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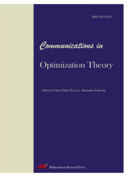

OCTONIONS, GAME EXTENSIONS, AND THE THREE-PLAYER GAME OF FIRMS

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Dedicated to the Memory of Prof. Ram U. Verma

Abstract. We present an octonionic representation of the payoff function for the three-player, two-strategy Game of Firms. The octonionic representation provides a computational capability that we exploit to analyze and identify the potential Nash equilibria of the Game of Firms and its extensions.

Keywords. Octonions; Game extensions; Payoff function; Nash equilibria.

2020 Mathematics Subject Classification. 91A06.

1. INTRODUCTION

Game Theory concerns the mathematics of strategic decision making. It provides a general framework within which both cooperation and competition among independent agents, or *players*, may be modeled and gives powerful tools for analyzing these models. A game is said to be a *non-cooperative* game if each player involved pursues his or her own interests which are partially conflicting with others'. Non-cooperative games are important tools which are widely used in economics, social sciences, political sciences, computer science, biology, and in other fields. The central problem in game theory is the identification of some special strategic choices for the players.

This work is an extension of a previously published paper [1] which examined the Two-Player, Two-Strategy Game of Firms described in [2]. In that paper, we considered two firms, Firm 1 and Firm 2, that have one open position each for which they are offering salaries $2a$ and $2b$, respectively. Two players (job seekers) are competing for these two positions. Each player can apply to only one of the positions and the players must simultaneously decide whether to apply to Firm 1 or to Firm 2 by shouting F_1 or F_2, respectively. If only one of the players applies for a position, then he or she gets hired. If both players apply for the same position, then the concerned firm hires one of them at random, each player being equally likely to be selected. This final strategic form is a 2×2 non zero-sum game with payoff function given

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by the bimatrix (1.1), where we denote Player I's strategy space by $\{F_1,\ F_2\}$ and Player II's strategy space by $\{F_1,\ F_2\}$.

$$\begin{array}{c} \\ F_1 \\ F_2 \end{array} \begin{array}{c} \begin{array}{cc} F_1 & F_2 \end{array} \\ \begin{pmatrix} (a,a) & (2a,2b) \\ (2b,2a) & (b,b) \end{pmatrix} \end{array}, \tag{1.1}$$

The analysis of this game depended on the values and relationship of and between the parameters a and b. To make sure that the two salaries are not too far out of line with each other, we imposed the constraints $2a > b > 0$ and $2b > a > 0$. In [1], we examined this game and its extensions. In particular, the use of unit quaternions proved to be a powerful tool in the identification of families of Nash equilibria in the extended games.

In this work, we consider a three-player, two-strategy version of this game. As before, two firms, Firm 1 and Firm 2, have each one open position for which they are offering salaries $6a$ and $6b$, respectively. This time, three players are competing for these two open positions. The reason for using salaries $6a$ and $6b$ is just to keep the calculations simple. We still impose the constraints $2a > b > 0$ and $2b > a > 0$. With the same set-up as in the two-player case, we obtain the final strategic form of a 3×2 non zero-sum game with payoff function given by Fig. 1, where we denote Player I's strategy space by $\{F_1,\ F_2\}$, Player II's strategy space by $\{F_1,\ F_2\}$, and Player III's strategy space by $\{F_1,\ F_2\}$.

		II	
		F_1	F_2
I	F_1	$(2a,2a,2a)$	$(3a,6b,3a)$
	F_2	$(6b,3a,3a)$	$(3b,3b,6a)$

Player III chooses F_1

		II	
		F_1	F_2
I	F_1	$(3a,3a,6b)$	$(6a,3b,3b)$
	F_2	$(3b,6a,3b)$	$(2b,2b,2b)$

Player III chooses F_2

FIGURE 1. Three-Player, Two-Strategy Game of Firms Model

In this game each player has access to exactly two *pure strategies*, namely F_1 (apply to Firm 1's position) and F_2 (apply to Firm 2's position). We call the sets $S_1 = \{F_1, F_2\}$, $S_2 = \{F_1, F_2\}$, and $S_3 = \{F_1, F_2\}$ the *pure strategy spaces* of Player I, Player II, and Player III, respectively. Players I, II, and III move simultaneously and select pure strategies F_i, F_j, and F_k, respectively, resulting in a strategic profile (F_i, F_j, F_k) from which Player l obtains outcome or payoff $P_l(F_i, F_j, F_k)$. For example, the use of the strategy profile (F_1, F_2, F_2) yields a payoff of $P_3(F_1, F_2, F_2) = 3b$ to Player III. The function $P_l : S_1 \times S_2 \times S_3 \longrightarrow \mathbb{R}$ is called the *payoff function* of Player l. Hence, a three-player, two-strategy, or a 3×2, *pure classical game* G is completely specified by the tuple $G = (S_1, S_2, S_3, P_1, P_2, P_3)$.

A fundamental goal in game theory is the identification of some special strategies or strategic profiles. For example, most players would love to identify a strategy that guarantees a maximal utility. As this is not always possible, a *security strategy*, that is, a strategic choice that guarantees an explicit lower bound to the utility received, is also sought. However, given a fixed strategy profile $(t^\star, u^\star) \in S_2 \times S_3$ of the opponents, Player I seeks a *best reply* strategy, that is, $s^\star \in S_1$ that delivers a utility as great as, if not greater, than any other strategy $s \in S_1$. In symbols

$$P_1(s^\star, t^\star, u^\star) \geq P_1(s, t^\star, u^\star) \quad \forall s \in S_1 \tag{1.2}$$

The situation when each player in the game has chosen such a strategy is of fundamental importance in the theory of games and gives rise to the concept of Nash equilibrium. A *Nash equilibrium*, or a *solution*, or just an *equilibrium* for G is a strategy profile $(x_1, x_2, x_3) \in S_1 \times S_2 \times S_3$ such that each x_i is a best reply to the pair (x_j, x_k) of opponents' strategies. In symbols, (x_1, x_2, x_3) is a Nash equilibrium if

$$P_1(x_1, x_2, x_3) \geq P_1(s, x_2, x_3) \quad \forall s \in S_1 \tag{1.3}$$

and

$$P_2(x_1, x_2, x_3) \geq P_2(x_1, t, x_3) \quad \forall t \in S_2. \tag{1.4}$$

and

$$P_3(x_1, x_2, x_3) \geq P_3(x_1, x_2, u) \quad \forall u \in S_3. \tag{1.5}$$

Other ways of expressing this concept include the observation that no player can increase his or her payoff by unilaterally deviating from his or her equilibrium strategy or that at the equilibrium a player's opponents are indifferent to that player's strategic choice.

As an example, consider the three-player dilemma game examined by Benjamin and Hayden [3], a type of a non-zero-sum game in which three players may each "cooperate" (C) with or "defect" (D), that is, betray the other players. This is a non-cooperative game and, therefore, the players are assumed to be *rational*, that is, the only concern of each individual player in the game is to optimize his or her own payoff without any concern for the other players' payoffs. The payoff function for the classical dilemma game considered is given in Fig. 2, where all three players use the same two element pure strategy space denoted by $\{C, D\}$.

		II	
		C	D
I	C	$(0,0,0)$	$(-9,1,9)$
	D	$(1,-9,-9)$	$(9,9,1)$

Player III chooses C

		II	
		C	D
I	C	$(-9,-9,1)$	$(1,9,9)$
	D	$(9,1,9)$	$(2,2,2)$

Player III chooses D

FIGURE 2. Three-Player Dilemma Game

Here, we note that for Player I the pure strategy D always delivers a higher outcome than the strategy C (say D *strongly dominates* C) and, similarly, for Players II and III the pure strategy D strongly dominates C. Hence the triple (D, D, D) is the only Nash equilibrium of the three-player dilemma game. Note that if all three players cooperate with each other, i.e. stay silent and do not betray each other by employing the strategy profile (C, C, C), then they receive the

Pareto optimal payoff $(0,0,0)$, that is, the game result from which no player can improve their payoff by deviating without another player being worse off.

Given a game G, a Nash equilibrium may not exist among the pure strategy profiles. As an example, consider the game of Simplified Poker Model, a 3×2 zero-sum game whose payoff function is given in Fig. 3.

		II	
		t_1	t_2
I	s_1	$(5,-10,5)$	$(0,0,0)$
	s_2	$(0,0,0)$	$(10,-5,-5)$

Player III chooses u_1

		II	
		t_1	t_2
I	s_1	$(-7,3,4)$	$(0,0,0)$
	s_2	$(0,0,0)$	$(-8,5,3)$

Player III chooses u_2

FIGURE 3. Three-Player Simplified Poker Model

Here one can easily show that there are no triple of pure strategies (x_i,x_j,x_k) such that each pure strategy in the triple is a best reply to the remaining two. Hence, this game has no equilibria in pure strategies.

Note that in the game G, each player has access to a two-point worth strategy space. Suppose that we now extend the game G by enlarging the domain and extending the payoff functions. A standard extension of this point is to consider for each player the set of *mixed strategies*, i.e. the set of real convex combinations of their pure strategies. For example, in the game above, Player I could observe a fair coin and decide to play s_1 if it falls Heads and s_2 if it falls Tails. Suppose that Player I uses his pure strategy s_1 with probability p, Player II uses his pure strategy t_1 with probability q, and Player III uses her pure strategy u_1 with probability r. Each player has now access to the set of mixed strategies. We denote the set of probability distributions over S_i by $\Delta(S_i)$ and define the mixed strategy spaces of the players as follows

$$\Delta(S_1)=\{ps_1+(1-p)s_2 \mid 0\leqslant p\leqslant 1\}\equiv[0,1] \tag{1.6}$$

$$\Delta(S_2)=\{qt_1+(1-q)t_2 \mid 0\leqslant q\leqslant 1\}\equiv[0,1] \tag{1.7}$$

$$\Delta(S_3)=\{ru_1+(1-r)u_2 \mid 0\leqslant r\leqslant 1\}\equiv[0,1] \tag{1.8}$$

Note that we can embed S_i into $\Delta(S_i)$ by considering the element $s\in S_i$ as mapped to the probability distribution which assigns 1 to s and 0 to everything else. Given $(p,q,r)\in\Delta(S_1)\times\Delta(S_2)\times\Delta(S_3)$, Player i obtains an expected outcome given by a probability distribution over the outcomes of G, that is, an element of $\Delta(ImP_i)$, the set of probability distributions over the image of P_i. Now the game G is extended to a new, larger game G^{mix}, the *mixed classical game associated to* G. The mixed classical game G^{mix} is therefore specified by the tuple $G^{mix}=(\Delta(S_1),\Delta(S_2),\Delta(S_3),E_1,E_2,E_3)$, where

$$E_i:[0,1]\times[0,1]\times[0,1]\longrightarrow\Delta(ImP_i) \tag{1.9}$$

is Player i's expected payoff function. More specifically

$$\begin{aligned}E_i(p,q,r) =& pqrP_i(s_1,t_1,u_1)+pq(1-r)P_i(s_1,t_1,u_2)+p(1-q)rP_i(s_1,t_2,u_1)\\ &+p(1-q)(1-r)P_i(s_1,t_2,u_2)+(1-p)qrP_i(s_2,t_1,u_1)\\ &+(1-p)q(1-r)P_i(s_2,t_1,u_2)+(1-p)(1-q)rP_i(s_2,t_2,u_1)\\ &+(1-p)(1-q)(1-r)P_i(s_2,t_2,u_2)\end{aligned}$$

or in matrix form

$$E_i(p,q,r)=\left(\begin{pmatrix}p & 1-p\end{pmatrix}M_1\begin{pmatrix}q\\ 1-q\end{pmatrix} \quad \begin{pmatrix}p & 1-p\end{pmatrix}M_2\begin{pmatrix}q\\ 1-q\end{pmatrix}\right)\begin{pmatrix}r\\ 1-r\end{pmatrix} \tag{1.10}$$

where

$$M_1=\begin{pmatrix}P_i(s_1,t_1,u_1) & P_i(s_1,t_2,u_1)\\ P_i(s_2,t_1,u_1) & P_i(s_2,t_2,u_1)\end{pmatrix} \quad \text{and} \quad M_2=\begin{pmatrix}P_i(s_1,t_1,u_2) & P_i(s_1,t_2,u_2)\\ P_i(s_2,t_1,u_2) & P_i(s_2,t_2,u_2)\end{pmatrix} \tag{1.11}$$

A triple of mixed strategies $(p^\star,q^\star,r^\star)$ is a Nash equilibrium in G^{mix} if

$$E_1(p^\star,q^\star,r^\star)\geq E_1(p,q^\star,r^\star) \quad \forall p\in[0,1] \tag{1.12}$$

and

$$E_2(p^\star,q^\star,r^\star)\geq E_2(p^\star,q,r^\star) \quad \forall q\in[0,1]. \tag{1.13}$$

and

$$E_3(p^\star,q^\star,r^\star)\geq E_3(p^\star,q^\star,r) \quad \forall r\in[0.1]. \tag{1.14}$$

While a Nash equilibrium may not exist in the classical pure game G, Nash's famous theorem [4] says that if the S_i's are all finite, then there always exists at least one equilibrium in G^{mix}. For example, the classical mixed game G^{mix} associated to the game depicted in Fig. 3 has two Nash equilibria $(p,q,r)=(0.894,0.488,0.204)$ and $(p,q,r)=(0.160,0.977,0.588)$ with expected payoffs to the players $(-2.221,0.315,1.906)$ and $(0.060,-0.743,0.683)$, respectively.

In the game G^{mix}, each player has access to a unit interval worth of strategies. As before, suppose Player I plays his first pure strategy with probability p, say, Player II plays his first strategy with probability q, say, and Player III plays her first pure strategy with probability r, say, then the resulting probability distribution over the outcomes of G is given in Table 1.

	t_1	t_2
s_1	pqr	$p(1-q)r$
s_2	$(1-p)qr$	$(1-p)(1-q)r$

(A) Player III uses u_1

	t_1	t_2
s_1	$pq(1-r)$	$p(1-q)(1-r)$
s_2	$(1-p)q(1-r)$	$(1-p)(1-q)(1-r)$

(B) Player III uses u_2

TABLE 1. Probability Distribution over ImP_i

We note that the maps E_i are not onto and therefore some probability distributions over the outcomes of P_i will be impossible to establish. For example, an easy exercise shows that the element of $\Delta(ImP_i)$ represented in Table 2 is not realizable by any choice of p, q, and r.

This observation motivates the search for a higher randomization apparatus. Since the game G has eight outcomes, we will utilize the eight-dimensional real division algebra of the octonions to establish a suitable extension of G^{mix}.

	t_1	t_2
s_1	1/2	0
s_2	0	0

(A) Player III uses u_1

	t_1	t_2
s_1	0	0
s_2	0	1/2

(B) Player III uses u_2

TABLE 2. An Element of $\Delta(ImP_i)$

2. REAL DIVISION ALGEBRAS OF THE QUATERNIONS AND OCTONIONS

There are exactly four real division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. The real numbers form a complete ordered field. The complex numbers are algebraically complete but not ordered. The quaternions are not commutative, and the octonions are both non-commutative and non-associative.

2.1. **Quaternions.**

The quaternions, denoted by $\mathbb{H}$, are a 4-dimensional normed division algebra over the real numbers. They are spanned by the identity 1 and three imaginary units i, j, and k. These fundamental units satisfy the so-called *Hamilton's relation*

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2.1}$$

A general quaternion q is of the form

$$q = a + bi + cj + dk, \tag{2.2}$$

where a, b, c, d are real numbers and i, j, and k satisfy Hamilton's relation. We can also express a general quaternion in the form

$$q = \alpha + \beta j, \tag{2.3}$$

where α and β are complex numbers. Throughout, we will work with the general quaternions $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_2 k$.

The sum of two quaternions is a new quaternion and we have

Definition 2.1. Addition with quaternions is component wise, that is,

$$p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (q_3 + q_3)k. \tag{2.4}$$

The product of two quaternions results in a new quaternion.

Definition 2.2. Multiplication with quaternions is polynomial subject to Hamilton's relation $i^2 = j^2 = k^2 = ijk = -1$, that is, for p and q given as above, we have

$$\begin{aligned} pq &= (p_0 + p_1 i + p_2 j + p_3 k)(q_0 + q_1 i + q_2 j + q_3 k) \\ &= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) + (p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2)i \\ &\quad + (pa_0 q_2 - p_1 q_3 + p_2 q_0 + p_3 q_1)j + (p_0 q_3 + p_1 q_2 - p_2 q_1 + p_3 q_0)k \end{aligned} \tag{2.5}$$

If $p = \alpha + \beta j$ and $q = \delta + \gamma j$, where α, β, δ, and γ, are complex numbers, then an alternative definition for the product of p and q is given by the map

$$((\alpha, \beta), (\delta, \gamma)) \longmapsto \left(\alpha\delta - \beta\overline{\gamma}, \alpha\gamma + \beta\overline{\delta}\right).$$

The length of a quaternion can be calculated the way we compute length in $\mathbb{R}^4$.

Definition 2.3. The *conjugate* of a quaternion p is defined as

$$\overline{p} = p_0 - p_1 i - p_2 j - p_3 k. \tag{2.6}$$

It is straightforward to verify the following properties.

1. The product $\overline{p}p = p_0^2 + p_1^2 + p_2^2 + p_3^2$ defines the square of a norm $||p||$ for the quaternion p. That is,
$$||p||^2 = \overline{p}p = p_0^2 + p_1^2 + p_2^2 + p_3^2. \tag{2.7}$$
2. The norm is multiplicative, that is, $||pq|| = ||p||||q||$ for all quaternions p and q.
3. For any nonzero quaternion q,
$$q^{-1} = \frac{\overline{q}}{||q||^2}. \tag{2.8}$$
This establishes $\mathbb{H} - \{0\}$ as a division algebra.
4. The set of unit quaternions $\mathbb{H}_1 = \{q \mid ||q||^2 = 1\}$ forms a subgroup of $\mathbb{H} - \{0\}$ under quaternionic multiplication and can be thought as the unit 3-sphere $\mathbb{S}^3$ living in $\mathbb{R}^4$.
5. Multiplication with quaternions is not commutative.
6. Multiplication with quaternions is associative.
7. The distributive laws hold.

In light of all the above properties, the quaternions form a *skew-field*, that is, a non-commutative field. In addition, as a real vector space, $\mathbb{H}$ can be identified with $\mathbb{R}^4$ via the map

$$(a_0 + a_1 i + a_2 j + a_3 k) \longmapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{2.9}$$

For more details on real division algebras in general and on quaternions in particular, the reader is referred to [5, 6].

2.2. **Octonions.** The octonions $\mathbb{O}$ are a non-associative, non-commutative, 8-dimensional, normed division algebra over the real numbers. One can derive the octonions from the set of quaternions $\mathbb{H}$ the way we obtain the set of complex numbers $\mathbb{C}$ from $\mathbb{R}^2$ as $\mathbb{C} = \mathbb{R} + \mathbb{R}i$, or the way we obtain $\mathbb{H}$ from $\mathbb{C}^2$ as $\mathbb{H} = \mathbb{C} + \mathbb{C}j$. They are spanned by the real number 1 and seven basic square roots of -1 that we denote by i_1, i_2, i_3, i_4, i_5, i_6, and i_7. A general octonion o has form

$$o = \sum_{j=0}^{7} a_j i_j \tag{2.10}$$

where the a_t's are real numbers, $i_0 = 1$ by convention, and the i_t's have the property that $i_t^2 = -1$ for all positive indices. Addition with octonions is component-wise exactly the way we add vectors in $\mathbb{R}^8$. Given any two distinct basic square roots of -1, i_r and i_s, say, there is a third i_t, so that these distinct three basic square roots of -1 satisfy

$$i_r^2 = i_s^2 = i_t^2 = i_r i_s i_t = -1 \tag{2.11}$$

Thus any pair of distinct basic square roots of -1 determines a quaternionic subspace. Up to order, there are exactly seven such choices. Therefore, there are seven "natural" quaternionic subspaces all together. Any such quaternionic subspaces intersect in a common copy of the complex numbers. Now consider the seven basic square roots of -1 as "points" and the seven quaternionic subspaces as “lines” these points are incident to. Thus the octonionic algebra satisfies the following two axioms of projective geometry:

1. Two points determine a line.
2. Two lines determine a point.

Not surprisingly, octonionic multiplication of the seven basic square roots of -1 is modeled along the 7 points, 7 lines projective plane shown in Fig. 4. We use the Fano plane to perform octonionic multiplication of the basic octonionic elements. In particular, multiplication with octonions is polynomial subject to $i_r^2 = i_s^2 = i_t^2 = i_r i_s i_t = -1$ if i_r, i_s, and i_t are cyclically ordered as shown in the Fano plane. One can also easily verify that octonionic multiplication is non-associative by considering, for example, the associated products $i_3(i_5 i_7)$ and $(i_3 i_5)i_7$. On one hand $i_3(i_5 i_7) = i_3 i_4 = i_6$. On the other hand, $(i_3 i_5)i_7 = i_2 i_7 = -i_6$. In general, associativity of basic octonionic elements fail up to a sign. Note that although octonionic multiplication is not associative, it is nevertheless *alternative*, that is, $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathbb{O}$. The octonionic *conjugate* of an octonion o is defined as $\overline{o} = a_0 - \sum_{j=1}^{7} a_j i_j$. The product $o\overline{o} = \sum_{j=0}^{7} a_j^2$ defines the square of the real-valued multiplicative norm $N(o)$ of the octonion o. More specifically, $N(o)^2 = o\overline{o} = \sum_{j=0}^{7} a_j^2$. Each non-zero octonion o possesses a non-zero inverse given by $o^{-1} = \overline{o}/[N(o)]^2$. A unit octonion has length 1 and the set of all unit octonions $\{o \mid N(o) = 1\}$ can be thought as the 7-sphere $\mathbb{S}^7$ living in $\mathbb{R}^8$. For any octonion o, we will denote by $\pi_k(o)$ the projection of the octonion o onto the subspace of $\mathbb{O}$ spanned by the vector basis element i_k with the convention that $i_0 = 1$. For example, $\pi_3(2 - 3i_1 + 5i_3 - 2i_7) = 5$. For us the octonions will be of great use due to the existence within a collection of 3-spheres that intersect in a common circle. Further these three spheres will be identified with the special unitary matrices $SU(2)$ and the common circle with the unitary matrices $U(1)$, all respecting octonionic multiplication. Our collection of 3-spheres is then given by the three copies of unit quaternions, meeting in a common copy of the unit complexes. Among the seven copies of quaternionic subspaces of $\mathbb{O}$, we are interested in three copies with a common embedded copy

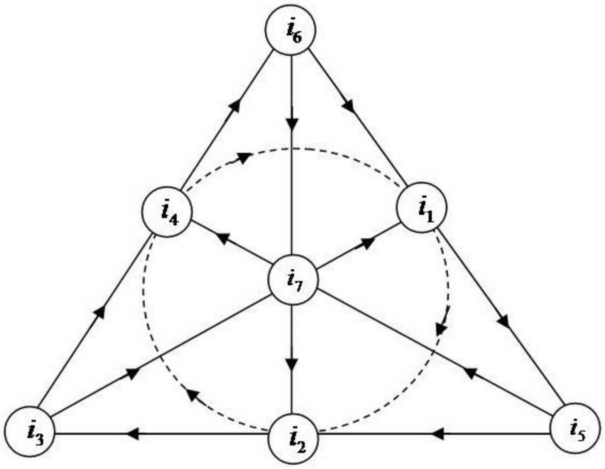

FIGURE 4. An edge oriented Fano plane.

of the complex numbers $\mathbb{C}$. For this we choose the quaternionic subspaces

$$\mathbb{H}_1 = \{a_0 + a_1 i_1 + a_2 i_2 + a_3 i_4 \mid a_j \in \mathbb{R}\} \tag{2.12}$$

$$\mathbb{H}_2 = \{b_0 + b_1 i_1 + b_2 i_5 + b_3 i_6 \mid b_j \in \mathbb{R}\} \tag{2.13}$$

$$\mathbb{H}_3 = \{c_0 + c_1 i_1 + c_2 i_3 + c_3 i_7 \mid c_j \in \mathbb{R}\} \tag{2.14}$$

which meet in the complex space $\{\alpha + \beta i_1 \mid \alpha, \beta \in \mathbb{R}\}$. We focus our attention on the unit $\mathbb{S}^3$'s in each of these four-dimensional copies of $\mathbb{H}$ and consider each such $\mathbb{S}^3$ as a "longitude" of the unit octonions which form a seven-dimensional sphere $\mathbb{S}^7 \subset \mathbb{O}$. Call $\mathbb{U}_j$ the set of unit octonions in $\mathbb{H}_j$.

3. UNIT OCTONIONS AS STRATEGIES

Consider a generic three-player, two-strategy game G whose payoff function is indicated in Fig. 5, where $(x_t, y_t, z_t) \in \mathbb{R}^3$ for all $t = 0, 1, 2, \cdots, 7$.

II

		t_1	t_2
I	s_1	(x_0, y_0, z_0)	(x_6, y_6, z_6)
	s_2	(x_4, y_4, z_4)	(x_3, y_3, z_3)

Player III chooses u_1

II

		t_1	t_2
I	s_1	(x_7, y_7, z_7)	(x_2, y_2, z_2)
	s_2	(x_5, y_5, z_5)	(x_1, y_1, z_1)

Player III chooses u_2

FIGURE 5. Three-Player Generic Game

Theorem 3.1. *Let G be the game described in Fig. 5. Then the associated pure quantum game G^Q is the three-player game in which Player i's strategy space is $\mathbb{U}_i$ and the payoff function for Player i is given by*

$$P_i^Q(s,t,u) = \sum_{j=0}^{7} [\pi_j(o(s,t,u))]^2 w_j \tag{3.1}$$

for some unit octonion o function of the unit octonions s, t, and u. Here $w = x$ when $i = 1$, $w = y$ when $i = 2$, and $w = z$ when $i = 3$. Therefore, the game G^Q is completely specified by the tuple $G^Q = (\mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3, P_1^Q, P_2^Q, P_3^Q)$.

The proofs of Theorem 3.1 and its corollary below can be found in [7, 8].

Corollary 3.2. *In the game G^Q if two of the players use pure quantum strategies which are represented by canonical octonionic basis elements and the third player employs a pure quantum strategy represented by a unit octonion, that is, if the players employ a pure quantum strategy profile (s,t,u) of the form*

$$(s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4, i_l, i_m) \tag{3.2}$$

$$(i_k, t_0 + t_1 i_1 + t_2 i_5 + t_3 i_6, i_m) \tag{3.3}$$

$$(i_k, i_l, u_0 + u_1 i_1 + u_2 i_3 + u_3 i_7) \tag{3.4}$$

where $k \in \{0,1,2,4\}$, $l \in \{0,1,5,6\}$, *and* $m \in \{0,1,3,7\}$, *then the payoff to Player i is given by*

$$P_i^Q(s,t,u) = \sum_{j=0}^{7} [\pi_j((st)u)]^2 w_j \tag{3.5}$$

The following lemma is a straightforward result from Theorem 3.1 and its proof is omitted.

Lemma 3.3. *In the game* G^Q *if the players use a quantum strategy profile represented by a triple of unit octonions* (s,t,u) *of the form* $(\alpha_0+\alpha_1 i_4, \beta_0+\beta_1 i_6, \gamma_0+\gamma_1 i_7)$ *or* $(\alpha_0 i_2+\alpha_1 i_4, \beta_0 i_5+\beta_1 i_6, \gamma_0 i_3+\gamma_1 i_7)$, *then the payoff to Player i is given by* $P_i^Q(s,t,u) = \sum_{j=0}^{7}[\pi_j((st)u)]^2 w_j$.

Next we show that the game G^Q is an extension of the game G.

Theorem 3.4. *The game* G^Q *is an extension of the game* G^{mix} *and, therefore, also an extension of the pure classical game G.*

Proof. To show that G^Q is an extension of G^{mix} it is sufficient to find a triple of three unit octonions $(s,t,u) \in \mathbb{U}_1 \times \mathbb{U}_2 \times \mathbb{U}_3$ such that $P_i^Q(s,t,u) = E_i(p,q,r)$ for all $i = 1$, 2, and 3 where $(p,q,r) \in [0,1] \times [0,1] \times [0,1]$. Let $s = \sqrt{p} + \sqrt{1-p}\, i_4$, $t = \sqrt{q} + \sqrt{1-q}\, i_6$, and $u = \sqrt{r} + \sqrt{1-r}\, i_7$. Then,

$$\begin{aligned}(st)u = \sqrt{pqr} &+ \sqrt{(1-p)(1-q)(1-r)}\, i_1 + \sqrt{p(1-q)(1-r)}\, i_2 + \sqrt{(1-p)(1-q)r}\, i_3 \\ &+ \sqrt{(1-p)qr}\, i_4 - \sqrt{(1-p)q(1-r)}\, i_5 + \sqrt{p(1-q)r}\, i_6 + \sqrt{pq(1-r)}\, i_7.\end{aligned}$$

Therefore

$$\begin{aligned}P_i^Q(s,t,u) &= \sum_{j=0}^{7} [\pi_j((st)u)]^2 w_j \\ &= pqrw_0 + (1-p)(1-q)(1-r)w_1 + p(1-q)(1-r)w_2 + (1-p)(1-q)rw_3 \\ &+ (1-p)qrw_4 + (1-p)q(1-r)w_5 + p(1-q)rw_6 + pq(1-r)w_7 \\ &= E_i(p,q,r),\end{aligned}$$

where $w = x$ if $i = 1$, $w = y$ if $i = 2$, and $w = z$ if $i = 3$. □

We say that two strategy triples represented by the unit octonions $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_2, \mu_2, \nu_2)$ are *equivalent* and write $(\lambda_1, \mu_1, \nu_1) \sim (\lambda_2, \mu_2, \nu_2)$ if the resulting outcomes from the use of both triples are equal, that is, $P_i^Q(\lambda_1, \mu_1, \nu_1) = P_i^Q(\lambda_2, \mu_2, \nu_2)$ for all $i = 1$, 2, and 3. We note that the strategy triples $(\pm 1, \pm 1, \pm 1), (\pm 1, \pm i_1, \pm i_1), (\pm i_1, \pm 1, \pm i_1), (\pm i_1, \pm i_1, \pm 1)$, $(\pm i_2, \pm i_5, \pm i_3)$, $(\pm i_2, \pm i_6, \pm i_7), (\pm i_4, \pm i_5, \pm i_7)$, and $(\pm i_4, \pm i_6, \pm i_3)$ are all equivalent and yield the payoffs (x_0, y_0, z_0) to Player I, II, and III, respectively. Therefore, there are 64 triples of unit octonions that are equivalent to the pure strategy profile (s_1, t_1, u_1). Similarly, we have 64 equivalent strategy triples to each of the 7 remaining triples of classical pure strategies (s_i, t_j, u_k). This means that in the game G^Q the players have 64 ways of playing each of the pure strategy profiles of G. The extended game G^Q will be of great interest to us due to the existence within of these equivalence classes of strategy profiles.

In the game G^Q, each player has access to a unit sphere worth of strategic space. With the use of unit octonions as strategies, now we can establish any probability distribution over the outcomes of the game G. For example, the element of $\Delta(ImP_i)$ depicted in Table 2 can be now realized by selecting the triplet $(\sqrt{2}/2 + \sqrt{2}/2 i_1, 1, 1)$ as $P_i^Q(\sqrt{2}/2 + \sqrt{2}/2 i_1, 1, 1) =$

$\sum_{j=0}^{7}[\pi_j((\sqrt{2}/2+\sqrt{2}/2i_1,1,1)(1))1)]^2 w_j = (1/2)w_0 + (1/2)w_1$. Note that there are many other ways one can establish the element of $\Delta(ImP_i)$ shown in Table 2.

It is worth noting here that nothing prohibits us from having the game G^Q play the role of G in the classical situation and by considering the probability distributions over $\mathbb{U}_j$, the set of unit octonions in $\mathbb{H}_j$, creating yet a larger game G^{mQ}, the *mixed quantum game associated to G*. Hence $G^{mQ} = (\Delta(\mathbb{U}_1),\Delta(\mathbb{U}_2),\Delta(\mathbb{U}_3),E_1^Q,E_2^Q,E_3^Q)$, where

$$E_i^Q : \Delta(\mathbb{U}_1)\times\Delta(\mathbb{U}_2)\times\Delta(\mathbb{U}_3) \longrightarrow \Delta(ImP_i)$$

is Player i's quantum expected payoff. If Player I, Player II, and Player III employ the probability distributions λ, μ, and ν over $\mathbb{U}_1$, $\mathbb{U}_2$, and $\mathbb{U}_3$, respectively, then Player i's expected payoff is given by

$$E_i^Q(\lambda,\mu,\nu) = \int_{\mathbb{U}_1\times\mathbb{U}_2\times\mathbb{U}_3} P_i^Q(s,t,u)d\lambda(s)d\mu(t)d\nu(u)$$

The space of probability distributions over the set of $\mathbb{U}_j$ is huge and complex. For simplicity, we will restrict our work to probability distributions supported on the four points spanning $\mathbb{U}_j$, that is, real convex combinations of 1, i_1, i_2, and i_4 for Player I, real convex combinations of 1, i_1, i_5, and i_6 for Player II, and real convex combinations of 1, i_1, i_3, and i_7 for Player III. For more details on game extensions, the reader is referred to [9].

4. APPLICATION

As an application of the theory discussed above, we consider the three-player, two-strategy game of Firms described in Fig.1. As opposed to the case of the two-player game, the three-player game requires that we establish an order between the parameters a and b when one undertakes the important task of identifying the Nash equilibria of the game G and its extensions G^{mix}, G^Q, and G^{mQ}. To see why we are required to consider various cases depending on the order between a and b, let us look at the following plausible scenario in the game G. Referring to Fig.1, if Player II and Player III use the pure strategies F_2 and F_1, respectively, then Player I can respond with pure strategy F_1 and will receive a payoff of $3a$ or he can respond with F_2 and will receive a payoff of $3b$. However, as there is no a definite order between a and b, Player I cannot determine which of the two outcomes $3a$ or $3b$ is maximal, and hence, will fail to come up with a best reply strategy. Thus this lack of order between a and b will cause a huge technical glitch in the game. The work below seeks to remedy this problem.

4.1. Firm 1 pays better than Firm 2.

In this section we analyze the case where Firm 1 pays better than Firm 2, that is, $6a > 6b$ or $a > b$. To further simplify the calculations, and without loss of generality, we select $a = 4$ and $b = 3$. We obtain the strategic form of the game with payoff function given in Fig.6. Note that this choice of a and b respects the prescribed conditions on a and b in the game, that is, $2a > b$ and $2b > a$.

4.1.1. *Nash Equilibria in G.*

A classical analysis of this game yields the following result.

Proposition 4.1. *The game G admits three Nash equilibria, namely the pure strategic triplets* (F_1,F_1,F_2), (F_1,F_2,F_1) *and* (F_2,F_1,F_1). *These equilibria pay out* $(12,12,18)$, $(12,18,12)$, *and* $(18,12,12)$ *to the players, respectively.*

		II	
		F_1	F_2
I	F_1	$(8,8,8)$	$(12,18,12)$
	F_2	$(18,12,12)$	$(9,9,24)$

Player III chooses F_1

		II	
		F_1	F_2
I	F_1	$(12,12,18)$	$(24,9,9)$
	F_2	$(9,24,9)$	$(6,6,6)$

Player III chooses F_2

FIGURE 6. Case 1: Firm 1 pays better than Firm 2 ($a > b$)

Proof. We will prove that the first pure strategic triplet is a Nash equilibrium. The proof that the remaining two are Nash equilibria is left as an exercise for the reader. Suppose that Player 1 and Player II use pure strategy F_1. If Player III responds with F_1, then she receives the amount 8. If she responds with F_2, then she receives the amount 18. Since $18 > 8$ and Player III is assumed to be a rational player, she will respond with her best reply F_2. Now, suppose Player I and Player III employ the pure strategies F_1 and F_2, respectively. Then, Player II can respond with F_1 and receives a payoff of 12 or he can respond with F_2 and receives a payoff of 9. Therefore, Player II's best reply will be F_1 for $12 > 9$. Finally, suppose that Player II and Player III stick with their equilibrium strategies. If Player I responds with F_1, he receives a payoff of 12. If he responds with F_2, he receives a payoff of 9. Therefore, Player I's best reply will be F_1 for $12 > 9$. This completes the proof that the strategic pair (F_1,F_1,F_2) is a Nash equilibrium. ☐

We note that in each of these three equilibria, two of the three players compete for the job position at Firm 1 while the third player is happy to seek the job position at Firm 2. This behavior is due to the fact that Firm 1 pays better than Firm 2.

4.1.2. *Nash Equilibria in G^{mix}.* If the players employ probability distributions over their pure classical strategies, then there is one Nash equilibrium in mixed classical strategies. More specifically, we have the following result.

Proposition 4.2. *The game G^{mix} admits the Nash equilibrium (p,q,r), where $p = q = r = \frac{15-3\sqrt{21}}{2} \approx 0.626$. This mixed strategy equilibrium pays out* 12.110 *to each player.*

Proof. Suppose that Player I uses his pure strategy F_1 with probability p, Player II uses his pure strategy F_1 with probability q, and Player III uses her pure strategy F_1 with probability r. Then Player I's expected payoff function is given by

$$E_1(p,q,r) = \left((p \quad 1-p)A_1 \begin{pmatrix} q \\ 1-q \end{pmatrix} \quad (p \quad 1-p)A_2 \begin{pmatrix} q \\ 1-q \end{pmatrix} \right) \begin{pmatrix} r \\ 1-r \end{pmatrix} \tag{4.1}$$

where

$$A_1 = \begin{pmatrix} 8 & 12 \\ 18 & 9 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 12 & 24 \\ 9 & 6 \end{pmatrix} \tag{4.2}$$

or

$$E_1(p,q,r) = p[8qr - 12q - 12r + 24] + (1-p)[6qr + 3q + 3r + 6]$$

In a similar manner, we find that Player II and Player III's expected payoff functions are given by

$$E_2(p,q,r) = q[8pr - 12p - 12r + 24] + (1-q)[6pr + 3p + 3r + 6]$$

and

$$E_3(p,q,r) = r[8pq - 12p - 12q + 24] + (1-r)[6pq + 3p + 3q + 6],$$

respectively. Then, the triple $(p,q,r) \in [0,1]^3$ is a Nash equilibrium in G^{mix} if

$$\begin{cases} \frac{\partial E_1}{\partial p} & = 0 \\ \frac{\partial E_2}{\partial q} & = 0 \\ \frac{\partial E_3}{\partial r} & = 0 \end{cases} \tag{4.3}$$

We obtain the unique solution $r = p = q = \frac{15-3\sqrt{21}}{2} \approx .626$. Finally, this mixed Nash equilibrium pays out $E_1(.626,.626,.626) = E_2(.626,.626,.626) = E_3(.626,.626,.626) \approx 12.110$ to each player. □

We note that at this mixed equilibrium each player selects about 63% of the time pure strategy F_1 and about 37% of the time pure strategy F_2. Again, the position with the higher salary is more attractive to the players.

4.1.3. *Nash Equilibria in G^Q.* A quantum analysis of the game G^Q yields the following result.

Proposition 4.3. *Let G be the game depicted in Fig.6 and let G^Q be its associated quantum game. Then the quantum strategy profiles represented by the triplets of unit octonions $(\pm 1, \pm i_5, \pm i_1)$ and $(\pm 1, \pm i_5, \pm i_7)$ are Nash equilibria in G^Q. Moreover, these equilibria yield to the players the payoffs (12,18,12) and (18,12,12), respectively.*

Proof. We show that $(1, i_5, i_1)$ is a Nash equilibrium. The proof that the remaining 15 triplets are Nash equilibria is left as an exercise for the reader. For this, take players II and III strategies as given and suppose that Player I responds with the unit octonion $s = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_4$. We obtain the octonionic associated product $(s \cdot 1) \cdot i_1 = -a_0 i_6 + a_1 i_5 + a_2 i_7 + a_3 i_3$. Then Player I's expected payoff is given by

$$P_1^Q(s, i_5, i_1) = 12a_0^2 + 9a_1^2 + 12a_2^2 + 9a_3^2 \tag{4.4}$$

Player I needs to maximize Equation (4.4) subject to the constraint that s must be a unit octonion, that is

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \tag{4.5}$$

Then Player I's best response is to select a unit octonion s such that $a_1 = a_3 = 0$, that is, $s = a_0 + a_2 i_2$ with $a_0^2 + a_2^2 = 1$. In particular, Player I can select $s = 1$. Next, suppose Player I responds with a mixed quantum strategy represented by a probability distribution μ over $\mathbb{U}_1$. Then Player I's expected payoff is given by

$$E_1^Q(\mu, i_5, i_1) = \int_{\mathbb{U}_1} P_1^Q(s, i_5, i_1) d\mu(s) = \int_{\mathbb{U}_1} (12a_0^2 + 9a_1^2 + 12a_2^2 + 9a_3^2) d\mu(s) \tag{4.6}$$

Player I seeks to maximize (4.6) subject to (4.5). Then Player I's best response is to select a μ that assigns a_0 and a_2 positive probabilities and zero to everything else. In particular, Player I can choose $s = 1$. In a similar manner, one can show that Player II's best response to the opponents' strategic profile $(1, i_1)$ is i_5 and Player III's best response to $(1, i_5)$ is i_1. Accordingly, the triplet $(1, i_5, i_1)$ is a Nash equilibrium that pays out $P_1^Q(1, i_5, i_1) = 12$ to Player I, $P_2^Q(1, i_5, i_1) = 18$ to Player II, and $P_3^Q(1, i_5, i_1) = 12$ to Player III. □

Note that the triplets $(\pm 1, \pm i_5, \pm i_1)$ and $(\pm 1, \pm i_5, \pm i_7)$ are equivalent to the pure strategy profiles (F_1, F_2, F_1) and (F_2, F_1, F_1), respectively. In other words, we recovered two of the three equilibria of the pure classical game G. We observe that in the game G^Q, Player III is no longer interested in competing for the job posting of Firm 2.

4.1.4. *Nash Equilibria in G^{mQ}.*

We will only consider probability distributions over the fundamental set of unit octonions for each player. Players I, II, and III can then use real convex combinations of the basis elements $\{1, i_1, i_2, i_4\}$, $\{1, i_1, i_5, i_6\}$, and $\{1, i_1, i_3, i_7\}$, respectively.

Proposition 4.4. *The game G^{mQ} admits a family of Nash equilibria where Player I uses the unit octonion $s = 1$, Player II uses the unit octonion $t = i_5$, and Player III uses any real convex combination of the unit octonions i_1 and i_7. That is, the triplet $(1, i_5, \alpha i_1 + \beta i_7)$, where α and β are in $[0, 1]$ and $\alpha + \beta = 1$, is a Nash equilibrium in G^{mQ}. This equilibrium pays out $(12\alpha + 18\beta, 18\alpha + 12\beta, 12)$ to Players I. II, and III, respectively.*

Proof. Take players II and III strategies as given and suppose that Player I responds with the unit octonion $s = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_4$. Then Player I's expected payoff is given by

$$\begin{aligned} E_1^Q(s, i_5, \alpha i_1 + \beta i_7) &= \alpha P_1^Q(s, i_5, i_1) + \beta \alpha P_1^Q(s, i_5, i_1) \\ &= \alpha[12a_0^2 + 9a_1^2 + 12a_2^2 + 9a_3^2] + \beta[12a_0^2 + 9a_1^2 + 12a_2^2 + 9a_3^2] \\ &= 12a_0^2 + 9a_1^2 + 12a_2^2 + 9a_3^2 \end{aligned} \tag{4.7}$$

Player I needs to maximize Equation (4.7) subject to the constraint that s must be a unit octonion as shown in (4.5). Then Player I's best response is to select a unit octonion s such that $a_1 = a_3 = 0$, that is, $s = a_0 + a_2 i_2$ with $a_0^2 + a_2^2 = 1$. In particular, Player I can select $s = 1$.

Next, take Players I and III strategies as given and suppose that Player II responds with the unit octonion $t = b_0 + b_1 i_1 + b_2 i_5 + b_3 i_6$. Then Player II's expected payoff is given by

$$\begin{aligned} E_2^Q(1, t, \alpha i_1 + \beta i_7) &= \alpha P_2^Q(s, i_5, i_1) + \beta P_2^Q(s, i_5, i_1) \\ &= \alpha[6b_0^2 + 8b_1^2 + 12b_2^2 + 9b_3^2] + \beta[12b_0^2 + 9b_1^2 + 12b_2^2 + 9b_3^2] \\ &= (6\alpha + 12\beta)b_0^2 + (8\alpha + 9\beta)b_1^2 + (12\alpha + 12\beta)b_2^2 + (9\alpha + 9\beta)b_3^2 \end{aligned} \tag{4.8}$$

Player II seeks to maximize (4.8) subject to the constraint that t must be a unit octonion, that is

$$b_0^2 + b_1^2 + b_2^2 + b_3^2 = 1 \tag{4.9}$$

Since the return $12\alpha + 12\beta = 12$ is the largest, Player II's best response is to select a unit octonion t such that $b_0 = b_1 = b_3 = 0$, that is, $t = i_5$.

Finally, take Players I and II strategies as given and suppose that Player III responds with the unit octonion $u = c_0 + c_1 i_1 + c_2 i_3 + c_3 i_7$. Then Player III's expected payoff is given by

$$P_3^Q(1, i_5, u) = 9c_0^2 + 12c_1^2 + 9c_2^2 + 12c_3^2 \tag{4.10}$$

Player III seeks to maximize (4.10) subject to the constraint that u must be a unit octonion, that is

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1 \tag{4.11}$$

Then Player III's best response is to select a unit octonion u such that $c_0 = c_2 = 0$, that is, $u = c_1 i_1 + c_3 i_7$, where $c_1^2 + c_3^2 = 1$. In particular, Player III can select the real convex combination $\alpha i_1 + \beta i_7$, where $\alpha = c_1^2$ and $\beta = c_3^2$. If Player III responds with a mixed quantum strategy represented by a probability distribution μ over $\mathbb{U}_3$. Then Player III's expected payoff is given by

$$E_3^Q(1, i_5, \mu) = \int_{\mathbb{U}_3} P_3^Q(1, i_5, u) d\mu(u) = \int_{\mathbb{U}_3} (9c_0^2 + 12c_1^2 + 9c_2^2 + 12c_3^2) d\mu(u) \tag{4.12}$$

Player III seeks to maximize (4.12) subject to (4.11). Then Player III's best response is to select a μ that assigns c_1 and c_3 positive probabilities and zero to everything else. In particular, Player III can choose $\mu = \alpha i_1 + \beta i_7$. Accordingly, $(1, i_5, \alpha i_1 + \beta i_7)$ is a Nash equilibrium that pays out $E_i^Q(1, i_5, \alpha i_1 + \beta i_7) = \alpha P_i^Q(1, i_5, i_1) + \beta P_i^Q(1, i_5, i_7) = \alpha w_6 + \beta w_4$ to Player i. This translates into the triplet payoffs to the players $(12\alpha + 18\beta, 18\alpha + 12\beta, 12)$. □

Define a *discrete distribution* as a mixed strategy that is supported on a finite number of points. One such distribution is the *special* discrete distribution where each player uses his pure strategy corresponding to a single octonionic basis element with probability 1/4. In particular, for Players I, II, and III these are the mixed quantum strategies given by $\lambda = \frac{1}{4} + \frac{1}{4} i_1 + \frac{1}{4} i_2 + \frac{1}{4} i_4$, $\mu = \frac{1}{4} + \frac{1}{4} i_1 + \frac{1}{4} i_5 + \frac{1}{4} i_6$, and $\nu = \frac{1}{4} + \frac{1}{4} i_1 + \frac{1}{4} i_3 + \frac{1}{4} i_7$, respectively.

Proposition 4.5. *The strategic profile (λ, μ, ν) is a Nash equilibrium in G^{mQ} that pays out the average of the classical individual payoffs or 12.25 to each player.*

Proof. Take Players II and III strategies as given and suppose that Player I responds with a pure quantum strategy represented by the unit octonion $s = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_4$. Then, the expected payoff to Player I is given by

$$E_1^Q(s, \mu, \nu) = \frac{1}{16} \sum_{l,m,n} P_1^Q(s, i_m, i_n),$$

where $m \in \{0, 1, 5, 6\}$ and $n \in \{0, 1, 3, 7\}$ with $i_0 = 1$. After some easy but tedious calculations, one can show that $E_1^Q(s, \mu, \nu) = 12.25$. Therefore Player I is indifferent between his pure quantum strategies.

Suppose now that Player I responds with a mixed quantum strategy σ which is a probability measure on $\mathbb{U}_1$. Then

$$E_1^Q(\sigma, \mu, \nu) = \int\limits_{\mathbb{U}_1 \times \mathbb{U}_2 \times \mathbb{U}_3} P_1^Q(s, t, u) d(\sigma \times \mu \times \nu)(s, t, u) \tag{4.13}$$

Apply Fubini's Theorem to obtain

$$\begin{aligned} E_1^Q(\sigma, \mu, \nu) &= \int\limits_{\mathbb{U}_1} \left[\int\limits_{\mathbb{U}_2 \times \mathbb{U}_3} P_1(s, t, u) d(\mu \times \nu)(t, u) \right] d\sigma(s) \\ &= \int\limits_{\mathbb{U}_1} \left(\frac{1}{8} \sum_{j=0}^{7} x_j \right) d\sigma(s) \\ &= \frac{1}{8} \sum_{j=0}^{7} x_j \sigma(\mathbb{U}_1) = \frac{1}{8} \sum_{j=0}^{7} x_j = 12.25. \end{aligned} \tag{4.14}$$

Hence, Player I is indifferent between all his mixed quantum strategies. Now, if we interchange the roles of the players, we obtain, by symmetry, the same conclusion. Therefore, the special discrete distribution is a Nash equilibrium in G^{mQ}. This equilibrium pays out an expected payoff of 12.25 to each player. $\square$

In the cases where $a = b$ or $a < b$, we will only state the results and omit the proofs. One can refer to the proofs given above which are typical when showing that a strategic profile is a Nash equilibrium.

4.2. **Firm 1 and Firm 2 offer the same salary.** In this section we analyze the case where Firm 1 and Firm 2 offer the same salary, that is, $6a = 6b$ or $a = b$. To further simplify the calculations, and without loss of generality, we select $a = b = 3$. We obtain the strategic form of the game with payoff function given in Fig.7. Note that this choice of a and b respects the prescribed conditions on a and b in the game, that is, $2a > b$ and $2b > a$.

		II	
		F_1	F_2
I	F_1	$(6,6,6)$	$(9,18,9)$
	F_2	$(18,9,9)$	$(9,9,8)$

Player III chooses F_1

		II	
		F_1	F_2
I	F_1	$(9,9,18)$	$(18,9,9)$
	F_2	$(9,18,9)$	$(6,6,6)$

Player III chooses F_2

FIGURE 7. Case 2: Firm 1 and Firm 2 offer the same salary ($a = b$)

4.2.1. *Nash Equilibria in G.* A classical analysis of this game yields the following result.

Proposition 4.6. *The game G admits six Nash equilibria, namely the pure strategic triplets* (F_1,F_1,F_2), (F_1,F_2,F_1), (F_1,F_2,F_2), (F_2,F_1,F_1), (F_2,F_1,F_2), *and* (F_2,F_2,F_1) .

Six out of the the eight possible strategic profiles are Nash equilibria. This abundance of equilibria is due to the fact that both positions are equally competitive since the salaries offered are the same.

4.2.2. *Nash Equilibria in* G^{mix}. If the players employ probability distributions over their pure classical strategies, then there is one Nash equilibrium.

Proposition 4.7. *The game* G^{mix} *admits the Nash equilibrium* (p,q,r), *where* $p = q = r = \frac{1}{2}$. *This mixed strategy equilibrium pays out* 10.5 *to each player.*

4.2.3. *Nash Equilibria in* G^Q. Throughout, we assume that $l \in \{0,1,2,4\}$, $m \in \{0,1,5,6\}$, and $n \in \{0,1,3,7\}$.

Proposition 4.8. *Every triplet of the form* (i_l,i_m,i_n) *with* $(i_l i_m)i_n = \pm i_6$ *is a Nash equilibrium in* G^Q. *These strategic profiles are all equivalent to the pure classical Nash equilibrium* (F_1,F_2,F_1).

By symmetry, one can generalize this proposition by observing that every triplet of the form (i_l,i_m,i_n) with $(i_l i_m)i_n \neq \pm 1$ and $(i_l i_m)i_n \neq i_1$ is in fact a Nash equilibrium in G^Q.

4.2.4. *Nash Equilibria in G^{mQ}.* Consider the real convex combinations $\lambda = \alpha_1 i_2 + \alpha_2 i_4$, $\mu = \beta_1 i_5 + \beta_2 i_6$, and $\nu = \gamma_1 i_3 + \gamma_2 i_7$. Then we have families of equilibria where one player uses a mix of two octonionic basis elements and the other two use the unit octonion 1.

Proposition 4.9. *The strategy profiles $(\lambda, 1, 1)$, $(1, \mu, 1)$, and $(1, 1, \nu)$ are Nash equilibria in G^{mQ}.*

4.3. **Firm 2 pays better than Firm 1.** In this section we analyze the case where Firm 2 pays better than Firm 1, that is, $6a < 6b$ or $a < b$. To further simplify the calculations, and without loss of generality, we select $a = 3$ and $b = 4$. We obtain the strategic form of the game with payoff function given in Fig.8. Note that this choice of a and b respects the prescribed conditions on a and b in the game, that is, $2a > b$ and $2b > a$.

		II	
		F_1	F_2
I	F_1	$(6,6,6)$	$(9,24,9)$
	F_2	$(24,9,9)$	$(12,12,18)$

Player III chooses F_1

		II	
		F_1	F_2
I	F_1	$(9,9,24)$	$(18,12,12)$
	F_2	$(12,18,12)$	$(8,8,8)$

Player III chooses F_2

FIGURE 8. Case 3: Firm 2 pays better than Firm 1($a < b$)

This case is the mirror image of case 1 if one interchanges the roles of a and b. For instance, the Nash equilibria in G are (F_1, F_2, F_2), (F_2, F_1, F_2), and (F_2, F_2, F_1). There is a unique Nash equilibrium (p, q, r) in G^{mix}, where $p = q = r = \frac{3\sqrt{21}-13}{2} \approx 0.371$. Furthermore, the triplets $(1, i_5, 1)$ and $(1, i_5, i_3)$ are Nash equilibria in G^{Q}. If Players I and II stick with the pure strategies 1 and i_5, respectively, and Player III uses a real convex combination of 1 and i_3, then we obtain a family of Nash equilibria in G^{mQ}.

5. CONCLUSION

In this paper, we have shown that the unit octonions are powerful tools that can be utilized in identifying the Nash equilibria in the extended games G^{Q} and G^{mQ}. In the case of the three-player, two-strategy game of Firms, the use of unit octonions proved to be useful in identifying families of Nash equilibria in both G^{Q} and G^{mQ}. The interest in the game of Firms originates from the constraints $2a > b > 0$ and $2b > a > 0$ and the fact that no order is specified between the numbers a and b in the two-player version. However, the analysis of the three-player version of the game requires that we establish an order between a and b.

A future direction of this work is to establish the complete classification of Nash equilibria in three-player, two-strategy octonionized quantum games. A best response analysis and the evidence obtained to date suggest a conjectural breakdown of the Nash equilibria into equilibria of types “pure, pure, pure” (each player uses a pure quantum strategy represented by a canonical octonionic basis element), “pure, pure, mix of two” up to permutations (two players use canonical octonionic basis elements, one player uses a mixed strategy supported on two orthonormal points), and "mix of two, mix of two, mix of two" (each player chooses a mixed quantum strategy supported on two canonical octonionic basis elements, each played with probability 1/2). Other types of equilibria may exist.

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