is, the set $\partial_C^w f(\bar{x})$ is not empty.

(i): Assume that $(\xi, r) \in \partial_C^w f(\bar{x})$, one gets for every $x \in X$,

$$f(x) + \delta(x,C) - f(\bar{x}) - \delta(\bar{x},C) \geq \langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\|,$$

which implies that

$$(f + \delta(.,C))(x) - (f + \delta(.,C))(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - r(\|x - \bar{x}\| + |f(x) - f(\bar{x})|).$$

Therefore,

$$\begin{array}{r} 0 \geq \langle (\xi, -1),\ (x - \bar{x}, (f + \delta(.,C))(x) - (f + \delta(.,C))(\bar{x})) \rangle \\ - r\|(x - \bar{x},\ f(x) - f(\bar{x}))\| \end{array} \quad (\forall x \in X),$$

which combined with the equalities $\operatorname{epi}(f + \delta(.,C)) = \operatorname{epi} f|_C$ and

$$N^a_{\operatorname{epi}(f+\delta(.,C))}(\bar{x}, (f + \delta(.,C))(\bar{x})) = N^a_{\operatorname{epi} f|_C}(\bar{x}, f(\bar{x}))$$

prove that $((\xi, -1), r) \in N^a_{\operatorname{epi} f|_C}(\bar{x}, f(\bar{x}))$, as required.

(ii): Suppose that $((\xi, \lambda), r) \in N^a_{\operatorname{epi} f|_C}(\bar{x}, \mu)$ with $(\bar{x}, \mu) \in \operatorname{epi} f|_C$. Since $\bar{x} \in C$, by applying the augmented normal cone notion, we arrive at the conclusion that $\lambda(s - \mu) - r|s - \mu| \leq 0$ for all $(\bar{x}, s) \in \operatorname{epi} f|_C$. We always have $f(\bar{x}) \leq \min\{s, \mu\}$ because $(\bar{x}, \mu),\ (\bar{x}, s) \in \operatorname{epi} f|_C$. In the case when $\mu \leq s$, it follows from the inequality above that $\lambda \leq r$. In the case when $s = f(\bar{x})$, it can be verified that $\lambda \geq -r$, and hence, $|\lambda| \leq r$, which completes the proof. □

Theorem 3.6. *(***Optimality conditions for global minimum***) Let a nonempty subset $K \subset C \subset X$, $\bar{x} \in K$ and let $l : X \to \mathbb{R} \cup \{+\infty\}$ be lower Lipschitz at $\bar{x}$. We have the following assertions*

(i) *If l has a global minimum on K at $\bar{x}$, then there exists a real number $r_0 \geq 0$ such that $(0, r) \in \partial^w l(\bar{x}) + N^a_C(\bar{x})$ for every $r \geq r_0$.*

(ii) *If $(0,0) \in \partial^w l(\bar{x}) + N^a_K(\bar{x})$, then l attains a global minimum on K at $\bar{x}$.*

Proof. Since l is lower Lipschitz at $\bar{x}$, it is weakly subdifferentiable at $\bar{x}$, that is, $\partial^w l(\bar{x}) \neq \emptyset$.

(i): Since l attains a global minimum on K at $\bar{x}$, it follows that $l(x) \geq l(\bar{x})$ for every $x \in K$. To finish the proof, we assume to the contrary, that for every $r > 0$, there exists $r_0 \geq r$ such that $(0, r_0) \notin \partial^w l(\bar{x}) + N^a_C(\bar{x})$. Because $\bar{x} \in K \subset C$, it is evident that $(0,0) \in N^a_C(\bar{x})$ and so $\partial^w l(\bar{x}) \subset \partial^w l(\bar{x}) + N^a_C(\bar{x})$. Therefore, $(0, r_0) \notin \partial^w l(\bar{x})$. By the weak subdifferential notion, there exists $x_0 \in X \setminus \{\bar{x}\}$ such that

$$l(x_0) - l(\bar{x}) < \langle 0, x_0 - \bar{x} \rangle - r_0\|x_0 - \bar{x}\| = -r_0\|x_0 - \bar{x}\|. \tag{3.5}$$

If $x_0 \in K \setminus \{\bar{x}\}$, then, it follows from (3.5) that $l(x_0) - l(\bar{x}) < 0$, which contradicts to f has a global minimum on K at $\bar{x}$. If $x_0 \in (X \setminus K) \setminus \{\bar{x}\}$, then one can achieve from (3.5) and $r_0 \geq r$ that

$$r \leq r_0 < \frac{l(\bar{x}) - l(x_0)}{\|x - \bar{x}\|} \ \ (\forall r > 0). \tag{3.6}$$

By the initial assumption, l is lower Lipschitz at $\bar{x}$, one can find a non-negative Lipschitz constant $L_0 > 0$ such that $l(x) - l(\bar{x}) \geq -L_0\|x - \bar{x}\|\ \ (\forall x \in X)$, which combined with (3.6) ensures that $r < L_0$ for every $r > 0$, this is a contradiction. In consequence, there exists $r_0 \geq 0$ such that $(0, r) \in \partial^w l(\bar{x}) + N^a_C(\bar{x})$ for every $r \geq r_0$.

(ii): Suppose that $(0,0) \in \partial^w l(\bar{x}) + N_K^a(\bar{x})$, which combined with (3.1) in Proposition 3.1 yields that $(0,0) \in \partial_K^w l(\bar{x})$. Thus, $l(x) - l(\bar{x}) \geq 0$ for all $x \in K$, which terminates the proof. □

Corollary 3.7. *Let a nonempty subset $K \subset X$, $\bar{x} \in K$ and let $l : X \to \mathbb{R} \cup \{+\infty\}$ be lower Lipschitz at $\bar{x}$. Then, if l has a global minimum on K at $\bar{x}$, then there exists a real number $r_0 \geq 0$ such that $(0,r) \in \partial^w l(\bar{x}) + N_K^a(\bar{x})$ for every $r \geq r_0$.*

Proof. The proof is straightforward from Theorem 3.6, as required. □

Theorem 3.8. *(***Optimality conditions via weak subdifferential***) Let a nonempty subset $K \subset C \subset X$, $\bar{x} \in K$ and let $f : X \to \mathbb{R}^n$ such that for every $P \in \mathscr{P}_n$, $P_0 f$ be lower Lipschitz at $\bar{x}$, where $P_0 f : X \to \mathbb{R}$ is defined by $(P_0 f)(x) := \langle P, f(x) \rangle$, $(\forall x \in X)$. We have the following assertions*

(i) *If f has a weakly efficient solution on K at $\bar{x}$, then there exist $P \in \mathscr{P}_n \setminus \{0\}$ and a real number $r_0 \geq 0$ such that*

$$\begin{cases} y_2 - y_1 \in \mathbb{R}^n_{++} \Longrightarrow P(y_1) < P(y_2), \\ (0,r) \in \partial^w (P_0 f)(\bar{x}) + N_C^a(\bar{x}) \ (\forall r \geq r_0). \end{cases} \tag{3.7}$$

(ii) *If there exist $P \in \mathscr{P}_n \setminus \{0\}$ such that*

$$\begin{cases} y_2 - y_1 \in \mathbb{R}^n_{++} \Longrightarrow P(y_1) < P(y_2), \\ (0,0) \in \partial^w (P_0 f)(\bar{x}) + N_K^a(\bar{x}). \end{cases} \tag{3.8}$$

Then f has a weakly efficient solution on K at $\bar{x}$.

(iii) *If there exist $P \in \mathscr{P}_n \setminus \{0\}$ such that*

$$\begin{cases} y_2 - y_1 \in \mathbb{R}^n_{+} \setminus \{0\} \Longrightarrow P(y_1) < P(y_2), \\ (0,0) \in \partial^w (P_0 f)(\bar{x}) + N_K^a(\bar{x}). \end{cases} \tag{3.9}$$

Then f has an efficient solution on K at $\bar{x}$.

Proof. By hypotheses, for any $P \in \mathscr{P}_n$, the scalar function $P_0 f$ is lower Lipschitz at $\bar{x}$, which ensures it is weakly subdifferentiable at $\bar{x}$, i.e., the set $\partial^w (P_0 f)(\bar{x})$ is not empty.

(i): Since f has a weakly efficient solution on K at $\bar{x}$, it entails from Theorem 3.1 [6] that there exist $P \in \mathscr{P}_n \setminus \{0\}$ such that if $y_2 - y_1 \in \mathbb{R}^n_{++}$, then $P(y_1) < P(y_2)$ and further $\langle P, f(x) - f(\bar{x}) \rangle \geq 0$ $(\forall x \in K)$. Thus $\langle P, f(x) \rangle \geq \langle P, f(\bar{x}) \rangle$ $(\forall x \in K)$. We have $P_0 f$ attains a global minimum on K at $\bar{x}$, which combined with Theorem 3.6 (i) yields that there exists a real number $r_0 \geq 0$ such that the system (3.7) above is fulfilled.

(ii): Assume that there is $P \in \mathscr{P}_n \setminus \{0\}$ satisfying the system (3.8). Then, we have $(0,0) \in \partial^w (P_0 f)(\bar{x}) + N_K^a(\bar{x})$ which yields the existence of $(\xi, r) \in N_K^a(\bar{x})$ such that $(\xi, r) \in -\partial^w (P_0 f)(\bar{x})$. For all $x \in K$, it results that $\langle P, f(x) - f(\bar{x}) \rangle \geq -\big(\langle \xi, x - \bar{x} \rangle - r\|x - \bar{x}\|\big) \geq 0$. This together with the result "$y_2 - y_1 \in \mathbb{R}^n_{++} \Longrightarrow P(y_1) < P(y_2)$" one can achieve that $f(x) - f(\bar{x}) \notin -\mathbb{R}^n_{++}$ for all $x \in K$, means that $\bar{x}$ being a weakly efficient solution of f.

(iii): Analogously to the proof of case (ii) with observing that the hypotheses "$y_2 - y_1 \in \mathbb{R}^n_+ \setminus \{0\} \Longrightarrow P(y_1) < P(y_2)$" one can reach the result $f(x) - f(\bar{x}) \notin -\mathbb{R}^n_+ \setminus \{0\}$ for any $x \in K$, i.e., $\bar{x}$ being an efficient solution of f, which completes the proof. □

Corollary 3.9. *Let a nonempty subset $K \subset X$, $\bar{x} \in K$ and let $f : X \to \mathbb{R}^n$ such that for every $P \in \mathscr{P}_n$, $P_0 f$ be lower Lipschitz at $\bar{x}$. Then, if f has a weakly efficient solution on K at $\bar{x}$, then there exist $P \in \mathscr{P}_n \setminus \{0\}$ and a real number $r_0 \geq 0$ such that*

$$\begin{cases} y_2 - y_1 \in \mathbb{R}^n_{++} \Longrightarrow P(y_1) < P(y_2), \\ (0,r) \in \partial^w (P_0 f)(\bar{x}) + N^a_K(\bar{x}) \ (\forall r \geq r_0). \end{cases}$$

Proof. Analogously to the proof of Theorem 3.8, and completes the proof. □

Theorem 3.10. *(**Optimality conditions via augmented weak subdifferential**) Let a nonempty subset $K \subset X$, $\bar{x} \in K$ and let $f : X \to \mathbb{R}^n$ such that for every $P \in \mathscr{P}_n$, $P_0 f$ be lower Lipschitz at $\bar{x}$. We have the following assertions*

(i) *If f has a weakly efficient solution on K at $\bar{x}$, then there exist $P \in \mathscr{P}_n \setminus \{0\}$ and a real number $r_0 \geq 0$ such that*

$$\begin{cases} y_2 - y_1 \in \mathbb{R}^n_{++} \Longrightarrow P(y_1) < P(y_2), \\ (0,P,r) \in \partial^w_a f(\bar{x}) \ (\forall r \geq r_0). \end{cases} \tag{3.10}$$

(ii) *If there exist $P \in \mathscr{P}_n \setminus \{0\}$ such that*

$$\begin{cases} y_2 - y_1 \in \mathbb{R}^n_{++} \Longrightarrow P(y_1) < P(y_2), \\ (0,P,0) \in \partial^w_a f(\bar{x}). \end{cases} \tag{3.11}$$

Then f has a weakly efficient solution on K at $\bar{x}$.

(iii) *If there exist $P \in \mathscr{P}_n \setminus \{0\}$ such that*

$$\begin{cases} y_2 - y_1 \in \mathbb{R}^n_{+} \setminus \{0\} \Longrightarrow P(y_1) < P(y_2), \\ (0,P,0) \in \partial^w_a f(\bar{x}). \end{cases} \tag{3.12}$$

Then f has an efficient solution on K at $\bar{x}$.

Proof. Analogously to the proof of case (i) in Theorem 3.6, it is not difficult to very that if $l : X \to \mathbb{R}$ has a global minimum on K at $\bar{x}$, then there exists a real number $r_0 \geq 0$ such that $(0,r) \in \partial^w l(\bar{x})$ for all $r \geq r_0$. Especially, for the case $l = P_0 f$, where $P \in \mathscr{P}_n \setminus \{0\}$, one has $(0,r) \in \partial^w l(\bar{x})$, i.e., for every $x \in K$, one obtains $\langle P, f(x) - f(\bar{x}) \rangle \geq -r\|x - \bar{x}\|$. Thus, $(0,P,r) \in \partial^w_a f(\bar{x})$. Then, in a similar idea as for proving Theorem 3.8 (i), we assert that if f has a weakly efficient solution on K at $\bar{x}$, then there exist $P \in \mathscr{P}_n \setminus \{0\}$ and a real number $r_0 \geq 0$ satisfying (3.10), and thus, (i) is fulfilled. On the other hand, we that $(0,P,0) \in \partial^w_a f(\bar{x})$ is equivalent to $(0,0) \in \partial^w (P_0 f)(\bar{x})$. Analogously to the proof of (ii) & (iii) in Theorem 3.8, the remain cases can be verified. □

4. Applications to a nonconvex mathematical programming problem

In this section, we derive some necessary and sufficient optimality conditions in terms of weak subdifferentials for the efficiency of a nonconvex mathematical programming problem with set, inequality and equality constraints in real normed space.

Now, we consider the following nonconvex mathematical programming problem with set, inequality and equality constraints:

$$\begin{aligned}
&\min f(x) = \big(f_1(x), f_2(x), \ldots, f_m(x)\big)\\
&\text{subject to } g_i(x) \geq 0,\ i = 1,2,\ldots,p,\\
&\qquad\qquad h_j(x) = 0,\ j = 1,2,\ldots,q,\\
&\qquad\qquad x \in C,
\end{aligned} \tag{MPPC}$$

where $f_i, g_j, h_k : X \to \mathbb{R}$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,p$; $k = 1,2,\ldots,q$ are given real-valued functions, and a nonempty subset $C \subset X$.

Definition 4.1. The feasible set of problem (MPPC) is denoted by K and is defined by

$$K := \{x \in C \,|\, g_j(x) \geq 0,\ j = 1,2,\ldots,p;\ h_k(x) = 0,\ k = 1,2,\ldots,q\}.$$

For each point $x \in K$ is called a feasible solution to the problem (MPPC).

For each $x \in X$, we denote by

$$\begin{aligned}
l(x) &:= \Big(l_1(x), l_2(x), \ldots, l_{p+2q}(x)\Big)\\
&:= \Big(g_1(x), \ldots, g_p(x), h_1(x), \ldots, h_q(x), -h_1(x), \ldots, -h_q(x)\Big)
\end{aligned}$$

where $l = (l_1, l_2, \ldots, l_{p+2q}) : X \to \mathbb{R}^{p+2q}$ is a vector-valued function. Then, the feasible set of problem (MPPC) can be re-written as

$$K := \{x \in C \,|\, l_i(x) \geq 0,\ i = 1,2,\ldots,p+2q\}.$$

Definition 4.2. A vector $\bar{x} \in K$ is said to be a weakly efficient solution to the problem (MPPC) if f has a weakly efficient solution on K at $\bar{x}$.

Definition 4.3. A vector $\bar{x} \in K$ is said to be an efficient solution to the problem (MPPC) if f has an efficient solution on K at $\bar{x}$.

In order to treat necessary and sufficient optimality conditions for the efficiency of problem (MPPC), the following fuzzy sum rules play an important role for our study in the sequel.

Theorem 4.4. *Let single-valued functions $f_i : X \to \mathbb{R} \cup \{+\infty\}$ be lower Lipschitz at $\bar{x} \in X$, $i = 1,2,\ldots,m$ and the non-negative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$. Then*

(i) *The functions $f_i,\ i = 1,2,\ldots,m$ are weakly subdifferentiable at $\bar{x} \in X$.*
(ii) *The sum function $\sum_{i=1}^m \alpha_i f_i$ is weakly subdifferentiable at $\bar{x} \in X$.*
(iii) *The following inclusion holds true:*

$$\sum_{i=1}^m \alpha_i \partial^w f_i(\bar{x}) \subset \partial^w \big(\sum_{i=1}^m \alpha_i f_i\big)(\bar{x}). \tag{4.1}$$

In addition, for any $\emptyset \neq C \subset X$, one also has

$$\sum_{i=1}^m \alpha_i \partial^w_C f_i(\bar{x}) \subset \partial^w_C \big(\sum_{i=1}^m \alpha_i f_i\big)(\bar{x}). \tag{4.2}$$

Proof. (i): Because the functions $f_i : X \to \mathbb{R} \cup \{+\infty\}$ are lower Lipschitz at $\bar{x} \in X,\ i = 1,2,\ldots,m$, there exist non-negative Lipschitz constants L_i such that $(0, L_i) \in \partial^w f_i(\bar{x}),\ i = 1,2,\ldots,m$. Thus, they are weakly subdifferentiable at $\bar{x} \in X$.

(ii): We always have

$$f_i(x) - f_i(\bar{x}) \geq -L_i\|x - \bar{x}\| \text{ for all } x \in X,\ i = 1,2,\ldots,m.$$

Taking $L = \sum_{i=1}^m \alpha_i$, one can obtain the result that $L \geq 0$ because $\alpha_i \geq 0,\ i = 1,2,\ldots,m$. Additionally, it is evident that

$$\begin{aligned}\sum_{i=1}^m \alpha_i f_i(x) - \sum_{i=1}^m \alpha_i f_i(\bar{x}) &= \sum_{i=1}^m \alpha_i \Big(f_i(x) - f_i(\bar{x})\Big)\\ &\geq -\sum_{i=1}^m \alpha_i L_i\|x - \bar{x}\| = -L\|x - \bar{x}\| \text{ for all } x \in X.\end{aligned}$$

By virtue of the lower Lipschitz function notion, $\sum_{i=1}^m \alpha_i f_i$ is lower Lipschitz at $\bar{x}$, and so, this sum function is weakly subdifferentiable at $\bar{x}$.

(iii): By (ii), we get the set $\partial^w\big(\sum_{i=1}^m \alpha_i f_i\big)(\bar{x})$ is not null. It can be verified that

$$\alpha_i \partial^w f_i(\bar{x}) \subset \partial^w(\alpha_i f_i)(\bar{x}),\ i = 1,2,\ldots,m. \tag{4.3}$$

We claim by induction the relations of order $m \geq 2$. In fact, together (4.3) with Proposition 2 [9] guarantees that

$$\alpha_1 \partial^w f_1(\bar{x}) + \alpha_2 \partial^w f_2(\bar{x}) \subset \partial^w(\alpha_1 f_1)(\bar{x}) + \partial^w(\alpha_2 f_2)(\bar{x}) \subset \partial^w(\alpha_1 f_1 + \alpha_2 f_2)(\bar{x}).$$

So, for $m = 2$ the inclusion (4.1) is fulfilled. Suppose that (4.1) is satisfied for every integer positive number $2 < k < m$, which means that the inclusion (4.1) holds for $m = k$, means that

$$\sum_{i=1}^k \alpha_i \partial^w f_i(\bar{x}) \subset \partial^w\big(\sum_{i=1}^k \alpha_i f_i\big)(\bar{x}).$$

For the case $m = k+1$, one can obtain the following result

$$\begin{aligned}\sum_{i=1}^{k+1} \alpha_i \partial^w f_i(\bar{x}) &= \sum_{i=1}^{k} \alpha_i \partial^w f_i(\bar{x}) + \alpha_{k+1} \partial^w f_{k+1}(\bar{x})\\ &\subset \partial^w\big(\sum_{i=1}^k \alpha_i f_i\big)(\bar{x}) + \partial^w(\alpha_{k+1} f_{k+1})(\bar{x}) \subset \partial^w\big(\sum_{i=1}^{k+1} \alpha_i f_i\big)(\bar{x}),\end{aligned}$$

which the conclusion above can be verified. □

We mention that the proof of the inclusion (4.2) is similar to the proof of the inclusion (4.1), where X is replaced by C.

Theorem 4.5. *Let vector-valued functions $f_i : X \to \mathbb{R}^n$ and $P_i \in \mathscr{P}_n$, $i = 1,2,\ldots,m$ such that $P_{i0} f_i$ be lower Lipschitz at $\bar{x} \in X$, $i = 1,2,\ldots,m$ and the non-negative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$. Then*

(i) *The functions $f_i,\ i = 1,2,\ldots,m$ are augmented weakly subdifferentiable at $\bar{x} \in X$.*
(ii) *The sum function $\sum_{i=1}^m \alpha_i f_i$ is augmented weakly subdifferentiable at $\bar{x} \in X$.*

(iii) *The following inclusion holds true:*

$$\sum_{i=1}^{m} \alpha_i \partial_a^w f_i(\bar{x}) \subset \partial_a^w \big(\sum_{i=1}^{m} \alpha_i f_i \big)(\bar{x}). \tag{4.4}$$

In addition, for any $\emptyset \neq C \subset X$, one also has

$$\sum_{i=1}^{m} \alpha_i \partial_{a,C}^w f_i(\bar{x}) \subset \partial_{a,C}^w \big(\sum_{i=1}^{m} \alpha_i f_i \big)(\bar{x}). \tag{4.5}$$

Proof. (i): By the initial assumptions one has the functions $P_{i0} f_i : X \to \mathbb{R}$ are lower Lipschitz at $\bar{x}$, $i = 1,2,\dots,m$, then there exist non-negative Lipschitz constants L_i such that

$$(0, L_i) \in \partial^w (P_{i0} f_i)(\bar{x}),\ i = 1,2,\dots,m,$$

or equivalently, $(0, P_i, L_i) \in \partial_a^w f_i(\bar{x})$, $i = 1,2,\dots,m$. Therefore, f_i, $i = 1,2,\dots,m$ are generalized weakly subdifferentiable at $\bar{x} \in X$.

(ii): We obtain that

$$\langle P_i, f_i(x) - f_i(\bar{x}) \rangle \geq -L_i \|x - \bar{x}\| \text{ for all } x \in X,\ i = 1,2,\dots,m.$$

By puting $L := \sum_{i=1}^{m} \alpha_i \geq 0$ is due to $\alpha_i \geq 0$, $i = 1,2,\dots,m$. In addition, it is easy to verify that

$$\begin{aligned}\sum_{i=1}^{m} \alpha_i (P_{i0} f_i)(x) - \sum_{i=1}^{m} \alpha_i (P_{i0} f_i)(\bar{x}) &= \sum_{i=1}^{m} \alpha_i \Big((P_{i0} f_i)(x) - (P_{i0} f_i)(\bar{x}) \Big) \\ &\geq -\sum_{i=1}^{m} \alpha_i L_i \|x - \bar{x}\| = -L\|x - \bar{x}\| \text{ for all } x \in X.\end{aligned}$$

Consequently, the sum function $\sum_{i=1}^{m} \alpha_i (P_{i0} f_i)$ is lower Lipschitz at $\bar{x}$ and thus, $\sum_{i=1}^{m} \alpha_i f_i$ is augmented weakly subdifferentiable at $\bar{x}$.

(iii): By using the inclusion (4.1) it follows that

$$\sum_{i=1}^{m} \alpha_i \partial_a^w f_i(\bar{x}) \subset \partial_a^w \big(\sum_{i=1}^{m} \alpha_i f_i \big)(\bar{x}).$$

In fact, let $(\xi_i, P_i, r_i) \in \partial_a^w f_i(\bar{x})$, $i = 1,2,\dots,m$ be arbitrary. It follows from the definition that

$$(\xi_i, r_i) \in \partial^w (P_{i0} f_i)(\bar{x}),\ i = 1,2,\dots,m.$$

Also one can obtain that

$$\sum_{i=1}^{m} \alpha_i (\xi_i, P_i, r_i) \in \sum_{i=1}^{m} \alpha_i \partial_a^w f_i(\bar{x}) \text{ and } \sum_{i=1}^{m} \alpha_i (\xi_i, r_i) \in \sum_{i=1}^{m} \alpha_i \partial^w (P_{i0} f_i)(\bar{x}),$$

which combined with the fuzzy sum rule in Theorem 4.4 one can reach the result

$$\sum_{i=1}^{m} \alpha_i \partial_a^w (P_{i0} f_i)(\bar{x}) \subset \partial_a^w \big(\sum_{i=1}^{m} \alpha_i P_{i0} f_i \big)(\bar{x}),$$

which guarantees that

$$\sum_{i=1}^{m} \alpha_i (\xi_i, r_i) \in \partial_a^w \big(\sum_{i=1}^{m} \alpha_i P_{i0} f_i \big)(\bar{x}).$$

Consequently,

$$\sum_{i=1}^{m} \alpha_i (\xi_i, P_i, r_i) \in \partial_a^w \big(\sum_{i=1}^{m} \alpha_i f_i \big)(\bar{x}),$$

and thus, the inclusion (4.4) is fulfilled. Finally, analogously to the argument above with observing the inclusion (4.2) replacing the inclusion (4.1), which entails the inclusion (4.5) is fulfilled too and the conclusion follows. □

Next, we provide some necessary and sufficient optimality conditions for the (weakly) efficient solutions of problem (MPPC) in terms of weak subdifferentials.

Theorem 4.6. *(**Necessary optimality condition**) Let $\bar{x} \in K$ and suppose that for any $P \in \mathscr{P}_m$, the real-valued functions $P_0 f, g_1, g_2, \ldots, g_p, h_1, h_2, \ldots, h_q$ are lower Lipschitz at $\bar{x}$. Then, if $\bar{x}$ is a weakly efficient solution to problem (MPPC), then there exist $P \in \mathscr{P}_m \setminus \{0\}$, $r_0 \in \mathbb{R}_+$, $\eta := (\eta_1, \eta_2, \ldots, \eta_p) \in \mathbb{R}^p_+$ and $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_q) \in \mathbb{R}^q$ satisfying*

$$(0, r) \in \partial^w (P_0 f)(\bar{x}) + \sum_{i=1}^{p} \partial^w (\eta_i g_i)(\bar{x}) + \sum_{j=1}^{q} \partial^w (\gamma_j h_j)(\bar{x}) + N^a_C(\bar{x}) \ \ (\forall r \geq r_0); \tag{4.6}$$

$$y_2 - y_1 \in \mathbb{R}^n_{++} \Longrightarrow P(y_1) < P(y_2); \tag{4.7}$$

$$\eta_i g_i(\bar{x}) = 0, \qquad i = 1, 2, \ldots, p; \tag{4.8}$$

$$\gamma_j h_j(\bar{x}) = 0, \qquad j = 1, 2, \ldots, q; \tag{4.9}$$

Proof. Under all the hypotheses of Theorem 4.6, for all $P \in \mathscr{P}_m$, by taking into account Theorem 4.4 (i), we assert that the single-valued functions $P_0 f, g_1, g_2, \ldots, g_p, h_1, h_2, \ldots, h_q$ are weakly subdifferentiable at $\bar{x}$. In view of Definition 4.2, the mapping $f = (f_1, f_2, \ldots, f_m)$ has a weakly efficient solution on K at $\bar{x}$. By taking into account Theorem 3.8 (i), there exist $P \in \mathscr{P}_m \setminus \{0\}$ such that the implication (4.7) is fulfilled and a non-negative real number r^P_0 satisfying

$$(0, r^P) \in \partial^w (P_0 f)(\bar{x}) + N^a_C(\bar{x}) \ \ (\forall r^P \geq r^P_0). \tag{4.10}$$

By picking $\eta := (\eta_1, \eta_2, \ldots, \eta_p) \in \mathbb{R}^p_+$ satisfying (4.8) and then one can achieve the conclusion that $\eta_i g_i$ $(i = 1, 2, \ldots, p)$ have a global minimum on K at $\bar{x}$. By directly applying Theorem 3.6, there would exist $r^{\eta_i}_0 \geq 0$ $(i = 1, 2, \ldots, p)$ such that

$$(0, r^{\eta_i}) \in \partial^w (\eta_i g_i)(\bar{x}) + N^a_C(\bar{x}) \ \ (\forall r^{\eta_i} \geq r^{\eta_i}_0) \ \ (i = 1, 2, \ldots, p),$$

which yields without loss of generality that

$$(0, r^g) := \sum_{i=1}^{p} (0, r^{\eta_i}) \in \sum_{i=1}^{p} \partial^w (\eta_i g_i)(\bar{x}) + N^a_C(\bar{x}) \ \ (\forall r^g \geq r^g_0 := \sum_{i=1}^{p} r^{\eta_i}_0). \tag{4.11}$$

Then, in a similar idea to the proof above, there exist $\gamma^{(1)} := (\gamma^{(1)}_1, \gamma^{(1)}_2, \ldots, \gamma^{(1)}_q) \in \mathbb{R}^q_+$ and $\gamma^{(2)} := (\gamma^{(2)}_1, \gamma^{(2)}_2, \ldots, \gamma^{(2)}_q) \in \mathbb{R}^q_+$ satisfying

$$(0, r^{h^{(1)}}) := \sum_{j=1}^{q} (0, r^{\gamma^{(1)}_j}) \in \sum_{j=1}^{q} \partial^w (\gamma^{(1)}_j h_j)(\bar{x}) + N^a_C(\bar{x}) \ \ (\forall r^{h^{(1)}} \geq r^{h^{(1)}}_0 := \sum_{j=1}^{q} r^{\gamma^{(1)}_j}_0), \tag{4.12}$$

$$(0, r^{h^{(2)}}) := \sum_{j=1}^{q} (0, r^{\gamma^{(2)}_j}) \in \sum_{j=1}^{q} \partial^w (-\gamma^{(2)}_j h_j)(\bar{x}) + N^a_C(\bar{x}) \ \ (\forall r^{h^{(2)}} \geq r^{h^{(2)}}_0 := \sum_{j=1}^{q} r^{\gamma^{(2)}_j}_0). \tag{4.13}$$

By taking $\gamma = \gamma^{(1)} - \gamma^{(2)} \in \mathbb{R}^q$, $r^h := r^{h^{(1)}} + r^{h^{(2)}}$ and $r^h_0 := r^{h^{(1)}}_0 + r^{h^{(2)}}_0$. We have

$$\gamma_j h_j(\bar{x}) = \gamma^{(1)}_j h_j(\bar{x}) - \gamma^{(2)}_j h_j(\bar{x}) = 0, \ \ j = 1, 2, \ldots, q,$$

which yields the condition (4.9) is fulfilled. Without loss of generality we can obtain from Theorem 4.4 (iii) for the case $m = 2$ the following result

$$(0, r^h) \in \sum_{j=1}^{q} \partial^w(\gamma_j h_j)(\bar{x}) + N_C^a(\bar{x}) \ (\forall r^h \geq r_0^h). \tag{4.14}$$

We set $r_0 := r_0^P + r_0^g + r_0^h \geq 0$, and without loss of generality one can reach the inclusion (4.6) from the inclusions (4.10), (4.11) and (4.14), which completes the proof. $\square$

To demonstrate the previous result, we may take an example as follows.

Example 4.7. Consider the nonconvex mathematical programming problem with set, inequality and equality constraints:

$$\begin{aligned} \text{minimize}\, & f(x) = \big(f_1(x),\ f_2(x),\ f_3(x)\big) \\ \text{subject to}\ & g_i(x) \geq 0,\, i = 1,2; \\ & h_j(x) = 0,\ j = 1,2; \\ & x = (x_1, x_2, x_3) \in C, \end{aligned} \tag{MPPC1}$$

where $C = \{x \in \mathbb{R}^3 \,|\, \|x\| \leq 1\}$, $f_1(x) = \begin{cases} 0 & \text{if } x \in C, \\ \|x\| & \text{otherwise,} \end{cases}$

$f_2(x) = \begin{cases} 0 & \text{if } x \in C, \\ 2\|x\| & \text{otherwise,} \end{cases}$ $f_3(x) = \begin{cases} 0 & \text{if } x \in C, \\ -2\|x\| & \text{otherwise,} \end{cases}$

$g_1(x) = \begin{cases} \sin(\|x\|\pi) & \text{if } \|x\| \leq 2, \\ -\|x\| & \text{otherwise,} \end{cases}$ $g_2(x) = \begin{cases} \cos(\frac{\|x\|}{2}\pi) & \text{if } \|x\| \leq 2, \\ -2\|x\| & \text{otherwise,} \end{cases}$

$h_1(x) = \begin{cases} 0 & \text{if } x \in C, \\ \|x\| + \|x\|^2 & \text{otherwise,} \end{cases}$ $h_2(x) = \begin{cases} 0 & \text{if } x \in C, \\ -\|x\| + \|x\|^2 & \text{otherwise.} \end{cases}$

For the illustration let us consider $\bar{x} = (0,0,0)$. An easy computation gives that $K = C$, and hence, $\bar{x} \in K$. It is not difficult to verify that for every $P \in \mathscr{P}_3$, the real-valued functions $P_0 f, g_1, g_2, h_1, h_2 : \mathbb{R}^3 \to \mathbb{R}$ are lower Lipschitz at $\bar{x}$. In other words, for all $x = (x_1, x_2, x_3) \in K$, one can achieve that $(f_1(x), f_2(x), f_3(x)) - (f_1(\bar{x}), f_2(\bar{x}), f_3(\bar{x})) = (0,0,0) \notin -\mathbb{R}^3_{++}$, we mean that $\bar{x} = (0,0,0)$ is a weakly efficient solution of (MPPC1). Applying Theorem 4.6, there exist $P \in \mathscr{P}_3 \setminus \{(0,0,0)\}$, $r_0 \in \mathbb{R}_+$, $\eta := (\eta_1, \eta_2) \in \mathbb{R}^2_+$ and $\gamma := (\gamma_1, \gamma_2) \in \mathbb{R}^2$ satisfying

$$(0, r) \in \partial^w(P_0 f)(\bar{x}) + \sum_{i=1}^{2} \partial^w(\eta_i g_i)(\bar{x}) + \sum_{j=1}^{2} \partial^w(\gamma_j h_j)(\bar{x}) + N_C^a(\bar{x}) \ \ (\forall r \geq r_0); \tag{4.15}$$

$$y_2 - y_1 \in \mathbb{R}^3_{++} \Longrightarrow P(y_1) < P(y_2); \tag{4.16}$$

$$\eta_i g_i(\bar{x}) = 0, \qquad i = 1,2; \tag{4.17}$$

$$\gamma_j h_j(\bar{x}) = 0, \qquad j = 1,2. \tag{4.18}$$

In fact, in this setting, one can take $P = (1,0,0) \in \mathscr{P}_3 \setminus \{(0,0,0)\}$, $r_0 = 0 \in \mathbb{R}_+$, $\eta = (0,0) \in \mathbb{R}^2_+$ and $\gamma = (0,0) \in \mathbb{R}^2$. Then, (4.16), (4.17) and (4.18) are fulfilled automatically. The mapping

$P_0 f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $(P_0 f)(x) = f_1(x)$ for all $x \in \mathbb{R}^3$. Thus, the relation of (4.15) is equivalent to the following one

$$(0,r) \in \partial^w f_1(\bar{x}) + \sum_{i=1}^{2} \partial^w(0)(\bar{x}) + \sum_{j=1}^{2} \partial^w(0)(\bar{x}) + N_C^a(\bar{x}) \quad (\forall r \geq 0). \tag{4.19}$$

By virtue of Example 2.5, it leads to

$$\partial^w f_1(\bar{x}) = N_C^a(\bar{x}),$$

which combined with (4.19) yields that

$$(0,r) \in \sum_{i=1}^{2} \partial^w(0)(\bar{x}) + \sum_{j=1}^{2} \partial^w(0)(\bar{x}) + 2N_C^a(\bar{x}) \quad (\forall r \geq 0). \tag{4.20}$$

It is plain that $\partial^w(0)(\bar{x}) = \{(\xi,r) \in \mathbb{R}^3 \times \mathbb{R}_+ \,|\, \|\xi\| \leq r\}$, $(0,0) \in \{(\xi,r) \in \mathbb{R}^3 \times \mathbb{R}_+ | \|\xi\| \leq r\}$, $(0,0) \in 2N_C^a(\bar{x})$, and thus,

$$\big\{(\xi,r) \in \mathbb{R}^3 \times \mathbb{R}_+ | \|\xi\| \leq r\big\} \subset \sum_{i=1}^{2} \partial^w(0)(\bar{x}) + \sum_{j=1}^{2} \partial^w(0)(\bar{x}) + 2N_C^a(\bar{x}).$$

For every $r \geq 0$, one has $(0,r) \in \big\{(\xi,r) \in \mathbb{R}^3 \times \mathbb{R}_+ | \|\xi\| \leq r\big\}$. which proves the relation (4.20), as it was checked.

Corollary 4.8. *Let $\bar{x} \in K$ and suppose that for any $P \in \mathscr{P}_m$, the real-valued functions $P_0 f$, $g_1, g_2, \ldots, g_p, h_1, h_2, \ldots, h_q$ are lower Lipschitz at $\bar{x}$. Then, if $\bar{x}$ is a weakly efficient solution to problem (MPPC), then there exist $P \in \mathscr{P}_m \setminus \{0\}$, $r_0 \in \mathbb{R}_+$, $\eta := (\eta_1, \eta_2, \ldots, \eta_p) \in \mathbb{R}_+^p$ and $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_q) \in \mathbb{R}^q$ satisfying (4.7), (4.8), (4.9) and*

$$(0,r) \in \partial^w(P_0 f)(\bar{x}) + \sum_{i=1}^{p} \partial^w(\eta_i g_i)(\bar{x}) + \sum_{j=1}^{q} \partial^w(\gamma_j h_j)(\bar{x}) + N_K^a(\bar{x}) \quad (\forall r \geq r_0). \tag{4.21}$$

Proof. It is an immediately corollary from Theorem 4.6, and the claim follows. □

Theorem 4.9. *(***Sufficient optimality condition for weak efficiency***) Let $\bar{x} \in K$ and suppose that there exist $P \in \mathscr{P}_m \setminus \{0\}$, $\eta := (\eta_1, \eta_2, \ldots, \eta_p) \in -\mathbb{R}_+^p$ and $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_q) \in \mathbb{R}^q$ such that all the following assertions are fulfilled:*

(i) *The real-valued functions $P_0 f, \eta_1 g_1, \eta_2 g_2, \ldots, \eta_p g_p, \gamma_1 h_1, \gamma_2 h_2, \ldots, \gamma_q h_q$ are weakly subdifferentiable at $\bar{x}$.*

(ii) *If $y_2 - y_1 \in \mathbb{R}_{++}^n$, then $P(y_1) < P(y_2)$.*

(iii) *The following relations hold true:*

$$(0,0) \in \partial^w(P_0 f)(\bar{x}) + \sum_{i=1}^{p} \partial^w(\eta_i g_i)(\bar{x}) + \sum_{j=1}^{q} \partial^w(\gamma_j h_j)(\bar{x}) + N_C^a(\bar{x}), \tag{4.22}$$

$$\eta_i g_i(\bar{x}) = 0, \qquad i = 1, 2, \ldots, p. \tag{4.23}$$

Then, $\bar{x}$ is a weakly efficient solution to problem (MPPC).

Proof. Assume that all the assumptions of Theorem 4.9 are fulfilled. Then, under (i) the sum function $P_0 f + \sum_{i=1}^{p} \eta_i g_i + \sum_{j=1}^{q} \gamma_j h_j$ is weakly subdifferentiable at $\bar{x}$, which yields that the set $\partial_K^w(P_0 f + \sum_{i=1}^{p} \eta_i g_i + \sum_{j=1}^{q} \gamma_j h_j)(\bar{x})$ is not empty. Since $K \subset C$, it holds that $N_C^a(\bar{x}) \subset N_K^a(\bar{x})$. Thus, under the relation (4.22), we have

$$(0,0) \in \partial^w(P_0 f)(\bar{x}) + \sum_{i=1}^{p} \partial^w(\eta_i g_i)(\bar{x}) + \sum_{j=1}^{q} \partial^w(\gamma_j h_j)(\bar{x}) + N_K^a(\bar{x}).$$

Making use of the fuzzy sum role in Theorem 4.4 and then this combined with the previous relation ensures that

$$(0,0) \in \partial^w(P_0 f + \sum_{i=1}^{p} \eta_i g_i + \sum_{j=1}^{q} \gamma_j h_j)(\bar{x}) + N_K^a(\bar{x}).$$

In view of Proposition 3.1, one can reach the following result

$$(0,0) \in \partial_K^w(P_0 f + \sum_{i=1}^{p} \eta_i g_i + \sum_{j=1}^{q} \gamma_j h_j)(\bar{x}).$$

By applying the weak subdifferential notion and moreover for every $x \in K$ guarantees that $\eta_i(g_i(x) - g_i(\bar{x})) \leq 0,\ i = 1,2,\ldots,p$ and $\gamma_j(h_j(x) - h_j(\bar{x})) = 0,\ j = 1,2,\ldots,q$, we have

$$\begin{aligned}\langle P, f(x) - f(\bar{x})\rangle &\geq (P_0 f + \sum_{i=1}^{p} \eta_i g_i + \sum_{j=1}^{q} \gamma_j h_j)(x) \\ &\quad - (P_0 f + \sum_{i=1}^{p} \eta_i g_i + \sum_{j=1}^{q} \gamma_j h_j)(\bar{x}) \geq 0,\end{aligned}$$

or equivalently,

$$\langle P, f(x) - f(\bar{x})\rangle \geq 0.$$

Under the condition (ii), one can obtain that $f(x) - f(\bar{x}) \notin -\mathbb{R}^m_{++}$ for every $x \in K$. Thus, $\bar{x}$ is a weakly efficient solution to problem (MPPC) and completes the proof. □

Theorem 4.10. *(***Sufficient optimality condition for efficiency***) Let $\bar{x} \in K$ and suppose that there exist $P \in \mathscr{P}_m \setminus \{0\}$, $\eta := (\eta_1, \eta_2, \ldots, \eta_p) \in -\mathbb{R}^p_+$ and $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_q) \in \mathbb{R}^q$ such that all the following assertions are fulfilled:*

(i) *The real-valued functions $P_0 f, \eta_1 g_1, \eta_2 g_2, \ldots, \eta_p g_p, \gamma_1 h_1, \gamma_2 h_2, \ldots, \gamma_q h_q$ are weakly subdifferentiable at $\bar{x}$.*

(ii) *If $y_2 - y_1 \in \mathbb{R}^n_+ \setminus \{0\}$, then $P(y_1) < P(y_2)$.*

(iii) *The following relations hold true:*

$$(0,0) \in \partial^w(P_0 f)(\bar{x}) + \sum_{i=1}^{p} \partial^w(\eta_i g_i)(\bar{x}) + \sum_{j=1}^{q} \partial^w(\gamma_j h_j)(\bar{x}) + N_C^a(\bar{x}), \tag{4.24}$$

$$\eta_i g_i(\bar{x}) = 0, \qquad i = 1,2,\ldots,p. \tag{4.25}$$

Then, $\bar{x}$ is an efficient solution to problem (MPPC).

Proof. Analogously to the proof of Theorem 4.9, which can be verified that

$$\langle P, f(x) - f(\bar{x})\rangle \geq 0.$$

Under the condition (ii), one can achieve that $f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\}$ for all $x \in K$. Therefore, $\bar{x}$ is an efficient solution to problem (MPPC) and terminates the proof. □

We remark that the results obtained in Theorem 4.9 and Theorem 4.10 are still true if the augmented normal cone $N^a_C(\bar{x})$ is removed and replaced by an other cone $N^a_K(\bar{x})$.

5. Conclusion

In this paper, we have established some new important characterizations for the weak subdifferential and provided the fuzzy sum rules for the (augmented) weak subdifferentials involving the class of lower Lipschitz functions. Additionally, we presented some necessary and sufficient optimality conditions for the weakly efficient solution and the efficient solution of a nonconvex mathematical programming problem having set, inequality and equality constraints in terms of the weak subdifferentials. It is important to remark that our obtained results in this paper have not been fully discovered yet. In the future, these necessary and sufficient optimality conditions may be used to construct algorithms for finding the (weakly) efficient solutions of a clas of nonconvex mathematical programs via the weak subdifferentials notion.

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