Commun. Optim. Theory 2024 (2024) 24 https://doi.org/10.23952/cot.2024.24

Communications in Optimization Theory
Available online at http://cot.mathres.org

DIFFERENTIAL INEQUALITIES AND DYNAMICAL SYSTEMS FOR FIXED POINT PROBLEMS

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In memory of my friend and co-author R. U Verma

Abstract. In [18] and [19], we have recently studied a behavior of the iterative processes to find fixed points of nonexpansive self-mappings $S : \Omega \to \Omega$ using both total asymptotically nonexpansive approximations $S_k : \Omega \to \Omega$ and total asymptotically weakly contractive approximations S_k, where Ω is a closed and convex set in a uniformly convex Banach space B. We proved there strong and weak convergence of the corresponding iterative consequences. In the present paper we investigate the dynamical systems (1.14) with so called total asymptotically weakly contractive approximating family of operators $S(t) : \Omega \to \Omega$ depending on continuous parameter $t \geq t_0 \geq 0$. Part of the results deals with nonexpansive approximating family of operators $S(t)$. All our proofs are based on the estimates of solutions of the differential inequalities to which most of the paper is devoted.

Keywords. Convergence on subsets; Differential inequalities; Dynamical systems, Total asymptotically nonexpansive and weakly contractive approximating families of operators.

2020 Mathematics Subject Classification. 37C25, 37N30, 47H10.

1. Introduction and Preliminaries

Let B be a real normed space with norm $\|\cdot\|$, $\Omega \subseteq B$ be a nonempty subset and $S : \Omega \to B$ be a continuous operator. Let us recall that S is said to be:

1) strongly contractive if there exists a constant $0 < q < 1$ such that

$$\|Sx - Sy\| \leq q\|x - y\| \quad \forall x, y \in \Omega,$$

2) nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in \Omega,$$

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Received June 21, 2023; Accepted September 18, 2023.

3) weakly contractive if there exists a continuous and strictly increasing function $\psi(t)$ defined on R^+ and positive on $R^+ \setminus \{0\}$, $\psi(0) = 0$, such that

$$\|Sx - Sy\| \leq \|x - y\| - \psi(\|x - y\|) \quad \forall x, y \in \Omega. \tag{1.1}$$

Different aspects of the strongly contractive and nonexpansive operators were widely investigated in the literature (for example, in [22, 23]). The class of weakly contractive operators satisfying (1.1) has been introduced in our work [10] (see further development in [11, 21, 26, 30], [33]-[38] and others). It is clear that the class of strongly contractive mappings is contained in the class of weakly contractive mappings and the class of weakly contractive mappings is contained in the class of nonexpansive mappings.

The fixed point problems, that is, problems of finding solutions of the equations $x = Sx$ were studied for a long time in detail namely for cases 1)-3). Suppose that the fixed point set of S

$$\mathscr{N} = \{x^* : x^* = Sx^*\} \neq \emptyset.$$

To find $x^* \in \mathscr{N}$, the authors of the numerous papers mainly dealt with discrete iterative schemes of two types: first the so-called method of successive approximations

$$x_{n+1} = Sx_n, \;\; n = 1, 2, ..., \;\; x_1 \in \Omega, \tag{1.2}$$

and later the more general Krasnoselskii-Mann style iterative scheme

$$x_{n+1} = x_n - \omega_n(x_n - Sx_n), \;\; n = 1, 2, ..., \;\; x_1 \in \Omega, \tag{1.3}$$

where $0 < \omega_n \leq 1$. Note that (1.2) is the particular case of (1.3) when $\omega_n = 1$ for all $n \geq 1$.

It is obvious that fixed point problems for expanding operators lose their meaning, so it was essential to find intermediate classes of mappings for which the principle of fixed points remains valid. This was done at the end of the last century. Goebel and Kirk introduced in [29] the very important class of asymptotically nonexpansive maps as follows:

4) the mapping $S : \Omega \to B$ is said to be asymptotically nonexpansive if

$$\|S^n x - S^n y\| \leq (1 + k_n)\|x - y\|, \quad n = 1, 2, ..., \forall x, y \in \Omega, \tag{1.4}$$

where S^n denotes n-degree of S, a sequence $\{k_n\} \subset [0, \infty)$ and $k_n \to 0$ as $n \to \infty$.

Later there were several modifications of (1.4). For a corresponding review, see [24] and our work [8], which was also a generalization and significant extension of (1.4). There, we presented the concepts of so-called total asymptotically nonexpansive mappings and total asymptotically weakly contractive maps by using Definitions 5) and 6), respectively:

5) the mapping $S : \Omega \to \Omega$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ with $k_n^{(1)}$, $k_n^{(2)} \to 0$ as $n \to \infty$, and strictly increasing and continuous functions $\phi(\xi) : R^+ \to R^+$ with $\phi(0) = 0$ such that for all $n \geq 1$ and for all $x, y \in \Omega$

$$\|S^n x - S^n y\| \leq \|x - y\| + k_n^{(1)}\phi(\|x - y\|) + k_n^{(2)}. \tag{1.5}$$

Definition 5) was generalized in [25, 32, 37, 39] and elsewhere.

6) the mapping $S : \Omega \to \Omega$ is called total asymptotically weakly contractive if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ with $k_n^{(1)}$, $k_n^{(2)} \to 0$ as $n \to \infty$, and strictly increasing and continuous functions $\phi(\xi), \psi(\xi) : R^+ \to R^+$ with $\phi(0) = \psi(0) = 0$ such that for all $n \geq 1$ and for all $x, y \in \Omega$

$$\|S^n x - S^n y\| \leq \|x - y\| + k_n^{(1)}\phi(\|x - y\|) - \psi(\|x - y\|) + k_n^{(2)}. \tag{1.6}$$

For the classes of the mappings 4)-6) the corresponding iterative process

$$x_{n+1} = (1-\omega_n)x_n + \omega_n S^n x_n, \quad x_1 \in \Omega, \quad n = 1,2,..., \tag{1.7}$$

was introduced in [40, 41] and explored by many authors (we refer to [8], [33]).

Along with (1.7) we considered the regularized successive approximation method

$$y_{n+1} = q_n z_0 - (1-q_n)S^n y_n, \quad z_0 \in \Omega, \;\; y_1 \in \Omega, \quad n = 1,2,...,$$

and its implicit version

$$y_n = q_n z_0 - (1-q_n)S^n y_n, \quad z_0 \in \Omega, \;\; y_1 \in \Omega, \quad n = 1,2,...,$$

where $\lim_{n\to\infty} q_n = 0$ and $\sum_{n=1}^{\infty} q_n = \infty$, and also fixed point problems with nonself-mappings $S : \Omega \to B$ (see [9, 15] and references within).

Once again, we note that in the inequalities (1.4)-(1.6) the asymptotic conditions are defined for degrees of the exact operator S and this operator organizes the iterative scheme (1.7). We proposed a completely different idea in the papers [18] and [19]. Namely, the original map $T : \Omega \to \Omega$ with a fixed point x^* is not known exactly, however, a sequence of approximating operators $\{T_k\}$ *is* known, and given in the form of total asymptotically nonexpansive approximations of T and total asymptotically weakly contractive approximations of T.

We presume further that B is a uniformly convex Banach space, B^* is a dual space, $J : B \to B^*$ is a normalize duality mapping [31, 16], $\Omega \subseteq B$ is a convex closed subset and $T : \Omega \to \Omega$ is a nonexpansive self-mapping. Let $\{T_k\}$ be a sequence of self-mappings $T_k : \Omega \to \Omega$, $k = 1,2,...$.

7) the sequence $\{T_k\}$ is called to be a total asymptotically nonexpansive approximation of T if there exist nonnegative real sequences $\{h_k,\, g_k,\, l_k,\, m_k\} \to 0$ as $k \to \infty$, continuous functions $\eta(\xi) : R^+ \to R^+$ and $\phi(\xi) : R^+ \to R^+$ with $\phi(0) = 0$ such that

$$\|T_k x - Tx\| \le h_k \eta(\|x\|) + g_k \quad \forall x \in \Omega \tag{1.8}$$

and

$$\|T_k x - T_k y\| \le \|x-y\| + l_k \phi(\|x-y\|) + m_k \quad \forall x,y \in \Omega. \tag{1.9}$$

8) the sequence of mappings $\{T_k\}$ is called to be a total asymptotically weakly contractive approximation of T if (1.8) is satisfied while (1.9) is replaced by

$$\|T_k x - T_k y\| \le \|x-y\| - p_k \psi(\|x-y\|) + l_k \phi(\|x-y\|) + m_k \quad \forall x,y \in \Omega,$$

where $\psi(\xi) : R^+ \to R^+$ is a continuous function with $\psi(0) = 0$ and $0 \le p_k \le \bar{p}$.

In [18] and [19] we studied a behavior of the iterative sequence

$$x_{k+1} = (1-\omega_k)x_k + \omega_k T_k x_k, \quad x_1 \in \Omega, \quad k = 1,2,..., \tag{1.10}$$

for the maps defined in 7) and 8) with some additional restrictions for the functions $\eta(\xi), \phi(\xi)$, and $\psi(\xi)$. In the present work, we attempt to build (as much as we are able) a continuous version of (1.10).

Definition 1.1. A family of mappings $\{S(t)\}$, $S(t) : \Omega \to \Omega$, $0 \le t_0 \le t < \infty$, is called a total asymptotically weakly contractive approximation of self-mappings $S : \Omega \to \Omega$ if there exist nonnegative functions $k(t)$, $l(t)$, $m(t)$, $h(t)$ and $g(t)$, a positive bounded function $p(t)$, continuous functions $\phi(\xi)$ and $\psi(\xi) : R^+ \to R^+$ with $\phi(0) = 0,\ \psi(0) = 0,\ \psi(\xi) \not\equiv 0$, and continuous nondecreasing $\eta(\xi) : R^+ \to R^+$ such that $\{k(t),\ l(t),\ m(t),\ h(t),\ g(t)\} \to 0 \ \ as \ \ t \to \infty$,

$$\|S(t)x - Sx\| \leq h(t)\eta(\|x\|) + g(t) \tag{1.11}$$

and for all $x, y \in \Omega$

$$\|S(t)x - S(t)y\| \leq (1 + k(t))\|x - y\| - p(t)\psi(\|x - y\|) + l(t)\phi(\|x - y\|) + m(t). \tag{1.12}$$

Note that in (1.12) the cases $p(t) \geq p > 0$ and $p(t) \to 0$ as $t \to \infty$ are significantly different from each other: the second case asymptotically gives some additional level of weakly contractive degeneration, which approaches (1.12) to (1.13) below.

It is not difficult to check that in (1.11) parametric functions $h(t)$ and $g(t)$ can not be simultaneously equal to zero for all $t_0 \leq t < \infty$. Otherwise, $\{S(t)\}$ and S coincide for all $x \in \Omega$ and then the convergence problem disappears. The inequalities of type (1.11) are widely used in the theory of ill-posed problems for perturbed mappings [16]. It is easy to note that if $\eta(\xi)$ is a bounded function or Ω is a bounded set then $S(t)x \to Sx$ uniformly for all $x \in \Omega$. Additionally, if the function $\phi(\xi)$ is bounded, then (1.12) implies

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in \Omega,$$

that is, S is a nonexpansive mapping on Ω. It is well known that in this case *the clearance operator* $F = I - S$ is demi-closed [16]. Let us recall that the map F satisfies the inequality

$$\langle Fx - Fy, J(x - y)\rangle \geq 0 \quad \forall x, y \in \Omega.$$

This means that F is an accretive mapping on the set Ω [16].

Now we present the statement with much weaker conditions:

Definition 1.2. A family of mappings $\{S(t)\}$, $S(t) : \Omega \to \Omega$, $0 \leq t_0 \leq t < \infty$, is called a total asymptotically nonexpansive approximation of self-mappings $S : \Omega \to \Omega$ if there exist nonnegative functions $k(t)$, $l(t)$, $m(t)$, $h(t)$ and $g(t)$, continuous functions $\phi(\xi) : R^+ \to R^+$ with $\phi(0) = 0$, a continuous nondecreasing function $\eta(\xi) : R^+ \to R^+$ such that $\{k(t),\ l(t),\ m(t),\ h(t),\ g(t)\} \to 0 \ \ as \ \ t \to \infty$, (1.11) is satisfied, and for all $x, y \in \Omega$

$$\|S(t)x - S(t)y\| \leq (1 + k(t))\|x - y\| + l(t)\phi(\|x - y\|) + m(t). \tag{1.13}$$

For the family of mappings $\{S(t)\}$, described in Definitions 1.1 and 1.2, we study the behavior of trajectories $x(t)$ of the following dynamical system:

$$\frac{dx(t)}{dt} = -\omega(t)\Big(x(t) - S(t)x(t)\Big), \ \ t \geq t_0 \geq 0, \quad x(t_0) = x_0 \in \Omega, \tag{1.14}$$

with $0 < \omega(t) \leq 1$ and

$$\int_{t_0}^{\infty} \omega(t)dt = \infty. \tag{1.15}$$

We suppose that trajectories $x(t)$ exist and are differentiable on the interval $[t_0, \infty)$. If $F(t) = I - S(t)$, then (1.14) is

$$\frac{dx(t)}{dt} = -\omega(t)F(t)x(t), \ \ t \geq t_0 \geq 0, \quad x(t_0) = x_0 \in \Omega.$$

The mapping $F(t)$ is called *the clearance operator* of $S(t)$ at the point t [13].

2. Differential Inequalities

In general, our research concerning fixed point problems is based on estimates of solutions to the following differential inequalities:

$$\frac{d\lambda(t)}{dt} \le \beta(t)\lambda(t) - \alpha(t)\psi(\lambda(t)) + \rho(t)\phi(\lambda(t)) + \gamma(t), \ t \ge t_0, \ \lambda(t_0) = \lambda_0, \tag{2.1}$$

and

$$\frac{d\lambda(t)}{dt} \le -\alpha(t)\zeta(t) + \gamma(t), \ t \ge t_0, \ \lambda(t_0) = \lambda_0, \tag{2.2}$$

where $\lambda(t)$ is a nonnegative differentiable function for all $t \ge t_0$, $\psi(\lambda)$, $\phi(\lambda)$ are positive continuous functions for all $\lambda > 0$ with $\psi(0) = 0$ and $\phi(0) = 0$, $\beta(t)$, $\rho(t)$, $\gamma(t)$ and $\zeta(t)$ are nonnegative for all $t \ge t_0$, and $\alpha(t)$ is a positive continuous function for all $t \ge t_0$. Assume that solutions of (2.1) and (2.2) exist. From now on, we assume that

$$\int_{t_0}^{\infty} \alpha(t)dt = \infty \tag{2.3}$$

and

$$\int_{t_0}^{\infty} \beta(t)dt < \infty. \tag{2.4}$$

We consider (2.1) in Section 2 and (2.2) in Section 3. In Section 4 we explore the special differential inequality

$$\frac{d\lambda(t)}{dt} \le -\alpha(t)\lambda(t) + \rho(t)\lambda^n(t), \ n \ge 0, \quad t \ge t_0 \ge 0, \ \lambda(t_0) = \lambda_0 > 0. \tag{2.5}$$

In Section 5 we apply differential inequalities to obtain the strong convergence theorems for the dynamical systems (1.14) with (1.15). Much weaker results (convergence on subsets) hold in Section 6 under very weak assumptions of type (1.13). Note that the inequality (2.5) is used in the literature to establish convergence of differential methods of high orders, for instance, the Newton-Kantorovich dynamical systems.

2.1 Differential Inequality (2.1) with $\phi(\lambda) \equiv 0$

a) First we consider the homogeneous nonlinear differential inequality (see [3]):

$$\frac{d\lambda(t)}{dt} \le -\alpha(t)\psi(\lambda(t)), \ t \ge t_0, \ \lambda(t_0) = \lambda_0. \tag{2.6}$$

Lemma 2.1. *Let $\lambda(t)$ be a non-negative and differentiable function satisfying inequality (2.6), where $\alpha(t)$ is a continuous positive function for all $t \ge t_0$ and $\psi(\lambda)$ is a positive continuous function for all $\lambda > 0$ with $\psi(0) = 0$. Then*

$$\lambda(t) \le \Phi^{-1}\Big(\Phi(\lambda_0) - \int_{t_0}^{t} \alpha(\tau)d\tau\Big), \tag{2.7}$$

where $\Phi(\lambda)$ is any antiderivative of the function $\dfrac{1}{\psi(\lambda)}$ and $\Phi^{-1}(z)$ is the inverse function to $\Phi(\lambda)$. Moreover, if (2.3) is true, then $\lambda(t) \to 0$ as $t \to \infty$.

Proof. So, $\Phi(\lambda) = \int \frac{d\lambda}{\psi(\lambda)} + C$ with an arbitrary constant C (without loss of generality we can set $C = 0$). From (2.6) we obtain the obvious inequality

$$\int_{t_0}^{t} \frac{d\lambda}{\psi(\lambda)} \leq -\int_{t_0}^{t} \alpha(\tau)d\tau,$$

therefore

$$\Phi(\lambda(t)) \leq \Phi(\lambda_0) - \int_{t_0}^{t} \alpha(\tau)d\tau.$$

Since $\frac{d\Phi(\lambda)}{d\lambda} = \frac{1}{\psi(\lambda)} > 0$ for all $\lambda > 0$, the function $\Phi(\lambda)$ is strictly increasing. It is well known that $\Phi^{-1}(z)$ also possesses this property. This implies (2.7). If (2.3) is fulfilled, then the conclusion $\lim_{t\to\infty} \lambda(t) = 0$ is obtained again by virtue of the properties of $\Phi(\lambda)$ and $\Phi^{-1}(z)$.

Let us present a simple example. Examine the following inequality [1, 17]:

$$\frac{d\lambda(t)}{dt} \leq -\alpha(t)\lambda^{\nu}(t), \quad \nu > 0, \ t \geq t_0, \ \lambda(t_0) = \lambda_0, \tag{2.8}$$

that is, in (2.6) $\psi(\lambda) = \lambda^{\nu}$, $\nu > 0$. It is not difficult to deduce:

$$\Phi(\lambda) = ln\lambda, \text{ if } \nu = 1 \text{ and } \Phi(\lambda) = \frac{\lambda^{1-\nu}}{1-\nu}, \text{ if } \nu \neq 1,$$

$$\Phi^{-1}(z) = exp(z), \text{ if } \nu = 1 \text{ and } \Phi^{-1}(z) = [(1-\nu)z]^{1/1-\nu}, \text{ if } \nu \neq 1,$$

$$\lambda(t) \leq \lambda_0 exp\left(-\int_{t_0}^{t} \alpha(\tau)d\tau\right), \text{ if } \nu = 1, \tag{2.9}$$

and

$$\lambda(t) \leq \left(\lambda_0^{1-\nu} + (\nu-1)\int_{t_0}^{t} \alpha(\tau)d\tau\right)^{1/1-\nu}, \text{ if } \nu \neq 1. \tag{2.10}$$

Note that if in (2.8) $\nu < 1$, then in (2.10) $\lambda(t) \to 0$ as $t \to \bar{t}$, where $\bar{t} \geq t_0$ is a finite number.

b) Next we investigate the more general nonlinear homogeneous differential inequality

$$\frac{d\lambda(t)}{dt} \leq \beta(t)\lambda(t) - \alpha(t)\psi(\lambda(t)), \ t \geq t_0, \ \lambda(t_0) = \lambda_0. \tag{2.11}$$

Lemma 2.2. *Let $\lambda(t)$ be a non-negative and differentiable function, $\beta(t)$ be a non-negative function and $\alpha(t)$ be a continuous positive function for all $t \geq t_0$. Assume that inequality (2.11) is satisfied, where $\psi(\lambda)$ is a positive continuous and nondecreasing function for all $\lambda > 0$ with $\psi(0) = 0$. If (2.4) is fulfilled, then*

$$\lambda(t) \leq C_0\Phi^{-1}\left(\Phi(\lambda_0) - C_0^{-1}\int_{t_0}^{t} \alpha(\tau)d\tau\right), \tag{2.12}$$

where a positive constant C_0 is defined by the estimate

$$exp\left(\int_{t_0}^{\infty} \beta(t)dt\right) \leq C_0 \tag{2.13}$$

and $\Phi(\lambda) = \int \frac{d\lambda}{\psi(\lambda)}$. Moreover, if (2.3) is fulfilled, then $\lim_{t\to\infty} \lambda(t) = 0$.

Proof. First, by (2.4) there exists a constant $C_0 > 0$ such that

$$1 \leq exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big) \leq C_0. \tag{2.14}$$

In (2.11) we provide the following replacement:

$$\lambda(t) = \mu(t)exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big), \tag{2.15}$$

where $\mu(t)$, $\infty > t \geq t_0$, is some non-negative and differentiable function. Then $\mu(t_0) = \lambda(t_0) = \lambda_0$ and

$$\frac{d\lambda(t)}{dt} = \frac{d\mu(t)}{dt}\, exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big) + \mu(t)\beta(t)\, exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big).$$

On the other hand, by (2.11) and (2.15)

$$\frac{d\lambda(t)}{dt} \leq \beta(t)\mu(t)\, exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big) - \alpha(t)\psi\Big(\mu(t)\, exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big)\Big).$$

From this, it follows that

$$\frac{d\mu(t)}{dt} \leq -\alpha(t)\Big(exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big)\Big)^{-1}\psi\Big(\mu(t)exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big)\Big). \tag{2.16}$$

Therefore, due to (2.13) and the nondecreasing property of $\psi(\lambda)$ we have

$$\frac{d\mu(t)}{dt} \leq -C_0^{-1}\alpha(t)\psi(\mu(t)).$$

Consequently,

$$\mu(t) \leq \Phi^{-1}\Big(\Phi(\mu_0) - C_0^{-1}\int_{t_0}^{t} \alpha(\tau)d\tau\Big).$$

Through (2.15) one obtains (2.12), and (2.3) implies the limit result $\lim_{t\to\infty}\lambda(t) = 0$. The lemma is proved.

c) We study now the inhomogeneous linear differential inequality [1, 3, 16]:

$$\frac{d\lambda(t)}{dt} \leq -\alpha(t)\lambda(t) + \gamma(t), \;\; t \geq t_0, \;\; \lambda(t_0) = \lambda_0. \tag{2.17}$$

Lemma 2.3. *Assume that a non-negative and differentiable function $\lambda(t)$ and non-negative continuous function $\gamma(t)$ both satisfy the inequality (2.17), where $\alpha(t)$ is a continuous positive function for all $t \geq t_0$. Then the estimate*

$$\lambda(t) \leq \lambda_0 exp\Big(-\int_{t_0}^{t} \alpha(\tau)d\tau\Big) + \int_{t_0}^{t} \gamma(\theta)exp\Big(-\int_{\theta}^{t} \alpha(\tau)d\tau\Big)d\theta \tag{2.18}$$

holds. If (2.3) is carried out and

$$\lim_{t\to\infty}\frac{\gamma(t)}{\alpha(t)} = 0, \tag{2.19}$$

then $\lambda(t) \to 0$ as $t \to \infty$.

Proof. Multiplying both parts of (2.17) by

$$z(t) = exp\,\Big(\int_{t_0}^{t} \alpha(s)ds\Big)$$

we obtain

$$\frac{d}{dt}\Big(\lambda(t)z(t)\Big) \leq \gamma(t)z(t).$$

Then

$$\lambda(t)z(t) \leq \lambda(t_0) + \int_{t_0}^{t} \gamma(\tau)z(\tau)d\tau,$$

which is equivalent to (2.18). The first term on the right-hand side of (2.18) approaches zero by the condition (2.3). Let us find the limit of the second term as $t \to \infty$. Denote the anti-derivative of $\alpha(t)$ by $\bar{\alpha}(t)$. If the integral

$$\int_{t_0}^{\infty} \gamma(\theta)e^{\bar{\alpha}(\theta)}d\theta \tag{2.20}$$

is divergent then by applying L'Hôpital's rule and (2.19), one obtains

$$\lim_{t\to\infty} \int_{t_0}^{t} \gamma(\theta)e^{-\int_{\theta}^{t}\alpha(s)ds}d\theta = \lim_{t\to\infty} \frac{\int_{t_0}^{t}\gamma(\theta)e^{\bar{\alpha}(\theta)}d\theta}{e^{\bar{\alpha}(t)}} = \lim_{t\to\infty}\frac{\gamma(t)}{\alpha(t)} = 0. \tag{2.21}$$

If the integral (2.20) is convergent then

$$\lim_{t\to\infty} \int_{t_0}^{t} \gamma(\theta)e^{-\int_{\theta}^{t}\alpha(s)ds}d\theta = 0,$$

again by (2.3). The assertion of the lemma is true (see also Section 4).

d) Similarly to item b) for the differential inequality

$$\frac{d\lambda(t)}{dt} \leq \beta(t)\lambda(t) - \alpha(t)\lambda(t) + \gamma(t), \;\; t \geq t_0, \;\; \lambda(t_0) = \lambda_0, \tag{2.22}$$

the following statement is effective:

Lemma 2.4. *Let $\lambda(t)$, $\beta(t)$, $\alpha(t)$ and $\gamma(t)$ be as in Lemmas 2.2 and 2.3. Suppose that (2.4) and (2.19) are satisfied. Then for a solution $\lambda(t)$ of (2.22) the estimate*

$$\lambda(t) \leq C_0\lambda_0 exp\Big(-\frac{1}{C_0}\int_{t_0}^{t}\alpha(\tau)d\tau\Big) + C_0\int_{t_0}^{t}\gamma(\theta)exp\Big(-\frac{1}{C_0}\int_{\theta}^{t}\alpha(\tau)d\tau\Big)d\theta$$

is valid for any $t \geq t_0$ and $\lambda(t) \to 0$ as $t \to \infty$.

Proof. Using (2.15) and (2.16) we set

$$\frac{d\mu(t)}{dt} \leq -\alpha(t)\Big(exp\Big(\int_{t_0}^{t}\beta(\tau)d\tau\Big)\Big)^{-1}\Big(\mu(t)exp\Big(\int_{t_0}^{t}\beta(\tau)d\tau\Big)\Big) + \gamma(t)\Big(exp\Big(\int_{t_0}^{t}\beta(\tau)d\tau\Big)\Big)^{-1}.$$

Therefore

$$\frac{d\mu(t)}{dt} \leq -{C_0}^{-1}\alpha(t)\mu(t) + \gamma(t),$$

where C_0 is defined in (2.13). By virtue of (2.18)

$$\mu(t) \leq \mu(t_0)exp\Big(-\frac{1}{C_0}\int_{t_0}^{t}\alpha(\tau)d\tau\Big) + \int_{t_0}^{t}\gamma(\theta)exp\Big(-\frac{1}{C_0}\int_{\theta}^{t}\alpha(\tau)d\tau\Big)d\theta.$$

It only remains to apply (2.3), (2.13) and (2.15).

The particular case of (2.22) with $\gamma(t) \equiv 0$ gives

$$\lambda(t) \leq C_0 \lambda(t_0) exp\Big(-C_0{}^{-1} \int_{t_0}^{t} \alpha(\tau) d\tau \Big).$$

A more exact estimate is obtained from the inequality

$$\frac{d\lambda(t)}{dt} \leq \beta(t)\lambda(t) - \alpha(t)\lambda(t),$$

if we use (2.9).

e) Next we provide the following nonlinear differential inequality:

$$\frac{d\lambda(t)}{dt} \leq -\alpha(t)\psi(\lambda(t)) + \gamma(t), \;\; t \geq t_0, \;\; \lambda(t_0) = \lambda_0. \tag{2.23}$$

Lemma 2.5. *Assume that a non-negative and differentiable function $\lambda(t)$ satisfies the differential inequality (2.23), where for all $t \geq t_0$ the function $\alpha(t)$ is continuous and positive, $\gamma(t)$ is non-negative and continuous, $\psi(\lambda)$ is positive, continuous, and increasing for all $\lambda > 0$, and $\psi(0) = 0$. If (2.3) and (2.19) are fulfilled, then $\lambda(t) \to 0$ as $t \to \infty$*

Proof. For this result see [3, 16]. For each $t \geq t_0$ there are two possibilities:

$$H_1 : \psi(\lambda(t)) \; < \; q(t) \tag{2.24}$$

or

$$H_2 : \psi(\lambda(t)) \geq q(t), \tag{2.25}$$

where

$$q(t) = \frac{1}{\mathscr{A}(t)} + \frac{\gamma(t)}{\alpha(t)}$$

and $\mathscr{A}(t)$ is defined as

$$\mathscr{A}(t) = \int_{t_0}^{t} \alpha(\tau) d\tau. \tag{2.26}$$

We denote the sets

$$\mathscr{T}_1 = \{t \in T \;\mid\; H_1 \text{ is true}\} \;\; and \;\; \mathscr{T}_2 = \{t \in T \;\mid\; H_2 \text{ is true}\}. \tag{2.27}$$

In more detail:

$$\mathscr{T}_1^i = \{t \in (t_i^1, \bar{t}_i^1) \subseteq \mathscr{T}_1\}, \quad \mathscr{T}_1 = \cup_i T_1^i, \;\; i = 1, 2, ..., \bar{k}\,, \tag{2.28}$$

$$\mathscr{T}_2^j = \{t \in [t_j^2, \bar{t}_j^2] \subseteq \mathscr{T}_2\}, \quad \mathscr{T}_2 = \cup_j T_2^j, \;\; j = 1, 2, ... \bar{l}\,. \tag{2.29}$$

Sets $\mathscr{T}_1^i$ and $\mathscr{T}_2^j$ are alternating. It is easy to see that $\mathscr{T}_1 \cup \mathscr{T}_2 = T = [t_0, \infty)$. The case $\mathscr{T}_1 = T$ is also possible. Let us prove that $\mathscr{T}_1$ is always an unbounded set. We assume the contrary. Then there exists $t = \tau_1$ such that for all $t \geq \tau_1$ the hypothesis H_2 holds, and (2.23) yields the inequality

$$\frac{d\lambda(t)}{dt} \leq -\frac{\alpha(t)}{\mathscr{A}(t)} \quad \forall t \geq \tau_1. \tag{2.30}$$

Hence,

$$\lambda(t) \leq \lambda(\tau_1) - \int_{\tau_1}^{t} \frac{\alpha(s)}{\mathscr{A}(s)} ds. \tag{2.31}$$

By virtue of the Cauchy integral criterion, we show that

$$\lim_{t\to\infty} \int_{\tau_1}^{t} \frac{\alpha(s)}{\mathscr{A}(s)} ds = \lim_{t\to\infty} ln \frac{\mathscr{A}(t)}{\mathscr{A}(\tau_1)} = \infty. \tag{2.32}$$

It can be now seen from (2.31) that there exists a point $t = \tau_2$, for which $\lambda(\tau_2) < 0$. This contradicts the condition of the lemma. Consequently, the positive function $\psi(\lambda(t)) \to 0$ as $t \to \infty$ and $t \in \mathscr{T}_1$. Now the convergence of $\lambda(t)$ to zero as $t \in \mathscr{T}_1$ and $t \to \infty$ is guaranteed due to the properties of $\psi(t)$, namely,

$$\lambda(t) < \psi^{-1}\Big(q(t)\Big) \ \ \forall t \in \mathscr{T}_1. \tag{2.33}$$

Note that the last interval of T always belongs to $\mathscr{T}_1$, therefore $\bar{l} = \bar{k} - 1$ and only two cases are possible:

$$T = \mathscr{T}_1^1 \cup \mathscr{T}_2^1 \cup \mathscr{T}_1^2 \cup \mathscr{T}_2^2 \cup \ldots \cup \mathscr{T}_2^{\bar{k}-1} \cup \mathscr{T}_1^{\bar{k}} \tag{2.34}$$

and

$$T = \mathscr{T}_2^1 \cup \mathscr{T}_1^1 \cup \mathscr{T}_2^2 \cup \mathscr{T}_1^2 \cup \ldots \cup \mathscr{T}_2^{\bar{k}-1} \cup \mathscr{T}_1^{\bar{k}}. \tag{2.35}$$

Suppose case (2.34). By (2.30), on each set $T_2^j = [t_j^2, \bar{t}_j^2]$ the function $\lambda(t)$ strongly decreases because of $\dfrac{d\lambda(t)}{dt} < 0$. Thus, $\lambda(t) \leq \lambda(t_j^2)$ for all $t \in [t_j^2, \bar{t}_j^2]$. Since the function $\lambda(t)$ is continuous, we conclude without loss of generality that $\lambda(t) \leq \lambda(\bar{t}_{j-1}^1)$ on the interval $[t_j^2, \bar{t}_j^2]$. Therefore

$$\lambda(t) < \psi^{-1}\Big(q(\bar{t}_{j-1}^1)\Big) \ \ \forall t \in [t_j^2, \bar{t}_j^2]. \tag{2.36}$$

Note that all $q(\bar{t}_{j-1}^1) \to 0$ as $t \to \infty$.

If case (2.35), then on the interval $[t_0, \bar{t}_1^2] = [t_1^2, \bar{t}_1^2]$ the function $\lambda(t)$ strictly decreases from $\lambda(t_0)$ to $\lambda(\bar{t}_1^2)$. This must occur on the bounded interval $[t_0, \bar{t}]$, where $\bar{t}$ is determined by (2.31) as a solution of the inequality

$$\int_{t_0}^{t} \frac{\alpha(s)}{\mathscr{A}(s)} ds \leq \lambda(t_0)$$

with respect to t. Starting from $\bar{t}$ we return to the previous case (2.34). Finally, the lemma is proved by (2.33) and (2.36).

Remark 2.6. Earlier in [3] we obtained not only the convergence $\lambda(t) \to 0$ as $t \to \infty$, but also different non-asymptotic estimates for the rate of convergence. This part is long and difficult, therefore, we do not present it in this review.

Our next step is to obviate the increasing property of $\psi(t)$ (see [4]).

Lemma 2.7. *Let $\lambda(t)$ be a non-negative and differentiable function satisfying the differential inequality (2.23), where function $\alpha(t)$ is continuous and positive for $t \geq t_0$ while $\gamma(t)$ is continuous and non-negative. If $\psi(\lambda)$ is a continuous positive function for $\lambda > 0$ with $\psi(0) = 0$, there exist constants $c > 0$ and $\lambda_+ > 0$ such that $\psi(\lambda) \geq c$ for all $\lambda \geq \lambda_+$ and zero is its unique* *limit point on the interval* $[0,\lambda_+]$. *If (2.3) is fulfilled and (2.19) monotonically decreases to 0, then* $\lim_{t\to\infty}\lambda(t)=0$.

Proof. Using the alternative (2.24), (2.25), definitions (2.27)-(2.29) and also (2.34) and (2.35), we first prove that $\mathscr{T}_1$ is an unbounded set. It is clear from the previous lemma: either $\mathscr{T}_1 = T$ or intervals of T belong to $\mathscr{T}_1$ and $\mathscr{T}_2$ and alternate such that always the set $\mathscr{T}_1^{\bar{k}} = [t_1^{\bar{k}},\infty)$. By the hypothesis H_1, at each interval of $\mathscr{T}_1$ the function $\psi(\lambda(t))$ is estimated from above by the monotonically decreasing function and $\psi(\lambda(t))\to 0$ as $t\to\infty$ and $t\in\mathscr{T}_1$. Therefore, due to the properties of $\psi(\lambda)$, $\lambda(t)\to 0$ for such t. In turn, by virtue of the hypothesis H_2, $\lambda(t)$ is estimated by the (2.31) at each point $t\in[t_j^2,\bar{t}_j^2]\subset\mathscr{T}_2$. The right hand side of (2.31) monotonically decreases on interval $[t_j^2,\bar{t}_j^2]$. This means that $\lambda(t)\le\lambda(t_j^2)$ for all $t\in[t_j^2,\bar{t}_j^2]$. In addition, let us note that $\psi(\lambda(t_j^2)) = q(t_j^2)$. Since $q(t_j^2)$ monotonically decreases as j increases, we deduce that $\lim_{t\to\infty}\lambda(t)=0$. The proof is complete.

f) We further consider the inequality

$$\frac{d\lambda(t)}{dt}\le\beta(t)\lambda(t)-\alpha(t)\psi(\lambda(t))+\gamma(t),\ \ t\ge t_0,\ \ \lambda(t_0)=\lambda_0. \tag{2.37}$$

Lemma 2.8. *Assume that a non-negative and differentiable function* $\lambda(t)$ *and non-negative functions* $\beta(t)$ *and* $\gamma(t)$ *satisfy the inequality (2.37), where* $\alpha(t)$ *is a continuous positive function for all* $t\ge t_0$, *and* $\psi(\lambda)$ *is a positive continuous and increasing function for all* $\lambda>0$ *with* $\psi(0)=0$. *If (2.3), (2.4) and (2.19) are fulfilled, then* $\lambda(t)\to 0$ *as* $t\to\infty$.

Proof. The methods of Lemma 2.2 give the following inequality for (2.37):

$$\frac{d\mu(t)}{dt}\le-\Big[\alpha(t)\psi\Big(\mu(t)exp\Big(\int_{t_0}^{t}\beta(\tau)d\tau\Big)\Big)+\gamma(t)\Big]\Big(exp\Big(\int_{t_0}^{t}\beta(\tau)d\tau\Big)\Big)^{-1}.$$

Since $\psi(\lambda)$ is nondecreasing function and (2.13) is valid, we see that

$$\frac{d\mu(t)}{dt}\le-C_0^{-1}\alpha(t)\psi(\mu(t))+\gamma(t).$$

The proof follows from Lemma 2.5 and (2.15).

Remark 2.9. Another version of Lemma 2.8 is Corollary 2.13 below.

g) The very important question about the behavior of solutions of differential inequalities with a constant $\gamma(t)=\varepsilon>0$ has not previously been studied. Here, we will consider only one case of (2.23) where $\alpha(t)=\alpha>0$:

$$\frac{d\lambda(t)}{dt}\le-\alpha\psi(\lambda(t))+\varepsilon,\ \ t\ge t_0,\ \ \lambda(t_0)=\lambda_0. \tag{2.38}$$

In fact, we will establish the continuous version of Lemma 2.5 from [11].

Lemma 2.10. *Assume that in the differential inequality (2.38)* $\lambda(t)$ *is non-negative and differentiable function,* $\psi(\lambda)$ *is a strictly increasing function. Then there exists* $\bar{t}\ge t_0$. *such that the estimates (2.40)-(2.42) below are satisfied for all* $t\ge\bar{t}$.

Proof. Consider the alternative:

$$H_1:\psi(\lambda(t))\ <\ \frac{1}{\alpha t}+\frac{\varepsilon}{\alpha}\ \ or\ \ H_2:\psi(\lambda(t))\ \ge\frac{1}{\alpha t}+\frac{\varepsilon}{\alpha}. \tag{2.39}$$

Define the sets (2.27)-(2.29) and $T = \mathscr{T}_1 \cup \mathscr{T}_2 = [t_0, \infty)$. Like in item e) we prove that $\mathscr{T}_1$ is an unbounded set. Suppose the contrary. Then there exists $t = \tau_1$ such that H_2 is fulfilled for all $t \geq \tau_1$ and (2.38) gives the inequality

$$\frac{d\lambda(t)}{dt} \leq -\frac{1}{t} \quad \forall t \geq \tau_1.$$

This yields the following inequality for $t \geq \tau_1$:

$$\lambda(t) \leq \lambda(\tau_1) - \int_{\tau_1}^{t} \frac{ds}{s} = \lambda(\tau_1) - lnt + ln\tau_1,$$

which is impossible because $\lambda(t) \geq 0$ for all $t \geq t_0$. Thus, the hypothesis H_1 is carried out on subsets $\mathscr{T}_1^i \subset T$ and then

$$\lambda(t) < \psi^{-1}\Big(\frac{1}{\alpha t} + \frac{\varepsilon}{\alpha}\Big) \ \forall t \in \mathscr{T}_1. \tag{2.40}$$

Let us return to (2.34) and (2.35). For each $t \in \mathscr{T}_2^j = [t_j^2, \bar{t}_j^2]$ we obtain $\lambda(t) \leq \lambda(t_j^2)$, moreover,

$$\lambda(t_j^2) < \psi^{-1}\Big(\frac{1}{\alpha t_j^2} + \frac{\varepsilon}{\alpha}\Big) \tag{2.41}$$

or

$$\lambda(t_j^2) < \psi^{-1}\Big(\frac{1}{\alpha \bar{t}_{j-1}^2} + \frac{\varepsilon}{\alpha}\Big). \tag{2.42}$$

If $t_0 \in \mathscr{T}_1^1$, then the estimate (2.40) is true. If $t_0 \in \mathscr{T}_2^1 = [t_1^2, \bar{t}_1^2] = [t_0, \bar{t}_1^2]$, then for all $t \in \mathscr{T}_2^1$ one obtains $\lambda(t) \leq \lambda(t_0)$, and (2.40)-(2.42) are satisfied for at least all $t > \bar{t}_1^2$.

Remark 2.11. If the function $\lambda(t)$ satisfying (2.38) has a limit λ^* as $t \to \infty$, then inequalities (2.40)-(2.42) guaranty the estimate $\lambda^* < \psi^{-1}\Big(\dfrac{\varepsilon}{\alpha}\Big)$. However, we can not assert as before that $\lambda^* = 0$.

2.2 Differential Inequality (2.1) with $\phi(\lambda) \not\equiv 0$

We study the differential inequality

$$\frac{d\lambda(t)}{dt} \leq -\alpha(t)\psi(\lambda(t)) + \rho(t)\phi(\lambda(t)) + \gamma(t), \ \ t \geq t_0, \ \ \lambda(t_0) = \lambda_0, \tag{2.43}$$

In Lemma 2.12 below we use the following notation:

$$c_1 = max\Big\{\frac{\rho(t)}{\alpha(t)}, \ t \geq t_0\Big\}, \ c_2 = max\Big\{\frac{\gamma(t)}{\alpha(t)}, \ t \geq t_0\Big\}, \ \phi_1(\lambda) = c_1\phi(\lambda) + c_2. \tag{2.44}$$

Lemma 2.12. *Let $\lambda(t)$ be a non-negative and differentiable function satisfying the differential inequality (2.43), where $\rho(t)$ and $\alpha(t)$ are bounded positive functions, $\gamma(t)$ is continuous and non-negative, $\phi(\lambda) : R^+ \to R^+$ is a continuous function and $\psi(\lambda) : R^+ \to R^+$ is an increasing continuous function with $\phi(0) = \psi(0) = 0$. Suppose that there exists a constant $M \geq 0$ such that*

$\phi_1(\lambda) \le \psi(\lambda)$ for all $\lambda \ge M$ and the equation $\phi_1(\lambda) = \psi(\lambda)$ has no more than one root λ_* on the set $[0,\infty)$. Let (2.3) be fulfilled and

$$\lim_{t\to\infty} \frac{\rho(t)+\gamma(t)}{\alpha(t)} = 0. \tag{2.45}$$

Then $\lim_{t\to\infty} \lambda(t) = 0$.

Proof. One of the following must occur for each $t \in T = [t_0,\infty)$: either

$$H_1: \quad -\alpha(t)\psi(\lambda(t)) + \rho(t)\phi(\lambda(t)) + \gamma(t) \ge 0,$$

or

$$H_2: \quad -\alpha(t)\psi(\lambda(t)) + \rho(t)\phi(\lambda(t)) + \gamma(t) < 0.$$

Define the sets (2.27)-(2.29). As before, it is clear that $T = \mathscr{T}_1 \cup \mathscr{T}_2 = [t_0,\infty)$ and there exists a case $\mathscr{T}_1 = T$.

i) Suppose that the equation $\psi(\lambda) = \phi_1(\lambda)$ has exactly one root λ_* on the set $[0,\infty)$. We see from the conditions of this lemma that $\phi_1(\lambda) \le \psi(\lambda)$ if $\lambda \ge \lambda_*$ and $\phi_1(\lambda) \ge \psi(\lambda)$ if $\lambda \le \lambda_*$. a) First, consider $t_0 \in \mathscr{T}_1$, that is,

$$-\alpha(t_0)\psi(\lambda(t_0)) + \rho(t_0)\phi(\lambda(t_0)) + \gamma(t_0) \ge 0.$$

Then at each point $t \in [t_i^1, \bar{t}_i^1] \subset \mathscr{T}_1$ we have

$$\begin{aligned} 0 &\le \rho(t)\phi(\lambda(t)) - \alpha(t)\psi(\lambda(t)) + \gamma(t) \\ &\le \alpha(t)\Big(\frac{\rho(t)}{\alpha(t)}\phi(\lambda(t)) - \psi(\lambda(t)) + \frac{\gamma(t)}{\alpha(t)}\Big) \\ &\le \alpha(t)\Big(c_1\phi(\lambda(t)) - \psi(\lambda(t)) + c_2\Big). \end{aligned} \tag{2.46}$$

Since $\alpha(t) > 0$, one gets

$$\phi_1(\lambda(t)) = c_1\phi(\lambda(t)) + c_2 \ge \psi(\lambda(t)).$$

This means that

$$\lambda(t) \le \lambda_* \quad \forall t \in [t_i^1, \bar{t}_i^1]. \tag{2.47}$$

Consider now the interval $[t_j^2, \bar{t}_j^2] \subset \mathscr{T}_2$. Recall that the function $\phi_1(\lambda)$ is positive and continuous. Denote

$$M_{max} = max\{\phi_1(\lambda),\ 0 \le \lambda \le \lambda_*\}.$$

Taking into account that $\psi(\lambda)$ is a continuous and increasing function with $\psi(0) = 0$, it is easy to see that on the set $[0,\infty)$

$$\phi_1(\lambda) \le M_{max} + \psi(\lambda). \tag{2.48}$$

We now estimate $\lambda(t)$ for $t \in [t_j^2, \bar{t}_j^2]$ from the differential inequalities

$$\frac{d\lambda(t)}{dt} \le \rho(t)\phi(\lambda(t)) - \alpha(t)\psi(\lambda(t)) + \gamma(t) < 0. \tag{2.49}$$

Obviously $\lambda(t) \le \lambda(t_j^2)$ and by (2.47) $\lambda(t) \le \lambda_*$ for all $t \in [t_j^2, \bar{t}_j^2] \subset \mathscr{T}_2$. At an arbitrary point $t \in [t_i^1, \bar{t}_i^1] \subset \mathscr{T}_1$ the function $\lambda(t)$ is estimated by (2.46) and (2.47). If $t \in [t_{j+1}^2, \bar{t}_{j+1}^2]$, then $\lambda(t) \le \lambda_*$ because of (2.49). Continuing this process further we obtain $\lambda(t) \le \lambda_*$ for any $t \in T$.

b) Now let $t_0 \in \mathscr{T}_2$. Then (2.49) is fulfilled on the set $[t_0, \bar{t}_1^2]$ and we conclude that $\lambda(t) \le \lambda_0$ for all $t \in [t_0, \bar{t}_1^2]$. On the next interval $[t_1^1, \bar{t}_1^1]$ the hypothesis H_1 holds, therefore, $\lambda(t) \le \lambda_*$ similar to (2.47). On the set $[t_2^2, \bar{t}_2^2] \subset \mathscr{T}_2$ we again obtain the inequalities (2.49), hence $\lambda(t) \le \lambda_*$ for all $t \in [t_2^2, \bar{t}_2^2]$, etc. on each interval $[t_j^2, \bar{t}_j^2]$, which alternates with $[t_{j+1}^1, \bar{t}_{j+1}^1]$. Thus the following estimate holds:

$$\lambda(t) \le max\{\lambda_0, \lambda_*\} \quad \forall t \in T. \tag{2.50}$$

ii) Suppose now that the equation $\psi(\lambda) = \phi_1(\lambda)$ has no roots on the set $[0, \infty)$. Since there exists $M > 0$ such that $\phi_1(\lambda) \le \psi(\lambda)$ for all $\lambda \ge M$ and $\psi(0) = \phi(0) = 0$, this situation can only arise in the case of $c_2 = 0$. This implies the following assertion: the hypothesis H_1 does not appear for any $t \ge t_0$. Otherwise there exists $t \ge t_0$ such that $\phi_1(\lambda(t)) \ge \psi(\lambda(t))$. Therefore, the hypothesis H_2 is valid on the whole set T. Like b) we can show that $\lambda(t) \le \lambda(t_0) = \lambda_0$ for all $t \in T$. Hence, the general estimate remains in the form (2.50).

Since $\psi(\lambda)$ is an increasing function, from (2.48) we deduce the estimate

$$\begin{aligned} \phi(\lambda) &\le c^{-1}(M_{max} + \psi(\lambda)) - c_2 \\ &\le c^{-1}(M_{max} + \psi(K)) = C, \end{aligned}$$

where $K = max\{\lambda_0, \lambda_*\}$. As a result, we obtain the following differential inequality:

$$\frac{d\lambda(t)}{dt} \le -\alpha(t)\psi(\lambda(t)) + C\rho(t) + \gamma(t).$$

Together with (2.3), (2.45) and Lemma 2.5 this complete the proof.

Under the conditions of Lemma 2.12, the function $\psi(\lambda)$ must grow faster at infinity than the function $c_1\phi(\lambda) + c_2$. Let us now formulate the particularly important case of (2.1) with $\phi(\lambda) = \lambda$:

$$\frac{d\lambda(t)}{dt} \le -\alpha(t)\psi(\lambda(t)) + \rho(t)\lambda(t) + \gamma(t), \;\; t \ge t_0, \;\; \lambda(t_0) = \lambda_0. \tag{2.51}$$

Corollary 2.13. *Let $\lambda(t)$ be a non-negative and differentiable function satisfying the differential inequality (2.51), where $\rho(t)$ and $\alpha(t)$ are bounded positive functions, $\gamma(t)$ is continuous and non-negative, and $\psi(\lambda) : R^+ \to R^+$ is increasing continuous function with $\psi(0) = 0$. Assume that c_1 and c_2 is defined by (2.44) with $\phi_1(\lambda) = c_1\lambda + c_2$. Suppose also that there exists a constant $M \ge 0$ such that $\phi_1(\lambda) \le \psi(\lambda)$ for all $\lambda \ge M$ and the equation $\phi_1(\lambda) = \psi(\lambda)$ has no more than one root λ_* on the set $[0, \infty)$. In addition, if (2.3) and (2.45) are satisfied, then $\lim\limits_{t\to\infty} \lambda(t) = 0$.*

Remark 2.14. If we compare Corollary 2.13 and Lemma 2.8 for the inequality (2.51) with $\beta(t) = \rho(t)$, it can be seen that the first statement essentially has a weaker condition $\lim\limits_{t\to\infty} \dfrac{\rho(t)}{\alpha(t)} = 0$ in place of (2.4). At the same time, in Corollary 2.13 we assumed at least linear growth of $\psi(\lambda)$ at infinity.

3. Differential Inequality (2.2)

Lemma 3.1. *Assume that a non-negative and differentiable function $\lambda(t)$ satisfies the inequality (2.2), $\gamma(t)$ and $\zeta(t)$ are non-negative, and $\alpha(t)$ is a positive continuous function for all $t \ge t_0$.*

Let

$$\int_{t_0}^{\infty} \gamma(t)dt < \infty. \tag{3.1}$$

Then the function $\lambda(t)$ *is bounded for all* $t \geq t_0$.

Proof The proof is straightforward. We have from (2.2) that for any $t \geq t_0$

$$\lambda(t) \leq \lambda(t_0) - \int_{t_0}^{t} \alpha(\tau)\zeta(\tau)d\tau + \int_{t_{10}}^{t} \gamma(\tau)d\tau.$$

This implies the estimate $\lambda(t) \leq \lambda(t_0) + \int_{t_{10}}^{t} \gamma(\tau)d\tau$ because $\int_{t_0}^{t} \alpha(\tau)\zeta(\tau)d\tau \geq 0$. It follows that

$$\lambda(t) \leq \lambda(t_0) + \int_{t_{10}}^{\infty} \gamma(\tau)d\tau < \infty.$$

Lemma 3.2. *Assume that a non-negative and differentiable function* $\lambda(t)$ *satisfies the inequality (2.2),* $\gamma(t)$ *and* $\zeta(t)$ *are nonnegative continuous functions, and* $\alpha(t)$ *is a positive continuous function for all* $t \geq t_0$. *Let (2.3) and (3.1) hold. Then there exists an unbounded subset* $\mathscr{T}^1 \subset T = [t_0, \infty)$ *such that* $\lim\limits_{t\to\infty} \zeta(t) = 0$ *for all* $t \in \mathscr{T}^1$.

Proof. To begin, note that for any $\bar{t} \geq t_0$

$$\begin{aligned}
\int_{t_0}^{\bar{t}} \alpha(t)\zeta(t)dt &\leq -\int_{t_0}^{\bar{t}} \frac{d\lambda(t)}{dt}dt + \int_{t_0}^{\bar{t}} \gamma(t)dt \\
&\leq \lambda(t_0) - \lambda(\bar{t}) + \int_{t_0}^{\bar{t}} \gamma(t)dt < \infty.
\end{aligned}$$

Since $\lambda(t)$ is bounded for all $t \geq t_0$, we have $\displaystyle\int_{t_0}^{\infty} \alpha(t)\zeta(t)dt < \infty$. If the lemma is false, then there exists $\sigma > 0$ and $\bar{t} \geq t_0$ such that $\zeta(t) \geq \sigma$ for all $t \geq \bar{t}$. Therefore

$$\int_{\bar{t}}^{\infty} \alpha(t)\zeta(t)dt \geq \sigma \int_{\bar{t}}^{\infty} \alpha(t)dt,$$

that contradicts (2.3).

The next two statements demonstrate that the convergence of $\lambda(t) \to 0$ only occurs on some subsets of the set $T = [t_0, \infty)$.

Lemma 3.3. *Assume the non-negative and differentiable function* $\lambda(t)$ *satisfies the inequality (2.2),* $\gamma(t)$ *and* $\zeta(t)$ *are non-negative,* $\alpha(t)$ *is a positive continuous function for all* $t \geq t_0$, *and let (2.3) and (3.1) hold. Then there exists an unbounded set* $\mathscr{T}_1 \subset T = [t_0, \infty)$ *such that* $\zeta(t) \to 0$ *as* $t \to \infty$ *and all* $t \in \mathscr{T}_1$, *with the monotone estimate* H_1 *in (3.2).*

Proof. Consider for all $t \geq t_0$ the following alternative:

$$H_1: \ \ \zeta(t) \leq \frac{1}{\mathscr{A}(t)} \quad or \quad H_2: \ \ \zeta(t) > \frac{1}{\mathscr{A}(t)}, \tag{3.2}$$

where $\mathscr{A}(t)$ is defined by (2.26). As in Lemma 2.5 define the sets (2.27)-(2.29). In the general case $\mathscr{T}_1^k$ alternates with $\mathscr{T}_2^k$, $k = 1, 2, 3, \dots$, however $\mathscr{T}_1 = T$ is also possible. It is clear that $\mathscr{T}_1 \cup \mathscr{T}_2 = T = [t_0, \infty)$. We claim that set $\mathscr{T}_1$ is unbounded. If it is bounded, then there exists $t = \tau_1$ such that

$$\zeta(t) > \frac{1}{\mathscr{A}(t)} \quad \forall t \geq \tau_1,$$

and one gets

$$\frac{d\lambda(t)}{dt} < -\frac{\alpha(t)}{\mathscr{A}(t)} + \gamma(t) \ \ \forall t \geq \tau_1.$$

Then

$$\lambda(t) < \lambda(\tau_1) - \int_{\tau_1}^{t} \frac{\alpha(\tau)}{\mathscr{A}(\tau)} d\tau + \int_{\tau_1}^{t} \gamma(\tau) d\tau.$$

Using (3.1) and (2.32) we conclude that there exist $\bar{t} \geq \tau_1$ such that $\lambda(t)$ becomes negative for all $t \geq \bar{t}$, contradicting the conditions of the lemma. Thus, there exists a necessarily unbounded subset $\mathscr{T}_1 \subseteq T = [t_0, \infty)$ such that (3.2) holds for all $t \in \mathscr{T}_1$. This estimate is monotonically decreasing on the set $\mathscr{T}_1$ because $\mathscr{A}(t)$ is a strictly increasing function. Note, however, that the same behavior of $\zeta(t)$ on the set $\mathscr{T}_2$ cannot be expected. The proof is true.

Continuing, we obviate the requirement of (3.1) to present a more general lemma.

Lemma 3.4. *Assume that a non-negative and differentiable function $\lambda(t)$ satisfies the inequality (2.2), $\gamma(t)$ and $\zeta(t)$ are nonnegative, and $\alpha(t)$ is a positive continuous function for all $t \geq t_0$. Suppose that (2.3) and (2.19) hold. There then exists an unbounded set $\mathscr{T}_1 \subseteq T = [t_0, \infty)$ such that $\zeta(t) \to 0$ as $t \to \infty$ and all $t \in \mathscr{T}_1$ with the estimate (3.3). If the function in (2.19) monotonically tends to zero, then (3.3) gives a monotonically decreasing estimate on $\mathscr{T}_1$.*

Proof following the pattern of the previous lemma. For each $t \geq t_0$ either

$$H_1: \ \ \zeta(t) \leq \frac{1}{\mathscr{A}(t)} + \frac{\gamma(t)}{\alpha(t)} \tag{3.3}$$

or

$$H_2: \ \ \zeta(t) > \frac{1}{\mathscr{A}(t)} + \frac{\gamma(t)}{\alpha(t)},$$

where $\mathscr{A}(t)$ is defined by (2.26). Define again the sets (2.27)-(2.29). We will prove that $\mathscr{T}_1$ is an unbounded set. Suppose $\mathscr{T}_1$ is bounded. Then there exists $t = \tau_1$ such that

$$\zeta(t) > \frac{1}{\mathscr{A}(t)} + \frac{\gamma(t)}{\alpha(t)} \ \ \ \forall t \geq \tau_1.$$

The inequality (2.2) for all $t \geq \tau_1$ implies

$$\frac{d\lambda(t)}{dt} < -\frac{\alpha(t)}{\mathscr{A}(t)} \tag{3.4}$$

and (3.4) gives

$$\lambda(t) < \lambda(\tau_1) - \int_{\tau_1}^{t} \frac{\alpha(s)}{\mathscr{A}(s)} ds, \ \ \forall t > \tau_1.$$

By virtue of the Cauchy integral criterion, we again have (2.32). As in Lemma 2.5 we come to a contradiction with the condition that $\lambda(t) \geq 0$ for all $t \geq t_0$. Thus, $\mathscr{T}_1$ is unbounded set. By the hypothesis H_1, at each interval of $\mathscr{T}_1$ the function $\zeta(t)$ is estimated from above as in (3.3). Since $\mathscr{T}_1$ is an unbounded set, we conclude that $\zeta(t) \to 0$ as $t \to \infty$ and all $t \in \mathscr{T}_1$. If $f(t) = \dfrac{\gamma(t)}{\alpha(t)}$ is monotonically decreasing, then the estimate (3.3) is also monotonically decreasing because $\mathscr{A}(t)$ is a strictly increasing function. The lemma is true.

Recall that under the conditions of Lemma 3.4, $\mathscr{T}_1$ is an unbounded set and $\mathscr{T}_2$ is bounded. Let $\mathscr{T}_1^k \subset \mathscr{T}_1$ and $\mathscr{T}_2^k \subset \mathscr{T}_2$, $k = 1, 2, ...\bar{k}$. Introduce, for example, (2.28). On each interval $\mathscr{T}_2^j$ the function $\lambda(t)$ is bounded and monotonically decreasing by the inequality (3.4). However, its value on the left boundary of $\mathscr{T}_2^j$ is not defined by our methods. On each interval $\mathscr{T}_1^{j+1}$ the function $\zeta(t)$ monotonically decreases and its value on the left boundary of $\mathscr{T}_1^{j+1}$ is less than its value on the right boundary of $\mathscr{T}_1^j$. Therefore $\lim_{t\to\infty} \zeta(t) = 0$ for all $t \in \mathscr{T}_1$.

The following statement requires very strong assumptions (cf. [12]).

Lemma 3.5. *Assume that a non-negative and differentiable function $\lambda(t)$ satisfies the inequality (2.2), $\alpha(t)$ is a positive continuous function, $\lim\limits_{t\to\infty} \alpha(t) = 0$, $\zeta(t)$ is a positive continuous and differentiable function, (2.3) is fulfilled,*

$$\int_{t_0}^{\infty} \alpha(t)\zeta(t)dt < \infty$$

and there exists a constant $\theta > 0$ such that

$$\Big|\frac{d\zeta(t)}{dt}\Big| \leq \theta\alpha(t). \tag{3.5}$$

Then $\lim\limits_{t\to\infty} \zeta(t) = 0$.

Proof. Note first of all that $\dfrac{d\zeta(t)}{dt}$ is the integrable function on any interval $[t_1, t_2] \subset T = [t_0, \infty)$. Then the function $\Big|\dfrac{d\zeta(t)}{dt}\Big|$ is also integrable on this interval and the inequality

$$\Big|\int_{t_1}^{t_2} \frac{d\zeta(t)}{dt} dt\Big| \leq \int_{t_1}^{t_2} \Big|\frac{d\zeta(t)}{dt}\Big| dt$$

follows.

We know from Lemma 3.3 that there exists some piecewise continuous function $\zeta^{(1)}(t) \subset \zeta(t)$ defined on the unbounded subset $\mathscr{T}_1 \subset T$ such that $\zeta^{(1)}(t) \leq \dfrac{1}{\mathscr{A}(t)}$ for all $t \in \mathscr{T}_1$. If the result of this lemma does not hold, then there exists a constant $\sigma > 0$ and another piecewise continuous function $\zeta^{(2)}(t) \subset \zeta(t)$ defined on some unbounded subset $\mathscr{P} \subset T$ such that $\zeta^{(2)}(t) \geq \sigma$ for all $t \in \mathscr{P}$. In this case, we are able to construct a third piecewise continuous function $\zeta^{(3)}(t) \subset \zeta(t)$ defined on some unbounded subset $\mathscr{R} \subset \mathscr{T}$ with the following selection rule for the argument $t \in T$:

$$t = \tau_0 = \min\{t \geq t_0 : \ \zeta(t) \geq \sigma\}$$
$$t = \tau_1 = \min\{t \geq \tau_0 : \ \zeta(t) \leq 2^{-1}\sigma\},$$
$$t = \tau_2 = \min\{t \geq \tau_1 : \ \zeta(t) \geq \sigma\}$$
$$t = \tau_3 = \min\{t \geq \tau_2 : \ \zeta(t) \leq 2^{-1}\sigma\},$$
$$t = \tau_4 = \min\{t \geq \tau_3 : \ \zeta(t) \geq \sigma\},$$
$$etc...$$
$$t = \tau_{2k+1} = \min\{t \geq \tau_{2k} : \ \zeta(t) \leq 2^{-1}\sigma\},$$
$$t = \tau_{2k+2} = \min\{t \geq \tau_{2k+1} : \ \zeta(t) \geq \sigma\},$$

$$etc... .$$

It is clear that

$$\zeta(t) \geq 2^{-1}\sigma, \quad \tau_{2k} \leq t \leq \tau_{2k+1}. \tag{3.6}$$

Since $\int_{t_0}^{\infty} \alpha(t)\zeta(t)dt < \infty$, by (3.6) we get

$$\begin{aligned} \int_{t_0}^{\infty} \alpha(t)\zeta(t)dt &\geq \sum_{k}^{\infty} \int_{t_{2k}}^{\tau_{2k+1}} \alpha(t)\zeta(t)dt \\ &\geq 2^{-1}\sigma \sum_{k}^{\infty} \int_{t_{2k}}^{\tau_{2k+1}} \alpha(t)dt, \end{aligned} \tag{3.7}$$

that is, the series in (3.7) is convergent. Therefore,

$$\int_{t_{2k}}^{\tau_{2k+1}} \alpha(t)dt \to 0, \quad k \to \infty. \tag{3.8}$$

On the other hand, we have $\zeta(\tau_{2k}) \geq \sigma$ and $\zeta(\tau_{2k+1}) \leq 2^{-1}\sigma$, so that by virtue of (3.5)

$$\begin{aligned} \frac{\sigma}{2} \leq \zeta(\tau_{2k}) - \zeta(\tau_{2k+1}) &\leq \left| \int_{\tau_{2k}}^{\tau_{2k+1}} \frac{d\zeta(t)}{dt} dt \right| \\ &\leq \int_{\tau_{2k}}^{\tau_{2k+1}} \left| \frac{d\zeta(t)}{dt} \right| dt \\ &\leq \theta \int_{\tau_{2k}}^{\tau_{2k+1}} \alpha(t)dt, \quad \forall k \geq 0. \end{aligned}$$

This contradicts (3.8). Thus, $\lim_{t\to\infty} \zeta(t) = 0$. The proof is complete.

4. Differential Inequalities (2.5)

Now we investigate the differential inequality (2.5) under the conditions of Section 2 for $\lambda(t)$, $\alpha(t)$ and $\rho(t)$ (see, for example, Lemma 2.12).

Lemma 4.1. *For inequality (2.5) the estimate*

$$\lambda(t) \leq \left[\lambda_0^{1-n} V^{-1}(t) - (n-1) \int_{t_0}^{t} \rho(\theta) exp\Big((n-1) \int_{\theta}^{t} \alpha(s)ds\Big) d\theta\right]^{-\frac{1}{n-1}} \tag{4.1}$$

holds, where

$$V(t) = exp\Big((1-n) \int_{t_0}^{t} \alpha(s)ds\Big).$$

Proof. Introduce the differentiable function $y(t)$ and the replacement

$$\lambda(t) = y^{\frac{1}{1-n}}.$$

We have

$$\frac{d\lambda}{dy} = \frac{1}{1-n} y^{\frac{n}{1-n}}.$$

Using the simple equality $\frac{d\lambda}{dt} = \frac{d\lambda}{dy}\frac{dy}{dt}$, from (2.5) we calculate

$$\frac{1}{1-n} y^{\frac{n}{1-n}} \frac{dy(t)}{dt} \leq -\alpha(t) y^{\frac{1}{1-n}} + \rho(t) y^{\frac{n}{1-n}},$$

which gives

$$\frac{dy(t)}{dt} \geq (n-1)\alpha(t) y^{\frac{1}{1-n}} y^{\frac{n}{n-1}} - (n-1)\rho(t) y^{\frac{n}{1-n}} y^{\frac{n}{n-1}}.$$

Thus,

$$\frac{dy(t)}{dt} \geq (n-1)\alpha(t)y(t) - (n-1)\rho(t).$$

It is not difficult to see that

$$\frac{d}{dt}\Big[y(t)V(t)\Big] = V(t)\frac{dy(t)}{dt} - (n-1)\alpha(t)y(t)V(t).$$

Then

$$\frac{d}{dt}\Big[y(t)V(t)\Big] \geq -(n-1)\rho(t)V(t)$$

and we obtain

$$y(t)V(t) - y(t_0) \geq (1-n)\int_{t_0}^{t} \rho(\theta)V(\theta)d\theta.$$

From this, it follows that

$$y(t) \geq y(t_0)V^{-1}(t) + (1-n)\int_{t_0}^{t} \rho(\theta)V(\theta)V^{-1}(t)d\theta,$$

that is

$$y(t) \geq y(t_0)V^{-1}(t) + (1-n)\int_{t_0}^{t} \rho(\theta) exp\Big((n-1)\int_{\theta}^{t} \alpha(s)ds\Big)d\theta.$$

Thus, we have (4.1) for all $t \geq t_0$.

Let us note several partial cases:

1. If in the inequality (2.5) $\rho(t) \equiv 0$, then (4.1) gives

$$\lambda(t) \leq \Big[\lambda_0^{1-n} V^{-1}(t)\Big]^{-\frac{1}{n-1}},$$

which leads to the estimate

$$\lambda(t) \leq \lambda_0 \, exp\Big(-\int_{t_0}^{t} \alpha(s)ds\Big) \quad \forall t \geq t_0.$$

It coincides with (2.7) and (2.18) in the cases $\psi(\lambda) = \lambda$ and $\gamma(t) \equiv 0$, respectively. If (2.3) is fulfilled, then $\lambda(t) \to 0$ as $t \to \infty$.

2. If in (2.5) $\rho(t) \not\equiv 0$ and $n = 0$, then it is the inequality

$$\frac{d\lambda(t)}{dt} \leq -\alpha(t)\lambda(t) + \rho(t), \;\; t \geq t_0, \;\; \lambda(t_0) = \lambda_0.$$

The estimate (4.1) implies

$$\lambda(t) \leq \lambda_0 exp\Big(-\int_{t_0}^{t}\alpha(\tau)d\tau\Big) + \int_{t_0}^{t}\rho(\theta)exp\Big(-\int_{\theta}^{t}\alpha(\tau)d\tau\Big)d\theta, \ \ t \geq t_0.$$

It coincides with (2.18) if $\gamma(t)$ is replaced with $\rho(t)$. If (2.19) is fulfilled, then $\lambda(t) \to 0$ as $t \to \infty$.

3. If in (2.5) $\rho(t) \not\equiv 0$ and $n = 1$, then the right hand side of (4.1) is degenerated because of its degree, which equals $-\dfrac{1}{n-1}$. It is also clear from the corresponding differential inequality

$$\frac{d\lambda(t)}{dt} \leq -\Big(\alpha(t) - \rho(t)\Big)\lambda(t),$$

which yields the estimate

$$\lambda(t) \leq \lambda_0 exp\Big(-\int_{t_0}^{t}(\alpha(\tau) - \rho(\tau))d\tau\Big). \tag{4.2}$$

In view of (2.3) the convergence or divergence of $\lambda(t)$ to 0 in (4.2) depend on the function $\rho(t)$. It is easy to see that the condition $\rho(t) > \alpha(t)$ for all $t \geq \bar{t} \geq t_0$ sends the right hand side of (4.2) to ∞ as $t \to \infty$, while the inverse condition $\rho(t) < \alpha(t)$ for all $t \geq \bar{t} \geq t_0$ leads to the convergence: $\lim_{t\to\infty} \lambda(t) = 0$.

4. If in (2.5) $\rho(t) \not\equiv 0$ and $n = 2$, then Lemma 4.1 for the inequality

$$\frac{d\lambda(t)}{dt} \leq -\alpha(t)\lambda(t) + \rho(t)\lambda^2(t), \quad t \geq t_0 \geq 0, \ \ \lambda(t_0) = \lambda_0 > 0, \tag{4.3}$$

asserts that

$$\lambda(t) \leq \Big[\lambda_0^{-1} exp\Big(\int_{t_0}^{t}\alpha(s)ds\Big) - \int_{t_0}^{t}\rho(\theta)exp\Big(\int_{\theta}^{t}\alpha(s)ds\Big)d\theta\Big]^{-1} \quad \forall t \geq t_0. \tag{4.4}$$

The right hand side of (4.4) tends to zero only if $\rho(t)$ tends to 0. Let us give an example. Assume that $\alpha(t) = \dfrac{1}{t}$ and $\rho(t) = \bar{\rho} > 0$, that is, we consider the inequality

$$\frac{d\lambda(t)}{dt} \leq -\frac{1}{t}\lambda(t) + \bar{\rho}\lambda^2(t), \ \ t \geq t_0 \geq 0, \ \ \lambda(t_0) = \lambda_0 > 0,$$

the first part in the square brackets of (4.4)

$$\lambda_0^{-1} exp\Big(\int_{t_0}^{t}\alpha(s)ds\Big) = \frac{t}{\lambda_0 t_0}, \tag{4.5}$$

and the second part

$$\int_{t_0}^{t}\rho(\theta)exp\Big(\int_{\theta}^{t}\alpha(s)ds\Big)d\theta = t\bar{\rho}\ln\frac{t}{t_0}.$$

Thus, (4.4) gives the estimate:

$$\lambda(t) \leq \Big(\frac{t}{\lambda_0 t_0} - t\bar{\rho}\ln\frac{t}{t_0}\Big)^{-1} \quad \forall t \geq t_0. \tag{4.6}$$

It is clear that if $t > t_0 \exp(\bar{\rho}\lambda_0 t_0)^{-1}$, then the right hand side of (4.6) becomes negative. In this case, we cannot conclude that $\lim_{t\to\infty} \lambda(t) = 0$.

The following two simple examples illustrate the convergence of $\lambda(t)$ to 0, when $\rho(t)$ is not a constant and $\lim_{t\to\infty}\rho(t)=0$:

a) Let $\alpha(t)=\frac{1}{t}$ and $\rho(t)=\frac{\bar\rho}{t}$, where $t\ge t_0>0$ and $\bar\rho>0$. This means that we consider the differential inequality (4.3) in the form:

$$\frac{d\lambda(t)}{dt}\le -\frac{1}{t}\lambda(t)+\frac{\bar\rho}{t}\lambda^2(t),\ \ t\ge t_0\ge 0,\ \ \lambda(t_0)=\lambda_0>0.$$

We calculate the second part in the square brackets of (4.4):

$$\int_{t_0}^{t}\rho(\theta)exp\Big(\int_{\theta}^{t}\alpha(s)ds\Big)d\theta=\bar\rho\; t\int_{t_0}^{t}\frac{d\theta}{\theta^2}=-\bar\rho\;\Big(1-\frac{t}{t_0}\Big).$$

Taking into account (4.5), one gets

$$\lambda(t)\le\Big(\frac{t}{\lambda_0 t_0}-\frac{\bar\rho\; t}{t_0}+\bar\rho\Big)^{-1}\quad \forall t\ge t_0. \tag{4.7}$$

If $\bar\rho<\frac{1}{\lambda_0}$, then right hand side of (4.7) tends to 0 as $t\to\infty$, hence, $\lim_{t\to\infty}\lambda(t)=0$.

b) Now let $\alpha(t)=\frac{\alpha}{t}$ and $\rho(t)=\frac{\bar\rho}{t}$, where α and $\bar\rho$ are positive constants. They involve (4.3) as it follows

$$\frac{d\lambda(t)}{dt}\le -\frac{\alpha}{t}\lambda(t)+\frac{\bar\rho}{t}\lambda^2(t),\ \ t\ge t_0\ge 0,\ \ \lambda(t_0)=\lambda_0>0.$$

Then the first part in the square brackets of (4.4) is $\frac{1}{\lambda_0}\Big(\frac{t}{t_0}\Big)^{\alpha}$ and the second part

$$\int_{t_0}^{t}\rho(\theta)exp\Big(\int_{\theta}^{t}\alpha(s)ds\Big)d\theta=\frac{\bar\rho}{\alpha}\Big(\frac{t}{t_0}\Big)^{\alpha}-\frac{\bar\rho}{\alpha}.$$

Thus, we obtain

$$\lambda(t)\le\Big[\frac{1}{\lambda_0}\Big(\frac{t}{t_0}\Big)^{\alpha}-\frac{\bar\rho}{\alpha}\Big(\frac{t}{t_0}\Big)^{\alpha}+\frac{\bar\rho}{\alpha}\Big]^{-1}\quad \forall t\ge t_0.$$

If $\bar\rho<\frac{\alpha}{\lambda_0}$, then $\lim_{t\to\infty}\lambda(t)=0$.

c) Assume that $\alpha(t)=\frac{1}{t}$ and $\rho(t)=\frac{\bar\rho}{t^2}$, where $t\ge t_0>0$ and $\bar\rho>0$. That is, we study the differential inequality

$$\frac{d\lambda(t)}{dt}\le -\frac{1}{t}\lambda(t)+\frac{\bar\rho}{t^2}\lambda^2(t),\ \ t\ge t_0\ge 0,\ \ \lambda(t_0)=\lambda_0>0.$$

As in item a) the second part in the square brackets of (4.4) is calculated as

$$\int_{t_0}^{t}\rho(\theta)exp\Big(\int_{\theta}^{t}\alpha(s)ds\Big)d\theta=\bar\rho\; t\int_{t_0}^{t}\frac{d\theta}{\theta^3}=-\frac{\bar\rho}{2}\;\Big(\frac{1}{t}-\frac{t}{t_0^2}\Big).$$

Therefore

$$\lambda(t)\le\Big(\frac{t}{\lambda_0 t_0}-\frac{\bar\rho\; t}{2t_0^2}+\frac{\bar\rho}{2t}\Big)^{-1}\quad \forall t\ge t_0.$$

If $\bar\rho<\frac{2t_0}{\lambda_0}$, then $\lim_{t\to\infty}\lambda(t)=0$.

5. If in (2.5) $n=3$ and $\rho(t)\not\equiv 0$, then the inequality

$$\frac{d\lambda(t)}{dt}\le -\alpha(t)\lambda(t)+\rho(t)\lambda^3(t),\quad t\ge t_0\ge 0,\ \lambda(t_0)=\lambda_0>0$$

gives the estimate (4.1) in the form of

$$\lambda(t)\le\Big[\lambda_0^{-2}exp\Big(2\int_{t_0}^t\alpha(s)ds\Big)-2\int_{t_0}^t\rho(\theta)exp\Big(2\int_{\theta}^t\alpha(s)ds\Big)d\theta\Big]^{-1/2}.$$

As an example, consider $\alpha(t)=\dfrac{1}{t}$ and $\rho(t)=\dfrac{\bar\rho}{t}$, where $\bar\rho>0$. We obtain

$$\lambda(t)\le\Big(\frac{t^2}{\lambda_0^2 t_0{}^2}-\frac{\bar\rho\, t^2}{2t_0^2}+\frac{\bar\rho}{2}\Big)^{-1/2}\quad\forall t\ge t_0. \tag{4.8}$$

If $\bar\rho<\dfrac{2}{\lambda_0^2}$, then $\lambda(t)\to 0$ as $t\to\infty$. It is possible to similarly investigate other examples of item 4 and $n\ge 4$.

5. Dynamic systems with total asymptotically weakly contractive approximations of operators

In this Section we study the dynamical system (1.14), where $S(t)$ is a total asymptotically weakly contractive approximating family of nonexpansive operators S (see Definition 1.1).

Since the set Ω is convex and closed, $S(t):\Omega\to\Omega$ for all $t\ge t_0$, and $0<\omega(t)\le 1$, it is not difficult to see that the dynamical system

$$\frac{dx(t)}{dt}+x(t)=P_\Omega\Big(x(t)-\omega(t)(x(t)-S(t)x(t))\Big)$$

is equivalent to (1.14), $\dfrac{dx(t)}{dt}+x(t)\in\Omega$, and by means of [20] $x(t)\in\Omega$ for all $t\ge t_0$.

Let us recall that we denoted a fixed point set of S by $\mathscr{N}$, i.e., $\mathscr{N}:=\{x\in\Omega: Sx=x\}$. We posited that $\mathscr{N}\neq\emptyset$ and $x^*\in\mathscr{N}$.

If $J:B\to B^*$ is a normalized duality mapping in a uniformly convex Banach space B, then the following equality for dual products is true:

$$\Big\langle\frac{d(x(t)-x^*)}{dt},J(x(t)-x^*)\Big\rangle=-\,\omega(t)\Big\langle x(t)-S(t)x(t),J(x(t)-x^*)\Big\rangle. \tag{5.1}$$

Using the formula $\dfrac{d\|w(t)\|^2}{dt}=2\Big\langle\dfrac{dw(t)}{dt},Jw(t)\Big\rangle$, we rewrite (5.1) as

$$\frac{d\|x(t)-x^*\|^2}{dt}=-\,2\omega(t)\Big\langle x(t)-S(t)x(t),J(x(t)-x^*)\Big\rangle. \tag{5.2}$$

It is easy to check the equality

$$\begin{aligned}\Big\langle x(t)-S(t)x(t),J(x(t)-x^*)\Big\rangle &= \Big\langle F(t)x(t)-F(t)x^*,J(x(t)-x^*)\Big\rangle\\ &+\ \Big\langle F(t)x^*-Fx^*,J(x(t)-x^*)\Big\rangle,\end{aligned}$$

which together with (5.2) implies

$$\begin{aligned}\frac{d\|x(t)-x^*\|^2}{dt} = &- 2\omega(t)\Big\langle F(t)x(t)-F(t)x^*, J(x(t)-x^*)\Big\rangle\\ &- 2\omega(t)\Big\langle F(t)x^*-Fx^*, J(x(t)-x^*)\Big\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d\|x(t)-x^*\|^2}{dt} &\leq -2\omega(t)\Big\langle F(t)x(t)-F(t)x^*, J(x(t)-x^*)\Big\rangle\\ &+ 2\omega(t)\ \|F(t)x^*-Fx^*\|\|x(t)-x^*\|.\end{aligned} \tag{5.3}$$

Let us estimate the dual product in (5.3) from below, as

$$\begin{aligned}\Big\langle F(t)x(t)-F(t)x^*, J(x(t)-x^*)\Big\rangle &= \|x(t)-x^*\|^2-\langle S(t)x(t)-S(t)x^*, J(x(t)-x^*)\rangle\\ &\geq \|x(t)-x^*\|^2-\|S(t)x(t)-S(t)x^*\|\|x(t)-x^*\|,\end{aligned}$$

and we have

$$\begin{aligned}\frac{d\|x(t)-x^*\|^2}{dt} &\leq 2\omega(t)\Big(\|S(t)x(t)-S(t)x^*\|\|x(t)-x^*\|-\|x(t)-x^*\|^2\Big)\\ &+ 2\omega(t)\ \|F(t)x^*-Fx^*\|\|x(t)-x^*\|.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d\|x(t)-x^*\|}{dt} &\leq \omega(t)\Big(\|S(t)x(t)-S(t)x^*\|-\|x(t)-x^*\|\Big)\\ &+ \omega(t)\|F(t)x^*-Fx^*\|.\end{aligned}$$

Furthermore, Definition 1.1 gives us

$$\|F(t)x^*-Fx^*\|\leq h(t)\eta(\|x^*\|)+g(t).$$

It is clear from the conditions of the function $\eta(\xi)$ that there exists a constant $C_0>0$ such that $\eta(\|x^*\|)\leq C_0$ for any $x^*\in\mathscr{N}$. Then

$$\|F(t)x^*-Fx^*\|\leq C_0h(t)+g(t).$$

Now (1.13) in Definition 1.1 yields the following differential inequality:

$$\begin{aligned}\frac{d\|x(t)-x^*\|}{dt} &\leq \omega(t)\Big(k(t)\|x(t)-x^*\|-p(t)\psi(\|x(t)-x^*\|)\\ &+ l(t)\phi(\|x(t)-x^*\|)+m(t)\Big)+\omega(t)\Big(h(t)C_0+g(t)\Big).\end{aligned} \tag{5.4}$$

Setting $\lambda(t)=\|x(t)-x^*\|$, we get

$$\begin{aligned}\frac{d\lambda(t)}{dt} &\leq \omega(t)\Big(k(t)\lambda(t)-p(t)\psi(\lambda(t))+l(t)\phi(\lambda(t))+m(t)\Big)\\ &+ \omega(t)\Big(h(t)C_0+g(t)\Big).\end{aligned}$$

Below, we present strong convergence theorems supported by the lemmas of Subsections 2.1 and 2.2.

We start by considering the following particular case of (1.12):

$$\|S(t)x - S(t)y\| \le (1+k(t))\|x-y\| - p(t)\psi(\|x-y\|) + m(t), \ \forall x,y \in \Omega, \tag{5.5}$$

For $\omega(t)$ in (1.14) and $p(t)$ in (5.5) introduce the condition

$$\int_{t_0}^{\infty} \omega(t)p(t)dt = \infty. \tag{5.6}$$

1. Suppose that (1.11) is valid and (5.5) holds in the form of

$$\|S(t)x - S(t)y\| \le \|x-y\| - p(t)\psi(\|x-y\|) + m(t). \tag{5.7}$$

Then (5.4) becomes the following:

$$\frac{d\|x(t)-x^*\|}{dt} \le -\omega(t)p(t)\psi(\|x(t)-x^*\| + \omega(t)\Big(m(t)+h(t)C_0+g(t)\Big). \tag{5.8}$$

By setting $\lambda(t) = \|x(t)-x^*\|$ again, we obtain

$$\frac{d\lambda(t)}{dt} \le -\omega(t)p(t)\psi(\lambda(t)) + \omega(t)\Big(m(t)+h(t)C_0+g(t)\Big). \tag{5.9}$$

Theorem 5.1. *Assume that $\Omega \subseteq B$ is a closed convex set and $S : \Omega \to \Omega$. Let $S(t) : \Omega \to \Omega$ for each $t \ge t_0 \ge 0$ be a total asymptotically weakly contractive approximating family of S, where in (5.7) $\psi(\xi) : R^+ \to R^+$ is a continuous and increasing function with $\psi(0) = 0$, the functions $m(t)$, $h(t)$ and $g(t)$ are nonnegative, and $p(t)$ is a positive bounded function. Suppose that $\{m(t), h(t), g(t)\} \to 0$ as $t \to \infty$ and (5.6) is fulfilled. Starting from an arbitrary $x(t_0) = x_0 \in \Omega$, define trajectory $x(t)$ by the dynamical system (1.14) with the condition*

$$\lim_{t\to\infty} \frac{m(t)+h(t)+g(t)}{p(t)} = 0. \tag{5.10}$$

Then $\lim_{t\to\infty} x(t) = x^*$.

Proof. Denoting in (5.9) $\alpha(t) = \omega(t)p(t)$ and $\gamma(t) = \omega(t)(m(t)+h(t)C_0+g(t))$ and using (5.6) and (5.10), we conclude that all the conditions of Lemma 2.5 are fulfilled. Thereby we show that $\lambda(t) \to 0$ as $t \to \infty$. The theorem holds.

Remark 5.2. If $p(t) \ge \bar{p}$, where $\bar{p}$ is a positive constant, then the condition (1.15) replaces (5.6).

Under the conditions of Theorem 5.1, if $\psi(\xi) = \xi$ in (5.7), then the inequality

$$\frac{d\|x(t)-x^*\|}{dt} \le -\omega(t)p(t)\|x(t)-x^*\| + \omega(t)\Big(m(t)+h(t)C_0+g(t)\Big)$$

yields the following estimate of the convergence rate:

$$\begin{aligned} \|x(t)-x^*\| \ &\le\ 2^{-1}\|x(t_0)-x^*\| exp\Big(-\int_{t_0}^{t}\omega(\tau)p(\tau)d\tau\Big) \\ &+\ 2^{-1}\int_{t_0}^{t}\omega(\theta)\gamma(\theta) exp\Big(-\int_{\theta}^{t}\omega(\tau)p(\tau)d\tau\Big)d\theta, \end{aligned}$$

where $\gamma(t) = m(t)+h(t)C_0+g(t)$. This statement is supported by Lemma 2.3.

We now obviate the increasing property of $\psi(t)$.

Theorem 5.3. *Let $\Omega \subseteq B$ be a closed convex set and $S : \Omega \to \Omega$ and $S(t) : \Omega \to \Omega$ for each $t \geq t_0 \geq 0$ be a total asymptotically weakly contractive approximating family of S, where in (1.11) and (5.7) $\psi(\xi) : R^+ \to R^+$ is continuous function with $\psi(0) = 0$, the functions $m(t)$, $h(t)$ and $g(t)$ are nonnegative, and $p(t)$ is a positive bounded function. Suppose that $\{m(t), h(t), g(t)\} \to 0$ as $t \to \infty$ and (5.6) is fulfilled. Assume that there exist constants $c > 0$ and $\xi_+ > 0$ such that $\psi(\xi) \geq c$ for all $\xi \geq \xi_+$. Starting from an arbitrary $x(t_0) = x_0 \in \Omega$ define trajectory $x(t)$ by the dynamical system (1.14) with the condition (5.10). Then $\lim_{t\to\infty} x(t) = x^*$.*

Proof. As in previous theorem, the inequality (5.8) leads to (5.9). From (5.6) and (5.10) we conclude that all the conditions of Lemma 2.7 are met. Therefore the result is true.

2. Suppose now that (1.11) is valid and consider the inequality (5.5). In this case (5.4) implies

$$\begin{aligned} \frac{d\|x(t)-x^*\|}{dt} \;\leq\;& 2^{-1}\omega(t)\Big(k(t)\|x(t)-x^*\| - p(t)\psi(\|x(t)-x^*\|)\Big) \\ +\;& \omega(t)\Big(m(t)+h(t)C_0+g(t)\Big). \end{aligned} \tag{5.11}$$

Assume in addition to (5.6) that

$$\int_{t_0}^{\infty} \omega(t)k(t)dt < \infty. \tag{5.12}$$

Theorem 5.4. *Let $\Omega \subseteq B$ be a closed convex set and $S : \Omega \to \Omega$. Let $S(t) : \Omega \to \Omega$ for each $t \geq t_0 \geq 0$ be a total asymptotically weakly contractive approximating family of S, where in (1.11) and (5.5) $\psi(\xi) : R^+ \to R^+$ is a continuous and increasing function with $\psi(0) = 0$, the functions $k(t)$, $m(t)$, $h(t)$ and $g(t)$ are nonnegative, and $p(t)$ is a positive bounded function. Suppose that $\{k(t), m(t), h(t), g(t)\} \to 0$ as $t \to \infty$ and both (5.6) and (5.12) are fulfilled. Starting from an arbitrary $x(t_0) = x_0 \in \Omega$ define trajectory $x(t)$ by the dynamical system (1.14) with the condition (5.10). Then $\lim_{t\to\infty} x(t) = x^*$.*

Proof uses Lemma 2.8 for the differential inequality

$$\frac{d\lambda(t)}{dt} \;\leq\; \omega(t)\Big(k(t)\lambda(t) - p(t)\psi(\lambda(t)) + m(t) + h(t)C_0 + g(t)\Big),$$

which follows from (5.11) with $\lambda(t) = \|x(t)-x^*\|$.

We again obviate the increasing property of $\psi(t)$.

Theorem 5.5. *Assume that $\Omega \subseteq B$ is a closed convex set and $S : \Omega \to \Omega$. Let $S(t) : \Omega \to \Omega$ for each $t \geq t_0 \geq 0$ be a total asymptotically weakly contractive approximating family of S, where in (1.11) and (5.5) $\psi(\xi) : R^+ \to R^+$ is continuous function with $\psi(0) = 0$, the functions $k(t)$, $m(t)$, $h(t)$ and $g(t)$ are nonnegative, and $p(t)$ is a positive bounded function. Suppose that $\{k(t), m(t), h(t), g(t)\} \to 0$ as $t \to \infty$ and both (5.6) and (5.12) are fulfilled. Suppose there exist constants $c > 0$ and $\bar{\xi} > 0$ such that $\psi(\xi) \geq c$ for all $\xi \geq \bar{\xi}$. Starting from an arbitrary $x(t_0) = x_0 \in \Omega$ define trajectory $x(t)$ by the dynamical system (1.14) with the condition (5.10). Then $\lim_{t\to\infty} x(t) = x^*$.*

The proof of this theorem is based on Lemma 2.7.

Remark 5.6. In Theorems 5.4 and 5.5 we require the condition (5.12). It can be replaced by a stronger condition for $\psi(t)$ on infinity (see Corollary 5.8).

3. Assume that (1.12) is now given in the form:

$$\|S(t)x - S(t)y\| \le \|x-y\| - p(t)\psi(\|x-y\|) + l(t)\phi(\|x-y\|) + m(t). \tag{5.13}$$

Then

$$\begin{aligned}\frac{d\|x(t)-x^*\|}{dt} &\le 2^{-1}\omega(t)\Big(-p(t)\psi(\|x(t)-x^*\|) + l(t)\phi(\|x(t)-x^*\|)\Big)\\ &+ \omega(t)\Big(m(t) + h(t)C_0 + g(t)\Big).\end{aligned}$$

We introduce the following notation:

$$c_1 = max\Big\{\frac{l(t)}{p(t)},\ t \ge t_0\Big\},\ c_2 = max\Big\{\frac{m(t)}{p(t)},\ t \ge t_0\Big\},\ \phi_1(\xi) = c_1\phi(\xi) + c_2. \tag{5.14}$$

Theorem 5.7. *Let $\Omega \subseteq B$ be a closed convex set and $S : \Omega \to \Omega$ and $S(t) : \Omega \to \Omega$ for each $t \ge t_0 \ge 0$ be a total asymptotically weakly contractive approximating family of S, where in (1.11) and (5.13) $\phi(\xi) : R^+ \to R^+$ is continuous function with $\phi(0) = 0$, $\psi(\xi) : R^+ \to R^+$ is a continuous and increasing function with $\psi(0) = 0$, the functions $l(t)$, $m(t)$, $h(t)$ and $g(t)$ are nonnegative, and $p(t)$ is a positive bounded function. Assume that $\{l(t), m(t), h(t), g(t)\} \to 0$ as $t \to \infty$, (5.6) is fulfilled, and there exists a constant $M \ge 0$ such that $\phi_1(\xi) \le \psi(\xi)$ for all $\xi \ge M$ and the equation $\phi_1(\xi)=\psi(\xi)$ has no more than one root ξ_* on the set [0,∞). Starting from an arbitrary $x(t_0) = x_0 \in \Omega$ define trajectory $x(t)$ by the dynamical system (1.14) with the condition*

$$\lim_{t\to\infty} \frac{l(t) + m(t) + h(t) + g(t)}{p(t)} = 0.$$

Then $\lim_{t\to\infty} x(t) = x^*$.

Lemma 2.12 is used to prove this theorem. The general case of (1.12) with $k(t) \not\equiv 0$ is studied by analogy with Theorem 5.4.

4. Let us consider again the dynamical system (1.14) for (1.11) and (5.5), which is equivalent to the inequality

$$\|S(t)x - S(t)y\| \le \|x-y\| - p(t)\psi(\|x-y\|) + k(t)\|x-y\| + m(t) \tag{5.15}$$

for all $x, y \in \Omega$. It is particular case of (5.13) and in Theorem 5.7 we can put $l(t) = k(t)$ and $\phi(\lambda) = \lambda$. Then

$$\begin{aligned}\frac{d\|x(t)-x^*\|}{dt} &\le 2^{-1}\omega(t)\Big(-p(t)\psi(\|x(t)-x^*\|) + k(t)\|x(t)-x^*\|\Big)\\ &+ \omega(t)\Big(m(t) + h(t)C_0 + g(t))\Big).\end{aligned}$$

As in (5.14) we define:

$$c_1 = max\Big\{\frac{k(t)}{p(t)},\ t \ge t_0\Big\},\ c_2 = max\Big\{\frac{m(t)}{p(t)},\ t \ge t_0\Big\},\ \phi_1(\xi) = c_1\xi + c_2.$$

Then the following statement is correct:

Corollary 5.8. *Let $\Omega \subseteq B$ be a closed convex set and $S : \Omega \to \Omega$ and $S(t) : \Omega \to \Omega$ for each $t \geq t_0 \geq 0$, $S(t)$ be a total asymptotically nonexpansive approximation of S, where in (1.11) and (5.15) $\psi(\xi) : R^+ \to R^+$ is a continuous and increasing function with $\psi(0) = 0$, the functions $k(t)$, $m(t)$, $h(t)$ and $g(t)$ are nonnegative, and $p(t)$ is a positive bounded function. Suppose that $\{k(t), m(t), h(t), g(t)\} \to 0$ as $t \to \infty$, (5.6) is fulfilled, and there exists a constant $M \geq 0$ such that $\phi_1(\xi) \leq \psi(\xi)$ for all $\xi \geq M$ and the equation $\psi(\xi) = c_1\xi + c$ has no more than one root ξ_* on the set $[0,\infty)$. Starting from an arbitrary $x(t_0) = x_0 \in \Omega$ define trajectory $x(t)$ by the dynamical system (1.14) with the condition*

$$\lim_{t\to\infty} \frac{k(t) + m(t) + h(t) + g(t)}{p(t)} = 0.$$

Then $\lim\limits_{t\to\infty} x(t) = x^*$.

6. Dynamic Systems with Perturbed Nonexpansive Operators

In this Section, we study the dynamical system (1.14), where $S(t)$ is a perturbed approximating family of nonexpansive mapping S. The latter weakens the operator condition so much that it is impossible to guarantee even weak convergence of any trajectories $x(t)$ to the fixed point set $\mathscr{N}$. The only exception is in the paper [13]. However, it deals with an exactly given nonexpansive operator S and in (1.14) $0 < \omega \leq \omega(t) \leq 1$. One of the main problems lies in establishing a priori boundedness of $x(t)$. This can be proved in some rare cases, shown later. Another problem with our chosen method is that the condition (1.11) in Theorems 6.5 and 6.6 no longer applies, which leads to substantially weaker assertions.

Next, we present a very important auxiliary assertion, given without proof in [7]:

Lemma 6.1. *If $F = I - S$ with a nonexpansive mapping S, then for all $x, y \in B$ such that $\|x\| \leq R$ and $\|y\| \leq R$, the following estimate is satisfied:*

$$\langle Fx - Fy, J(x-y)\rangle \geq 2L^{-1}R^2\delta_B\Big(\frac{\|Fx - Fy\|}{4R}\Big), \tag{6.1}$$

where $\delta_B(\varepsilon)$ is the modulus of convexity of the uniformly convex Banach space B and $1 < L \leq 1.7$ is the Figiel's constant [28, 6].

Proof. In [14] (see also [6], p.22), following the lower parallelogram inequality we established:

$$2\|v\|^2 + 2\|w\|^2 - \|v+w\|^2 \geq 4\mathscr{R}^2\delta_B\Big(\frac{\|v-w\|}{2\mathscr{R}}\Big) \quad \forall v, w \in B,$$

where $\mathscr{R} = \sqrt{2^{-1}(\|v\|^2 + \|w\|^2)}$. It is equivalent to

$$\Big\|\frac{v+w}{2}\Big\|^2 \leq \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|^2 - \mathscr{R}^2\delta_B\Big(\frac{\|v-w\|}{2\mathscr{R}}\Big) \quad \forall v, w \in B.$$

The following proposition was proved in [36] (see Lemmas 3.4 and 3.5 in [6]):

If a convex functional $\varphi(x)$ defined on convex closed set $\Omega \subseteq B$ satisfies the inequality

$$\varphi(\frac{1}{2}v + \frac{1}{2}w) \leq \frac{1}{2}\varphi(v) + \frac{1}{2}\varphi(w) - \kappa(||v-w||),$$

where $\kappa(r) \geq 0,\ \kappa(\bar{r}) > 0$ for some $\bar{r} > 0$, then $\varphi(v)$ is uniformly convex functional with the modulus of convexity $\delta(t) = 2\kappa(t)$ and

$$\varphi(w) \geq \varphi(v) + < l(v), w - v > + 2\kappa(||v - w||)$$

for all $l(v) \in \partial\varphi(v)$. Here $\partial\varphi(v)$ is the set of all support functionals (the set of all subgradients) of $\varphi(v)$ at the point $v \in \Omega$.

We can apply this statement to get

$$\|v\|^2 \leq \|w\|^2 + 2\langle v - w, Jv\rangle - 2\mathscr{R}^2\delta_B\Big(\frac{\|v-w\|}{2\mathscr{R}}\Big) \quad \forall v, w \in B.$$

Introduce $v = x - y$ and $w = x - y - Fx + Fy$ for all $x, y \in B$. Then

$$\|x-y\|^2 \leq \|x - y - Fx + Fy\|^2 + 2\langle Fx - Fy, J(x-y)\rangle - 2\mathscr{R}^2\delta_B\Big(\frac{\|Fx - Fy\|}{2\mathscr{R}}\Big)$$

with

$$\mathscr{R} = \sqrt{2^{-1}(\|x-y\|^2 + \|Sx - Sy\|^2)}.$$

Let $\|x\| \leq R$ and $\|y\| \leq R$. Since $S = I - F$ is a nonexpansive operator, it is obvious that $\mathscr{R} \leq \|x-y\| \leq 2R$. Next, we require the following (Figiel's) inequality:

$$\varepsilon^2\delta_B(\eta) \geq (4L)^{-1}\eta^2\delta_B(\varepsilon) \quad \forall \eta \geq \varepsilon > 0.$$

Take $\eta = (2\mathscr{R})^{-1}\|Fx - Fy\|$ and $\varepsilon = (4R)^{-1}\|Fx - Fy\|$ with $\eta \geq \varepsilon$. Then

$$2\mathscr{R}^2\delta_B\Big(\frac{\|Fx - Fy\|}{2\mathscr{R}}\Big) \geq 2L^{-1}R^2\delta_B\Big(\frac{\|Fx - Fy\|}{4R}\Big).$$

From this

$$\begin{aligned}\|Sx - Sy\|^2 &= \|x - y - Fx + Fy\|^2 \\ &\geq \|x-y\|^2 - 2\langle Fx - Fy, J(x-y)\rangle + 2L^{-1}R^2\delta_B\Big(\frac{\|Fx - Fy\|}{4R}\Big).\end{aligned}$$

The last gives (6.1). The lemma is proved.

Remark 6.2. It can be show by the same way that if $\|v\| \leq R$ and $\|w\| \leq R$, then

$$\Big\|\frac{v+w}{2}\Big\|^2 \leq \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|^2 - L^{-1}R^2\delta_B\Big(\frac{\|v-w\|}{2R}\Big) \quad \forall v, w \in B.$$

This inequality means that the functional $\varphi(x) = \|x\|^2$ for all $x \in B$ is uniformly convex on any bounded set in a uniformly convex Banach space B.

Remark 6.3. Lemma 6.1 with arbitrary $x, y \in B$ is proved in an analogous fashion.

Continuing, we give two propositions including a proof of the boundedness of $x(t)$:

Theorem 6.4. *Let $\{S(t)\}$, $0 \leq t_0 \leq t < \infty$, $S(t) : \Omega \to \Omega$, be a family of asymptotically nonexpansive approximations of $S : \Omega \to \Omega$ with (1.11) and (1.13) as*

$$\|S(t)x - S(t)y\| \leq (1 + l(t))\|x - y\| + m(t). \tag{6.2}$$

We assume that in the dynamical system (1.14)

$$\int_{t_0}^{\infty} \omega(t)\Big(l(t) + m(t) + h(t) + g(t)\Big)dt < \infty. \tag{6.3}$$

Then its solution $x(t)$ is bounded for all $t \geq t_0$ by a constant $\bar{C}$, $\|x(t)\| \leq \bar{C}$. There exists an unbounded subset $\mathscr{T}^1 \subset T = [t_0, \infty]$ such that for all $t \in \mathscr{T}^1$

$$\lim_{t\to\infty} \delta_B\Big(\frac{\|x(t) - Sx(t)\|}{4R}\Big) = 0, \tag{6.4}$$

where $R = max\{\|x^*\|, \bar{C}\}$, with the monotone estimate

$$\delta_B\Big(\frac{\|x(t) - Sx(t)\|}{4R}\Big) \leq \frac{1}{\mathscr{D}(t)}, \quad \mathscr{D}(t) = L^{-1}R\int_{t_0}^{t} \omega(\tau)d\tau. \tag{6.5}$$

Proof. From (5.4) we have

$$\frac{d\|x(t) - x^*\|}{dt} \leq \omega(t)l(t)\|x(t) - x^*\| + \omega(t)\Big(m(t) + h(t)C_0 + g(t)\Big), \tag{6.6}$$

where $\eta(\|x^*\|) \leq C_0$. Denoting

$$\lambda(t) = \|x(t) - x^*\|, \;\; \beta(t) = \omega(t)l(t)$$

and

$$\gamma(t) = \omega(t)\Big(m(t) + h(t)C_0 + g(t)\Big),$$

from (6.6) it follows that

$$\frac{d\lambda(t)}{dt} \leq \beta(t)\lambda(t) + \gamma(t), \;\; t \geq t_0, \;\; \lambda(t_0) = \lambda_0.$$

It is clear that for all $t \geq t_0$ the function $\lambda(t)$ is non-negative and differentiable, $\gamma(t)$ is a non-negative continuous function, and $\beta(t)$ is a continuous positive function satisfying the inequalities $\int_{t_0}^{\infty} \gamma(t)dt < \infty$ and $\int_{t_0}^{\infty} \beta(t)dt < \infty$. Then (2.18) gives

$$\|x(t) - x^*\| \leq \|x_0 - x^*\| exp\Big(\int_{t_0}^{t} \beta(\tau)d\tau\Big) + \int_{t_0}^{t} \gamma(\theta) exp\Big(\int_{\theta}^{t} \beta(\tau)d\tau\Big)d\theta. \tag{6.7}$$

By virtue of the condition (6.3) there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$exp\Big(\int_{t_0}^{\infty} \beta(\tau)d\tau\Big) \leq C_1 \;\; \text{and} \;\; \int_{t_0}^{\infty} \gamma(\tau)d\tau \leq C_2.$$

Then from (6.7) we derive

$$\|x(t) - x^*\| \leq C_1(\|x_0 - x^*\| + C_2),$$

therefore

$$\|x(t)\| \leq C_1(\|x_0 - x^*\| + C_2) + \|x^*\| = \bar{C}.$$

Since the function $\eta(\xi)$ is non-decreasing, it follows from (1.11)

$$\|S(t)x(t) - Sx(t)\| \leq h(t)\eta(\bar{C}) + g(t). \tag{6.8}$$

Let us present now (1.14) in the form:

$$\frac{dx(t)}{dt} = -\omega(t)\Big(x(t) - Sx(t)\Big) - \omega(t)\Big(Sx(t) - S(t)x(t)\Big), \, t \geq t_0, \, x(t_0) = x_0 \in \Omega.$$

Since $Fx(t) = x(t) - Sx(t)$ and $Fx^* = 0$, similar to (5.3) one gets

$$\begin{aligned} \frac{d\|x(t)-x^*\|^2}{dt} &\leq -2\omega(t)\Big\langle Fx(t)-Fx^*, J(x(t)-x^*)\Big\rangle \\ &+ 2\omega(t)\|S(t)x(t)-Sx(t)\|\|x(t)-x^*\|. \end{aligned}$$

This implies

$$\begin{aligned} \frac{d\|x(t)-x^*\|}{dt} &\leq -\omega(t)\|x(t)-x^*\|^{-1}\Big\langle Fx(t)-Fx^*, J(x(t)-x^*)\Big\rangle \\ &+ \omega(t)\|S(t)x(t)-Sx(t)\|. \end{aligned} \tag{6.9}$$

Let $R = \max\{\|x^*\|, \bar{C}\} = \bar{C}$. By Lemma 6.1, in our case

$$\langle Fx(t)-Fx^*, J(x(t)-x^*)\rangle \geq 2L^{-1}R^2\delta_B\Big(\frac{\|Fx(t)-Fx^*\|}{4R}\Big) \tag{6.10}$$

with $Fx^* = 0$. Then (6.8), (6.9), and (6.10) give

$$\frac{d\|x(t)-x^*\|}{dt} \leq -2L^{-1}R\omega(t)\delta_B\Big(\frac{\|x(t)-Sx(t)\|}{4R}\Big) + \omega(t)\Big(h(t)\eta(\bar{C}) + g(t)\Big). \tag{6.11}$$

Denoting now

$$\lambda(t) = \|x(t)-x^*\|, \ \ \alpha(t) = 2L^{-1}R\omega(t),$$
$$\gamma(t) = \omega(t)\Big(h(t)\eta(\bar{C}) + g(t)\Big), \quad \zeta(t) = \delta_B\Big(\frac{\|x(t)-Sx(t)\|}{4R}\Big),$$

we obtain from (6.11) the differential inequality (2.2). It follows from (6.3) that

$$\int_{t_0}^{\infty} \omega(t)\Big(h(t)\eta(\bar{C}) + g(t)\Big)dt < \infty. \tag{6.12}$$

By (6.12) and Lemma 3.3 we conclude that the theorem is true.

Theorem 6.5. *Assume that in dynamical system (1.14), the approximation family $S(t)$ of nonexpansive operator $S : \Omega \to \Omega$ satisfies the following uniform condition at each point $x \in \Omega$:*

$$\|S(t)x - Sx\| \leq h(t) \tag{6.13}$$

and

$$\int_{t_0}^{\infty} \omega(t)h(t)dt < \infty. \tag{6.14}$$

Then a solution $x(t)$ is bounded for all $t \geq t_0$ and there exists an unbounded subset $\mathscr{T}^1 \subset T = [t_0, \infty]$ such that (6.4) holds for all $t \in \mathscr{T}^1$.

Proof. By Lemma 6.1, if $F = I - S$ with a nonexpansive mapping S and arbitrary $x, y \in \Omega$ then

$$\langle Fx - Fy, J(x-y)\rangle \geq 0.$$

Therefore from (6.13) and (6.9) one obtains

$$\frac{d\|x(t)-x^*\|}{dt} \leq \omega(t)h(t).$$

Under the condition (6.14) there exists a constant $C > 0$ such that

$$\|x(t)-x^*\| \leq \|x_0 - x^*\| + \int_{t_0}^{\infty} \omega(t)h(t)dt \leq C.$$

Therefore solutions $x(t)$ of the system (1.14) are bounded for all $t \geq t_0$, that is, there exists a constant $\bar{C} > 0$ such that $\|x(t)\| \leq \bar{C}$. Let $\max\{\|x^*\|, \bar{C}\} = R$.

By Lemma 6.1 again, if $F = I - S$ with a nonexpansive mapping S, then (6.10) holds. From (6.9) we obtain

$$\frac{d\|x(t) - x^*\|}{dt} \leq -\omega(t) 2L^{-1} R \delta_B\Big(\frac{\|Fx(t) - Fx^*\|}{4R}\Big) + \omega(t) h(t).$$

Next, it remains only to use Lemma 3.2. The proof is finished.

We suppose in the next theorem that (1.14) has a bounded solution $x(t)$, that is, there exists a constant $\bar{C} > 0$ such that $\|x(t)\| \leq \bar{C}$.

Theorem 6.6. *Assume that in dynamical system (1.14) the approximation family $S(t)$ of nonexpansive operator $S : \Omega \to \Omega$ at each point $x \in \Omega$ satisfies inequality (1.11). Then there exists an unbounded set $\mathscr{T}^1 \subset T = [t_0, \infty)$ such that (6.4) holds for all $t \in \mathscr{T}^1$ with the estimate*

$$\delta_B\Big(\frac{\|x(t) - Sx(t)\|}{4R}\Big) \leq \frac{1}{\mathscr{D}(t)} + h(t)\eta(\bar{C}) + g(t), \quad \mathscr{D}(t) = LR\int_{t_0}^{t} \omega(\tau) d\tau. \tag{6.15}$$

Proof. Using the same definition of $\lambda(t)$, $\alpha(t)$, $\gamma(t)$ and $\zeta(t)$ as in Theorem 6.4, apply Lemma 3.4 to (6.11). Since the inequality (1.11) assumes that $h(t) \to 0$ and $g(t) \to 0$ as $t \to \infty$, we obtain

$$\lim_{t\to\infty} \frac{\gamma(t)}{\alpha(t)} = \lim_{t\to\infty} \Big(h(t)\eta(\bar{C}) + g(t)\Big) = 0.$$

Thus, the estimate (6.15) is fulfilled and the limit relation (6.4) holds. The rest of the proof follows the pattern of the proof of Lemma 3.4.

Corollary 6.7. *Under the conditions of Theorems 6.4 - 6.6 for all $t \in \mathscr{T}^1$.*

$$\lim_{t\to\infty} \|x(t) - Sx(t)\| = 0. \tag{6.16}$$

Moreover, $\lim_{t\to\infty} \|x(t) - S(t)x(t)\| = 0.$

Proof. It is known (see [6, 27]) that in a uniformly convex Banach space B the modulus of convexity $\delta_B(\varepsilon)$ is well defined on the interval [0, 2], continuous, increasing (not strictly in the general case), and $\delta_B(0) = 0$ and $0 < \delta_B(\varepsilon) < 1$ if $0 < \varepsilon < 2$. These properties prove (6.16). Furthermore, using (6.2) or (6.13) we can write

$$\begin{aligned}
\|x(t) - S(t)x(t)\| &\leq \|x(t) - Sx(t)\| + \|S(t)x(t) - Sx(t)\| \\
&\leq \|x(t) - Sx(t)\| + h(t)\eta(\bar{C}) + g(t).
\end{aligned}$$

The result follows from (1.11) because $\lim_{t\to\infty}(h(t) + g(t)) = 0$.

Remark 6.8. If there exists strictly increasing function $\tilde{\delta}_B(\varepsilon)$ such that the modulus of convexity $\delta_B(\varepsilon) \geq \tilde{\delta}_B(\varepsilon)$ on the interval [0,2], then instead of the estimate (6.5) one has

$$\|x(t) - Sx(t)\| \leq 4R\tilde{\delta}_B^{-1}\Big(\frac{1}{\mathscr{D}(t)}\Big) \quad \forall t \in \mathscr{T}^1,$$

where $\tilde{\delta}_B^{-1}(.)$ is the inverse function to $\tilde{\delta}_B(\varepsilon)$. For example, in the spaces $B = l^p$ and $B = L^p$, $1 < p < \infty$, any $\delta_B(\varepsilon)$ has such $\tilde{\delta}_B(\varepsilon)$ (see [16], p.48). By analogy, one can consider (6.15).

Of course, it is not expected that the statements of this Section remain valid in the general case (1.13).

Acknowledgements
The author thanks the referee for his painstaking reading of this paper.

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