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PARAMETRIC OPTIMALITY AND DUALITY RESULTS FOR NONDIFFERENTIABLE L-UNIVEX MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS

TADEUSZ ANTCZAK

Faculty of Mathematics and Computer Science, University of Łódź, Poland

Dedicated to the Memory of Prof. Ram U. Verma

Abstract. In this work, a class of nondifferentiable multiobjective fractional programming problems with locally Lipschitz functions is considered. Parametric Karush-Kuhn-Tucker necessary conditions are established for such nonsmooth extremum problems via Mordukhovich subdifferentials of the involved functions. Moreover, a new concept of generalized convexity, namely, L-univexity is introduced via the notion of Mordukhovich subdifferential, and by employing it sufficient optimality conditions for the considered problem are derived. Further, for the aforesaid nondifferentiable multiobjective fractional programming problem, its parametric vector dual problem is defined and several duality results are proved also under L-univexity assumptions. The parametric optimality and duality results established in the paper for such nondifferentiable multiobjective fractional programming problems generalize the similar results existing in the literature.

Keywords. Multiobjective fractional programming problem; Mordukhovich subdifferential; Pareto solution; L-univexity; Parametric Schaible duality.

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1. Introduction

Multiobjective programming problems (also known as vector optimization problems) which are optimization problems involving several conflicting objectives have been the subject of extensive study in the recent literature. Multiobjective fractional programming problems refers to vector optimization problems where the objective functions are quotients. Such extremum problems are commonly encountered in many areas of human activity, among others including engineering, mechanics, science management, portfolio selection, physics, cutting and stock, game theory, optimal control, and others. Therefore, due to the application of such multicriteria optimization problems in so many areas of human activity, there is a need in the optimization

*Corresponding author.
E-mail address: tadeusz.antczak@wmii.uni.lodz.pl
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theory for optimality conditions and duality results for more and more new classes of extremum problems of this type.

Recently, optimality conditions and duality results have been studied for nonsmooth multiobjective fractional programming problems under various kinds of generalized convexity notions (see, for example, [1], [2], [3], [4], [6], [12], [13], [15], [16], [19], [22], [25], [26], [27], [28], and many others). Antczak [2] proved both parametric and non-parametric necessary and sufficient optimality conditions and duality results for the considered generalized fractional programming under appropriate (p,r)-invexity assumptions. Liu [16] established the Kuhn-Tucker necessary and sufficient conditions for an efficient solution of multiobjective fractional programming problems containing (F,ρ)-convex functions and duality results for the defined Bector type dual problem. Kuk et al. [13] established generalized Karush-Kuhn-Tucker necessary and sufficient optimality conditions and derived duality theorems for nonsmooth multiobjective fractional programming problems containing V-ρ-invex functions. Using the properties of limiting subdifferential vectors and a separation theorem in convex analysis, Soleimani-damaneh [21] derived some necessary and sufficient optimality conditions for the considered nonsmooth fractional multiple objective problems. Further, Soleimani-damaneh [22] studied fractional multiobjective optimization problems in special Banach spaces and, derived necessary optimality conditions by utilizing Mordukhovich subdifferential, as well as established sufficient optimality conditions for considered multicriteria optimization problems under the notion of KT-(p,r)-invexity introduced by Antczak [2]. Under the introduced concept of nearly invex functions for locally Lipschitz vector-valued functions, Kim et al. [12] derived generalized sufficient optimality conditions and proved weak and strong duality theorems for the multiobjective fractional optimization problem involving nearly invex functions. Kim [11] introduced the concept of generalized invexity for a fractional function and then he proved the sufficient optimality conditions and several duality results for the considered nonsmooth multiobjective fractional programming problems involving locally Lipschitz functions. In [19], Nobakhtian considered nonsmooth multiobjective fractional programming problems with mixed constraints and established for them the optimality conditions and several mixed duality results under various generalized invexity assumptions. Lai and Ho [15] studied a subdifferentiable multiobjective fractional programming problem and established sufficient optimality conditions and parametric duality results under exponential V-r-invexity hypotheses. In his book, Verma [25] presented a smooth and unified transitional framework from generalized fractional programming, with a finite number of variables and a finite number of constraints, to semi-infinite fractional programming, where a number of variables are finite but with infinite constraints. Chuong and Kim [7] derived optimality conditions and duality relations that are expressed in terms of limiting/Mordukhovich subdifferentials for nonsmooth multiobjective fractional programming problems. Based on the idea of L-invex-infine functions defined in terms of the limiting/Mordukhovich subdifferential of locally Lipschitz functions, Jayswal et al. [10] obtained sufficient optimality conditions for the considered nonsmooth minimax fractional programming problem. Antczak and Verma [3] proved both parametric necessary optimality conditions and, under the introduced concept of (b,Ψ,Φ,ρ)-univexity, also sufficient optimality conditions for a new class of nondifferentiable multiobjective fractional programming problems. For a bibliography of fractional programming, see, for example, Stancu-Minasian [23].

In this work, we prove optimality and duality results for a new class of nondifferentiable multiobjective fractional programming problems with both inequality and equality constraints. Namely, we use the limiting/Mordukhovich subdifferential as a tool in proving the main results in the paper. In order to prove the necessary optimality conditions for the considered nonsmooth multiobjective fractional programming problem, we also use the Dinkelbach parametric approach and the weighting method. Then, we establish the parametric Karush-Kuhn-Tucker optimality conditions for a feasible solution to be a weak Pareto solution in the considered nonsmooth multiobjective fractional programming problem which are formulated via the limiting/Mordukhovich subdifferential of real-valued functions. In proving the aforesaid optimality conditions, we also use the nonsmooth version of Fermat's rule and the sum rule for limiting/Mordukhovich subdifferentials given, for example, in [18]. Further, we also introduce the concept of a nonsmooth L-univex optimization problem for the analyzed multiobjective fractional programming problem. Then, we prove the sufficient optimality for such a nondifferentiable vector optimization problem. The optimality results established in the paper are illustrated by the example of a nondifferentiable multiobjective fractional programming problem which is L-univex and, moreover, they are compared to the similar optimality results existing in the literature. It is known that the optimality conditions and calculus rules in terms of Mordukhovich subdifferentials provide more sharp results than those given in terms of the Clarke subdifferential. Further, for the considered nonsmooth multiobjective fractional programming problem, we define its vector parametric dual problem in the sense of Schaible and we prove several duality theorems between these nondifferentiable multicriteria optimization problems also under L-univexity assumptions.

2. PRELIMINARIES

The following convention for equalities and inequalities will be used in the paper.

Let R^n be an n-dimensional Euclidean space and R^n_+ be its non-negative orthant. For any $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

Unless otherwise stated, X be a given Banach space whose a norm is always denoted by $\|\cdot\|$. Its dual space is denoted by X^* and the canonical pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$.

A real-valued function $f : X \to R$ is said to be locally Lipschitz at $\overline{x} \in X$ on X if there exist a positive constant $K_{\overline{x}} > 0$ and a neighborhood $U(\overline{x})$ of $\overline{x}$ such that the inequality $|f(x) - f(y)| \leqq K_{\overline{x}} \|x - y\|$ is satisfied for every $x, y \in U(\overline{x})$, where $\|\cdot\|$ denotes a norm in X.

The set $dom(f) := \{x \in X : |f(x)| < \infty\}$ is called the effective domain of f.

Definition 2.1. The polar cone of a set $Q \subset X$ is defined by $Q^0 = \{x^* \in X : \langle x^*, x \rangle \leqq 0, \forall x \in Q\}$.

Definition 2.2. The functional $F : X \to R$ is superlinear if

i) $F(x + u) \geqq F(x) + F(u), \ \forall x, u \in X$
ii) $F(\alpha x) = \alpha F(x), \forall x \in X, \forall \alpha \geqq 0$.

Definition 2.3. [18] Given a multifunction $F : X \rightrightarrows X^*$. Then, the notation

$$\underset{x\to\bar{x}}{Lim\sup} F(x) := \{u^* \in X^* : \exists \text{ sequences } x_n \to \bar{x} \text{ and } u_n^* \overset{w^*}{\to} u^* \text{ with } u^* \in F(x_n), \forall n \in N\}$$

signifies the sequential Painlev´e-Kuratowski upper/outer limit with respect to the norm topology of X and the weak* topology of X^*, where the notation $\overset{w^*}{\to}$ denotes the convergence in the weak* topology of X^* and N denotes the set of all natural numbers.

Definition 2.4. [7], [18] Let $\Omega \subset X$ be closed around $\bar{x} \in \Omega$, i.e. there is a neighborhood U of $\bar{x}$ such that $\Omega \cap clU$ is closed. The Fréchet normal cone to Ω at $\bar{x}$ is defined by

$$\widehat{N}(\bar{x};\Omega) := \left\{ x^* \in X^* : \underset{x\overset{\Omega}{\to}\bar{x}}{Lim\sup} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0 \right\},$$

where $x \overset{\Omega}{\to} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x};\Omega) := \varnothing$.

Definition 2.5. [18] The limiting/Mordukhovich normal cone $N(\bar{x};\Omega)$ to Ω at $\bar{x} \in \Omega$ is obtained from Fréchet normal cones by taking the sequential Painlevé-Kuratowski upper limits as follows

$$N(\bar{x};\Omega) := \underset{x\overset{\Omega}{\to}\bar{x}}{Lim\sup} \widehat{N}(x;\Omega).$$

If $\bar{x} \notin \Omega$, we put $N(\bar{x};\Omega) := \varnothing$.

Definition 2.6. [7], [18] The negative polar of the Mordukhovich normal cone to Ω at $\bar{x} \in \Omega$ is denoted by $N^{M-}(\bar{x};\Omega)$, i.e.,

$$N^{M-}(\bar{x};\Omega) := N(\bar{x};\Omega)^0 = \{x^* \in X^* : \langle x^*, d\rangle \leqq 0 \ \forall d \in N(\bar{x};\Omega)\}.$$

Definition 2.7. [7] The Clarke tangent cone to Ω at $\bar{x} \in \Omega$, denoted by $T^C(\bar{x};\Omega)$, is defined by

$$T^C(\bar{x};\Omega) := \{d \in X : \forall \{x_n\} \subset \Omega,\ x_n \to \bar{x}, \forall \{t_n\} \subset (0,\infty), t_n \to 0, \forall \{d_n\} \subset X,\ d_n \to d \text{ with } x_n + t_n d_n \in \Omega \ \forall n \in N\}.$$

Definition 2.8. [7] The negative polar of the Clarke tangent cone $T^C(\bar{x};\Omega)$, denoted by $N^{C-}(\bar{x};\Omega)$, is called the Clarke normal cone to Ω at $\bar{x}$, i.e.,

$$N^{C-}(\bar{x};\Omega) := T^C(\bar{x};\Omega)^0 = \left\{x^* \in X^* : \langle x^*, x\rangle \leqq 0 \ \forall x \in T^C(\bar{x};\Omega)\right\}.$$

It is known in the literature (see, for example, [7], [18]) that

$$N(\bar{x};\Omega) \subset N^{C-}(\bar{x};\Omega). \tag{2.1}$$

Definition 2.9. The epigraph of an extended real-valued function $\varphi : X \to \overline{R} := R \cup \{\pm\infty\}$ is defined by $epi\varphi := \{(x,\alpha) \in X \times R : \alpha \geqq \varphi(x)\}$.

Definition 2.10. [18] The limiting/Mordukhovich subdifferential of $\varphi : X \to \overline{R} := R \cup \{\pm\infty\}$ at $\bar{x} \in X$ with $|\varphi(\bar{x})| < \infty$ is defined by $\partial^M \varphi(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in N((\bar{x};\varphi(\bar{x}));epi\varphi)\}$.

Remark 2.11. If $|\varphi(\bar{x})| = \infty$, then one puts $\partial^M \varphi(\bar{x}) := \varnothing$.

Definition 2.12. [5] The Clarke subdifferential of $\varphi : X \to \overline{R} := R \cup \{\pm\infty\}$ at $\bar{x} \in X$ with $|\varphi(\bar{x})| < \infty$ is defined by $\partial^C \varphi(\bar{x}) := \left\{x^* \in X^* : (x^*, -1) \in N^{C-}((\bar{x};epi\varphi(\bar{x}));epi\varphi)\right\}$.

If $|\varphi(\bar{x})| = \infty$, then one puts $\partial^M \varphi(\bar{x}) = \partial^C \varphi(\bar{x}) = \varnothing$.

Remark 2.13. It follows from (2.1) and Definitions 2.10 and 2.12 that $\partial^M \varphi(\bar{x}) \subset \partial^C \varphi(\bar{x})$ for any $\bar{x} \in X$.

For the set Ω, let us consider now the indicator function $\delta(\cdot;\Omega)$ defined by

$$\delta(x;\Omega) := \begin{cases} 0 & \text{if} \quad x \in \Omega, \\ \infty & \text{if} \quad x \notin \Omega. \end{cases}$$

It is known (see Mordukhovich [18]) that there is the relation between the Mordukhovich normal cone and the limiting subdifferential of the indicator function and it has the following form

$$N(x;\Omega) = \partial^M \delta(x;\Omega), \ \forall x \in \Omega. \tag{2.2}$$

Definition 2.14. [18] A set $\Omega \subset X$ is sequentially normally compact (SNC) at $\bar{x} \in \Omega$ if, for any sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$ and $x_k^* \xrightarrow{w^*} 0$ with $x_k^* \in \widehat{N}(x_k;\Omega)$, one has $\|x_k^*\| \to 0$ as $k \to \infty$.

Remark 2.15. If Ω is closed around $\bar{x}$, then $\varepsilon_k \downarrow 0$ can be omitted in Definition 2.14. Obviously, this property is automatically satisfied in finite dimensional spaces. For various sufficient conditions ensuring the fulfillment of the SNC property, the reader is referred to [18].

Now, we re-call the intersection rule for the Mordukhovich normal cone under the assumption that the SNC property is fulfilled.

Lemma 2.16. [18] *Assume that Ω_1, $\Omega_2 \subset X$ are closed around $\bar{x} \in \Omega_1 \cap \Omega_2$ and that at least one of the sets Ω_1, Ω_2 is SNC at this point. If $N(\bar{x};\Omega_1) \cap (-N(\bar{x};\Omega_2)) = \{0\}$, then $N(\bar{x};\Omega_1 \cap \Omega_2) \subset N(\bar{x};\Omega_1) + N(\bar{x};\Omega_2)$.*

Now, we present the nonsmooth version of Fermat's rule (see, e.g., [11, Proposition 1.114]) which is an important tool using in many applications.

Theorem 2.17. *If $\bar{x}$ is a local minimizer of φ, then $0 \in \partial^M \varphi(\bar{x})$.*

Finally in this section, we invoke the sum rule for limiting subdifferentials which will be used in the sequel.

Lemma 2.18. *Let $\varphi_i : X \to \overline{R}$, $i = 1,\dots,k$, $k \geqq 2$, be lower semicontinuous around $\bar{x} \in X$ and let all but one of these functions be locally Lipschitz at $\bar{x}$. Then,*

$$\partial^M(\varphi_1 + \dots + \varphi_k)(\bar{x}) \subset \partial^M \varphi_1(\bar{x}) + \dots. + \partial^M \varphi_k(\bar{x}).$$

3. Multiobjective fractional programming

In the paper, we consider the following multiobjective fractional programming problem

$$\begin{aligned} \varphi(x) = (\varphi_1(x),\dots,\varphi_k(x)) = \left(\tfrac{f_1(x)}{q_1(x)},\dots,\tfrac{f_k(x)}{q_k(x)}\right) \to V\text{-}\min \\ \text{subject to} \quad g(x) := (g_1(x),\dots,g_m(x)) \leqq 0, \\ h(x) := (h_1(x),\dots,h_p(x)) = 0, \\ x \in Q, \end{aligned} \tag{MFP}$$

where Q is a nonempty SNC subset of X, $f_i : X \to R$, $q_i : X \to R$, $i \in I = \{1,\dots,k\}$, $g_j : X \to R$, $J = \{1,\dots,m\}$, $h_s : X \to R$, $S = \{1,\dots,p\}$, are locally Lipschitz functions on X such that $f_i(x) \geqq$

0, $q_i(x) > 0$ for every $i \in I$ and all $x \in X$. We call (MFP) the original multiobjective fractional programming problem.

Let $\Omega := \left\{x \in Q : g_j(x) \leqq 0, \quad j \in J,\ h_s(x) = 0,\ s \in S\right\}$ denote the set of all feasible solutions of (MFP). We also define the set of active inequality constraints at a feasible solution $\overline{x}$ by

$$J(\overline{x}) := \left\{ j \in J : g_j(\overline{x}) = 0 \right\}.$$

Definition 3.1. A feasible solution $\overline{x}$ is said to be a Pareto solution (an efficient solution) for (MFP) if and only if there is no other solution $x \in \Omega$ such that $\varphi(x) \leq \varphi(\overline{x})$.

Definition 3.2. A feasible solution $\overline{x}$ is said to be a weak Pareto solution (a weakly efficient solution) for (MFP) if and only if there is no other solution $x \in \Omega$ such that $\varphi(x) < \varphi(\overline{x})$.

Definition 3.3. It is said that the considered multiobjective fractional programming problem (MFP) satisfies the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) at $\overline{x} \in \Omega$ if there do not exist $\overline{\mu}_j \geqq 0$, $j \in J(\overline{x})$, and $\overline{\vartheta}_s \geqq 0$, $s \in S(\overline{x}) := \{s \in S : h_s(\overline{x}) = 0\}$, such that $\sum_{j \in J(\overline{x})} \overline{\mu}_j + \sum_{s \in S(\overline{x})} \overline{\vartheta}_s \neq 0$, and

$$0 \in \sum_{j=1}^{m} \overline{\mu}_j \partial^M g_j(\overline{x}) + \sum_{s=1}^{p} \overline{\vartheta}_s \left(\partial^M h_s(\overline{x}) \cup \partial^M (-h_s)(\overline{x})\right) + N(\overline{x}; Q).$$

We use the parametric method introduced by Dinkelbach [9] for solving the considered multiobjective fractional programming problem (MFP). Namely, in order to prove the parametric necessary optimality conditions for the considered multiobjective fractional programming problem (MFP), we use the parametric approach which is based on the method introduced by Crouzeix et al. [8] for minimax fractional programming problems. Then, for the considered multiobjective fractional programming problem (MFP), we define the associated parametric nonfractional multiobjective programming problem (MP_v) in the parameter v as follows:

$$\begin{aligned} &V\text{-minimize } \left(f_1(x) - v_1 q_1(x), \dots, f_k(x) - v_k q_k(x)\right) \\ &\text{subject to } \quad g_j(x) \leqq 0,\ j \in J = \{1, \dots m\}, \\ &\qquad\qquad\quad h_s(x) = 0,\ s \in S = \{1, \dots, p\}, \\ &\qquad\qquad\quad x \in Q. \end{aligned} \qquad (\text{MP}_v)$$

For the auxiliary multiobjective programming problem (MP_v) defined above, the following result is true:

Lemma 3.4. [3], [19] *Let $\overline{x} \in \Omega$ be a weak Pareto solution of the considered multiobjective fractional programming problem (MFP). Then $\overline{x}$ is also a weak Pareto solution in the nonfractional multiobjective programming problem ($\text{MP}_{\overline{v}}$) with $\overline{v}_i = \varphi_i(\overline{x})$, that is, $\overline{v}_i = \frac{f_i(\overline{x})}{q_i(\overline{x})}$, $i \in I$.*

Now, for the parametric nonfractional multiobjective programming problem (MP_v), we use the weighting method (see, for example [17]). Therefore, we define for (MP_v) its weighting problem as follows

$$\begin{aligned} &\text{minimize } \textstyle\sum_{i=1}^{k} \lambda_i \left(f_i(x) - v_i q_i(x)\right) \\ &\qquad\qquad x \in \Omega, \end{aligned} \qquad (\text{P}_v)$$

where $\lambda_i \geqq 0$, $i = 1, \dots, k$ and $\sum_{i=1}^{k} \lambda_i = 1$.

Theorem 3.5. *An optimal solution of ($P_{\overline{v}}$) is a weak Pareto solution of ($MP_{\overline{v}}$) with $\overline{v}_i = \varphi_i(\overline{x})$, that is, $\overline{v}_i = \frac{f_i(\overline{x})}{q_i(\overline{x})}$, $i \in I$. If the weighting coefficients λ_i, $i = 1,\ldots,k$, are positive, then an optimal solution of ($P_{\overline{v}}$) is a Pareto solution of ($MP_{\overline{v}}$).*

Proof. Let $\overline{x} \in \Omega$ be a minimizer of the weighting problem ($P_{\overline{v}}$), where $\overline{v}_i = \varphi_i(\overline{x})$, that is, $\overline{v}_i = \frac{f_i(\overline{x})}{q_i(\overline{x})}$, $i \in I$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto solution of ($MP_{\overline{v}}$). Then, there exists other $\widetilde{x} \in \Omega$ such that

$$f_i(\widetilde{x}) - \overline{v}_i q_i(\widetilde{x}) < f_i(\overline{x}) - \overline{v}_i q_i(\overline{x}),\ i = 1,\ldots,k.$$

Since $\lambda_i \geqq 0$, $i = 1,\ldots,k$ and $\sum_{i=1}^{k} \lambda_i = 1$, at least one weighting coefficient λ_i is positive. Hence, the above inequalities yield that the inequality

$$\sum_{i=1}^{k} \lambda_i \left(f_i(\widetilde{x}) - \overline{v}_i q_i(\widetilde{x})\right) < \sum_{i=1}^{k} \lambda_i \left(f_i(\overline{x}) - \overline{v}_i q_i(\overline{x})\right)$$

holds, which contradicts the assumption that $\overline{x} \in \Omega$ is a minimizer of the weighting problem ($P_{\overline{v}}$). The proof of efficiency is similar and, therefore, it is omitted. □

The following theorem gives the Karush-Kuhn-Tucker optimality conditions (KKT) for an optimal solution of the weighting problem ($P_{\overline{v}}$). Hence, we obtain the Karush-Kuhn-Tucker necessary conditions for ($MP_{\overline{v}}$) and, thus, the Karush-Kuhn-Tucker (KKT) necessary conditions for the original multiobjective fractional programming problem (MFP).

Theorem 3.6. *Let $\overline{x} \in \Omega$ be an optimal solution in the weighting problem ($P_{\overline{v}}$). Further, assume that the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) is satisfied at $\overline{x}$. Then, there exist $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$, $\overline{\vartheta} \in R^p$ and $\overline{v} \in R^k$ such that*

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i \left(\partial^M f_i(\overline{x}) - \overline{v}_i \partial^M q_i(\overline{x})\right) + \sum_{j=1}^{m} \overline{\mu}_j \partial^M g_j(\overline{x}) + \sum_{s=1}^{p} \overline{\vartheta}_s \left(\partial^M h_s(\overline{x}) \cup \partial^M (-h_s)(\overline{x})\right) + N(\overline{x}; Q), \tag{3.1}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0,\ j \in J, \tag{3.2}$$

$$\overline{\lambda}_i \geq 0,\ i \in I,\ \sum_{i=1}^{k} \overline{\lambda}_i = 1,\ \overline{\mu}_j \geqq 0,\ j \in J,\ \overline{\vartheta}_s \geqq 0,\ s \in S. \tag{3.3}$$

Proof. Let $\overline{x} \in \Omega$ be an optimal solution in the weighting problem ($P_{\overline{v}}$). This means that $\overline{x} \in \Omega$ is an optimal solution of the following unconstrained optimization problem

$$\text{minimize} \sum_{i=1}^{k} \lambda_i \left(f_i(x) - \overline{v}_i q_i(x)\right) + \delta(x; \Omega). \quad \text{(P)}$$

Applying the nonsmooth version of Fermat's rule (see Theorem 2.17), we get

$$0 \in \partial^M \left(\sum_{i=1}^{k} \lambda_i \left(f_i(\overline{x}) - \overline{v}_i q_i(\overline{x})\right) + \delta(\overline{x}; \Omega) \right).$$

Since the function $\sum_{i=1}^{k}\lambda_i\left(f_i(\bar{x})-\bar{v}_i q_i(\bar{x})\right)$ is Lipschitz continuous around $\bar{x}$ and the function $\delta(\bar{x};\Omega)$ is lower semicontinuous around $\bar{x}$, by the sum rule (see Lemma 2.18), we obtain

$$0\in\sum_{i=1}^{k}\lambda_i\left(\partial^M f_i(\bar{x})-\bar{v}_i\partial^M q_i(\bar{x})\right)+\partial^M\left(\delta(\bar{x};\Omega)\right).$$

Hence, by (2.2), it follows that

$$0\in\sum_{i=1}^{k}\lambda_i\left(\partial^M f_i(\bar{x})-\bar{v}_i\partial^M q_i(\bar{x})\right)+N(\bar{x};\Omega). \tag{3.4}$$

Now, let us denote

$$\Omega_0:=\left\{x\in X: g_j(x)\leqq 0,\quad j\in J,\ h_s(x)=0, s\in S\right\}.$$

Thus, one has $\Omega=\Omega_0\cap Q$. By assumption, (CQ) is fulfilled at $\bar{x}$. Hence, it follows that there do not exist $\mu_j\geqq 0$, $j\in J(\bar{x})$, and $\vartheta_s\geqq 0$, $s\in S(\bar{x})$, such that $\sum_{j\in J(\bar{x})}\overline{\mu}_j+\sum_{s\in S(\bar{x})}\overline{\vartheta}_s\neq 0$ and

$$0\in\sum_{j\in J(\bar{x})}\overline{\mu}_j\partial^M g_j(\bar{x})+\sum_{s\in S(\bar{x})}\overline{\vartheta}_s\left(\partial^M h_s(\bar{x})\cup\partial^M(-h_s)(\bar{x})\right).$$

Therefore, by Corollary 4.36 [18], one has

$$N(\bar{x};\Omega_0)\subset\left\{\sum_{j\in J(\bar{x})}\overline{\mu}_j\partial^M g_j(\bar{x})+\sum_{s\in S(\bar{x})}\overline{\vartheta}_s\left(\partial^M h_s(\bar{x})\cup\partial^M(-h_s)(\bar{x})\right),\right. \tag{3.5}$$
$$\left.\mu_j\geqq 0, j\in J(\bar{x}), \vartheta_s\geqq 0, s\in S(\bar{x})\right\}.$$

Since Q is the SNC subset of X and (CQ) is fulfilled at $\bar{x}$, by Lemma 2.18, one has

$$N(\bar{x};\Omega)=N(\bar{x};\Omega_0\cap Q)\subset N(\bar{x};\Omega_0)+N(\bar{x};Q). \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}0\in\textstyle\sum_{i=1}^{k}\overline{\lambda}_i\left(\partial^M f_i(\bar{x})-\bar{v}_i\partial^M q_i(\bar{x})\right)+\sum_{j\in J(\bar{x})}\overline{\mu}_j\partial^M g_j(\bar{x})+\\ \textstyle\sum_{s\in S(\bar{x})}\overline{\vartheta}_s\left(\partial^M h_s(\bar{x})\cup\partial^M(-h_s)(\bar{x})\right)+N(\bar{x};Q).\end{aligned} \tag{3.7}$$

Now, letting $\overline{\mu}_j=0$, $j\notin J(\bar{x})$, and $\overline{\vartheta}_s=0$, $s\notin S(\bar{x})$, we obtain that (3.7) implies the KKT condition (3.1). Moreover, it is not difficult to note that also the KKT condition (3.2) is satisfied. This completes the proof of this theorem. □

Remark 3.7. The Karush-Kuhn-Tucker conditions (3.1)-(3.3) are, in fact, the necessary optimality conditions for $\bar{x}\in\Omega$ to be a weak Pareto solution in the original multiobjective fractional programming problem (MFP). Indeed, since $\bar{x}\in\Omega$ is an optimal solution in the weighting problem ($P_{\bar{v}}$), by Theorem 3.5, is a weak Pareto solution in ($MP_{\bar{v}}$). Then, by Lemma 3.4, $\bar{x}$ is also a weak Pareto solution in the considered multiobjective fractional programming problem (MFP). Then, we obtain that the Karush-Kuhn-Tucker conditions (3.1)-(3.3) are necessary conditions for $\bar{x}\in\Omega$ to be a weak Pareto solution in (MFP).

Now, we prove the sufficiency of the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) under appropriate L-univexity hypotheses. Then, we define the concept of generalized convexity-affineness type for locally Lipschitz functions as follows on the lines of Chuong [7].

Let $\Phi_f : R \to R^k$, $\Phi_q : R \to R^k$, where $\Phi_{f_i} : R \to R$, $\Phi_{q_i} : R \to R$, $i \in I$, $\Phi_g : R \to R^m$, where $\Phi_{g_j} : R \to R$, $j \in J$, $\Phi_h : R \to R^p$, where $\Phi_{h_s} : R \to R$, $s \in S$, $b_f : X \times X \to R^k$, $b_q : X \times X \to R^k$, where $b_{f_i} : X \times X \to R_+ \setminus \{0\}$, $b_{q_i} : X \times X \to R_+ \setminus \{0\}$, $i \in I$, $b_g : X \times X \to R^m$, where $b_{g_j} : X \times X \to R_+ \setminus \{0\}$, $j \in J$, $b_h : X \times X \to R^p$, where $b_{h_s} : X \times X \to R_+ \setminus \{0\}$, $s \in S$.

Definition 3.8. It is said that $(f, -q, g, h)$ is L-univex-infine at $\bar{x} \in Q$ on $Q \subset X$ (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h) if, for all $x \in Q$ and any $\xi_i \in \partial^M f_i(\bar{x})$, $i \in I$, $-\beta_i \in \partial^M(-q_i)(\bar{x})$, $i \in I$, $\zeta_j \in \partial^M g_j(\bar{x})$, $j \in J$, $\varsigma_s \in \partial^M h_s(\bar{x}) \cup \partial^M(-h_s)(\bar{x})$, $s \in S$, there exists $d \in N(\bar{x}; Q)^0$ such that

$$b_{f_i}(x,\bar{x})\Phi_{f_i}(f_i(x) - f_i(\bar{x})) \geqq \langle \xi_i, d \rangle, \ i \in I, \tag{3.8}$$

$$b_{q_i}(x,\bar{x})\Phi_{q_i}(-q_i(x) - (-q_i(\bar{x}))) \geqq \langle -\beta_i, d \rangle, \ i \in I, \tag{3.9}$$

$$b_{g_j}(x,\bar{x})\Phi_{g_j}\left(g_j(x) - g_j(\bar{x})\right) \geqq \left\langle \zeta_j, d \right\rangle, \ j \in J, \tag{3.10}$$

$$b_{h_s}(x,\bar{x})\Phi_{h_s}(h_s(x) - h_s(\bar{x})) = w_s \langle \varsigma_s, d \rangle, \ s \in S, \tag{3.11}$$

where $w_s = 1$ (whenever $\varsigma_s \in \partial^M h_s(\bar{x})$) or $w_s = -1$ (whenever $\varsigma_s \in \partial^M(-h_s)(\bar{x})$). If (3.8)-(3.11) are satisfied for all $\bar{x} \in Q$, then it is said that $(f, -q, g, h)$ is L-univex-infine on Q (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h).

Definition 3.9. It is said that $(f, -q, g, h)$ is strictly L-univex-infine at $\bar{x} \in Q$ on $Q \subset X$ if, for all $x \in Q$, $(x \neq \bar{x})$, and any $\xi_i \in \partial^M f_i(\bar{x})$, $i \in I$, $\beta_i \in \partial^M(-q_i)(\bar{x})$, $i \in I$, $\zeta_j \in \partial^M g_j(\bar{x})$, $j \in J$, $\varsigma_s \in \partial^M h_s(\bar{x}) \cup \partial^M(-h_s)(\bar{x})$, $s \in S$, there exists $d \in N(\bar{x}; Q)^0$ such that

$$b_{f_i}(x,\bar{x})\Phi_{f_i}(f_i(x) - f_i(\bar{x})) > \langle \xi_i, d \rangle, \ i \in I, \tag{3.12}$$

$$b_{q_i}(x,\bar{x})\Phi_{q_i}(-q_i(x) - (-q_i(\bar{x}))) > \langle -\beta_i, d \rangle, \ i \in I, \tag{3.13}$$

$$b_{g_j}(x,\bar{x})\Phi_{g_j}\left(g_j(x) - g_j(\bar{x})\right) \geqq \left\langle \zeta_j, d \right\rangle, \ j \in J, \tag{3.14}$$

$$b_{h_s}(x,\bar{x})\Phi_{h_s}(h_s(x) - h_s(\bar{x})) = w_s \langle \varsigma_s, d \rangle, \ s \in S, \tag{3.15}$$

where $w_s = 1$ (whenever $\varsigma_s \in \partial^M h_s(\bar{x})$) or $w_s = -1$ (whenever $\varsigma_s \in \partial^M(-h_s)(\bar{x})$). If (3.12)-(3.15) are satisfied for all $\bar{x} \in Q$, $(x \neq \bar{x})$, then it is said that $(f, -q, g, h)$ is strictly L-univex-infine on Q (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h).

Theorem 3.10. *Let $\bar{x} \in \Omega$ and the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) be fulfilled at $\bar{x}$ with Lagrange multipliers $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ and $\overline{\vartheta} \in R^p$. Further, assume that $(f, -q, g, h)$ is L-univex-infine at $\bar{x}$ on Ω (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h), where $\Phi_{f_i}(\cdot) = \Phi_{q_i}(\cdot) = \Phi_i(\cdot)$, $i \in I$, each function Φ_i, $i \in I$, is an increasing superlinear function with $\Phi_i(0) = 0$, each function Φ_{g_j}, $j \in J(\bar{x})$, is an increasing nonnegative homogenous function with $\Phi_{g_j}(0) = 0$, each function Φ_{h_s}, $s \in S$, satisfies the condition $\Phi_{h_s}(0) = 0$, and, moreover, $b_{f_i}(x,\bar{x}) = b_{q_i}(x,\bar{x}) = b_i(x,\bar{x})$, $i \in I$, for all $x \in \Omega$. Then $\bar{x}$ is a weak Pareto solution of (MFP).*

Proof. Let $\bar{x} \in \Omega$ and the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) be fulfilled at $\bar{x}$ with Lagrange multipliers $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ and $\overline{\vartheta} \in R^p$. Then, by (3.1), there exist $\xi_i^* \in \partial^M f_i(\bar{x})$, $i \in I$, $-\beta_i^* \in \partial^M(-q_i)(\bar{x})$, $i \in I$, $\zeta_j^* \in \partial^M g_j(\bar{x})$, $j \in J$, $\varsigma_s^* \in \partial^M h_s(\bar{x}) \cup \partial^M(-h_s)(\bar{x})$, $s \in S$, such that

$$-\left(\sum_{i=1}^{k} \overline{\lambda}_i \left(\xi_i^* + \bar{v}_i \beta_i^*\right) + \sum_{j=1}^{m} \overline{\mu}_j \zeta_j^* + \sum_{s=1}^{p} \overline{\vartheta}_s \varsigma_s^*\right) \in N(\bar{x}; Q). \tag{3.16}$$

Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto optimal solution of (MFP). Then, by Definition 3.2, there exists $\widetilde{x} \in \Omega$ such that

$$\varphi(\widetilde{x}) < \varphi(\overline{x}). \tag{3.17}$$

Using the definition of φ, by $\overline{v} = \varphi(\overline{x})$, (3.17) gives

$$f_i(\widetilde{x}) - \overline{v}_i q_i(\widetilde{x}) < f_i(\overline{x}) - \overline{v}_i q_i(\overline{x}),\ i = 1, ..., k. \tag{3.18}$$

By assumption, each function Φ_i, $i \in I$, is an increasing function on R and $b_i(\widetilde{x}, \overline{x}) > 0$, $i \in I$. Then, (3.18) yields

$$b_i(\widetilde{x}, \overline{x})\Phi_i\left(f_i(\widetilde{x}) - f_i(\overline{x}) + \overline{v}_i(-q_i)(\widetilde{x}) - \overline{v}_i(-q_i)(\overline{x})\right) < \Phi_i(0) = 0,\ i = 1, ..., k. \tag{3.19}$$

Also by assumption, each function Φ_i, $i \in I$, is a superlinear function on R. Then, by Definition 2.2, (3.19) gives

$$b_i(\widetilde{x}, \overline{x})\Phi_i\left(f_i(\widetilde{x}) - f_i(\overline{x})\right) + b_i(\widetilde{x}, \overline{x})\Phi_i\left(\overline{v}_i(-q_i)(\widetilde{x}) - \overline{v}_i(-q_i)(\overline{x})\right) < 0,\ i = 1, ..., k. \tag{3.20}$$

Since $(f, -q, g, h)$ is L-univex-infine at $\overline{x} \in \Omega$ on $\Omega \subset X$ and Definition 3.8, there exists $d \in N(\overline{x}; Q)^0$ such that that the inequalities

$$b_i(\widetilde{x}, \overline{x})\Phi_i\left(f_i(\widetilde{x}) - f_i(\overline{x})\right) \geqq \langle \xi_i, d\rangle,\ i \in I, \tag{3.21}$$

$$b_i(\widetilde{x}, \overline{x})\Phi_i\left(-q_i(\widetilde{x}) - (-q_i(\overline{x}))\right) \geqq \langle -\beta_i, d\rangle,\ i \in I, \tag{3.22}$$

$$b_{g_j}(\widetilde{x}, \overline{x})\Phi_{g_j}\left(g_j(\widetilde{x}) - g_j(\overline{x})\right) \geqq \langle \zeta_j, d\rangle,\ j \in J(\overline{x}), \tag{3.23}$$

$$b_{h_s}(\widetilde{x}, \overline{x})\Phi_{g_j}\left(h_s(\widetilde{x}) - h_s(\overline{x})\right) = w_s\langle \varsigma_s, d\rangle,\ s \in S(\overline{x}) \tag{3.24}$$

hold for any $\xi_i \in \partial^M f_i(\overline{x})$, $-\beta_i \in \partial^M(-q_i)(\overline{x})$, $i \in I$, $\zeta_j \in \partial^M g_j(\overline{x})$, $j \in J(\overline{x})$, $\varsigma_s \in \partial^M h_s(\overline{x}) \cup \partial^M(-h_s)(\overline{x})$, $s \in S$. Multiplying (3.22) by $\overline{v}_i$ and taking into account that each function Φ_i, $i \in I$, is superlinear (then, it is a nonnegative homogenous), by Definition 2.2, we get

$$b_i(\widetilde{x}, \overline{x})\Phi_i\left(\overline{v}_i(-q_i)(\widetilde{x}) - (\overline{v}_i(-q_i)(\overline{x}))\right) \geqq \langle -\overline{v}_i\beta_i, d\rangle,\ i \in I. \tag{3.25}$$

Multiplying each inequality (3.21) and each inequality (3.25) by $\overline{\lambda}_i$, $i \in I$, each inequality (3.23) by $\overline{\mu}_j$, $j \in J$, and each equation (3.24) by $\overline{\vartheta}_s$, $s \in S$, we obtain that the inequality

$$\sum_{i=1}^{k}\left[b_i(\widetilde{x}, \overline{x})\overline{\lambda}_i\Phi_i\left(f_i(\widetilde{x}) - f_i(\overline{x})\right) + b_i(\widetilde{x}, \overline{x})\overline{\lambda}_i\Phi_i\left(\overline{v}_i(-q_i)(\widetilde{x}) - \overline{v}_i(-q_i)(\overline{x})\right)\right] +$$
$$\sum_{j \in J(\overline{x})} b_{g_j}(\widetilde{x}, \overline{x})\overline{\mu}_j\Phi_{g_j}\left(g_j(\widetilde{x}) - g_j(\overline{x})\right) + \sum_{s \in S(\overline{x})} \frac{b_{h_s}(\widetilde{x}, \overline{x})}{w_s}\overline{\vartheta}_s\Phi_{h_s}\left(h_s(\widetilde{x}) - h_s(\overline{x})\right) \geqq$$
$$\left\langle \sum_{i=1}^{k}\overline{\lambda}_i(\xi_i - \overline{v}_i\beta_i) + \sum_{j=1}^{m}\overline{\mu}_j\zeta_j + \sum_{s=1}^{p}\overline{\vartheta}_s\varsigma_s, d\right\rangle \tag{3.26}$$

holds for any $\xi_i \in \partial^M f_i(\overline{x})$, $-\beta_i \in \partial^M(-q_i)(\overline{x})$, $i \in I$, $\zeta_j \in \partial^M g_j(\overline{x})$, $j \in J(\overline{x})$, $\varsigma_s \in \partial^M h_s(\overline{x}) \cup \partial^M(-h_s)(\overline{x})$, $s \in S$. Hence, (3.26) is also satisfied for $\xi_i^* \in \partial^M f_i(\overline{x})$, $i \in I$, $-\beta_i^* \in \partial^M(-q_i)(\overline{x})$, $i \in I$, $\zeta_j^* \in \partial^M g_j(\overline{x})$, $j \in J$, $\varsigma_s^* \in \partial^M h_s(\overline{x}) \cup \partial^M(-h_s)(\overline{x})$, $s \in S$. Since $d \in N(\overline{x}; Q)^0$, by Definition 2.1 and (3.16), one has

$$\left\langle \sum_{i=1}^{k}\overline{\lambda}_i(\xi_i^* - \overline{v}_i\beta_i) + \sum_{j=1}^{m}\overline{\mu}_j\zeta_j^* + \sum_{s=1}^{p}\overline{\vartheta}_s\varsigma_s^*, d\right\rangle \geqq 0. \tag{3.27}$$

Combining (3.26) and (3.27), and using the assumption that each function Φ_{g_j}, $j \in J$, is a nonnegative homogenous function, we get

$$\sum_{i=1}^{k}\left[b_i(\widetilde{x},\overline{x})\overline{\lambda}_i\Phi_i\left(f_i(\widetilde{x})-f_i(\overline{x})\right)+b_i(\widetilde{x},\overline{x})\overline{\lambda}_i\Phi_i\left(\overline{v}_i\left(-q_i\right)\left(\widetilde{x}\right)-\overline{v}_i\left(-q_i\right)\left(\overline{x}\right)\right)\right]+$$

$$\sum_{j\in J(\overline{x})}b_{g_j}(\widetilde{x},\overline{x})\Phi_{g_j}\left(\overline{\mu}_j g_j(\widetilde{x})-\overline{\mu}_j g_j(\overline{x})\right)+\sum_{s\in S(\overline{x})}\frac{b_{h_s}(\widetilde{x},\overline{x})}{w_s}\overline{\vartheta}_s\Phi_{h_s}\left(h_s(\widetilde{x})-h_s(\overline{x})\right)\geqq 0.$$

By Karush-Kuhn-Tucker necessary optimality condition (3.2) and $\widetilde{x},\overline{x}\in\Omega$, it follows that

$$\sum_{i=1}^{k}\left[b_i(\widetilde{x},\overline{x})\overline{\lambda}_i\Phi_i\left(f_i(\widetilde{x})-f_i(\overline{x})\right)+b_i(\widetilde{x},\overline{x})\overline{\lambda}_i\Phi_i\left(-q_i(\widetilde{x})-\left(-q_i(\overline{x})\right)\right)\right]+ \tag{3.28}$$

$$\sum_{j\in J(\overline{x})}b_{g_j}(\widetilde{x},\overline{x})\Phi_{g_j}\left(\overline{\mu}_j g_j(\widetilde{x})\right)+\sum_{s\in S(\overline{x})}\frac{b_{h_s}(\widetilde{x},\overline{x})}{w_s}\overline{\vartheta}_s\Phi_{h_s}\left(0\right)\geqq 0.$$

Since each function Φ_{g_j}, $j\in J(\overline{x})$, is an increasing nonnegative homogenous function with $\Phi_{g_j}(0)=0$ and each function Φ_{h_s}, $s\in S(\overline{x})$, satisfies the condition $\Phi_{h_s}(0)=0$, (3.28) yields

$$\sum_{i=1}^{k}b_i(\widetilde{x},\overline{x})\overline{\lambda}_i\left[\Phi_i\left(f_i(\widetilde{x})-f_i(\overline{x})\right)+\Phi_i\left(\overline{v}_i\left(-q_i\right)\left(\widetilde{x}\right)-\overline{v}_i\left(-q_i\right)\left(\overline{x}\right)\right)\right]\geqq 0. \tag{3.29}$$

By assumption, each function Φ_i, $i\in I$, is a superlinear function on R. Then, by Definition 2.2, (3.29) implies that the inequality

$$\sum_{i=1}^{k}b_i(\widetilde{x},\overline{x})\overline{\lambda}_i\left[\Phi_i\left(f_i(\widetilde{x})-f_i(\overline{x})+\overline{v}_i\left(-q_i\right)\left(\widetilde{x}\right)-\overline{v}_i\left(-q_i\right)\left(\overline{x}\right)\right)\right]\geqq 0$$

holds, contradicting (3.20). This completes the proof of this theorem. □

Theorem 3.11. *Let $\overline{x}\in\Omega$ and the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) be fulfilled at $\overline{x}$ with Lagrange multipliers $\overline{\lambda}\in R^k$, $\overline{\mu}\in R^m$ and $\overline{\vartheta}\in R^p$. Further, assume that $(f,-q,g,h)$ is strictly L-univex-infine at $\overline{x}$ on Ω (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h), where $\Phi_{f_i}(\cdot)=\Phi_{q_i}(\cdot)=\Phi_i(\cdot)$, $i\in I$, each function Φ_i, $i\in I$, is an increasing superlinear function with $\Phi_i(0)=0$, each function Φ_{g_j}, $j\in J(\overline{x})$, is an increasing nonnegative homogenous function with $\Phi_{g_j}(0)=0$, each function Φ_{h_s}, $s\in S$, satisfies the condition $\Phi_{h_s}(0)=0$, and, moreover, $b_{f_i}(x,\overline{x})=b_{q_i}(x,\overline{x})=b_i(x,\overline{x})$, $i\in I$, for all $x\in\Omega$. Then $\overline{x}$ is a Pareto solution of (MFP).*

Now, we compare the Karush-Kuhn-Tucker necessary optimality conditions expressed in terms of the Mordukhovich subdifferentials to analogous the Karush-Kuhn-Tucker necessary optimality conditions expressed in terms of the Clarke subdifferentials.

Example 3.12. We consider the following multiobjective fractional programming problem

$$\begin{aligned} &V\text{-minimize}\ \left(\frac{f_1(x)}{q_1(x)},\ldots,\frac{f_k(x)}{q_k(x)}\right)\\ &g_1(x)=\arctan\left(-\left|x\right|\right)\leqq 0, \qquad \text{(MFP1)}\\ &h_1(x)=0,\end{aligned}$$

where $f_i(x) = \ln\left(x^2 + |x| + i^2\right), i = 1, \dots, k$, $q_i(x) = \ln\left(x^2 + |x| + 2 + i\right)$, where k is any integer, h_1 is a real-valued function such that $h_1(x) = 0$ for any $x \in R$. Note that $Q = R$ and $\bar{x} = 0$ is a feasible solution in the considered multiobjective fractional programming problem (MFP1). Then, one has $N(\bar{x}, Q) = \{0\}$ and $N(\bar{x}, Q)^0 = R$. Further, by Definition 2.10, we have $\partial^M f_i(\bar{x}) = \{-1, 1\}$, $\partial^M q_i(\bar{x}) = \{-1, 1\}$, $i = 1, \dots, k$, $\partial^M g_1(\bar{x}) = \{-1, 1\}$, $\partial^M h_1(\bar{x}) \cup \partial^M(-h_1)(\bar{x}) = \{0\}$. In order to show that $\bar{x} = 0$ is a Pareto solution of (MFP1), we utilize the optimality conditions deduced in Theorem 3.11. It is not difficult to note that there exist Lagrange multipliers $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$, $\overline{\vartheta} \in R^p$ and $\bar{v} \in R_+$ such that the Karush-Kuhn-Tucker optimality conditions (3.1)-(3.3) are fulfilled at $\bar{x}$ with these Lagrange multipliers. Now, we show, by Definition 3.9, that $(f, -q, g, h)$ is strictly L-univex-infine at $\bar{x} \in \Omega$ on Ω. Let us set $\Phi_{f_i} : R \to R$, $\Phi_{q_i} : R \to R$, $\Phi_{f_i}(a) = \Phi_{q_i}(a) = \Phi_i(a) = e^a$, $b_{f_i}(x, \bar{x}) = b_{q_i}(x, \bar{x}) = b_i(x, \bar{x}) = 1$, $i = 1, \dots, k$, $\Phi_{g_1} : R \to R$, $\Phi_{g_1}(a) = \tan(a)$, $b_{g_1}(x, \bar{x}) = \frac{1}{k^2}$, $\Phi_{h_1} : R \to R$, $\Phi_{h_1}(a) = a$, $b_{h_1}(x, \bar{x}) = 1$, then, by taking $d = \frac{\Phi_{g_1}(g_1(x)) - \Phi_{g_1}(g_1(\bar{x}))}{k^2 \zeta_1}$, where $\zeta_1 \in \partial^M g_1(\bar{x})$, we obtain that (3.12)-(3.15) are satisfied for all $x \in \Omega$, $(x \neq \bar{x})$. Then, we conclude, by Definition 3.9, that $(f, -q, g, h)$ is strictly L-univex-infine at $\bar{x} \in \Omega$ on Ω. Hence, by Theorem 3.11, it follows that $\bar{x} = 0$ is a Pareto solution of (MFP1).

Remark 3.13. Note that we can not apply the optimality conditions under the assumption that $(f, -q, g, h)$ is strictly L-invex-infine in the context the definition introduced by Chuong and Kim [7] for nonsmooth vector optimization problems. Hence, the sufficient optimality conditions deduced in Theorem 3.11 under L-univex-infiniteness are relevant for a wider class of nondifferentiable multiobjective fractional programming problems than the corresponding sufficient optimality conditions under L-invex-infiniteness. Note also that the aforesaid conditions are useful for a larger class of nondifferentiable multiobjective fractional programming problems than under the concept of invex-infine functions introduced by Sach et al. [20]. In fact, one has $T^C(\bar{x}; Q) = R$, $\partial^C g_1(\bar{x}) = [-1, 1]$ and $\partial^C h_1(\bar{x}) = \{0\}$. If we take $x \in Q \setminus \{0\}$ and $\zeta_1 = 0 \in [-1, 1] = \partial^C g_1(\bar{x})$, then the inequality $g_1(x) - g_1(\bar{x}) \geqq \langle \zeta_1, d \rangle$ is not satisfied for any $d \in T^C(\bar{x}; Q)$. This means, by the definition introduced by Sach et al. [20], that the functions constituting (MFP1) are not invex-infine at $\bar{x}$ on Q.

4. Parametric duality

In this section, for the considered multiobjective fractional programming problem (MFP), we study the parametric duality model defined by

$$\text{maximize } v$$

$$\begin{aligned}
0 \in \textstyle\sum_{i=1}^{k} \lambda_i \left(\partial^M f_i(u) - v_i \partial^M q_i(u)\right) + \textstyle\sum_{j=1}^{m} \mu_j \partial^M g_j(u) \\
+ \textstyle\sum_{s=1}^{p} \vartheta_s \left(\partial^M h_s(u) \cup \partial^M(-h_s)(u)\right) + N(u, Q), \\
\lambda_i \left(f_i(u) - v_i q_i(u)\right) \geqq 0, \ i \in I, \\
\mu_j g_j(u) \geqq 0, \ j \in J, \\
\vartheta_s h_s(u) \geqq 0, \ s \in S, \\
u \in Q, \lambda \geq 0, \textstyle\sum_{i=1}^{k} \lambda_i = 1, \mu \geqq 0, \vartheta \geqq 0, h(u) \in \left(\vartheta - \Delta(0, \|\vartheta\|)\right)^0,
\end{aligned} \tag{PVD}$$

where $\Delta(0, \|\vartheta\|) = \{\gamma \in R^p : \|\gamma\| = \|\vartheta\|\}$. We denote by Γ the set of all feasible solutions $(u, \lambda, \mu, \vartheta, v) \in Q \times R_+^k \times R_+^m \times R_+^p \times R_+^k$ in the parametric dual problem (PVD), that is, all points $(u,\lambda,\mu,\vartheta,v)\in Q\times R^k_+\times R^m_+\times R^p_+\times R^k_+$ for which all constraints of (PVD) are fulfilled. Further, we denote by U the set $U=\{u\in Q:(u,\lambda,\mu,\vartheta,v)\in\Gamma\}$.

In proving various parametric duality results between (MFP) and (PVD), we also use the concept of nonsmooth L-univexity introduced in this paper.

Theorem 4.1. *(Weak duality). Let x and $(u,\lambda,\mu,\vartheta,v)$ be any feasible solutions in (MFP) and (PVD), respectively. Further, assume that $(f,-q,g,h)$ is L-univex-infine at u on $\Omega\cup U$ (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h), where $b_{f_i}(\cdot,\cdot)=b_{q_i}(\cdot,\cdot)=b_i(\cdot,\cdot)$, $\Phi_{f_i}(\cdot)=\Phi_{q_i}(\cdot)=\Phi_i(\cdot)$, $i\in I$, each function Φ_i, $i\in I$, is an increasing superlinear function with $\Phi_i(0)=0$, each function Φ_{g_j}, $j\in J$, is an increasing nonnegative homogenous function with $\Phi_{g_j}(0)=0$, each function Φ_{h_s}, $s\in S$, satisfies the condition $\Phi_{h_s}(0)=0$. Then, $\varphi(x)\nless v$.*

Proof. Let x and $(u,\lambda,\mu,\vartheta,v)$ be any feasible solutions in (MFP) and (PVD), respectively. Since $(u,\lambda,\mu,\vartheta,v)\in\Gamma$ satisfies all constraints of (PVD), there exist $\xi_i^*\in\partial^M f_i(u)$, $i\in I$, $\beta_i^*\in\partial^M(-q_i)(u)$, $i\in I$, $\zeta_j^*\in\partial^M g_j(u)$, $j\in J$, $\varsigma_s^*\in\partial^M h_s(u)\cup\partial^M(-h_s)(u)$, $s\in S$, such that

$$-\left(\sum_{i=1}^k\lambda_i(\xi_i^*-v_i\beta_i^*)+\sum_{j=1}^m\mu_j\zeta_j^*+\sum_{s=1}^p\vartheta_s\varsigma_s^*\right)\in N(u;Q). \tag{4.1}$$

We proceed by contradiction. Suppose, contrary to the result, that

$$\varphi(x)<v.$$

Hence, by the definition of φ and by the assumptions that each function Φ_i, $i\in I$, is an increasing function satisfying the condition $\Phi_i(0)=0$, $b_i(x,u)>0$, it follows that

$$\sum_{i=1}^k b_i(x,u)\lambda_i\Phi_i(f_i(x)-v_iq_i(x))<0. \tag{4.2}$$

By assumption, $(f,-q,g,h)$ is L-univex-infine at $u\in\Gamma$ on $\Omega\cup U$ (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h). Then, by Definition 3.8, for all $x\in\Omega$ and any $\xi_i\in\partial^M f_i(u)$, $i\in I$, $\beta_i\in\partial^M(-q_i)(u)$, $i\in I$, $\zeta_j\in\partial^M g_j(u)$, $j\in J$, $\varsigma_s\in\partial^M h_s(u)\cup\partial^M(-h_s)(u)$, $s\in S$, there exists $d\in N(u;Q)^0$ such that

$$b_i(x,u)\Phi_i(f_i(x)-f_i(u))\geqq\langle\xi_i,d\rangle,\ i\in I, \tag{4.3}$$

$$b_i(x,u)\Phi_i(-q_i(x)-(-q_i(u)))\geqq\langle-\beta_i,d\rangle,\ i\in I, \tag{4.4}$$

$$b_{g_j}(x,u)\Phi_{g_j}\left(g_j(x)-g_j(u)\right)\geqq\left\langle\zeta_j,d\right\rangle,\ j\in J, \tag{4.5}$$

$$b_{h_s}(x,u)\Phi_{g_j}(h_s(x)-h_s(u))=w_s\langle\varsigma_s,d\rangle,\ s\in S, \tag{4.6}$$

where $w_s=1$ (whenever $\varsigma_s\in\partial^M h_s(u)$) or $w_s=-1$ (whenever $\varsigma_s\in\partial^M(-h_s)(u)$). Then, multiplying each inequality (4.3) and each inequality (4.4) by λ_i, $i\in I$, and by v_i, $i\in I$, each inequality (4.5) by μ_j, $j\in J$, and each equation (4.6) by ϑ_s, $s\in S$, using that each function Φ_i, $i\in I$, is nonnegative homogenous, we get that the inequality

$$\sum_{i=1}^k b_i(x,u)\lambda_i\left[\Phi_i(f_i(x)-f_i(u))+\Phi_i(v_i(-q_i)(x)-v_i(-q_i)(u))\right]+$$

$$\sum_{j\in J(u)}b_{g_j}(x,u)\mu_j\Phi_{g_j}\left(g_j(x)-g_j(u)\right)+\sum_{s\in S}b_{h_s}(x,u)\frac{\vartheta_s}{w_s}\Phi_{h_s}(h_s(x)-h_s(u))\geqq$$

$$\left\langle \sum_{i=1}^{k} \lambda_i (\xi_i - v_i \beta_i) + \sum_{j=1}^{m} \mu_j \zeta_j + \sum_{s=1}^{p} \vartheta_s \varsigma_s, d \right\rangle \tag{4.7}$$

holds for any $\xi_i \in \partial^M f_i(u)$, $-\beta_i \in \partial^M(-q_i)(u)$, $i \in I$, $\zeta_j \in \partial^M g_j(u)$, $j \in J$, $\varsigma_s \in \partial^M h_s(u) \cup \partial^M(-h_s)(u)$, $s \in S$. Hence, (4.7) is also satisfied for $\xi_i^* \in \partial^M f_i(u)$, $i \in I$, $-\beta_i^* \in \partial^M(-q_i)(u)$, $i \in I$, $\zeta_j^* \in \partial^M g_j(u)$, $j \in J$, $\varsigma_s^* \in \partial^M h_s(u) \cup \partial^M(-h_s)(u)$, $s \in S$. Since $d \in N(\bar{x}; Q)^0$, by Definition 2.1 and (4.1), we have

$$\left\langle \sum_{i=1}^{k} \lambda_i (\xi_i^* - v_i \beta_i^*) + \sum_{j=1}^{m} \mu_j \zeta_j^* + \sum_{s=1}^{p} \vartheta_s \varsigma_s^*, d \right\rangle \geqq 0. \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$\sum_{i=1}^{k} b_i(x,u)\lambda_i [\Phi_i(f_i(x) - f_i(u)) + \Phi_i(v_i(-q_i)(x) - v_i(-q_i)(u))] +$$

$$\sum_{j=1}^{m} b_{g_j}(x,u)\Phi_{g_j}(\mu_j g_j(x) - \mu_j g_j(u)) + \sum_{s=1}^{p} b_{h_s}(x,u)\frac{\vartheta_s}{w_s}\Phi_{h_s}(h_s(x) - h_s(u)) \geqq 0.$$

Using $x \in \Omega$ and $(u, \lambda, \mu, \vartheta, v) \in \Gamma$ together with the assumptions that each function Φ_{g_j}, $j \in J$, satisfies $\Phi_{g_j}(0) = 0$, $b_{g_j}(x,u) > 0$, $j \in J$, each function Φ_i, $i \in I$, is a superlinear function, we get

$$\sum_{i=1}^{k} b_i(x,u)\lambda_i \Phi_i(f_i(x) - v_i q_i(x) - (f_i(u) - v_i q_i(u))) +$$

$$\sum_{s=1}^{p} b_{h_s}(x,u)\frac{\vartheta_s}{w_s}\Phi_{h_s}(-h_s(u)) \geqq 0.$$

Since $f_i(u) - v_i q_i(u) \geqq 0$ for any $i \in I$, and each function Φ_i, $i \in I$, is an increasing function, the above inequality gives

$$\sum_{i=1}^{k} b_i(x,u)\lambda_i \Phi_i(f_i(x) - v_i q_i(x)) + \sum_{s=1}^{p} b_{h_s}(x,u)\frac{\vartheta_s}{w_s}\Phi_{h_s}(-h_s(u)) \geqq 0.$$

Now, let us set $\gamma_s = \frac{\vartheta_s}{w_s}$, $s \in S$. Note that since $\gamma = (\gamma_1, ..., \gamma_p) \in R^p$ and $\|\gamma\| = \|\vartheta\|$, therefore, $\gamma \in \Delta(0, \|\vartheta\|)$. Then, the above inequality can be rewritten as follows

$$\sum_{i=1}^{k} b_i(x,u)\lambda_i \Phi_i(f_i(x) - v_i q_i(x)) + \sum_{s=1}^{p} b_{h_s}(x,u)\vartheta_s \Phi_{h_s}(-h_s(u))$$

$$-\sum_{s=1}^{p} b_{h_s}(x,u)(\vartheta_s - \gamma_s)\Phi_{h_s}(-h_s(u)) \geqq 0.$$

Since each function Φ_{h_s}, $s \in S$, is a nonnegative homogenous function, we get

$$\sum_{i=1}^{k} b_i(x,u)\lambda_i \Phi_i(f_i(x) - v_i q_i(x)) + \tag{4.9}$$

$$\sum_{s=1}^{p} b_{h_s}(x,u)\Phi_{h_s}(-\vartheta_s h_s(u)) - \sum_{s=1}^{p} b_{h_s}(x,u)\Phi_{h_s}((\vartheta_s - \gamma_s)(-h_s(u))) \geqq 0.$$

By assumption, each function Φ_{h_s}, $s \in S$, is an increasing function satisfying $\Phi_{h_s}(0) = 0$. From $(u, \lambda, \mu, \vartheta, v) \in \Gamma$, it follows that $(\vartheta_s - \gamma_s)(-h_s(u)) \geqq 0$, $s \in S$. Hence, (4.9) implies that the following inequality

$$\sum_{i=1}^{k} b_i(x,u)\lambda_i \Phi_i\left(f_i(x) - v_i q_i(x)\right) \geqq 0$$

holds, contradicting (4.2). This completes the proof of this theorem. □

If we assume slightly stronger assumptions, then we obtain stronger result.

Theorem 4.2. *(Weak duality). Let x and $(u, \lambda, \mu, \vartheta, v)$ be any feasible solutions in (MFP) and (PVD), respectively. Further, assume that $(f, -q, g, h)$ is strictly L-univex-infine at u on $\Omega \cup U$ (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h), where $b_{f_i}(\cdot,\cdot) = b_{q_i}(\cdot,\cdot) = b_i(\cdot,\cdot)$, $\Phi_{f_i}(\cdot) = \Phi_{q_i}(\cdot) = \Phi_i(\cdot)$, $i \in I$, each function Φ_i, $i \in I$, is an increasing superlinear function with $\Phi_i(0) = 0$, each function Φ_{g_j}, $j \in J$, is an increasing nonnegative homogenous function with $\Phi_{g_j}(0) = 0$, each function Φ_{h_s}, $s \in S$, satisfies the condition $\Phi_{h_s}(0) = 0$. Then, $\varphi(x) \nleq v$.*

Theorem 4.3. *(Strong duality). Let $\overline{x} \in \Omega$ be a weak Pareto solution (a Pareto solution) in (MFP) and the constraint qualification be satisfied for (MFP). Then there exist $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$, $\overline{\vartheta} \in R^p$ and $\overline{v} \in R^k$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}, \overline{v}\right)$ is feasible in (PVD) and the objective functions of (MFP) and (PVD) are equal at these points. If all hypotheses of the weak duality theorem are also satisfied, then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}, \overline{v}\right)$ is a weakly efficient solution (an efficient solution) of a maximum type in (PVD).*

Proof. Let $\overline{x} \in \Omega$ be a weak Pareto solution in (MFP) and the constraint qualification be satisfied for (MFP). Then, by Theorem 3.6 (see also Remark 3.7), there exist $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$, $\overline{\vartheta} \in R^p$ and $\overline{v} \in R^k$ such that the conditions (3.1)-(3.3) are fulfilled. Moreover, since $\overline{x} \in \Omega$, we have that $h_s(\overline{x}) = 0$ for all $s \in S$. Hence, this implies that $\left\langle \overline{\vartheta} - \gamma, h(\overline{x}) \right\rangle = 0$ for all $\gamma \in R^p$ satisfying the condition $\|\gamma\| = \left\|\overline{\vartheta}\right\|$. This means that $h(\overline{x}) \in \left(\overline{\vartheta} - \Delta\left(0, \left\|\overline{\vartheta}\right\|\right)\right)^0$. Hence, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}, \overline{v}\right)$ is feasible in (PVD) and the objective functions of (MFP) and (PVD) are equal at these points. Further, assume that all hypotheses of the weak duality theorem are satisfied. We proceed by contradiction. Suppose, contrary to the result, that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}, \overline{v}\right)$ is not a weak efficient solution of a maximum type in (PVD). Then, there exists other $\left(\widetilde{u}, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\vartheta}, \widetilde{v}\right) \in \Gamma$ such that $\overline{v} < \widetilde{v}$. Since $\overline{v} = \varphi(\overline{x})$, the above inequality implies that $\varphi(\overline{x}) < \widetilde{v}$. Since $\overline{x} \in \Omega$ and $\left(\widetilde{u}, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\vartheta}, \widetilde{v}\right) \in \Gamma$, this is a contradiction to weak duality (Theorem 4.1). □

Theorem 4.4. *(Converse duality). Let $\left(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}, \overline{v}\right) \in \Gamma$ be a weakly efficient solution of a maximum type (an efficient solution of a maximum type) in (PVD) such that $\overline{u} \in \Omega$. Further, assume that $(f, -q, g, h)$ is (strictly) L-univex-infine at $\overline{u}$ on $\Omega \cup U$ (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h), where $\Phi_{f_i}(\cdot) = \Phi_{q_i}(\cdot) = \Phi_i(\cdot)$, $i \in I$, each function Φ_i, $i \in I$, is an increasing superlinear function with $\Phi_i(0) = 0$, each function Φ_{g_j}, $j \in J(\overline{x})$, is an increasing nonnegative homogenous function with $\Phi_{g_j}(0) = 0$, each function Φ_{h_s}, $s \in S$, satisfies the condition $\Phi_{h_s}(0) = 0$, and, moreover, $b_{f_i}(x, \overline{u}) = b_{q_i}(x, \overline{u}) = b_i(x, \overline{u})$, $i \in I$, for all $x \in \Omega$. Then $\overline{u}$ is a weak Pareto solution (a Pareto solution) in (MFP).*

Proof. Suppose, contrary to the result, that $\overline{u}$ is not a weak Pareto solution in (MFP). Then, by Definition 3.1, there exists $\widetilde{x} \in \Omega$ such that $\varphi(\widetilde{x}) < \varphi(\overline{u})$. Hence, this inequality implies that $f_i(\widetilde{x}) - \overline{v}_i q_i(\widetilde{x}) < 0$, $i \in I$. Since each function Φ_i, $i \in I$, is an increasing superlinear function with $\Phi_i(0) = 0$ and $b_i(\widetilde{x},\overline{u}) > 0$, $i \in I$, we get

$$\sum_{i=1}^{k} b_i(\widetilde{x},\overline{u})\,\overline{\lambda}_i \Phi_i\left(f_i(\widetilde{x}) - \overline{v}_i q_i(\widetilde{x})\right) < 0. \tag{4.10}$$

In the similar way as in the proof of Theorem 4.1, under the assumption of L-univexity of $(f,-q,g,h)$, we get that the inequality

$$\sum_{i=1}^{k} b_i(\widetilde{x},\overline{u})\overline{\lambda}_i\left[\Phi_i\left(f_i(\widetilde{x}) - f_i(\overline{u})\right) + \Phi_i\left(\overline{v}_i(-q_i)(\widetilde{x}) - \overline{v}_i(-q_i)(\overline{u})\right)\right] + \tag{4.11}$$

$$\sum_{j=1}^{m} b_{g_j}(\widetilde{x},\overline{u})\overline{\mu}_j\Phi_{g_j}\left(g_j(\widetilde{x}) - g_j(\overline{u})\right) + \sum_{s=1}^{p} b_{h_s}(\widetilde{x},\overline{u})\frac{\overline{\vartheta}_s}{w_s}\Phi_{h_s}\left(h_s(\widetilde{x}) - h_s(\overline{u})\right) \geqq$$

$$\left\langle \sum_{i=1}^{k} \overline{\lambda}_i(\xi_i + \overline{v}_i\beta_i) + \sum_{j=1}^{m} \overline{\mu}_j\zeta_j + \sum_{s=1}^{p} \overline{\vartheta}_s\varsigma_s, d \right\rangle$$

holds for any $\xi_i \in \partial^M f_i(\overline{u})$, $-\beta_i \in \partial^M(-q_i)(\overline{u})$, $i \in I$, $\zeta_j \in \partial^M g_j(\overline{u})$, $j \in J$, $\varsigma_s \in \partial^M h_s(\overline{u}) \cup \partial^M(-h_s)(\overline{u})$, $s \in S$, and some $d \in N(\overline{u};Q)^0$. Since $\left(\overline{u},\overline{\lambda},\overline{\mu},\overline{\vartheta},\overline{v}\right) \in \Gamma$, using the first constraint of (PVD), there exist $\xi_i^* \in \partial^M f_i(\overline{u})$, $i \in I$, $\beta_i^* \in \partial^M(-q_i)(\overline{u})$, $i \in I$, $\zeta_j^* \in \partial^M g_j(\overline{u})$, $j \in J$, $\varsigma_s^* \in \partial^M h_s(\overline{u}) \cup \partial^M(-h_s)(\overline{u})$, $s \in S$, such that

$$-\left(\sum_{i=1}^{k} \overline{\lambda}_i(\xi_i^* + \overline{v}_i\beta_i^*) + \sum_{j=1}^{m} \overline{\mu}_j\zeta_j^* + \sum_{s=1}^{p} \overline{\vartheta}_s\varsigma_s^*\right) \in N(\overline{u};Q). \tag{4.12}$$

Combining (4.11) and (4.12), and the properties of Φ_i, $i \in I$, Φ_{g_j}, $j \in J$, we obtain

$$\sum_{i=1}^{k} b_i(\widetilde{x},\overline{u})\overline{\lambda}_i\left[\Phi_i\left(f_i(\widetilde{x}) - \overline{v}_i q_i(\widetilde{x}) - (f_i(\overline{u}) - \overline{v}_i q_i(\overline{u}))\right)\right] + \tag{4.13}$$

$$\sum_{j=1}^{m} b_{g_j}(\widetilde{x},\overline{u})\Phi_{g_j}\left(\overline{\mu}_j g_j(\widetilde{x}) - \overline{\mu}_j g_j(\overline{u})\right) + \sum_{s=1}^{p} b_{h_s}(\widetilde{x},\overline{u})\frac{\overline{\vartheta}_s}{w_s}\Phi_{h_s}\left(h_s(\widetilde{x}) - h_s(\overline{u})\right) \geqq 0.$$

Since each function Φ_i, $i \in I$, is an increasing function, each function Φ_{g_j}, $j \in J$, is an increasing nonnegative function with $\Phi_{g_j}(0) = 0$ and by $\widetilde{x} \in \Omega$, we get

$$\sum_{i=1}^{k} b_i(\widetilde{x},\overline{u})\overline{\lambda}_i\left[\Phi_i\left(f_i(\widetilde{x}) - \overline{v}_i q_i(\widetilde{x})\right)\right] + \sum_{s=1}^{p} b_{h_s}(\widetilde{x},\overline{u})\frac{\overline{\vartheta}_s}{w_s}\Phi_{h_s}\left(-h_s(\overline{u})\right) \geqq 0. \tag{4.14}$$

In the similar way as in the proof of Theorem 4.1, (4.14) implies that the inequality

$$\sum_{i=1}^{k} b_i(\widetilde{x},u)\lambda_i\Phi_i\left(f_i(\widetilde{x}) - v_i q_i(\widetilde{x})\right) \geqq 0$$

holds, contradicting (4.10). This completes the proof of this theorem. □

Theorem 4.5. *(Strict converse duality). Let $\overline{x} \in \Omega$ be a weak Pareto solution in (MFP) and $\left(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}, \overline{v}\right)$ be a weakly efficient solution of a maximum type in (PVD) and all hypotheses of strong duality (Theorem 4.3) be fulfilled. Further, assume that $(f, -q, g, h)$ is L-univex-infine at $\overline{u}$ on $\Omega \cup U$ (with respect to Φ_f, Φ_q, Φ_g, Φ_h, b_f, b_q, b_g, b_h), where $\Phi_{f_i}(\cdot) = \Phi_{q_i}(\cdot) = \Phi_i(\cdot)$, $i \in I$, each function Φ_i, $i \in I$, is an increasing superlinear function with $\Phi_i(0) = 0$, each function Φ_{g_j}, $j \in J(\overline{x})$, is an increasing nonnegative homogenous function with $\Phi_{g_j}(0) = 0$, each function Φ_{h_s}, $s \in S$, satisfies the condition $\Phi_{h_s}(0) = 0$, and, moreover, $b_{f_i}(\overline{x}, \overline{u}) = b_{q_i}(\overline{x}, \overline{u}) = b_i(\overline{x}, \overline{u})$, $i \in I$. Then, $\overline{x} = \overline{u}$, that is, $\overline{u}$ is a weak Pareto solution of (MFP).*

Proof. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \neq \overline{u}$. Since all hypotheses of strong duality are fulfilled, by Theorem 4.3, one has

$$\varphi(\overline{x}) = \overline{v}. \tag{4.15}$$

In the similar way as in Theorem 4.1, we obtain that the following inequality

$$\sum_{i=1}^{k} b_i(\overline{x}, \overline{u}) \overline{\lambda}_i \Phi_i \left(f_i(\overline{x}) - \overline{v}_i q_i(\overline{x}) \right) \geqq 0$$

holds. Since $b_i(\overline{x}, \overline{u}) > 0$ for any $i \in I$, the above inequality implies that there exists at least one $i \in I$ such that $f_i(\overline{x}) - \overline{v}_i q_i(\overline{x}) > 0$. This means that the inequality $\frac{f_i(\overline{x})}{q_i(\overline{x})} = \varphi_i(\overline{x}) > \overline{v}_i$ is satisfied for at least one $i \in I$, contradicting (4.15). This completes the proof of this theorem. □

5. CONCLUSIONS

In this work, optimality and duality results have been analyzed for a new class of nonsmooth multiobjective fractional programming problems in which the involved functions are locally Lipschitz. Namely, the parametric Karush-Kuhn-Tucker necessary optimality conditions have been formulated for such nondifferentiable vector optimization problems via Mordukhovich subdifferentials of the involved functions. Since the limiting/Mordukhovich subdifferential of a real-valued function at a given point is contained in the Clarke subdifferential of such a function at the corresponding point (see [18]), the necessary conditions formulated in terms of the limiting/Mordukhovich subdifferential are sharper than the corresponding ones expressed in terms of the Clarke subdifferential. Moreover, sufficient optimality conditions for the considered nondifferentiable multiobjective fractional programming problem are established by utilizing the new concept of L-univexity, which is introduced in this paper by the notion of limiting subdifferential for locally Lipschitz functions. Comparing them to the similar sufficient optimality conditions formulated in the literature for nonsmooth multiobjective fractional programming problems in terms the limiting/Mordukhovich subdifferentials of the involved functions, we have shown that they are applicable for a larger class of such nonsmooth optimization problems. Further, for the considered nonsmooth multiobjective fractional programming problem, its vector parametric Schaible dual problem has been defined and several duality theorems have been proved also under L-univexity assumptions. Thus, optimality conditions and duality theorems in this work generalize similar results existing in optimization literature to a new class of nonconvex nonsmooth multiobjective rational programming problems.

It seems that the techniques employed in this paper can be used in proving similarly results for other classes of multiobjective fractional programming problems. We shall investigate these problems in the subsequent papers.

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