Commun. Optim. Theory 2024 (2024) 25 https://doi.org/10.23952/cot.2024.25

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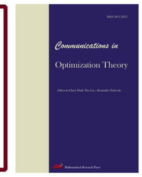

INVERTIBLE AND FREDHOLM OPERATORS IN SPACES OF REAL INTERPOLATION

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Dedicated to the memory of Professor Petr P. Zabreiko

Abstract. In this paper we suggest a new method for investigation of stability of Fredholm properties of operators on interpolation Banach spaces constructed by the real interpolation method. The method consists of two closely related steps. In the first step, invertible operators are investigated. In the second step, the results obtained for invertible operators are applied to investigation of Fredholm operators. This approach allows us to obtain results on stability of kernels and cokernels of Fredholm operators in the spaces constructed by the real interpolation method. A characterization of maximal intervals of parameter θ for which an operator $T: \vec{X}_{\theta,q} \to \vec{Y}_{\theta,q}$ is Fredholm is also obtained.

Keywords. Fredholm operator; Interpolation method; Invertible operator.

2020 Mathematics Subject Classification. 46B70, 47A53.

1. Introduction

A linear bounded operator $T: X \to Y$ from a Banach space X to a Banach space Y is called a Fredholm operator if it has a finite dimensional kernel, that is, $n(T) := \dim(\ker T) < \infty$ and its range $T(X)$ has a finite codimension in Y, that is, $d(T) := \dim(Y/T(X)) < \infty$. The quantity $i(T) := n(T) - d(T)$ is called the index of the operator T. It is clear that a set of Fredholm operators is a natural extension of a set of invertible operators. Note also that Fredholm operators are very important in analysis and in PDE's in particular.

Let us recall some basic properties of Fredholm operators. It is well known that from $d(T) < \infty$ it follows that $T(X)$ is a closed subspace of Y (see, for example, [1]). It is also known that if an operator T is a Fredholm operator and a bounded linear operator $S\colon X \to Y$ is such that $\|S-T\|_{X\to Y} < \varepsilon$, where $\varepsilon = \varepsilon(T) > 0$ is small enough, then the operator S is also Fredholm with $i(S) = i(T)$ and $n(S) \leq n(T)$, $d(S) \leq d(T)$. Moreover, if we consider the conjugate operator

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Received: July 2, 2023; Accepted: November 25, 2023.

$T^*\colon Y^* \to X^*$ of the Fredholm operator T, then T^* is also Fredholm and $n(T^*) = d(T)$ and $d(T^*) = n(T)$.

Throughout the paper by a Banach couple $\vec{X} = (X_0, X_1)$ we mean two Banach spaces X_0, X_1 that are linearly and continuously embedded in some Hausdorff linear space $\mathscr{X}$. Let $\theta \in (0,1)$ and $q \in [1,\infty]$. As usual, by $\vec{X}_{\theta,q} = (X_0, X_1)_{\theta,q}$ we denote the interpolation space constructed by the real method of interpolation and by $[\vec{X}]_\theta = [X_0, X_1]_\theta$ the interpolation space constructed by the complex method (see [8]).

We recall that a couple $\vec{X} = (X_0, X_1)$ is called regular if $X_0 \cap X_1$ is dense in X_i, $i = 0,1$. An interpolation functor F is called regular if for any couple $\vec{X}$ the space $X_0 \cap X_1$ is dense in $F(\vec{X})$. Functors $F(\vec{X}) = [\vec{X}]_\theta$ and $F(\vec{X}) = \vec{X}_{\theta,q}$ are regular for $\theta \in (0,1)$ and $q \in [1,\infty)$.

Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ be a bounded linear operator from a Banach couple (X_0, X_1) to a Banach couple (Y_0, Y_1), i.e., $T\colon X_0 + X_1 \to Y_0 + Y_1$ is a linear operator and the restriction of T to X_i $(i = 0,1)$ is a bounded linear operator to Y_i $(i = 0,1)$. Henceforth we only consider such type of operators. Everywhere below we use the notation $\ker T := \{x \in X_0 + X_1\colon Tx = 0\}$.

The following remarkable result on local stability of Fredholm operators for the complex method of interpolation was proved by I. Shneiberg (see [18]) in 1974.

Theorem 1.1. (*Shneiberg*) *Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$. Suppose that for some $\theta_* \in (0,1)$ the operator $T\colon [\vec{X}]_{\theta_*} \to [\vec{Y}]_{\theta_*}$ is Fredholm. Then there exists $\varepsilon > 0$ such that for any $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ the operator $T\colon [\vec{X}]_\theta \to [\vec{Y}]_\theta$ is Fredholm with an index equals to the index of $T\colon [\vec{X}]_{\theta_*} \to [\vec{Y}]_{\theta_*}$.*

In the same paper ([18], Lemma 6) I. Shneiberg also proved that for all $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$, we have

$$n(T\colon [\vec{X}]_\theta \to [\vec{Y}]_\theta) \le n(T\colon [\vec{X}]_{\theta_*} \to [\vec{Y}]_{\theta_*}), \tag{1.1}$$

$$d(T\colon [\vec{X}]_\theta \to [\vec{Y}]_\theta) \le d(T\colon [\vec{X}]_{\theta_*} \to [\vec{Y}]_{\theta_*}). \tag{1.2}$$

K.-H. Förster and K. Günther showed that (1.1) and (1.2) are in fact equalities (see [11]). It is also clear that from Theorem 1.1 and the inequalities (1.1) and (1.2) it follows that if an operator $T\colon [\vec{X}]_{\theta_*} \to [\vec{Y}]_{\theta_*}$ is invertible then for any $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ the operator $T\colon [\vec{X}]_\theta \to [\vec{Y}]_\theta$ is also invertible.

For the real method analogs of Shneiberg's result were obtained by M. Zafran (see [19]) in 1980 for invertible operators and by W. Cao and Y. Sagher (see [10]) in 1990 for Fredholm operators. In 1998 M. Krause (see [14]) proved the following more general result.

Theorem 1.2. (*Krause*) *Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$. Suppose that for some $\theta_* \in (0,1)$ and $q_* \in [1,\infty)$ the operator $T\colon \vec{X}_{\theta_*,q_*} \to \vec{Y}_{\theta_*,q_*}$ is Fredholm. Then there exists $\varepsilon > 0$ such that for any $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ and $q \in [1,\infty)$ the operator $T\colon \vec{X}_{\theta,q} \to \vec{Y}_{\theta,q}$ is Fredholm with the same index and kernel, i.e.,*

$$i(T\colon \vec{X}_{\theta,q} \to \vec{Y}_{\theta,q}) = i(T\colon \vec{X}_{\theta_*,q_*} \to \vec{Y}_{\theta_*,q_*}),$$
$$\ker T \cap \vec{X}_{\theta,q} = \ker T \cap \vec{X}_{\theta_*,q_*}.$$

Moreover, there exists a finite dimensional space $M \subset Y_0 \cap Y_1$ independent of $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ and $q \in [1,\infty)$ such that $\vec{Y}_{\theta,q} = M \oplus T(\vec{X}_{\theta,q})$.

Let us note that Shneiberg's theorem (which deals with the complex method) does not prove the local stability of kernels and cokernels of Fredholm operators, and Krause's theorem (which deals with the real method) proves the local stability for the case of $q<\infty$ but not for $q=\infty$.

In this paper, we suggest a new strategy for investigation of Fredholm operators on interpolation spaces constructed by the real interpolation method. This strategy is based on the reduction theorem that reduces investigation of Fredholm operators to investigation of invertible operators. This approach allows us to prove new and interesting results.. In particular, we generalize Krause's theorem for the important and non-trivial case $q=\infty$. We also use our results to characterize maximal intervals of Fredholmness in terms of generalized dilation indices.

We would like to note that the use of dilation indices was inspired by the remarkable work due to S. Ivanov and N. Kalton [13] on interpolation of subspaces. Note also that the connection between the problem of description of maximal Fredholm intervals and the problem of interpolation of subspaces was considered in [3].

2. Invertible operators in spaces of real interpolation

Let $T\colon (X_0,X_1)\to(Y_0,Y_1)$ be a bounded linear operator between Banach couples and let a parameter $q_*\in[1,\infty]$ be fixed. In [15] (see also [19] for $1\le q_*<\infty$) it was shown that if $T\colon \vec{X}_{\theta_*,q_*}\to\vec{Y}_{\theta_*,q_*}$ is invertible, then there exists $\delta>0$ such that for any $\theta\in(\theta_*-\delta,\theta_*+\delta)$ the operator $T\colon \vec{X}_{\theta,q_*}\to\vec{Y}_{\theta,q_*}$ is invertible. So the set of all $\theta\in(0,1)$ for which the operator $T\colon \vec{X}_{\theta,q_*}\to\vec{Y}_{\theta,q_*}$ is invertible is open. A similar result for the complex method was obtained by Shneiberg in [18].

Definition 2.1. Let $T\colon (X_0,X_1)\to(Y_0,Y_1)$. An interval $(a,b)\subset(0,1)$ is said to be an interval of invertibility of T for the real method with a parameter q_* if for any $\theta\in(a,b)$ the operator$T\colon \vec{X}_{\theta,q_*}\to\vec{Y}_{\theta,q_*}$ is invertible.

Let us start with the following theorem.

Theorem 2.2. *Let (a,b) be an interval of invertibility of an operator $T\colon (X_0,X_1)\to(Y_0,Y_1)$ for the real method with a parameter $q_*\in[1,\infty]$. Then for any $\theta_0,\theta_1\in(a,b)$ and any interpolation functor G the operator $T\colon G(\vec{X}_{\theta_0,q_*},\vec{X}_{\theta_1,q_*})\to G(\vec{Y}_{\theta_0,q_*},\vec{Y}_{\theta_1,q_*})$ is invertible.*

Proof. Let $\theta_0,\theta_1\in(a,b)$. Since the operator

$$T\colon (\vec{X}_{\theta_0,q_*},\vec{X}_{\theta_1,q_*})\to(\vec{Y}_{\theta_0,q_*},\vec{Y}_{\theta_1,q_*})$$

is invertible on the end spaces, i.e., the operators $T\colon \vec{X}_{\theta_i,q_*}\to\vec{Y}_{\theta_i,q_*}$, $i=0,1$, are invertible, then according to Proposition 1 in [4] it is sufficient to prove the injectivity of the operator

$$T\colon \vec{X}_{\theta_0,q_*}+\vec{X}_{\theta_1,q_*}\to\vec{Y}_{\theta_0,q_*}+\vec{Y}_{\theta_1,q_*}. \tag{2.1}$$

To prove its injectivity it is enough to show that on the interval (a,b) the operator T has the property of "local injectivity", i.e., for any $\theta_*\in(a,b)$ there exists $\varepsilon=\varepsilon(\theta_*)>0$ such that for any $\mu_0,\mu_1\in(\theta_*-\varepsilon,\theta_*+\varepsilon)$ the operator

$$T\colon \vec{X}_{\mu_0,q_*}+\vec{X}_{\mu_1,q_*}\to\vec{Y}_{\mu_0,q_*}+\vec{Y}_{\mu_1,q_*} \tag{2.2}$$

is injective. Indeed, suppose that on (a,b) the operator T has the property of "local injectivity" but the operator (2.1) is not injective. Then there exists a non-zero element $x\in\vec{X}_{\theta_0,q_*}+\vec{X}_{\theta_1,q_*}$

such that $Tx = 0$. Hence, $x = x_0 + x_1$, $x_0 \in \vec{X}_{\theta_0,q_*}$, $x_1 \in \vec{X}_{\theta_1,q_*}$ and so from $Tx = 0$, we have $y = Tx_0 = T(-x_1) \in \vec{Y}_{\theta_0,q_*} \cap \vec{Y}_{\theta_1,q_*}$. Then from the reiteration theorem it follows that

$$y \in \bigcap_{\theta\in[\theta_0,\theta_1]} \vec{Y}_{\theta,q_*}.$$

By our hypothesis the operator $T\colon \vec{X}_{\theta,q_*} \to \vec{Y}_{\theta,q_*}$ is invertible for any $\theta \in [\theta_0,\theta_1]$, so there exists a family $\{x_\theta\}_{\theta\in[\theta_0,\theta_1]}$ of elements $x_\theta \in \vec{X}_{\theta,q_*}$ such that $Tx_\theta = y$. Let us now observe that from "local injectivity" of T, it follows that for any $\theta_* \in [\theta_0,\theta_1]$ there exists an interval $(\theta_* - \varepsilon, \theta_* + \varepsilon)$ such that the elements x_θ are equal for all $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon) \cap [\theta_0,\theta_1]$. Now using compactness of the interval $[\theta_0,\theta_1]$, we can conclude that $x_{\theta_0} = x_{\theta_1}$. It remains to note that $x_{\theta_0} = x_0$ and $x_{\theta_1} = -x_1$. Therefore $x = x_0 + x_1 = 0$, i.e., from "local injectivity" follows injectivity of the operator (2.1). So we only need to prove "local injectivity". Let $\theta_* \in (a,b)$ then there exists $\delta > 0$ such that the operator $T\colon (X_0,X_1)_{\theta,q_*} \to (Y_0,Y_1)_{\theta,q_*}$ is invertible for any $\theta \in (\theta_* - \delta, \theta_* + \delta)$. Now let us consider the couples (A_0,A_1), (B_0,B_1), where

$$A_0 = \vec{X}_{\theta_*-\delta,q_*} + \vec{X}_{\theta_*+\delta,q_*},\ \ A_1 = \vec{X}_{\theta_*-\delta,q_*} \cap \vec{X}_{\theta_*+\delta,q_*},$$
$$B_0 = \vec{Y}_{\theta_*-\delta,q_*} + \vec{Y}_{\theta_*+\delta,q_*},\ \ B_1 = \vec{Y}_{\theta_*-\delta,q_*} \cap \vec{Y}_{\theta_*+\delta,q_*}.$$

From the calculations in [16] and [17], based on K-divisibility (see Theorems 3.3.15 and 3.8.7 in [9]), it follows that for $\lambda \in (0,1/2)$ we have

$$(A_0,A_1)_{\lambda,q_*} = \vec{X}_{\theta_*-(1-2\lambda)\delta,q_*} + \vec{X}_{\theta_*+(1-2\lambda)\delta,q_*},$$
$$(B_0,B_1)_{\lambda,q_*} = \vec{Y}_{\theta_*-(1-2\lambda)\delta,q_*} + \vec{Y}_{\theta_*+(1-2\lambda)\delta,q_*}$$

and

$$(A_0,A_1)_{1/2,q_*} = \vec{X}_{\theta_*,q_*},\ (B_0,B_1)_{1/2,q_*} = \vec{Y}_{\theta_*,q_*}. \tag{2.3}$$

Since $T\colon (A_0,A_1) \to (B_0,B_1)$, from (2.3) follows the invertibility of the operator

$$T\colon (A_0,A_1)_{1/2,q_*} \to (B_0,B_1)_{1/2,q_*}.$$

So there exists $\varepsilon_1 > 0$ such that for all $\lambda \in (1/2 - \varepsilon_1, 1/2 + \varepsilon_1)$ the operator

$$T\colon (A_0,A_1)_{\lambda,q_*} \to (B_0,B_1)_{\lambda,q_*}$$

is invertible. Hence, if we take $\varepsilon \in (0, 2\delta\varepsilon_1)$ then $\lambda = 1/2 - \varepsilon/2\delta \in (1/2 - \varepsilon_1, 1/2)$ and the operator

$$T\colon \vec{X}_{\theta_*-\varepsilon,q_*} + \vec{X}_{\theta_*+\varepsilon,q_*} \to \vec{Y}_{\theta_*-\varepsilon,q_*} + \vec{Y}_{\theta_*+\varepsilon,q_*}$$

is invertible. From the reiteration theorem we deduce that for any $\mu_0,\mu_1 \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ the following inclusion holds:

$$\vec{X}_{\mu_0,q_*} + \vec{X}_{\mu_1,q_*} \subset \vec{X}_{\theta_*-\varepsilon,q_*} + \vec{X}_{\theta_*+\varepsilon,q_*}.$$

Thus, the operator (2.2) is injective and so "local injectivity" is proved. □

From Theorem 2.2 and the reiteration theorem for the real method immediately follows:

Corollary 2.3. *If (a,b) is an interval of invertibility for an operator $T\colon (X_0,X_1) \to (Y_0,Y_1)$ for the real method for some parameter $q_* \in [1,\infty]$, then (a,b) is an interval of invertibility for the real method for any parameter $q \in [1,\infty]$.*

Remark 2.4. Theorem 2.2 also follows from Theorem 4.9 and Proposition 4.2 in [7]. However, the proof presented above is much simpler than the proof in [7]. Corollary 2.3 also follows from the results in [4].

3. Fredholm operators in spaces of real interpolation

In this section we apply our results on invertibility of operators to investigation of Fredholm operators. Below we need the following technical lemma (see Lemma 3.1 in [6]).

Lemma 3.1. *Let X and Y be Banach spaces such that $X \subset Y$ and X is dense in Y. Then for any closed subspace V of Y with $\dim(Y/V) < \infty$ there exists a finite dimensional subspace M of X with $\dim M = \dim(Y/V)$ such that $Y = M \oplus V$ and $X = M \oplus (X \cap V)$. In particular, we have $\dim(Y/V) = \dim(X/(X \cap V))$.*

Our main tool in this section is the following theorem.

Theorem 3.2. (*Reduction theorem*) *Let $\vec{X}$, $\vec{Y}$ be regular couples and F be a regular functor. Let $T\colon (X_0,X_1) \to (Y_0,Y_1)$. Suppose that $T\colon F(\vec{X}) \to F(\vec{Y})$ is Fredholm. Then there exist couples $(\widetilde{X}_0,\widetilde{X}_1)$, $(\widetilde{Y}_0,\widetilde{Y}_1)$ and an invertible operator $\widetilde{T}\colon F(\widetilde{X}_0,\widetilde{X}_1) \to F(\widetilde{Y}_0,\widetilde{Y}_1)$ such that if $\widetilde{T}\colon G(\widetilde{X}_0,\widetilde{X}_1) \to G(\widetilde{Y}_0,\widetilde{Y}_1)$ is invertible for some interpolation functor G then the operator $T\colon G(\vec{X}) \to G(\vec{Y})$ is Fredholm with the index equal to the index of $T\colon F(\vec{X}) \to F(\vec{Y})$.*

Proof. The proof repeats the reasoning given in Theorem 1.2 in [6]. For the convenience of the reader we show here the construction of the couples $(\tilde{X}_0, \tilde{X}_1), (\tilde{Y}_0, \tilde{Y}_1)$ and the operator $\tilde{T}$. The construction consists of several steps.

Since $T\colon F(X_0,X_1) \to F(Y_0,Y_1)$ is Fredholm therefore $\dim(\ker T \cap F(X_0,X_1)) < \infty$. Hence, there exists a closed subspace $U_{\vec{X}}$ of X_0+X_1 such that $X_0+X_1 = (\ker T \cap F(X_0,X_1)) \oplus U_{\vec{X}}$. From regularity of (X_0,X_1) and Lemma 3.1 it follows that there exists a finite dimensional subspace $M_{\vec{X}} \subset X_0 \cap X_1$ such that

$$X_0+X_1 = M_{\vec{X}} \oplus U_{\vec{X}}. \tag{3.1}$$

Then we define $\widetilde{X}_i$ by $\widetilde{X}_i := X_i \cap U_{\vec{X}}$ for $i = 0,1$.

From Fredholmness of $T\colon F(X_0,X_1) \to F(Y_0,Y_1)$, we have $\dim(F(Y_0,Y_1)/T(F(X_0,X_1))) < \infty$. Since $Y_0 \cap Y_1$ is dense in $F(Y_0,Y_1)$ then by Lemma 3.1 there exists a finite dimensional space $M_{\vec{Y}} \subset Y_0 \cap Y_1$ such that $F(Y_0,Y_1) = M_{\vec{Y}} \oplus T(F(X_0,X_1))$. As $M_{\vec{Y}}$ is finite dimensional then in Y_0+Y_1 there exists a closed subspace $U_{\vec{Y}}$ which is a complement to $M_{\vec{Y}}$, i.e.,

$$Y_0+Y_1 = M_{\vec{Y}} \oplus U_{\vec{Y}}. \tag{3.2}$$

Let us now define the space $\widetilde{Y}_i$ by $\widetilde{Y}_i := Y_i \cap U_{\vec{Y}}$ for $i = 0,1$. To construct the operator $\widetilde{T}$, we consider the projections $P_{\vec{X}}\colon X_0+X_1 \to U_{\vec{X}}$ and $P_{\vec{Y}}\colon Y_0+Y_1 \to U_{\vec{Y}}$ with the corresponding kernels $M_{\vec{X}}$ and $M_{\vec{Y}}$ (see (3.1) and (3.2)). We define the operator $\widetilde{T}$ by $\widetilde{T} := P_{\vec{Y}} T P_{\vec{X}} : \widetilde{X}_0+\widetilde{X}_1 \to \widetilde{Y}_0+\widetilde{Y}_1$. ☐

Below we need the following lemma (see also [12] and [2]).

Lemma 3.3. *Suppose that couples $\vec{X}$, $\vec{Y}$ are ordered and regular, i.e., $X_0 \subset X_1$, $Y_0 \subset Y_1$, X_0 is dense in X_1 and Y_0 is dense in Y_1. Suppose that an operator $T\colon (X_0,X_1) \to (Y_0,Y_1)$ is such that the operators $T\colon X_i \to Y_i$, $i = 0,1$, are Fredholm with equal indices. Then*

$$\ker T \cap X_0 = \ker T \cap X_1 \tag{3.3}$$

and

$$T(X_0) = T(X_1) \cap Y_0. \tag{3.4}$$

Moreover, there exists a finite dimensional space $M \subset Y_0$ such that $Y_i = M \oplus T(X_i)$, $i = 0, 1$.

Proof. Let $n_i = \dim(\ker T \cap X_i)$, $i = 0, 1$. Then from the inclusion

$$\ker T \cap X_0 \subset \ker T \cap X_1 \tag{3.5}$$

it follows that $n_0 \leq n_1$. Since Y_0 is dense in Y_1 and the operator $T\colon X_1 \to Y_1$ is Fredholm, Lemma 3.1 with $X = Y_0$, $Y = Y_1$ and $V = T(X_1)$ yields that the codimension of $T(X_1)$ in Y_1 is equal to the codimension of $T(X_1) \cap Y_0$ in Y_0. Clearly,

$$T(X_0) \subset T(X_1) \cap Y_0 \tag{3.6}$$

and so $d_0 := \dim(T(X_0)/Y_0) \geq d_1 := \dim(T(X_1)/Y_1)$. From equality of indices of the operators $T\colon X_i \to Y_i$, $i = 0, 1$, we have $n_0 - n_1 = d_0 - d_1$. Hence from the inequalities $n_0 \leq n_1$, $d_0 \geq d_1$ it follows that $n_0 = n_1$ and $d_0 = d_1$. Consequently, in the inclusions (3.5) and (3.6) we have the equality of the indicated linear spaces, i.e., we have proved (3.3) and (3.4).

Finally, let us choose M such that $Y_0 = M \oplus T(X_0)$. As the codimension of $T(X_0)$ in Y_0 is equal to the codimension of $T(X_1)$ in Y_1 and $T(X_0) = T(X_1) \cap Y_0$ then $Y_1 = M \oplus T(X_1)$. This completes the proof. □

In the next theorem we consider a family $\{F_\theta\}$ of interpolation functors F_θ, $\theta \in (0,1)$, such that

(a) functors F_θ are of order θ, i.e., for any couple $\vec{X}$ holds $\vec{X}_{\theta,1} \subset F_\theta(\vec{X}) \subset \vec{X}_{\theta,\infty}$;
(b) functors F_θ are regular and satisfy the reiteration theorem $F_\lambda(F_{\theta_0}, F_{\theta_1}) = F_{(1-\lambda)\theta_0 + \lambda\theta_1}$;
(c) if $T\colon (X_0, X_1) \to (Y_0, Y_1)$ and $T\colon F_{\theta_*}(\vec{X}) \to F_{\theta_*}(\vec{Y})$ is invertible, then there exists $\varepsilon > 0$ such that for any interpolation functor G the operator

$$T\colon G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X})) \to G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}(\vec{Y}))$$

is invertible.

Note that from Theorem 2.2 it follows that the family of functors of real interpolation $F_\theta(\cdot) = (\cdot)_{\theta,q_*}$ with a fixed parameter $q_* \in [1,\infty)$ satisfies the conditions (a)-(c).

Our main result in this section is the following theorem.

Theorem 3.4. *Let $\{F_\theta\}, 0 < \theta < 1$, be a family of interpolation functors that satisfies the above conditions* (a)-(c)*. Suppose that $T\colon (X_0, X_1) \to (Y_0, Y_1)$ and $T\colon F_{\theta_*}(\vec{X}) \to F_{\theta_*}(\vec{Y})$ is Fredholm. Then there exists $\varepsilon > 0$ such that for any interpolation functor G*

(i) *the operator*

$$T\colon G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X})) \to G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}(\vec{Y}))$$

is Fredholm with the index equal to the index of $T\colon F_{\theta_*}(\vec{X}) \to F_{\theta_*}(\vec{Y})$;

(ii) *the spaces*

$$W = \ker T \cap G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X})), \tag{3.7}$$

$$WW = T(G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X}))) \cap (Y_0 \cap Y_1) \tag{3.8}$$

do not depend on the functor G;

(iii) *there exists a finite dimensional space $M \subset Y_0 \cap Y_1$ independent of G such that*

$$G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}(\vec{Y})) = M \oplus T(G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X}))).$$

Proof. (i) From the reiteration theorem and regularity of functors F_θ, $\theta \in (0,1)$, it follows that without loss of generality we can assume that the couples (X_0, X_1), (Y_0, Y_1) are regular. Let $(\tilde{X}_0, \tilde{X}_1)$, $(\tilde{Y}_0, \tilde{Y}_1)$ and $\tilde{T}$ be the couples and the operator constructed in Theorem 3.2. In this case the operator

$$\tilde{T}\colon F_{\theta_*}(\tilde{X}_0, \tilde{X}_1) \to F_{\theta_*}(\tilde{Y}_0, \tilde{Y}_1)$$

is invertible. Then from the condition (c) for the family of functors F_θ it follows that there exists $\varepsilon > 0$ such that for any interpolation functor G the operator

$$\tilde{T}\colon G(F_{\theta_*-\varepsilon}(\tilde{X}_0, \tilde{X}_1), F_{\theta_*+\varepsilon}(\tilde{X}_0, \tilde{X}_1)) \to G(F_{\theta_*-\varepsilon}(\tilde{Y}_0, \tilde{Y}_1), F_{\theta_*+\varepsilon}(\tilde{Y}_0, \tilde{Y}_1))$$

is invertible. So from Theorem 3.2 it follows that the operator

$$T\colon G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X})) \to G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}(\vec{Y}))$$

is Fredholm with the index equal to the index of $T\colon F_{\theta_*}(\vec{X}) \to F_{\theta_*}(\vec{Y})$.

(ii) From (i) it follows that the operator

$$T\colon F_{\theta_*-\varepsilon}(\vec{X}) + F_{\theta_*+\varepsilon}(\vec{X}) \to F_{\theta_*-\varepsilon}(\vec{Y}) + F_{\theta_*+\varepsilon}(\vec{Y})$$

is Fredholm with the index equal to the index of $T\colon F_{\theta_*}(\vec{X}) \to F_{\theta_*}(\vec{Y})$. Let us consider a couple (A_0, A_1) where

$$A_0 = G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X})),\ \ A_1 = F_{\theta_*-\varepsilon}(\vec{X}) + F_{\theta_*+\varepsilon}(\vec{X})$$

and a couple (B_0, B_1) where

$$B_0 = G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}(\vec{Y})),\ \ B_1 = F_{\theta_*-\varepsilon}(\vec{Y}) + F_{\theta_*+\varepsilon}(\vec{Y}).$$

Clearly, these couples are ordered. As $X_0 \cap X_1 \subset A_0 \subset A_1$ and $X_0 \cap X_1$ is dense in A_1 therefore the couple (A_0, A_1) is regular. Similarly, the couple (B_0, B_1) is also regular. Moreover, from (i) we see that the operator $T\colon (A_0, A_1) \to (B_0, B_1)$ satisfies the conditions of Lemma 3.3. Hence, $\ker T \cap A_1 = \ker T \cap A_0$, i.e.,

$$\ker T \cap (F_{\theta_*-\varepsilon}(\vec{X}) + F_{\theta_*+\varepsilon}(\vec{X})) = \ker T \cap G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X})).$$

As the left-hand side of this equality does not depend on the interpolation functor G therefore the space W (see (3.7)) is independent of G. Moreover, from Lemma 3.3 it also follows that $T(A_1) \cap B_0 = T(A_0)$, i.e.,

$$T(F_{\theta_*-\varepsilon}(\vec{X}) + F_{\theta_*+\varepsilon}(\vec{X})) \cap G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}\vec{Y})) = T(G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X}))).$$

If we intersect both parts of this equality with $Y_0 \cap Y_1$ then from the embedding

$$Y_0 \cap Y_1 \subset G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}(\vec{Y})) = B_0$$

we obtain

$$T(F_{\theta_*-\varepsilon}(\vec{X}) + F_{\theta_*+\varepsilon}(\vec{X})) \cap (Y_0 \cap Y_1) = T(G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X}))) \cap (Y_0 \cap Y_1).$$

As the left-hand side does not depend on G therefore the space WW (see (3.8)) is independent of G. The proof of the statement (ii) is complete.

(iii) From Lemma 3.1 with $X = Y_0 \cap Y_1$, $Y = F_{\theta_*-\varepsilon}(\vec{Y}) + F_{\theta_*+\varepsilon}(\vec{Y})$ and $V = T(F_{\theta_*-\varepsilon}(X_0, X_1) + F_{\theta_*+\varepsilon}(X_0, X_1))$, we obtain that there exists a finite dimensional space $M \subset Y_0 \cap Y_1$ such that

$$F_{\theta_*-\varepsilon}(\vec{Y}) + F_{\theta_*+\varepsilon}(\vec{Y}) = M \oplus T(F_{\theta_*-\varepsilon}(\vec{X}) + F_{\theta_*+\varepsilon}(\vec{X})).$$

We can see that the codimension of $T(F_{\theta_*-\varepsilon}(\vec{X}) + F_{\theta_*+\varepsilon}(\vec{X}))$ in $F_{\theta_*-\varepsilon}(\vec{Y}) + F_{\theta_*+\varepsilon}(\vec{Y})$ (i.e., the codimension of $T(A_1)$ in B_1) is equal to $\dim M$.

From (i) it follows that the indices of the operators $T\colon A_0 \to B_0$ and $T : A_1 \to B_1$ are equal, so from the equality of kernels $\ker T \cap A_0 = \ker T \cap A_1$ (see (ii)) we obtain that the codimension of $T(A_0)$ in B_0 is equal to the codimension of $T(A_1)$ in B_1. Hence, the codimension of $T(A_0)$ in B_0 is equal to $\dim M$. As

$$M \subset Y_0 \cap Y_1 \subset G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}(\vec{Y})) = B_0$$

and

$$M \cap T(A_0) \subset M \cap T(A_1) = \{0\}$$

therefore $B_0 = M \oplus T(A_0)$, i.e.,

$$G(F_{\theta_*-\varepsilon}(\vec{Y}), F_{\theta_*+\varepsilon}(\vec{Y})) = M \oplus T(G(F_{\theta_*-\varepsilon}(\vec{X}), F_{\theta_*+\varepsilon}(\vec{X}))).$$

Moreover, from the construction of M it is clear that M is independent of the functor G. □

To formulate the next result we need the following definition.

Definition 3.5. Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$. An interval $(a, b) \subset (0, 1)$ is said to be an interval of Fredholmness of the operator T for the real method with a parameter q_* if the operator $T\colon (X_0, X_1)_{\theta,q_*} \to (Y_0, Y_1)_{\theta,q_*}$ is Fredholm for all $\theta \in (a, b)$.

The next theorem proves stability of kernels and cokernels of an operator T on a Fredholm interval for the real method. Note also that this theorem generalizes the result due to M. Krause [14] to the important case of $q = \infty$.

Theorem 3.6. *Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ and let (a, b) be an interval of Fredholmness of the operator T for the real method with a parameter $q_* \in [1, \infty)$. Then for any $\theta \in (a, b)$ and $q \in [1, \infty]$ the operator $T\colon \vec{X}_{\theta,q} \to \vec{Y}_{\theta,q}$ is Fredholm and the spaces*

$$W = \ker T \cap \vec{X}_{\theta,q} \text{ and } WW = T(\vec{X}_{\theta,q}) \cap (Y_0 \cap Y_1)$$

are independent of $\theta \in (a, b)$ and $q \in [1, \infty]$. Moreover, there exists a finite dimensional space $M \subset Y_0 \cap Y_1$ such that

$$\vec{Y}_{\theta,q} = M \oplus T(\vec{X}_{\theta,q}) \tag{3.9}$$

for all $\theta \in (a, b)$ and $q \in [1, \infty]$.

Proof. Note that for any $\theta_* \in (a, b)$ the operator $T\colon \vec{X}_{\theta_*,q_*} \to \vec{Y}_{\theta_*,q_*}$ is Fredholm. Applying Theorem 3.4 with functors of real interpolation $F_\theta(\cdot) = (\cdot)_{\theta,q_*}$ and $G(\cdot) = (\cdot)_{\lambda,q}$, $q \in [1, \infty]$, we obtain that there exists $\varepsilon = \varepsilon(\theta_*) > 0$ such that for all $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ the spaces

$$W = \ker T \cap \vec{X}_{\theta,q} \text{ and } WW = T(\vec{X}_{\theta,q}) \cap (Y_0 \cap Y_1)$$

are independent of $\theta \in (\theta_* - \varepsilon(\theta_*), \theta_* + \varepsilon(\theta_*))$ and $q \in [1, \infty]$. Let $\theta_0, \theta_1 \in (a, b)$, then

$$[\theta_0, \theta_1] \subset \bigcup_{\theta_* \in [\theta_0, \theta_1]} (\theta_* - \varepsilon(\theta_*), \theta_* + \varepsilon(\theta_*)).$$

From compactness of the interval $[\theta_0, \theta_1]$ it follows that this interval can be covered by a finite number of constructed above intervals $(\theta_* - \varepsilon(\theta_*), \theta_* + \varepsilon(\theta_*))$ with $\theta_* \in [\theta_0, \theta_1]$. From this and from independence of W and WW of θ and q on each interval $(\theta_* - \varepsilon(\theta_*), \theta_* + \varepsilon(\theta_*))$ it immediately follows that the spaces W and WW are independent of θ on the interval $[\theta_0, \theta_1]$. Since $\theta_0, \theta_1 \in (a,b)$ are arbitrary we have independence of W and WW of θ and q on the whole interval (a,b).

Now let us prove that there exists a finite dimensional space $M \subset Y_0 \cap Y_1$ independent of $\theta \in (a,b)$, with the property (3.9). First let us note that from Fredholmness of the operator T it follows that for any $\theta \in (a,b)$ the space $T(\vec{X}_{\theta,q})$ is closed in $\vec{Y}_{\theta,q}$ and therefore from $Y_0 \cap Y_1 \subset \vec{Y}_{\theta,q}$ we obtain that the subspace

$$WW = T(\vec{X}_{\theta,q}) \cap (Y_0 \cap Y_1)$$

is closed in $Y_0 \cap Y_1$.

Now, if in the statement (iii) in Theorem 3.4 we set $G(\vec{X}) = \vec{X}_{1/2,q}$ then for any $\theta \in (a,b)$, we obtain the existence of a finite dimensional space $M_\theta \subset Y_0 \cap Y_1$ such that

$$\vec{Y}_{\theta,q} = M_\theta \oplus T(\vec{X}_{\theta,q}).$$

If we intersect the spaces in both sides of this equality with $Y_0 \cap Y_1$ we obtain

$$Y_0 \cap Y_1 = M_\theta \oplus WW.$$

Hence WW has a finite codimension in $Y_0 \cap Y_1$. As WW does not depend on $\theta \in (a,b)$ and $q \in [1,\infty]$ therefore there exists a finite dimensional space M that is also independent of $\theta \in (a,b)$ and q and such that

$$Y_0 \cap Y_1 = M \oplus WW.$$

Moreover, it is clear that for any $\theta \in (a,b)$ and $q \in [1,\infty]$ we have

$$\vec{Y}_{\theta,q} = M_\theta \oplus T(\vec{X}_{\theta,q}) \subset M \oplus T(\vec{X}_{\theta,q}) = \vec{Y}_{\theta,q}.$$

This gives us the required equality (3.9) and the proof is complete. □

Corollary 3.7. *If (a,b) is an interval of Fredholmness of an operator T for the real method with a parameter $q_* \in [1,\infty)$, then (a,b) is an interval of Fredholmness for any $q \in [1,\infty]$, and so the interval of Fredholmness is independent of $q_* \in [1,\infty)$.*

4. Characterization of Maximal Fredholm Intervals

Let an operator $T\colon (X_0,X_1) \to (Y_0,Y_1)$ be invertible on the end spaces, i.e., the operators $T\colon X_i \to Y_i$, $i = 0,1$, are invertible. Suppose also that for some parameters $\theta_* \in (0,1)$, $q \in [1,\infty)$ the operator $T\colon \vec{X}_{\theta_*,q} \to \vec{Y}_{\theta_*,q}$ is Fredholm. In this section we show that it is possible to characterize the maximal interval (a,b) that contains θ_* and such that for all $\theta \in (a,b)$ the operator $T\colon \vec{X}_{\theta,q} \to \vec{Y}_{\theta,q}$ is Fredholm. We would like to mention that the problem of characterization of a maximal interval of invertibility of an operator was solved in [4].

Below we need the notion of quotient operators for the case of couples. Let (X_0,X_1) be a couple and let $U \subset X_0 + X_1$ be a closed subspace. Then we can consider a quotient operator with the kernel U, i.e., an operator

$$\pi\colon X_0 + X_1 \to (X_0 + X_1)/U,$$

where $(X_0+X_1)/U$ (we also denote it by $\pi(X_0+X_1)$) is a quotient space with the quotient norm

$$\|\bar{x}\|_{\pi(X_0+X_1)} = \inf_{\pi(x)=\bar{x}} \|x\|_{X_0+X_1}.$$

Note that $X_i \cap U$ is a closed subspace of X_i, $i=0,1$. Therefore, on the space $\pi(X_i)$ we can consider the quotient norm $X_i/(X_i\cap U)$, i.e.,

$$\|\bar{x}\|_{\pi(X_i)} = \inf_{\pi(x)=\bar{x}} \|x\|_{X_i}, \ \ i=0,1.$$

The couple $(\pi(X_0),\pi(X_1))$ with these norms is said to be the quotient couple. Then with the equality of the norms (see [5]) we have

$$\pi(X_0)+\pi(X_1) = \pi(X_0+X_1).$$

It is not difficult to prove the next result.

Proposition 4.1. *Let $T\colon (X_0,X_1)\to(Y_0,Y_1)$. Suppose that $U\subset \ker T$ is a closed subspace of X_0+X_1 and $\pi\colon X_0+X_1\to(X_0+X_1)/U$ is a quotient map. Then there exists a unique bounded linear operator $S\colon(\pi(X_0),\pi(X_1))\to(Y_0,Y_1)$ such that*

$$T = S\pi.$$

To characterize maximal Fredholm intervals of an operator $T\colon (X_0,X_1)\to(Y_0,Y_1)$ we need to define linear spaces $V^0_{\theta,q}=V^0_{\theta,q}(T)$, $V^1_{\theta,q}=V^1_{\theta,q}(T)$. These spaces were introduced in [4] and they are defined by the formulas

$$V^0_{\theta,q}(T) = \left\{x\in\ker T : x=x_0+x_1,\ x_0\in\vec{X}_{\theta,q}\cap X_0,\ x_1\in X_1\right\},$$
$$V^1_{\theta,q}(T) = \left\{x\in\ker T : x=x_0+x_1,\ x_0\in X_0,\ x_1\in\vec{X}_{\theta,q}\cap X_1\right\}.$$

The spaces $V^0_{\theta,q}$ and $V^1_{\theta,q}$ can also be defined in terms of the K-functional (see [4]):

$$V^0_{\theta,q} = \left\{x\in\ker T : \left(\int_0^1 (t^{-\theta}K(t,x;X_0,X_1)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty\right\},$$
$$V^1_{\theta,q} = \left\{x\in\ker T : \left(\int_1^\infty (t^{-\theta}K(t,x;X_0,X_1)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty\right\}.$$

From these definitions it follows (see [4] and [5]) that

$$V^0_{\theta_1,q}\subset V^0_{\theta_0,q}, \quad V^1_{\theta_0,q}\subset V^1_{\theta_1,q} \text{ if } \theta_0<\theta_1 \tag{4.1}$$

and

$$\ker T\cap\vec{X}_{\theta,q} = V^0_{\theta,q}\cap V^1_{\theta,q}. \tag{4.2}$$

We also define a linear space $\widetilde{V}_{\theta,q}$ such that

$$\ker T = (V^0_{\theta,q}+V^1_{\theta,q})\oplus\widetilde{V}_{\theta,q}. \tag{4.3}$$

In [5] it was shown that in the case when $T\colon\vec{X}_{\theta,q}\to\vec{Y}_{\theta,q}$ is Fredholm the spaces $V^0_{\theta,q}\cap V^1_{\theta,q}$ and $\widetilde{V}_{\theta,q}$ are finite dimensional and the operator T can be decomposed into three operators

$$T = T_3T_2T_1, \tag{4.4}$$

where T_1 is a quotient operator with the kernel $U_1 = V^0_{\theta,q} \cap V^1_{\theta,q}$ that maps the couple (X_0, X_1) to the couple $(T_1(X_0), T_1(X_1))$, the operator T_2 is a quotient operator with the kernel $U_2 = T_1(\widetilde{V}_{\theta,q})$ that maps the couple $(T_1(X_0), T_1(X_1))$ to the couple $(T_2T_1(X_0), T_2T_1(X_1))$ and T_3 is chosen in such a way that $T = T_3T_2T_1$ (see Proposition 4.1).

The next theorem proves stability of spaces $V^0_{\theta,q}$, $V^1_{\theta,q}$ on Fredholm intervals.

Theorem 4.2. *Suppose that an operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$ is invertible on the end spaces. Let (a, b) be a Fredholm interval of the operator T for the real method and $1 \le q < \infty$. Then the spaces $V^0_{\theta,q}$, $V^1_{\theta,q}$ are independent of $\theta \in (a, b)$.*

Proof. Let us first prove that

$$V^i_{\theta,q} = (V^i_{\theta,q} + \widetilde{V}_{\theta,q}) \cap (V^0_{\theta,q} + V^1_{\theta,q}),\ i = 0, 1. \tag{4.5}$$

Clearly,

$$V^i_{\theta,q} \subset (V^i_{\theta,q} + \widetilde{V}_{\theta,q}) \cap (V^0_{\theta,q} + V^1_{\theta,q}).$$

To prove the opposite embedding let $x \in (V^i_{\theta,q} + \widetilde{V}_{\theta,q}) \cap (V^0_{\theta,q} + V^1_{\theta,q})$. Then there exist decompositions

$$x = u + \widetilde{v}, \quad u \in V^i_{\theta,q}, \quad \widetilde{v} \in \widetilde{V}_{\theta,q},$$
$$x = v_0 + v_1,\ v_0 \in V^0_{\theta,q},\ \ v_1 \in V^1_{\theta,q}.$$

Hence

$$\widetilde{v} = v_0 + v_1 - u \in \widetilde{V}_{\theta,q} \cap (V^0_{\theta,q} + V^1_{\theta,q}).$$

From the definition of $\widetilde{V}_{\theta,q}$ it follows that $\widetilde{V}_{\theta,q} \cap (V^0_{\theta,q} + V^1_{\theta,q}) = \{0\}$, i.e., $\widetilde{v} = 0$, and therefore $x = u \in V^i_{\theta,q}$, which proves (4.5).

Let us now show that the linear space

$$V^0_{\theta,q} + V^1_{\theta,q}$$

is independent of $\theta \in (a, b)$, i.e., $V^0_{\theta_0,q} + V^1_{\theta_0,q} = V^0_{\theta_1,q} + V^1_{\theta_1,q}$ for any $\theta_0, \theta_1 \in (a, b)$. It is sufficient to show that the embedding

$$V^0_{\theta_0,q} + V^1_{\theta_0,q} \subset V^0_{\theta_1,q} + V^1_{\theta_1,q} \tag{4.6}$$

is valid for any arbitrary $\theta_0, \theta_1 \in (a, b)$. Let $x \in V^0_{\theta_0,q} + V^1_{\theta_0,q}$, which means that $x = v_0 + v_1$, where $v_i \in V^i_{\theta_0,q}$, $i = 0, 1$. From the definition of $V^i_{\theta_0,q}$, we get

$$v_0 = v^0_0 + v^1_0,\ v^0_0 \in X_0 \cap \vec{X}_{\theta_0,q},\ v^1_0 \in X_1,$$
$$v_1 = v^0_1 + v^1_1,\ v^0_1 \in X_0, \qquad v^1_1 \in X_1 \cap \vec{X}_{\theta_0,q}.$$

As $v_i \in \ker T$ then

$$Tv^0_i = -Tv^1_i \in T(\vec{X}_{\theta_0,q}) \cap (Y_0 \cap Y_1) = WW,\ i = 0, 1.$$

Defining $x_0 = v^0_0 + v^0_1$ and $x_1 = v^1_0 + v^1_1$ we have

$$x = v_0 + v_1 = (v^0_0 + v^0_1) + (v^1_0 + v^1_1) = x_0 + x_1,$$

where $x_i \in X_i$ $(i = 0, 1)$ and $Tx_0 = -Tx_1 \in WW$. According to Theorem 3.6 the space WW is independent of $\theta \in (a, b)$, i.e.,

$$WW = T(\vec{X}_{\theta_0,q}) \cap (Y_0 \cap Y_1) = T(\vec{X}_{\theta_1,q}) \cap (Y_0 \cap Y_1)$$

and therefore there exists $\tilde{x} \in \vec{X}_{\theta_1,q}$ such that

$$T\tilde{x} = Tx_0 = -Tx_1.$$

Let us consider a decomposition $x = (x_0 - \tilde{x}) + (x_1 + \tilde{x})$ with $x_0 - \tilde{x},\ x_1 + \tilde{x} \in \ker T$. Since $\tilde{x} \in \vec{X}_{\theta_1,q}$ therefore according to Propositions 4 and 2 from [4] there exists a decomposition $\tilde{x} = \tilde{x}_0 + \tilde{x}_1$ such that $\tilde{x}_i \in X_i \cap \vec{X}_{\theta_1,q}$. Hence,

$$x_0 - \tilde{x} = (x_0 - \tilde{x}_0) - \tilde{x}_1 \text{ and } x_1 + \tilde{x} = \tilde{x}_0 + (x_1 + \tilde{x}_1),$$

where

$$x_0 - \tilde{x}_0 \in X_0, -\tilde{x}_1 \in X_1 \cap \vec{X}_{\theta_1,q} \text{ and } \tilde{x}_0 \in X_0 \cap \vec{X}_{\theta_1,q},\ x_1 + \tilde{x}_1 \in X_1.$$

It means that $x_0 - \tilde{x} \in V^1_{\theta_1,q}$ and $x_1 + \tilde{x} \in V^0_{\theta_1,q}$, i.e.,

$$x = (x_0 - \tilde{x}) + (x_1 + \tilde{x}) \in V^0_{\theta_1,q} + V^1_{\theta_1,q}.$$

This proves the embedding (4.6) and therefore $V^0_{\theta,q} + V^1_{\theta,q}$ is independent of $\theta \in (a,b)$.

The fact that the space $V^0_{\theta,q} + V^1_{\theta,q}$ does not depend on $\theta \in (a,b)$ allows us to choose the space $\widetilde{V}_{\theta,q}$ in (4.3) independently of $\theta \in (a,b)$. So, everywhere below we suppose that the space $\widetilde{V}_{\theta,q}$ is independent of $\theta \in (a,b)$.

From (4.5) it follows that to prove the theorem it remains to show that the spaces

$$V^0_{\theta,q} + \widetilde{V}_{\theta,q},\ V^1_{\theta,q} + \widetilde{V}_{\theta,q} \tag{4.7}$$

are independent of $\theta \in (a,b)$. From (4.2) and Theorem 3.6 we see that $V^0_{\theta,q} \cap V^1_{\theta,q} = \ker T \cap \vec{X}_{\theta,q}$ is independent of θ on the whole interval (a,b). Since the space $\widetilde{V}_{\theta,q}$ is also independent of $\theta \in (a,b)$ we obtain that the operators T_1 and T_2 in (4.4), and, consequently, even T_3 are independent of $\theta \in (a,b)$. As shown in [5], the operator

$$T_3\colon (T_2T_1(X_0), T_2T_1(X_1))_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$$

is invertible for all $\theta \in (a,b)$. Therefore, from Theorems 11 and 14 in [4], it follows that

$$V^0_{\theta,q}(T_3) \oplus V^1_{\theta,q}(T_3) = \ker T_3 \tag{4.8}$$

and the spaces $V^0_{\theta,q}(T_3)$ and $V^1_{\theta,q}(T_3)$ are independent of $\theta \in (a,b)$.

From the definitions of the spaces $V^0_{\theta,q}(T)$ and $V^1_{\theta,q}(T)$ it is not difficult to prove that

$$T_2T_1(V^i_{\theta,q}(T)) \subset V^i_{\theta,q}(T_3),\ i = 0,1. \tag{4.9}$$

Therefore, from (4.8) it follows that

$$T_2T_1(V^0_{\theta,q}(T)) \oplus T_2T_1(V^1_{\theta,q}(T)) \subset \ker T_3.$$

On the other hand, $T_2T_1\colon X_0 + X_1 \to T_2T_1(X_0) + T_2T_1(X_1)$ is surjective therefore for any $z \in \ker T_3$ there exists $x \in X_0 + X_1$ such that $T_2T_1x = z$. Clearly, $x \in \ker T = V^0_{\theta,q} + V^1_{\theta,q} + \widetilde{V}_{\theta,q}$ and therefore $x = v_0 + v_1 + \tilde{v}$, where $v_i \in V^i_{\theta,q}$, $i = 0,1$, and $\tilde{v} \in \tilde{V}_{\theta,q}$. Since $T_1(\tilde{V}_{\theta,q}) = \ker T_2$ we have

$$z = T_2T_1x = T_2T_1v_0 + T_2T_1v_1.$$

Hence,

$$T_2T_1(V^0_{\theta,q}(T)) \oplus T_2T_1(V^1_{\theta,q}(T)) = \ker T_3.$$

Taking into account (4.8) and (4.9), we obtain

$$T_2T_1(V^i_{\theta,q}(T)) = V^i_{\theta,q}(T_3),\ i = 0,1.$$

So from $\ker(T_2T_1) = V^0_{\theta,q} \cap V^1_{\theta,q} + \widetilde{V}_{\theta,q}$ it follows that

$$\left\{x \in X_0 + X_1 : T_2T_1x \in V^i_{\theta,q}(T_3)\right\} = V^i_{\theta,q}(T) + \widetilde{V}_{\theta,q}(T),\ i = 0,1.$$

Since the space $V^i_{\theta,q}(T_3),\ i = 0,1$, and the operator T_2T_1 are independent of $\theta \in (a,b)$ therefore the space $V^i_{\theta,q}(T) + \widetilde{V}_{\theta,q}(T),\ i = 0,1$, is also independent of $\theta \in (a,b)$.

We have proved that the space (4.7) is independent of $\theta \in (a,b)$ and thus the proof of the theorem is complete. □

In [5] the spaces $V^0_{\theta,q}, V^1_{\theta,q}$ and $\tilde{V}_{\theta,q}$ were used to formulate the necessary and sufficient condition for Fredholmness of an operator $T : \vec{X}_{\theta,q} \to \vec{Y}_{\theta,q}$. The result was formulated in terms of generalized dilation indices. Let us remind the definitions of these indices.

Let us consider a set $\Omega \subset X_0 + X_1$. We denote by $\beta(\Omega)$ an infimum of all $\theta \in [0,1]$ for which there exists $\gamma = \gamma(\theta,\Omega) > 0$ such that for all $x \in \Omega \setminus \{0\}$ and all $0 < s \le t$, we have

$$\frac{K(s,x;X_0,X_1)}{K(t,x;X_0,X_1)} \ge \gamma \left(\frac{s}{t}\right)^{\theta}. \tag{4.10}$$

If in this definition we consider such parameters s and t that satisfy the inequalities $0 < s \le t \le 1$, then we obtain an index $\beta_0(\Omega)$ and if we consider such parameters s and t that satisfy the inequalities $1 \le s \le t$ then we obtain an index $\beta_\infty(\Omega)$.

Similarly, by $\alpha(\Omega)$ we denote the supremum of all $\theta \in [0,1]$ for which there exists $\gamma = \gamma(\theta,\Omega) > 0$ such that for all $x \in \Omega \setminus \{0\}$ and all $0 < s \le t$ we have

$$\frac{K(s,x;X_0,X_1)}{K(t,x;X_0,X_1)} \le \gamma \left(\frac{s}{t}\right)^{\theta}. \tag{4.11}$$

If instead of $0 < s \le t$ we consider such parameters s and t that satisfy the inequalities $0 < s \le t \le 1$ then we obtain an index $\alpha_0(\Omega)$ and if we consider such parameters s and t that satisfy $1 \le s \le t$ then we obtain an index $\alpha_\infty(\Omega)$.

In the case when the set Ω consists of one element, i.e., $\Omega = \{x\}$, we denote the indices $\alpha(\Omega)$, $\beta(\Omega)$ by $\alpha(x)$, $\beta(x)$ (similarly, we obtain the indices $\alpha_0(x)$, $\beta_0(x)$, $\alpha_\infty(x)$ and $\beta_\infty(x)$). Note that if $\Omega = \{0\}$, we put $\alpha(\Omega) = \alpha_0(\Omega) = \alpha_\infty(\Omega) = 1$ and $\beta(\Omega) = \beta_0(\Omega) = \beta_\infty(\Omega) = 0$.

Let $U \subset X_0 + X_1$ be a closed subspace, then if π is a quotient operator with a kernel U then using Proposition 5 in [5] we obtain that the K-functional of an element $\pi(x) \in \pi(X_0) + \pi(X_1)$ is equal to

$$K(t,\pi(x),\pi(X_0),\pi(X_1)) = \inf_{u \in U} K(t,x+u,X_0,X_1).$$

Thus, the indices for the set $\pi(\Omega)$ ($\Omega \subset X_0 + X_1$) with respect to the couple $(\pi(X_0),\pi(X_1))$ can be calculated in terms of the K-functional of the couple (X_0,X_1). We denote them by $\alpha(\Omega;U)$, $\beta(\Omega;U)$, $\alpha_0(\Omega;U)$, $\beta_0(\Omega;U)$, $\alpha_\infty(\Omega;U)$, and $\beta_\infty(\Omega;U)$.

Note that from the definition of the Fredholm interval for the real method with a parameter q it follows that if an operator $T : (X_0,X_1)_{\theta_*,q} \to (Y_0,Y_1)_{\theta_*,q}$, $1 \le q < \infty$, is Fredholm, then there exists a maximal Fredholm interval (a,b) that contains θ_*.

Theorem 4.3. *Suppose that an operator $T\colon (X_0,X_1)\to(Y_0,Y_1)$ is invertible on the end spaces and the operator $T\colon \vec{X}_{\theta_*,q}\to\vec{Y}_{\theta_*,q}, 1\le q<\infty$, is Fredholm. Let $(a,b)\ni\theta_*$ be a maximal Fredholm interval for the real method with a parameter q. Then $\theta\in(a,b)$ if and only if*

$$\beta_\infty(U)<\theta<\alpha_0(U), \tag{4.12}$$

$$\beta_0(V;U)<\theta<\alpha_\infty(V;U), \tag{4.13}$$

$$\beta(V^1_{\theta_*,q};U+V)<\theta<\alpha(V^0_{\theta_*,q};U+V), \tag{4.14}$$

where $U=V^0_{\theta_*,q}\cap V^1_{\theta_*,q}$ and $V=\tilde{V}_{\theta_*,q}$.

Proof. Necessity. Let $\theta\in(a,b)$. From Theorem 4.2 we see that $V^0_{\theta,q}=V^0_{\theta_*,q}$ and $V^1_{\theta,q}=V^1_{\theta_*,q}$ and therefore the space $\tilde{V}_{\theta,q}$ can be chosen equal to $\widetilde{V}_{\theta_*,q}$. Hence, Corollary 1 from [5] gives us that for θ the inequalities (4.12)-(4.14) are valid.

Sufficiency. We need to show that for any θ satisfying the inequalities (4.12)-(4.14), the operator $T\colon \vec{X}_{\theta,q}\to\vec{Y}_{\theta,q}$ is Fredholm. Since $T\colon \vec{X}_{\theta_*,q}\to\vec{Y}_{\theta_*,q}$ is Fredholm, the operator T can be decomposed as $T=T_3T_2T_1$ (see (4.4)), where T_1,T_2 are quotient operators with the kernels $V^0_{\theta_*,q}\cap V^1_{\theta_*,q}$ and $T_1(\widetilde{V}_{\theta_*,q})$, respectively. From Theorem 10 and Theorem 11 in [5] it follows that for any θ satisfying (4.12)-(4.13), the operators

$$T_1\colon (X_0,X_1)_{\theta,q}\to(T_1(X_0),T_1(X_1))_{\theta,q},$$
$$T_2\colon (T_1(X_0),T_1(X_1))_{\theta,q}\to(T_2T_1(X_0),T_2T_1(X_1))_{\theta,q}$$

are Fredholm. Note that from Theorem 12 in [5] we conclude that the operator T_3 defined on $(T_2T_1(X_0),T_2T_1(X_1))_{\theta_*,q}$ is invertible. Therefore, applying Theorem 14 from [4], we conclude that for any θ satisfying (4.14) the operator $T_3\colon (T_2T_1(X_0),T_2T_1(X_1))_{\theta,q}\to(Y_0,Y_1)_{\theta,q}$ is invertible. Consequently, the operator

$$T=T_3T_2T_1\colon (X_0,X_1)_{\theta,q}\to(Y_0,Y_1)_{\theta,q}$$

is Fredholm. This completes the proof. □

5. Characterization of Maximal Intervals of Invertibility

Theorem 4.3 generalizes Theorem 14 from [4], in which maximal intervals of invertibility were characterized. In this case $U=V^0_{\theta_*,q}\cap V^1_{\theta_*,q}=\{0\}$, $V=\widetilde{V}_{\theta_*,q}=\{0\}$ and we arrive at the following theorem.

Theorem 5.1. *Suppose that an operator $T\colon (X_0,X_1)\to(Y_0,Y_1)$ is invertible on the end spaces and the operator $T\colon \vec{X}_{\theta_*,q}\to\vec{Y}_{\theta_*,q}, 1\le q\le\infty$, is invertible. Let (a,b) be a maximal interval of invertibility of the operator T that contains θ_*. Then $\theta\in(a,b)$ if and only if*

$$a=\beta(V^1_{\theta_*,q})<\theta<\alpha(V^0_{\theta_*,q})=b. \tag{5.1}$$

The calculation of indices $\beta(V^1_{\theta_*,q})$ and $\alpha(V^0_{\theta_*,q})$ is a rather difficult problem. As we can see below, the next result allows us to simplify the calculations in some concrete situations.

Theorem 5.2. *Suppose that an operator $T\colon (X_0,X_1)\to(Y_0,Y_1)$ is invertible on the end spaces. Let (a_0,b_0) and (a_1,b_1) be two maximal intervals of invertibility of T for the real method with $a_0<b_0\le a_1<b_1$. If $\theta_0\in(a_0,b_0)$ and $\theta_1\in(a_1,b_1)$ then*

$$b_0=\alpha(V^0_{\theta_0,q}\cap V^1_{\theta_1,q}),\ \ a_1=\beta(V^0_{\theta_0,q}\cap V^1_{\theta_1,q}).$$

Proof. We need some simple properties of dilation indices that follow directly from the definitions (see (4.10), (4.11)):

(i) let $\Omega \subset X_0 + X_1$ and $\Omega \neq \{0\}$ then

$$\alpha(\Omega) \leq \beta(\Omega); \tag{5.2}$$

(ii) if $\Omega_0 \subset \Omega_1 \subset X_0 + X_1$ then

$$\alpha(\Omega_0) \geq \alpha(\Omega_1) \text{ and } \beta(\Omega_0) \leq \beta(\Omega_1). \tag{5.3}$$

Note that from (5.1) we have

$$a_i = \beta(V^1_{\theta_i,q}),\ b_i = \alpha(V^0_{\theta_i,q}),\ i = 0,1. \tag{5.4}$$

Let us first show that

$$V^0_{\theta_0,q} = V^0_{\theta_1,q} \oplus (V^0_{\theta_0,q} \cap V^1_{\theta_1,q}),\ \ V^1_{\theta_1,q} = V^1_{\theta_0,q} \oplus (V^0_{\theta_0,q} \cap V^1_{\theta_1,q}). \tag{5.5}$$

From invertibility of operators

$$T : \vec{X}_{\theta_i,q} \to \vec{Y}_{\theta_i,q},\ i = 0,1,$$

and Theorem 11 from [4] it follows that

$$\ker T = V^0_{\theta_i,q} \oplus V^1_{\theta_i,q},\ i = 0,1. \tag{5.6}$$

Hence,

$$V^0_{\theta_1,q} \cap (V^0_{\theta_0,q} \cap V^1_{\theta_1,q}) = \{0\},\ \ V^1_{\theta_0,q} \cap (V^0_{\theta_0,q} \cap V^1_{\theta_1,q}) = \{0\}$$

and therefore from the inclusions (4.1) we have

$$V^0_{\theta_1,q} \oplus (V^0_{\theta_0,q} \cap V^1_{\theta_1,q}) \subset V^0_{\theta_0,q},\ \ V^1_{\theta_0,q} \oplus (V^0_{\theta_0,q} \cap V^1_{\theta_1,q}) \subset V^1_{\theta_1,q}. \tag{5.7}$$

Now let us prove the opposite inclusions. According to (5.6) for any $x \in \ker T$ there exist decompositions

$$x = x^0_{\theta_0} + x^1_{\theta_0},\ \ x = x^0_{\theta_1} + x^1_{\theta_1}, \tag{5.8}$$

where $x^0_{\theta_0} \in V^0_{\theta_0,q}$, $x^1_{\theta_0} \in V^1_{\theta_0,q}$ and $x^0_{\theta_1} \in V^0_{\theta_1,q}$, $x^1_{\theta_1} \in V^1_{\theta_1,q}$. So $x^0_{\theta_0} + x^1_{\theta_0} = x^0_{\theta_1} + x^1_{\theta_1}$ and therefore the element

$$u = x^0_{\theta_0} - x^0_{\theta_1} = x^1_{\theta_1} - x^1_{\theta_0}$$

belongs to $V^0_{\theta_0,q} \cap V^1_{\theta_1,q}$. Since $x^0_{\theta_0} = x^0_{\theta_1} + u$ and $x^1_{\theta_1} = x^1_{\theta_0} + u$ therefore from the assumption that $x \in V^0_{\theta_0,q}$ we have $x = x^0_{\theta_0}$ in (5.8) and consequently we obtain the inclusions

$$V^0_{\theta_0,q} \subset V^0_{\theta_1,q} \oplus (V^0_{\theta_0,q} \cap V^1_{\theta_1,q}).$$

Similarly, from the assumption that $x \in V^1_{\theta_1,q}$ we have

$$V^1_{\theta_1,q} \subset V^1_{\theta_0,q} \oplus (V^0_{\theta_0,q} \cap V^1_{\theta_1,q}).$$

The obtained inclusions together with (5.7) give the equalities (5.5).

To prove the theorem we need to show that

$$\alpha(V^0_{\theta_0,q}) = \alpha(V^0_{\theta_0,q} \cap V^1_{\theta_1,q}),\ \ \beta(V^1_{\theta_1,q}) = \beta(V^0_{\theta_0,q} \cap V^1_{\theta_1,q}).$$

First let us note that the inclusions $V^0_{\theta_0,q}\cap V^1_{\theta_1,q}\subset V^0_{\theta_0,q}$, $V^0_{\theta_0,q}\cap V^1_{\theta_1,q}\subset V^1_{\theta_1,q}$ and the property (5.3) give us

$$\alpha(V^0_{\theta_0,q})\le\alpha(V^0_{\theta_0,q}\cap V^1_{\theta_1,q}), \tag{5.9}$$

$$\beta(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})\le\beta(V^1_{\theta_1,q}). \tag{5.10}$$

Since $V^0_{\theta_0,q}=V^0_{\theta_1,q}\oplus(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})$, any element $x\in V^0_{\theta_0,q}$ can be decomposed as $x=u+v$, where $u\in V^0_{\theta_1,q}$ and $v\in V^0_{\theta_0,q}\cap V^1_{\theta_1,q}$. Hence, from (5.10) and (5.4) it follows that

$$\beta(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})\le\beta(V^1_{\theta_1,q})=a_1<b_1=\alpha(V^0_{\theta_1,q})$$

and therefore Lemma 1 in [4] gives us the inequalities

$$\begin{aligned}\frac{K(s,u+v)}{K(t,u+v)}&\le\gamma_1\frac{K(s,u)+K(s,v)}{K(t,u)+K(t,v)}\\&\le\frac{\gamma_2(\frac{s}{t})^{\alpha(V^0_{\theta_1,q})-\varepsilon}K(t,u)+\gamma_3(\frac{s}{t})^{\alpha(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})-\varepsilon}K(t,v)}{K(t,u)+K(t,v)}\end{aligned}$$

for $0<s\le t$. Here γ_i, $i=1,2,3$, are positive constants independent of u,v,s and t. From the properties (5.2) and (5.3) we have

$$\alpha(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})\le\beta(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})\le\beta(V^1_{\theta_1,q})<\alpha(V^0_{\theta_1,q})$$

and therefore

$$\frac{K(s,x)}{K(t,x)}=\frac{K(s,u+v)}{K(t,u+v)}\le\gamma(\frac{s}{t})^{\alpha(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})-\varepsilon},$$

which means that

$$\alpha(V^0_{\theta_0,q})\ge\alpha(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})$$

and we have the equality in (5.9). Similarly, we can prove the equality in (5.10). Indeed, as $V^1_{\theta_1,q}=V^1_{\theta_0,q}\oplus(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})$ therefore an arbitrary element $x\in V^1_{\theta_1,q}$ can be decomposed as $x=u+v$, where $u\in V^1_{\theta_0,q}$ and $v\in V^0_{\theta_0,q}\cap V^1_{\theta_1,q}$. Hence, from (5.9) and (5.4) it follows that

$$\beta(V^1_{\theta_0,q})=a_0<b_0=\alpha(V^1_{\theta_0,q})\le\alpha(V^0_{\theta_0,q}\cap V^1_{\theta_1,q}).$$

So from Lemma 1 in [4] we have

$$\begin{aligned}\frac{K(s,u+v)}{K(t,u+v)}&\ge\gamma_1\frac{K(s,u)+K(s,v)}{K(t,u)+K(t,v)}\\&\ge\frac{\gamma_2(\frac{s}{t})^{\beta(V^1_{\theta_0,q})+\varepsilon}K(t,u)+\gamma_3(\frac{s}{t})^{\beta(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})+\varepsilon}K(t,v)}{K(t,u)+K(t,v)}\end{aligned}$$

for $0<s\le t$. Here γ_i, $i=1,2,3$, are positive constants independent of u,v,s and t. As $0<s\le t$ and

$$\beta(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})\ge\alpha(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})\ge\alpha(V^0_{\theta_0,q})>\beta(V^1_{\theta_0,q})$$

therefore

$$\frac{K(s,u+v)}{K(t,u+v)}\ge\gamma(\frac{s}{t})^{\beta(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})+\varepsilon}.$$

Hence, $\beta(V^1_{\theta_1,q})\le\beta(V^0_{\theta_0,q}\cap V^1_{\theta_1,q})$ and we obtain the equality in (5.10). This completes the proof of the theorem. □

Note that if $\dim\left(V^0_{\theta_0,q}\cap V^1_{\theta_1,q}\right)=1$ and $e\in V^0_{\theta_0,q}\cap V^1_{\theta_1,q}$, then in Theorem 5.2 we have $b_0=\alpha(e)$ and $a_1=\beta(e)$.

In [4] it was shown that if an operator $T\colon (X_0,X_1)\to(Y_0,Y_1)$ is invertible on the end spaces and $\dim\ker T=n$ then the number of maximal intervals of invertibility does not exceed $n+1$. The next theorem applies Theorem 5.2 to find the necessary conditions for the number of maximal intervals of invertibility to be exactly $n+1$.

Theorem 5.3. *Let $T\colon (X_0,X_1)\to(Y_0,Y_1)$ be an operator invertible on the end spaces with $\dim(\ker T)=n$. The set of invertibility consists of $(n+1)$ disjoint intervals if and only if there exists a basis $e_1,...,e_n$ of $\ker T$ such that*

$$0<\alpha(e_1)\le\beta(e_1)<\alpha(e_2)\le\beta(e_2)<\ldots<\alpha(e_n)\le\beta(e_n)<1.$$

Proof. Sufficiency follows immediately from Theorem 16 in [4]. To prove necessity, let the set of invertibility consists of $n+1$ disjoint intervals $(a_1,b_1),\ldots,(a_{n+1},b_{n+1})$, where $a_k<b_k\le a_{k+1}<b_{k+1}$, $k=1,\ldots,n$, and let $\theta_k\in(a_k,b_k)$. Note that from Theorem 5.1 one has

$$a_k=\beta(V^1_{\theta_k,q}),\ b_k=\alpha(V^0_{\theta_k,q})$$

and therefore $V^0_{\theta_k,q}\ne V^0_{\theta_{k+1},q}$, $V^1_{\theta_k,q}\ne V^1_{\theta_{k+1},q}$ for each $k=1,\ldots,n+1$. Hence, from (4.1), the equality $\ker T=V^0_{\theta_k,q}\oplus V^1_{\theta_k,q}$ (see Theorem 11 in [4]), and the fact that $\dim\ker T=n$ we obtain

$$\ker T=V^0_{\theta_1,q}\supset V^0_{\theta_2,q}\supset\ldots\supset V^0_{\theta_k,q}\supset\ldots\supset V^0_{\theta_{n+1},q}=\{0\},\tag{5.11}$$

$$\{0\}=V^1_{\theta_1,q}\subset V^1_{\theta_2,q}\subset\ldots\subset V^1_{\theta_k,q}\subset\ldots\subset V^1_{\theta_{n+1},q}=\ker T\tag{5.12}$$

with strict inclusions. Therefore, for the intervals (a_1,b_1) and (a_{n+1},b_{n+1}) Theorem 5.1 gives us

$$a_1=\beta(V^1_{\theta_1,q})=\beta(\{0\})=0,$$
$$b_{n+1}=\alpha(V^0_{\theta_{n+1},1})=\alpha(\{0\})=1.$$

Taking into account the equalities (5.5), from the inclusions (5.11)-(5.12) we have

$$V^0_{\theta_k,q}=V^0_{\theta_{k+1},q}\oplus(V^0_{\theta_k,q}\cap V^1_{\theta_{k+1},q}),\quad k\in\{1,\ldots,n\}.$$

Therefore,

$$\ker T=(V^0_{\theta_1,q}\cap V^1_{\theta_2,q})\oplus(V^0_{\theta_2,q}\cap V^1_{\theta_3,q})\oplus\cdots\oplus(V^0_{\theta_{n-1},q}\cap V^1_{\theta_n,q})$$

and $\dim(V^0_{\theta_k,q}\cap V^1_{\theta_{k+1},q})=1$, $k\in\{1,\ldots,n\}$. Let e_k be an arbitrary element from $V^0_{\theta_k,q}\cap V^1_{\theta_{k+1},q}$, $k=1,\ldots,n$. It is clear that $\{e_1,\ldots,e_n\}$ forms a basis in $\ker T$. Applying Theorem 5.2 we arrive at

$$b_k=\alpha(V^0_{\theta_k,q}\cap V^1_{\theta_{k+1},q})=\alpha(e_k)\text{ and }a_{k+1}=\beta(V^0_{\theta_k,q}\cap V^1_{\theta_{k+1},q})=\beta(e_k)$$

for each $k=1,\ldots,n$. Hence, as

$$0=a_1<b_1\le a_2<b_2\le\ldots\le a_{n+1}<b_{n+1}=1$$

we have constructed a basis $\{e_1,\ldots,e_n\}$ of $\ker T$ such that

$$0<\alpha(e_1)\le\beta(e_1)<\alpha(e_2)\le\beta(e_2)<\ldots<\alpha(e_n)\le\beta(e_n)<1.$$

□

We conclude the paper with the following.

Remark 5.4. The proof of the theorem shows that in the settings of Theorem 5.3 the set of invertibility of the operator T, i.e., the set of all $\theta \in (0,1)$ such that $T : \vec{X}_{\theta,q} \to \vec{Y}_{\theta,q}$ is invertible, is $(0,\alpha(e_1)) \cup (\beta(e_1),\alpha(e_2)) \cup \ldots \cup (\beta(e_n),1)$.

Acknowledgements
The third named author was supported by the National Science Centre, Poland, Project no. 2019/33/B/ST1/00165.

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