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FIXED POINT THEOREMS VIA VECTOR-VALUED MEASURES OF NONCOMPACTNESS WITH APPLICATIONS TO SEMILINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor Petr Petrovich Zabreiko

Abstract. In this paper, we prove some Darbo-Sadovskii type fixed point theorems associated with vector-valued measures of noncompactness in locally convex spaces. As an application of our results, we investigate the existence of mild solutions for a broad class of semilinear neutral differential equations.
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1. Introduction

The degree of noncompactness of a set is measured by means of functions called measures of noncompactness (MNCs in short). The first MNC was considered by Kuratowski [20] in 1930 in connection with problems of general topology. It was Darbo [9] who first realized the usefulness of the Kuratowski measure of noncompactness in fixed point theory. Since then, the interplay between MNCs and fixed point theory has, over the course of time, become more and more strong and fruitful. Essentially, MNCs have played a prominent role in the development of fixed point theory and its applications to the theories of differential, integral and integrodifferential equations [3, 6, 7, 11, 12, 13]. It is relevant to note that there is a number of definitions of MNCs which have appeared in the literature, over the years. We quote for instance the Hausdorff measure of noncompactness introduced by Goldenstein et al. in 1957, the inner Hausdorff measure of noncompactness and the Istratescu measure introduced by Istratescu in 1972 [3]. This notion afterwards got an abstract setting in which the Kuratowski and Hausdorff measures are only examples. We will mention here the axiomatic approach for measure of noncompactness, developed by Banas and Goebel [5] in 1980. A view to applications has motivated a useful extension of MNCs, namely the vector-valued MNCs. As stressed in

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[4], the use of MNCs which take values in an ordered vector space, has better effect in applications than the classical MNCs, which take values in $[0,\infty)$. Indeed it turns out that vector-valued MNCs provide a more flexible tool enabling to unblock some difficult situations (see Section 4). The main focus of this work is to contribute to highlighting the role of vector-valued measures of noncompactness in fixed point theory and its applications to the study of the solvability of ordinary and partial differential equations and systems. Specifically, we first prove some new fixed point theorems for mappings satisfying a Darbo-Sadovskii type condition with respect to a vector-measure of noncompactness. Furthermore, we show how this fixed point results may be used to investigate the solvability of a class of neutral differential equations.

Our paper is organized as follows. In preliminaries we give some basic definitions and facts concerning vector-valued measures of noncompactness as well as some results concerning the estimation of the measure of noncompactness of some integral operators. In the third section we prove some new Darbo-Sadovskii type fixed point theorems associated with vector-valued measures of noncompactness in locally convex spaces. Section four is devoted to applications of results from the third section to establish existence principles for a class of semilinear neutral differential equations. In the last section, we examine and discuss some key assumptions that we have used in the fourth section.

2. Preliminaries

In this section, we recall some basic definitions and results which will be needed in our further considerations. Let X be a locally convex space and $\mathscr{M}$ be a family of subsets of X. Let $(E,\preceq)$ be an ordered Banach space with a cone P. We reprodudce here the definition, slightly modified, of a (vector-valued) measure of noncompactness introduced in [3, 4, 19].

Definition 2.1. An operator $\phi:\mathscr{M}\to P$ is called a *measure of noncompactness (MNC for short)* if

(i) For all $\Omega\in\mathscr{M}$ we have $\overline{conv}(\Omega)\in\mathscr{M}$ and $\phi(\overline{conv}(\Omega))=\phi(\Omega)$.
(ii) For all $\Omega\in\mathscr{M}$ we have $\phi(\Omega)=0$ implies Ω is relatively compact.

The MNC ϕ is said to be :

(1) *x_0-stable* ($x_0\in X$), if $\{x_0\}\in\mathscr{M}$ and for all $\Omega\in\mathscr{M}$ we have $\Omega\cup\{x_0\}\in\mathscr{M}$ and $\phi(\Omega\cup\{x_0\})=\phi(\Omega)$.
(2) *Countably x_0-stable* ($x_0\in X$), if $\{x_0\}\in\mathscr{M}$ and for any countable set $\Omega\in\mathscr{M}$ we have $\Omega\cup\{x_0\}\in\mathscr{M}$ and $\phi(\Omega\cup\{x_0\})=\phi(\Omega)$.
(3) *Monotone,* if for all $\Omega\in\mathscr{M}$ and all $\Lambda\subset\Omega$ we have $\Lambda\in\mathscr{M}$ and $\phi(\Lambda)\leq\phi(\Omega)$.

We remark in passing that one can easily construct plenty of examples of both real-valued and vector-valued MNCs in different function spaces (see for instance [3, 5, 11, 12, 13, 18, 19] and the references therein). We limit ourselves here to giving one nontrivial example which we will use thereafter. To this end, let $(E,\|\cdot\|)$ be a Banach space and let $X:=C([a,b],E)$ be the Banach space of E-valued functions defined and continuous on the interval $[a,b]$, endowed with the usual norm of uniform convergence $\|x\|_\infty=\sup\limits_{t\in[a,b]}\|x(t)\|$. Let $\mathscr{B}(X)$ stands for the collection of all bounded subsets of X and consider the mappings $\psi_1,\psi_2\colon\mathscr{B}(X)\to[0,+\infty)$ given by :

$$\psi_1(B)=\sup_{t\in[a,b]}e^{-Lt}\chi(B(t))\quad(L\text{ is a positive constant}),$$

where $\chi(.)$ is the Hausdorff measure of noncompactness, defined on each bounded subset M of E by

$$\chi(M) = \inf\{r > 0, M \text{ can be covered with a finite number of balls of radius equal to } r\},$$

and

$$\psi_2(B) = \sup_{t\in[a,b]} \mod_C(B(t)),$$

where $\mod_C(B(t))$ is the modulus of equicontinuity of the set of functions B at point t given by the formula

$$\mod_C(B(t)) = \lim_{\delta\to 0}\left\{\sup_{x\in B}\{\sup\{\|x(t_1)-x(t_2)\| : t_1,t_2 \in (t-\delta,t+\delta)\}\}\right\}.$$

It is manifest that ψ_1 and ψ_2 fail to be MNCs in X. Now, let $\psi\colon \mathscr{B}(X) \to \mathbb{R}^2$ defined by

$$\psi(B) = \begin{pmatrix} \psi_1(B) \\ \psi_2(B) \end{pmatrix}, \quad B \in \mathscr{B}(X),$$

It is routine to check that ψ is a vector-valued MNC in the sense of Definition 2.1, which is monotone and x_0-stable for each $x_0 \in X$ (see [1, 3, 5, 19]).

We close this section by recalling a result regarding the estimation of the Hausdorff measure of noncompactness for a class of integral operators as well as other related results. For this purpose, let $(U(t))_{t\geq 0}$ be a strongly continuous semigroup of linear operators acting on a Banach space E and consider the integral operator $\mathscr{V}_0$ defined on $C([0,a];E)$ by:

$$(\mathscr{V}_0 x)(t) = \int_0^t U(t-s)x(s)ds, \quad \text{for} \quad t\in[0,a],$$

where $x \in C([0,a];E)$.

The following fundamental theorems are crucial for our further work.

Theorem 2.2. [19, Theorem 4.2.2]. *Let $(u_n)_{n\geq 1} \subset L^1([0,a],E)$ be integrably bounded, namely,*

$$\|u_n(t)\| \leq \nu(t) \text{ for all } n\geq 1 \text{ and a.e } t\in[0,a], \tag{2.1}$$

where $\nu \in L^1([0,a])$. Assume that

$$\chi((u_n(t))_{n\geq 1}) \leq q(t) \text{ for a.e } t\in[0,a], \tag{2.2}$$

where $q \in L^1([0,a])$. Then $\chi((\mathscr{V}_0 u_n(t))_{n\geq 1}) \leq 2M_a \int_0^t q(s)ds$, for all $t\in[0,a]$, where $M_a = \sup_{t\in[0,a]} \|U(t)\|$.

Theorem 2.3. [11, Theorem 3.10.] *Let $(u_n)_{n=1}^{\infty} \subset L^1([0,a],E)$ be as in (2.1). Assume that (2.2) holds. Then, for every $t\in[0,a]$ we have :*

$$\mod_C((\mathscr{V}_0 u_n(t))_{n\geq 1}) \leq 4M_a \int_0^t q(s)ds. \tag{2.3}$$

3. Fixed point theorems

This section of the paper contains our main results. Basically, we use some material from the previous section to prove a Darbo-Sadovskii type fixed point theorem in locally convex spaces. For this purpose, we recall the following result on mappings of a compact topological space into itself. The proof of this lemma can be found in [21].

Lemma 3.1. *Let T be a mapping of a compact topological space K into itself. Then there exists a nonempty subset $M \subseteq K$ such that $M = \overline{T(M)}$.*

Remark 3.2. Note that if T is continuous and K is Hausdorff space then $T(M)$ is compact, therefore closed. Hence $M = T(M)$.

Now, we are ready to state and prove the following sharpening of [4, Theorem 2.5].

Theorem 3.3. *Let $(E, \preceq)$ be a Banach space ordered by a cone P and $L \subset P$ be a complete lattice with $0 \in L$. Let X be a Fréchet space, C be a nonempty convex closed subset of X and $x_0 \in C$. Let $T: C \to C$ be a continuous mapping and $\phi: \mathscr{M} \to E$ be a monotone, countably x_0-stable MNC defined on a family $\mathscr{M}$ of subsets of X. Moreover, assume that*

(1) For all $\Omega \in \mathscr{M}$ with $\Omega \subset C$ we have $\phi(\Omega) \in L$.
(2) $T(C) \in \mathscr{M}$.
(3) There is an increasing mapping $A: L \to L$ such that $\phi(T\Omega) \preceq A(\phi(\Omega))$ whenever $\Omega \in \mathscr{M}$ is a countable subset of C.
(4) A does not have fixed points in $L \setminus \{0\}$.

Then T has at least one fixed point in C.

Proof. Consider the set $K := \{T^n(x_0), n = 0, 1, \cdots\}$ of iterates starting from x_0. Clearly $T(K) \cup \{x_0\} = K$. From our assumptions we infer that $K \in \mathscr{M}$ and $\phi(K) = \phi(T(K) \cup \{x_0\}) = \phi(T(K)) \preceq A(\phi(K))$. Therefore, from Tarski's fixed point theorem it follows that there is $u \in L$ such that $\phi(K) \preceq u$ and $Au = u$. Hence, $u = 0$ by condition (4) and therefore $\phi(K) = 0$. This implies that $\overline{K}$ is compact. Let Λ be the subset of $\overline{K}$ whose existence is insured by the Lemma 3.1 and denote by $\mathscr{F}$ the class of all closed and convex subsets Ω of C such that $\Omega \in \mathscr{M}$, $\Lambda \subset \Omega$ and $T(\Omega) \subset \Omega$. Put

$$\Gamma = \bigcap \{\Omega: \Omega \in \mathscr{F}\}, \quad \Sigma = \overline{conv}(T(\Gamma)).$$

Obviously, $\overline{conv}(T(C)) \in \mathscr{F}$ and $\Gamma \in \mathscr{M}$. Furthermore, it is easily seen that $T(\Gamma) \subset \Gamma$ and $\Sigma \in \mathscr{M}$. We now claim that $\Gamma = \Sigma$. Indeed, since $T(\Gamma) \subset \Gamma$, it follows that $\Sigma \subset \Gamma$. This implies $T(\Sigma) \subset T(\Gamma) \subset \Sigma$, so that $\Sigma \in \mathscr{F}$. Hence $\Gamma \subset \Sigma$ and therefore $\Gamma = \Sigma = \overline{conv}(T(\Gamma))$. Put $\Delta = \sup\{\phi(\Omega): \Omega \text{ is a countable subset of } \Gamma\}$. Let Ω be a countable subset of Γ. According to [22, Proposition 3.55], there is a countable $\Omega^* \subset \Gamma$ with $\Omega \subset \overline{conv}(T\Omega^*)$. Thus,

$$\phi(\Omega) \preceq \phi(\overline{conv}(T\Omega^*)) \preceq \phi(T\Omega^*) \preceq A(\phi(\Omega^*)) \preceq A(\phi(\Delta)).$$

Taking the supremum over all Ω we find $\Delta \preceq A(\Delta)$. Once more, it follows from Tarski's fixed point theorem that there is $u \in L$ such that $\phi(\Delta) \preceq u$ and $Au = u$. Hence, $u = 0$ by condition (4) and therefore $\phi(\Delta) = 0$. We claim that Γ is compact. Indeed, let (x_n) be a sequence in Γ and $S = \{x_n: n \in \mathbb{N}\}$. We have obviously that $\phi(S) \preceq \phi(\Delta)$ and so $\phi(S) = 0$. Hence, (x_n) has a convergent subsequence, which implies the claim. Then, by the Schauder-Tychonoff fixed point theorem, there exists $x \in \Gamma$ such that $Tx = x$.

□

Remark 3.4. Notice that Theorem 3.3 is stated for Fréchet spaces but it is worthwile to mention the result remains valid in quasicomplete, metrizable locally convex spaces.

From Theorem 3.3 one can derive several important and extremely pleasant results. We shall merely present here one corollary which we will subsequently use.

Corollary 3.5. *Let X be a Banach space and let C be a nonempty closed convex bounded subset of X. Let $T: C \to C$ be a continuous mapping and assume that there exist set functions $\mu_i\colon \mathscr{B}(X) \to [0,+\infty)$, $i = 1,\cdots,n$ satisfying:*

- *(P1) For each $i = 1,\cdots,n$ and for any bounded subset M of X we have $\mu_i(\overline{conv}(M)) = \mu_i(M)$;*
- *(P2) there is $x_0 \in C$ such that for each $i = 1,\cdots,n$ and for any bounded countable subset M of X we have $\mu_i(M \cup \{x_0\}) = \mu_i(M)$,*
- *(P3) For each $i = 1,\cdots,n$ we have $M_1 \subset M_2$ implies $\mu_i(M_1) \le \mu_i(M_2)$.*
- *(P4) If $\mu_i(M) = 0$ for all $i \in \{1,\cdots,n\}$, then M is relatively compact.*

Assume further that there is a nonnegative matrix $A \in \mathscr{M}_n(\mathbb{R}^+)$ with spectral radius $r(A) < 1$ such that for any countable set $M \subset C$ we have

$$\begin{pmatrix} \mu_1(TM) \\ \vdots \\ \mu_n(TM) \end{pmatrix} \preceq A \begin{pmatrix} \mu_1(M) \\ \vdots \\ \mu_n(M) \end{pmatrix}, \tag{3.1}$$

where $\preceq$ is the natural coordinatewise partial order on $\mathbb{R}^n$. Then T has at least one fixed point in C.

Remark 3.6. Since $A \in \mathscr{M}_n(\mathbb{R}^+)$ is a nonnegative matrix with spectral radius $r(A) < 1$, then it is readily seen that for each $X \succeq 0$ we have $X \preceq AX$ implies $X = 0$ (see for instance [23, Lemma 4.1]).

Proof. Apply Theorem 3.3 with $\mathscr{M} = \mathscr{B}(X)$, $L = [0,\mu_1(C)] \times \cdots \times [0,\mu_n(C)]$ and

$$\phi(\cdot) = \begin{pmatrix} \mu_1(\cdot) \\ \vdots \\ \mu_n(\cdot) \end{pmatrix}.$$

□

Remark 3.7. It is of interest that the contractivity condition (3.1) is only assumed to hold for countable sets. As a matter of fact, if the operator T involves integration of vector functions, estimates like (3.1) are only obtained for countable sets (see for instance Theorem 2.2 and Theorem 2.3).

Remark 3.8. It should be underlined that our fixed point results extend several earlier works including [4, 17] and many others and offer some new tools to deal with the existence of solutions to many differential and integral equations.

4. Application to neutral differential equations

The results we consider in this section are generalizations of the work done in [10]. Specifically, we shall discuss the existence of mild solutions to the following neutral differential equation

$$\begin{cases} \dfrac{d}{dt}\big(x(t)-g(t,x(t))\big)=A\big(x(t)-g(t,x(t))\big)+f(t,x(t)),\ t\in I, \\ \\ x(0)=x_0. \end{cases} \tag{4.1}$$

Here $I=[0,a]$, $a>0$, A is the generator of a strongly continuous semigroup $(U(t))_{t\geq 0}$ of linear operators defined on a Banach space E, $f,g\colon [0,a]\times E\longrightarrow E$ are suitably defined functions satisfying certain conditions to be specified later.

We should emphasize that Eq. (4.1) is an abstract formulation of many partial differential equations arising in the mathematical modeling of real world phenomena (see for instance [8, 14, 15, 16, 24, 25, 26] and the references therein).

In what follows we make the following assumptions :

(H1): The semigroup $(U(t))_{t\geq 0}$ is strongly continuous.

(H2): The function g maps $[0,a]\times E$ into E and there are nonnegative constants k_g and $c_g\in[0,1)$ such that for all $u,v\in E$ and for all $t,s\in[0,a]$ we have

$$\|g(t,u)-g(s,v)\|\leq k_g|t-s|+c_g\|u-v\|.$$

(H3):

(i) The map $f\colon [0,a]\times E\longrightarrow E$ satisfies Carathéodory conditions, that is, for almost every $t\in[0,a]$, the function $f(t,.):E\longrightarrow E$ is continuous and for all $x\in E$, the function $f(.,x):[0;a]\longrightarrow E$ is measurable.

(ii) There exists a function $h\colon [0,a]\times\mathbb{R}^+\to\mathbb{R}^+$ such that $h(.,s)\in L^1([0,a],\mathbb{R}^+)$ for every, $s\geq 0$, $h(t,\cdot)$ is continuous and increasing for a.e. $t\in[0,a]$, and $\|f(t,x)\|\leq h(t,\|x\|)$ for a.e. $t\in[0,a]$ and all $x\in X$, and for each positive scalar m there exists a continuous function $r\colon [0,a]\to\mathbb{R}^+$ such that

$$m+M_a\int_0^t h(s,r(s))ds\leq(1-c_g)r(t),\quad t\in[0,a]. \tag{4.2}$$

(iii) There exists a function $\beta\in L^1([0,a],\mathbb{R}^+)$such that for every bounded set $D\subseteq E$ we have $\chi(f(t,D))\leq\beta(t).\chi(D)$ for almost every $t\in[0,a]$.

Remark 4.1. We point out that (H3), (ii) implies that any composition $t\in[0,a]\to h(t,x(t))$ is measurable whenever $x\in C([0,a],\mathbb{R}^+)$. We refer the readers to [2] for more information on measurability of compositions. In addition, for any continuous function x we have $h(t,x(t))\leq h(t,\kappa)$ where $\kappa=\sup\limits_{t\in[0,a]}x(t)$. Thus, $h(\cdot,x(\cdot))\in L^1([0,a],\mathbb{R}^+)$ whenever $x\in C([0,a],\mathbb{R}^+)$.

Our main goal in the immediate sequel is to show the existence of solutions to Eq. (4.1) under the assumptions made above . Before doing so, it is appropriate to clarify the definition of solution we will consider.

Definition 4.2. A continuous function $x : [0,a] \longrightarrow E$ is said to be a mild solution to the neutral differential equation (4.1) if:

(i) $x(0) = x_0$,

(ii) $x(t) = U(t)(x_0 - g(0,x_0)) + g(t,x(t)) + \int_0^t U(t-s)f(s,x(s))ds$ for $t \in [0,a]$.

To allow the abstract formulation of our problem we consider the operators $T,S\colon C([0,a];E) \longrightarrow C([0,a];E)$ defined by

$$(Tx)(t) = U(t)(x_0 - g(0,x_0)) + \int_0^t U(t-s)f(s,x(s))ds = U(t)u_0 + \int_0^t U(t-s)f(s,x(s))ds,$$

and

$$(Sx)(t) = g(t,x(t)).$$

It is plainly visible that a fixed point of $T+S$ is a mild solution of equation (4.1). With this in mind, we shall show that operator $T+S$ satisfies all conditions of Corollary 3.5. This will be achieved in a series of lemmas.

Lemma 4.3. *There is a closed convex subset V of $C([0,a],E)$ such that $(S+T)(V) \subset V$.*

Proof. Let $r(t)$ the continuous function given by (4.2) for $m = \|g(t,0)\| + M_a\|u_0\|$. Let

$$V = \Big\{ x \in C([0,a],E) : \|x(t)\| \le r(t), \text{ for } t \in [0,a] \Big\}.$$

For any $x \in V$, we have

$$\begin{aligned} \|Tx(t) + Sx(t)\| &\le \|g(t,x(t))\| + \|U(t)(u_0)\| + \int_0^t \|U(t-s)f(s,x(s))\|ds \\ &\le \|g(t,0)\| + c_g\|x(t)\| + M_a\|u_0\| + M_a \int_0^t h(s,\|x(s)\|)ds \\ &\le \|g(t,0)\| + c_g r(t) + M_a\|u_0\| + M_a \int_0^t h(s,r(s))ds \\ &\le r(t), \end{aligned}$$

which is the desired result. □

Lemma 4.4. *For any bounded subset B of $C([0,a],E)$ we have*

(i) $\psi_2(S(B)) \le c_g \psi_2(B)$.

(ii) $\psi_1(S(B)) \le c_g \psi_1(B)$.

Proof. To prove (i), take any bounded subset B of $C([0,a],E)$ and pick up any $t \in [0,a]$. For any $x \in B$ and any $t_1,t_2 \in [0,a]$, we have

$$\|(Sx)(t_1) - (Sx)(t_2)\| = \|g(t_1,x(t_1)) - g(t_2,x(t_2))\| \le k_g|t_1 - t_2| + c_g\|x(t_1) - x(t_2)\|.$$

Thus,

$$\begin{aligned} &\sup_{x\in B}\{\sup\{\|S(x)(t_1) - S(x)(t_2)\| :\ t_1,t_2 \in (t-\delta,t+\delta)\}\} \\ \le\ & 2k_g\delta + c_g\left\{\sup_{x\in B}\{\sup\{\|x(t_1) - x(t_2)\| : t_1,t_2 \in (t-\delta,t+\delta)\}\}\right\}. \end{aligned}$$

Hence, letting δ tend to 0 we see that $\operatorname{mod}_C((SB)(t)) \leq c_g \operatorname{mod}_C(B(t))$ so that $\psi_2(S(B)) \leq c_g\psi_2(B)$. To prove (ii), let B be a bounded subset of $C([0,a],E)$. Pick up any $t \in [a,b]$ and take $\lambda > \chi(B(t))$. There exist $x_1,\cdots,x_n \in E$ such that $B(t) \subseteq \bigcup_{k=1}^{n} B(x_k,\lambda)$. Let $x \in B$. Then there exist $k \in \{1,\cdots,n\}$ and $z \in E$ with $\|z\| \leq \lambda$ such that $x(t) = x_k + z$. Hence,

$$\|(Sx)(t) - g(t,x_k)\| = \|g(t,x(t)) - g(t,x_k)\| \leq c_g\|x(t) - x_k\| = c_g\|z\| \leq c_g\lambda,$$

and therefore, $(SB)(t) \subseteq \bigcup_{k=1}^{n} B(g(t,x_k),c_g\lambda)$. Thus, $\chi((S(B))(t)) \leq c_g\lambda$ for all $t \in [0,a]$. Whence it follows that $\chi((SB)(t)) \leq c_g\chi(B(t))$ for all $t \in [0,a]$, which leads to the desired result. □

Lemma 4.5. *T and S are continuous on $C([0,a];E)$.*

Proof. The result follows by a routine argument. We first prove that T is continuous. To do this, let (u_n) be a sequence in $C([0,a],E)$ which converges to some u. From assumption (H3), (i) it follows that:

$$f(s,u_n(s)) \to f(s,u(s)), \text{ as } n \to +\infty.$$

Since $(u_n)_n$ is bounded, then there exists $N > 0$ such that $\|u_n\| \leq N$, for all $n \in \mathbb{N}$. Thus, $\|u\| \leq N$ and therefore

$$\|f(s,u_n(s)) - f(s,u(s))\| \leq 2h(s,N).$$

Using the dominated convergence theorem, one can conclude that T is continuous. Further, it is a straightforward matter to check that S is continuous. □

Lemma 4.6. *There is a real matrix A of second order with spectral radius $r(A) < 1$ such that for any countable set D of V, we have $\psi((T+S)D) \leq A\psi(D)$.*

Proof. Let $D = \{u_n\}$ be a countable subset of V and $F_n(t) = f(t,u_n(t))$. From our hypotheses we know that $\|F_n(t)\| \leq h(t,r(t))$ and

$$\chi(\{F_n(t)\}) \leq \beta(t)\chi(D(t)) \leq \beta(t)e^{Lt}\psi_1(D).$$

Referring to Theorem 2.2 we see that

$$\chi(\{\mathscr{V}_0F_n(t)\}) \leq 2M_a\psi_1(D)\int_0^t e^{Ls}\beta(s)ds,$$

Hence,

$$e^{-Lt}\chi(T(D)(t)) = e^{-Lt}\chi(\{\mathscr{V}_0F_n(t))\}) \leq 2M_a\psi_1(D)\int_0^t e^{-L(t-s)}\beta(s)ds$$

for all $t \in [0,a]$. Therefore,

$$\psi_1(T(D)) \leq 2M_a\lambda_2(L)\psi_1(D), \tag{4.3}$$

where $\lambda_2(L) = \sup_{t\in[0,a]}\int_0^t e^{-L(t-s)}\beta(s)ds$.

Furtheremore, from Theorem 2.3 we deduce that

$$mod_C(\{\mathscr{V}_0F_n(t)\}) \leq 4M_a\psi_1(D)\int_0^t e^{Ls}\beta(s)ds,$$

so that,

$$\psi_2(T(D)) \leq 4M_a e^{La} \|\beta\|_1 \psi_1(D). \tag{4.4}$$

Since $\lim_{L\to\infty} \lambda_2(L) = 0$ (see [11, Lemma 2.7]) we may choose L as large as we please so that $c_g + 2M_a\lambda_2(L) < 1$. Put

$$A = \begin{pmatrix} c_g + 2M_a\lambda_2(L) & 0 \\ 4M_a e^{aL}\|\beta\|_1 & c_g \end{pmatrix}.$$

It is at once clear that $r(A) = c_g + 2M_a\lambda_2(L) < 1$ and $\psi((T+S)(D) \leq A\psi(D)$. ☐

With these preliminaries out of the way we are in a position to state the main result of this section.

Theorem 4.7. *Assume that the conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied. Then Equation (4.1) has at least one continuous mild solution on $[0,a]$.*

Proof. The result follows from Corollary 3.5 on the basis of Lemmas 4.3, 4.4, 4.5 and 4.6. ☐

Remark 4.8. In contrast to [10] in our setting the semigroup generated by A need not be equicontinuous.

Remark 4.9. It should be observed that the use of vector-valued measures of noncompactness may serve to alleviate or eliminate the effect of some unsettling coefficients. For example, the constant $4M_a e^{aL}\|\beta\|_1$ appearing in the estimation (4.4), as large as it may be, has no bearing on our result.

5. Comments and Discussions

In this section, we shall examine and discuss a key assumption used in the previous section. As a matter of fact, we shall give some examples of function h for which (4.2) holds for some continuous function $r\colon [0,a] \to \mathbb{R}^+$. We start with the following result.

Proposition 5.1. *Let $h\colon [0,a]\times\mathbb{R}^+ \to \mathbb{R}^+$ be a mapping such that $h(.,u) \in L^1([0,a],\mathbb{R}^+)$ for every $u \geq 0$. Assume further that $\liminf_{u\to+\infty} \frac{1}{u}\int_0^a h(s,u)ds = \ell < +\infty$ and*

$$c_g + M_a\ell < 1. \tag{5.1}$$

Then for each positive scalar m there exists a continuous function $r\colon [0,a] \to \mathbb{R}^+$ satisfying (4.2).

Proof. Let m be a positive scalar and consider the set W of real numbers $r \geq 0$ which satisfy the inequality

$$r \leq \frac{1}{1-c_g}\big(m + M_a\int_0^a h(s,r)ds\big).$$

We claim that there exists a constant r_1 such that for all $r \in W$ we have $r \leq r_1$. If it is not the case, then there exists a sequence $r_n \in W$ with $r_n \to +\infty$ as $n \to +\infty$. Hence,

$$1 \leq \frac{1}{1-c_g}\big(\frac{m}{r_n} + \frac{M_a}{r_n}\int_0^a h(s,r_n)ds\big),$$

so that

$$1 \leq \frac{1}{1-c_g}\big(M_a \liminf_{u\to+\infty} \frac{1}{u}\int_0^a h(s,u)ds\big),$$

which contradicts (5.1). Choose $r_0 > r_1$, then, $r_0 \notin W$. Hence,

$$\frac{1}{1-c_g}\big(m + M_a \int_0^a h(s,r_0)ds\big) < r_0.$$

Thus, the condition (4.2) is satisfied for $r(t) = r_0$.

□

Example 5.2. Let $h(t,u) = e^{-t}\sqrt{u} + t^2$. Then,

$$\frac{1}{u}\int_0^a h(s,u)ds = \frac{1}{\sqrt{u}}(1-e^{-a}) + \frac{a^3}{3u},$$

which implies that

$$\lim_{u\to+\infty} \frac{1}{u}\int_0^a h(s,u)ds = 0.$$

As a convenient specialization of Proposition 5.1 we obtain the following result.

Corollary 5.3. *Let $h(t,u) = \theta(t)\Omega(u)$, where $\theta \in L^1([0,a],\mathbb{R}_+)$ and $\Omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Assume that $\liminf_{u\to+\infty} \frac{\Omega(u)}{u} = \ell < +\infty$ and $c_g + \|\theta\|_1 \ell < 1$, where $\|\theta\|_1 = \int_0^a \theta(s)ds$. Then for each positive scalar m there exists a continuous function $r\colon [0,a] \to \mathbb{R}^+$ satisfying (4.2).*

We next turn our attention to the case when h is affine. In this case, we show that the condition (5.1) can be dropped. It is appropriate to point out that this result does not follow from Proposition 5.1.

Proposition 5.4. *Let $h(t,u) = \alpha(t) + \gamma(t)u$, where $\alpha, \gamma \in L^1([0,a],\mathbb{R}^+)$. Then for each positive scalar m there exists a continuous function $r\colon [0,a] \to \mathbb{R}^+$ satisfying (4.2).*

Proof. Let $C([0,a],\mathbb{R})$ be the Banach space of continuous real-valued functions defined on $[0,a]$ endowed with the following Bielecki's norm

$$\|x\|_b = \sup_{t\in[0,a]} e^{-\rho t}|x(t)|.$$

Let P be the positive cone of $C([0,a],\mathbb{R})$ and m be a positive scalar. Define an operator $F : P \to P$ by

$$(F\varphi)(t) = m + c_g\varphi(t) + M_a \int_0^t (\alpha(s) + \gamma(s)\varphi(s))\,ds.$$

It is readily seen that

$$\begin{aligned} e^{-\rho t}|(F\varphi)(t) - (F\psi)(t)| &\leq c_g e^{-\rho t}|\varphi(t) - \psi(t)| + M_a e^{-\rho t}\int_0^t \gamma(s)|\varphi(s) - \psi(s)|ds \\ &\leq c_g e^{-\rho t}|\varphi(t) - \psi(t)| + M_a\|\varphi - \psi\|_b \int_0^t e^{-\rho(t-s)}\gamma(s)ds \end{aligned}$$

Consequently,

$$\|(F\varphi)(t)-(F\psi)(t)\|_b \leq (c_g + M_a\lambda_1(\rho))\,\|\varphi-\psi\|_b,$$

where $\lambda_1(\rho) = \sup_{t\in[0,a]} \int_0^t e^{-\rho(t_0-s)}\gamma(s)ds$. Since $\lim_{\rho\to\infty} \lambda_1(\rho) = 0$ (see [11, Lemma 2.7]) we may choose ρ_0 as large as we please so that $c_g + \lambda_1(\rho_0) < 1$. An appeal to the Banach contraction principle yields that there exists a nonnegative continuous function $r(\cdot)$ such that

$$r(t) = m + c_g r(t) + M_a \int_0^t (\alpha(s) + \gamma(s)r(s))ds. \tag{5.2}$$

This shows that the condition (4.2) is satisfied.

□

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