Commun. Optim. Theory 2024 (2024) 20 https://doi.org/10.23952/cot.2024.20

Communications in Optimization Theory
Available online at http://cot.mathres.org

# MOND-WEIR AND WOLFE TYPE DUALITY FOR NONSMOOTH MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS WITH EQUILIBRIUM CONSTRAINTS ON HADAMARD MANIFOLDS

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Dedicated to the Memory of Prof. Ram U. Verma

**Abstract.** This article deals with a certain category of nonsmooth multiobjective fractional programming problems with equilibrium constraints in the setting of Hadamard manifolds (abbreviated as, (NMFPPEC)). The generalized Guignard constraint qualification (abbreviated as, (GGCQ)) for (NMFPPEC) and Karush-Kuhn-Tucker (abbreviated as, KKT) type necessary criteria of Pareto efficiency for (NMFPPEC) are presented. Mond-Weir as well as Wolfe type dual models related to (NMFPPEC) are formulated. Weak, strong, and strict converse duality results are derived relating (NMFPPEC) and the respective dual models. Suitable non-trivial examples have been furnished to demonstrate the significance of the results established in this article. The results derived in the article extend and generalize several notable results previously existing in the literature.

**Keywords.** Duality; Equilibrium constraints; Hadamard manifolds; Multiobjective fractional programming.

**2020 Mathematics Subject Classification.** 90C34, 90C46, 90C48, 90C29.

## 1. Introduction

In the theory of mathematical programming, an optimization problem accompanied by certain complementarity constraints or variational inequality constraints is referred to as a mathematical programming problem with equilibrium constraints (in brief, (MPEC)). One of the first attempts in investigating such optimization problems is due to Harker and Pang [17], who explored the existence of efficient solutions for (MPECs). Due to its immense scope of applicability in numerous fields of science, technology, and engineering (see, for instance, [6, 28, 29]), (MPECs) have been studied by numerous authors in recent years. For further details and an updated survey of (MPEC) and its applications, we refer the readers to [22, 31, 35, 46] and the references cited therein.

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Received: May 16, 2023; Accepted: October 11, 2023.

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In the last few decades, it has been observed that numerous real-life problems emerging in various areas related to engineering, technology, and science can be formulated in a more effective way on manifold setting instead of Euclidean space, see [5, 37]. Further, extending and generalizing the methods of optimization from the setting of Euclidean spaces to the setting of manifolds have several crucial advantages. For instance, by appropriately using the notions of Riemannian geometry, several constrained mathematical optimization problems can be conveniently converted into unconstrained problems. Apart from this, numerous non-convex optimization problems can be converted into convex problems by employing Riemannian geometry perspective (see, for instance, [25, 27]). Furthermore, it is a common observation that numerous important constraints which naturally arise in certain mathematical programming problems have a relative interior that can be viewed as Hadamard manifolds, for instance, the hypercube $(0,1)^n$ endowed with the metric $Z^{-2}(I-Z)^{-2} = \operatorname{diag}\left(z_1^{-2}(1-z_1)^{-2}, \ldots, z_n^{-2}(1-z_n)^{-2}\right)$ and the set containing every symmetric positive definite matrix $\mathcal{S}_{++}^n$ with the metric $-\log\det X$ are Hadamard manifolds (see, for instance, [26]). As a result, a wider range of mathematical programming problems can be solved by formulating the problems in the framework of Riemannian and Hadamard manifolds. Some important concepts of optimization, such as convex sets and convex functions, have been generalized, and corresponding notions of geodesic convex sets and functions in manifold setting have been introduced (see, for instance, [30]). Further, Udrişte [37] generalized the notion of geodesic convex function in Riemannian manifolds, and provided the notions of geodesic pseudoconvex and quasiconvex function in the framework of manifolds. In recent times, various other notions and concepts involved in mathematical programming have been extended from Euclidean spaces to Riemannian and Hadamard manifolds by several authors; see, for instance, [11, 13, 18, 19, 47].

Several regularity and optimality criteria for (MPEC) were investigated by Chen and Florian [7]. Abadie constraint qualification for (MPEC) was studied by Flegel and Kanzow [9]. Optimality conditions for (MPEC) were explored by Ye [48]. Guignard constraint qualification and criteria of optimality for for (MPEC) were explored by Flegel and Kanzow [10]. KKT-type criteria for optimality as well as some duality results for multiobjective (MPEC), were deduced by Singh and Mishra [31]. Treanţă et al. [35] studied optimality conditions for multiobjective (MPEC) on Hadamard manifolds. Recently, Ghosh et al. [12] studied optimality conditions and duality for multiobjective fractional programming problems with equilibrium constraints on Hadamard manifolds.

It is a well-known fact that duality plays a very important role in the modern theory of optimization. Duality theory enables us to investigate a primal optimization problem from the perspective of a dual problem (see, for instance, [3, 23, 24]). In recent times, several researchers have explored duality theory for various optimization problems. For instance, duality models for vector-valued fractional control problems involving $(\rho, b)$-quasiinvexity were studied by Treanţă and Mititelu [36]. Several duality theorems for $(\rho, \psi, d)$-quasiinvex multiobjective optimization problems with interval-valued components were investigated by Treanţă [32]. Duality for multiobjective interval-valued variational control problems was explored by Treanţă [33]. Guo et al. [15] derived optimality conditions and duality results for a class of generalized convex interval-valued optimization problems. Mond-Weir and Wolfe-type dual models for set-valued fractional minimax problems in terms of contingent epi-derivative of second-order were investigated by Das et al. [8]. Several symmetric gH-derivative applications to dual interval-valued optimization problems were explored by Guo et al. [16]. Duality results for a class of constrained robust nonlinear optimization problems were investigated by Treanţă and Saeed [34]. Antczak et al. [2] studied efficiency conditions and duality results for a class of nonconvex non-differentiable multiobjective fractional variational control problems. Recently, optimality conditions and duality results for $E$-differentiable multiobjective programming involving $V-E$-type $I$ functions were investigated by Abdulaleem and Treanţă [1].

Motivated by the results established in [7, 31, 35, 41, 48], a category of nonsmooth multiobjective fractional programming problems with equilibrium constraints (NMFPPEC) is studied in the present article in the setting of Hadamard manifolds. The generalized Guignard constraint qualification (GGCQ) for (NMFPPEC) is presented. Further, the KKT type necessary criteria of optimality for (NMFPPEC) is presented. subsequently, Mond-Weir, as well as Wolfe-type dual models related to (NMFPPEC), are formulated. Weak, strong, and strict converse duality results are derived relating (NMFPPEC) and the respective dual models. Suitable non-trivial examples have been furnished to demonstrate the significance of the results established in this article. The novelty and the contributions of the present paper are twofold. Firstly, the duality results derived by [31] are extended to the class of (NMFPPEC) in the setting Hadamard manifolds by the results derived in this article. Secondly, the results deduced in this paper extend the corresponding results of [41] from the setting of multiobjective (MPEC) to a wider category of problems, namely, (NMFPPEC). In particular, we have extended the corresponding results of [12] smooth multiobjective fractional programming problems with equilibrium constraints to (NMFPPEC). To the best of our knowledge, duality models for (NMFPPEC) have not been explored before in the Hadamard manifold setting.

The remaining portion of the article unfolds in the following manner. We recollect some basic definitions and mathematical preliminaries that will be helpful in this article in Section 2. We define (NMFPPEC) in manifold setting and introduce (GGCQ) for (NMFPPEC) in Section 3. Further, we present KKT-type necessary criteria of optimality employing (GGCQ). In Section 4 and Section 5, Mond-Weir as well as Wolfe-type dual models related to (NMFPPEC) are formulated, respectively. Weak, strong and strict converse duality results are derived relating (NMFPPEC) and respective the dual models. Finally, in Section 6, we draw conclusions to our work in this article and further discuss some future courses of research.

## 2. Notation and Mathematical Preliminaries

The standard symbols $\mathbb{R}^n$ and $\mathbb{N}$ are employed to signify the Euclidean space having dimension $n$, and the set of all natural numbers, respectively. The non-negative orthant of $\mathbb{R}^n$, denoted by the notation $\mathbb{R}^n_+$, is defined as: $\mathbb{R}^n_+ := \{(z_1, z_2, \ldots, z_n) : z_k \geq 0, \forall k = 1, 2, \ldots, n\}$. We use the symbol $\langle \cdot, \cdot \rangle$ to signify the usual Euclidean inner product on the set $\mathbb{R}^n$. For arbitrary $\alpha, \beta \in \mathbb{R}^n$, the following notation for inequalities will be employed in the sequel:

$$\alpha \prec \beta \iff \alpha_k < \beta_k, \quad \forall k = 1, 2, \ldots, n.$$

$$\alpha \preceq \beta \iff \begin{cases} \alpha_k \leq \beta_k, & \text{for all } k = 1, 2, \ldots, n; \\ \alpha_r < \beta_r, & \text{for at least one } r \in \{1, 2, \ldots, n\}. \end{cases}$$

We shall be using the notation $\mathscr{M}$ to signify a smooth manifold having dimension $n$, where $n$ is any natural number. Let $y^* \in \mathscr{M}$ be arbitrary. The set that contains every tangent vector at the element $y^* \in \mathscr{M}$ is known as the tangent space at $y^*$, and is signified by $T_{y^*}\mathscr{M}$. For any element

$y^* \in \mathscr{M}$, $T_{y^*}\mathscr{M}$ is a real linear space, having a dimension $n$, $n \in \mathbb{N}$. In case we are restricted to real manifolds, $T_{y^*}\mathscr{M}$ is isomorphic to the $n$-dimensional Euclidean space $\mathbb{R}^n$. For any arbitrary subset $\mathcal{W} \subset T_{y^*}\mathscr{M}$, the closure and convex hull of $\mathcal{W}$ in $T_{y^*}\mathscr{M}$ is denoted by the symbols $\mathrm{cl}(\mathcal{W})$ and $\mathrm{co}(\mathcal{W})$, respectively.

A Riemannian metric, denoted by the notation $\mathscr{G}$ on the set $\mathscr{M}$ is a 2-tensor field that is symmetric as well as positive-definite. For every pair of elements $w_1, w_2 \in T_{y^*}\mathscr{M}$, the inner product of $w_1$ and $w_2$ is given by: $\langle w_1, w_2\rangle_{y^*} = \mathscr{G}_{y^*}(w_1, w_2)$, where the symbol $\mathscr{G}_{y^*}$ denotes the Riemannian metric at the element $y^* \in \mathscr{M}$. The norm corresponding to the inner product $\langle w_1, w_2\rangle_{y^*}$ is denoted by $\|\cdot\|_{y^*}$ (or simply, $\|\cdot\|$, when there is no ambiguity regarding the subscript).

Let $a, b \in \mathbb{R}$, $a < b$ and $\nu : [a,b] \to \mathscr{M}$ be any piecewise differentiable curve that joins the elements $y^*$ and $\hat{z}$ in $\mathscr{M}$. That is, we have $\nu(a) = y^*$, $\nu(b) = \hat{z}$. The length of the curve $\nu$ is denoted by the notation $l(\nu)$ and is defined in the following manner:

$$l(\nu) := \int_a^b \|\nu'(t)\| dt.$$

For any differentiable curve $\nu$, a vector field $Y$ is referred to be parallel along the curve $\nu$, provided that the following condition is satisfied $\nabla_{\nu'} Y = 0$. If $\nabla_{\nu'}\nu' = 0$, then $\nu$ is termed as a geodesic. If $\|\nu\| = 1$, then the curve $\nu$ is said to be normalised.

For any $y^* \in \mathscr{M}$, the exponential function $\exp_{y^*} : T_{y^*}\mathscr{M} \to \mathscr{M}$ is given by $\exp_{y^*}(\hat{w}) = \nu(1)$, where $\nu$ is a geodesic which satisfies $\nu(0) = y^*$ and $\nu'(0) = \hat{w}$. A Riemannian manifold $\mathscr{M}$ is referred to as geodesic complete, provided that the exponential function $\exp_u(v)$ is defined for every arbitrary $v \in T_p\mathscr{M}$ and $u \in \mathscr{M}$.

A Riemannian manifold is referred to as a Hadamard manifold (or, Cartan-Hadamard manifold) provided that $\mathscr{M}$ is simply connected, geodesic complete, as well as, has a nonpositive sectional curvature throughout. Henceforth, in our discussions, the notation $\mathscr{M}$ will always signify a Hadamard manifold of dimension $n$, unless it is specified otherwise.

Let $y^* \in \mathscr{M}$ be some arbitrary element lying in the Hadamard manifold $\mathscr{M}$. Then, the exponential function on the tangent space $\exp_{y^*} : T_{y^*}\mathscr{M} \to \mathscr{M}$ is a globally diffeomorphic function. Moreover, the inverse of the exponential function $\exp_{y^*}^{-1} : \mathscr{M} \to T_{y^*}\mathscr{M}$ satisfies $\exp_{y^*}^{-1}(y^*) = 0$. Furthermore, for every pair of arbitrary elements $y_1^*, y_2^* \in \mathscr{M}$, there will always exist some unique normalized minimal geodesic $\nu_{y_1^*, y_2^*} : [0,1] \to \mathscr{M}$, such that the geodesic $\nu$ satisfies the following:

$$\nu_{y_1^*, y_2^*}(\tau) = \exp_{y_1^*}(\tau \exp{}_{y_1^*}^{-1}(y_2^*)), \quad \forall \tau \in [0,1].$$

Thus, every Hadamard manifold $\mathscr{M}$ of dimension $n$ is diffeomorphic to the corresponding $n$-dimensional Euclidean space $\mathbb{R}^n$. Unless specified otherwise, throughout the remaining part of the paper, we shall use the symbol $\mathscr{M}$ to denote a Hadamard manifold of dimension $n$.

Now, we recall the definition of locally Lipschitz function in the setting of Hadamard manifolds (see, for instance, [19]).

**Definition 2.1.** Let $\mathcal{G} \subseteq \mathscr{M}$ and $\Psi : \mathcal{G} \to \mathbb{R}$ be a real valued function. Then $\Psi$ is referred to as a locally Lipschitz at $z \in \mathcal{A}$ with rank $K$ ($K \in \mathbb{R}$, $K > 0$), if for every $z_1, z_2$ in some open neighborhood of $z$, the following is satisfied:

$$|\Psi(z_1) - \Psi(z_2)| \leq K\omega(z_1, z_2),$$

where $\omega(z_1, z_2)$ is the Riemannian distance between the points $z_1$ and $z_2$ on $\mathcal{G}$.

**Remark 2.2.** (a) If $\Psi$ is locally Lipschitz at every element $z \in \mathcal{G}$, then $\Psi$ is said to be locally Lipschitz on the set $\mathcal{A}$.

(b) It is worthwhile to note that several functions which are not locally Lipschitz in Euclidean space setting, can be considered as locally Lipschitz in the setting of Hadamard manifolds. For instance, let us consider the set $\mathscr{H}$ defined as:

$$\mathscr{H} := \{z = (z_2, z_2) \in \mathbb{R}^2 : z_1 > 0, z_2 > 0\}.$$

Consider the real valued function $\Psi : \mathscr{H} \to \mathbb{R}$ defined as follows:

$$\Psi(z) = \sum_{i=1}^{n} \ln (z_i),$$

for every $z = (z_1, \ldots, z_n) \in \mathscr{H}$. One can verify that the function $\Psi$ is not a locally Lipschitz function on the set $\mathscr{H}$ in the usual Euclidean sense. However, $\mathscr{H}$ can be considered as a Hadamard manifold by endowing the set $\mathscr{H}$ with the Riemannian metric given by

$$\langle u, v \rangle_y = \langle \mathscr{G}(y) u, v \rangle, \quad \forall u, v \in T_y \mathscr{H} = \mathbb{R}^2, \ y \in \mathscr{H},$$

where the symbol $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^2$ and

$$\mathscr{G}(y) = \begin{pmatrix} \frac{1}{y_1^2} & 0 \\ 0 & \frac{1}{y_2^2} \end{pmatrix}.$$

Then, it can be verified that the function $\Psi$ is locally Lipschitz with rank 1 on the set $\mathscr{H}$ in the setting of manifolds (see, for instance, [11]).

The following definitions are taken from [4].

**Definition 2.3.** Let us consider that $\Psi : \mathscr{M} \to \mathbb{R}$ be a real-valued and locally Lipschitz function defined on a Hadamard manifold $\mathscr{M}$. Let $x, y \in \mathscr{M}$ be arbitrary elements. Then, the generalized directional derivative of $\Psi$ at $y$ in the direction $v \in T_y \mathscr{M}$, is denoted by the symbol $\Psi^\circ(y; v)$, and is defined as follows:

$$\Psi^\circ(y; v) := \limsup_{x \to y, t \downarrow 0} \frac{\Psi\left(\exp_x t \left(d \exp_y\right)_{\exp_y^{-1} x} v\right) - \Psi(x)}{t},$$

where $\left(d \exp_y\right)_{\exp_y^{-1} x} : T_{\exp_y^{-1} x} (T_y \mathscr{M}) \simeq T_y \mathscr{M} \to T_x \mathscr{M}$ is the differential of the exponential function at $\exp_y^{-1} x$.

**Definition 2.4.** Let $\Psi : \mathscr{M} \to \mathbb{R}$ be any real-valued and locally Lipschitz function defined on a Hadamard manifold $\mathscr{M}$. Then, the generalized gradient (in other words, Clarke subdifferential) of $\Psi$ at $y \in \mathscr{M}$ is a subset $\partial_c \Psi(y)$ of $T_y \mathscr{M}$, and is defined by

$$\partial_c \Psi(y) := \left\{ \zeta \in T_y \mathscr{M} \mid \Psi^\circ(y; v) \geq \langle \zeta, v \rangle, \quad \forall v \in T_y \mathscr{M} \right\}.$$

**Remark 2.5.** (a) $\partial_c \Psi(y)$ is a nonempty, convex, compact subset of $T_y \mathscr{M}$ (see, for instance, [19]).

(b) If $\mathscr{M}$ is finite dimensional, then $\partial_c \Psi(y)$ is upper semicontinuous at $y$ (see, for instance, [19]).

Now, we recall the following lemma from [14] that will be useful in the sequel.

**Lemma 2.6.** *Let $\Psi : \mathscr{M} \to \mathbb{R}$ be a real-valued locally Lipschitz function defined on a Hadamard manifold $\mathscr{M}$. Let $\mathscr{D}_\Psi$ be the set of all points at which the function $\Psi$ is differentiable on $\mathscr{M}$. Then, $\mathscr{D}_\Psi$ is dense in $\mathscr{M}$, and*

$$\partial_c \Psi(y) = \operatorname{co}\left\{ \lim_{n\to\infty} \operatorname{grad}(y_n) : \{y_n\} \subseteq \mathscr{D}_\Psi,\ y_n \to y \right\}.$$

The following definition is from Udrişte [37].

**Definition 2.7.** Any subset $\mathcal{G}$ of a Hadamard manifold $\mathscr{M}$ is said to be a geodesic convex set in $\mathscr{M}$, if for every pair of distinct points $z_1, z_2 \in \mathcal{G}$ and for any geodesic $\gamma_{z_1,z_2} : [0,1] \to \mathscr{M}$ joining the points $z_1$ and $z_2$, we have

$$\gamma_{z_1,z_2}(\sigma) \in \mathcal{G}, \quad \forall \sigma \in [0,1],$$

where, $\gamma_{z_1,z_2}(\sigma) = \exp_{z_1}\left(\sigma \exp_{z_1}^{-1} z_2\right)$.

The following definition of geodesic convexity is from Barani [4].

**Definition 2.8.** Let $\Psi : \mathcal{G} \to \mathbb{R}$ be a locally Lipschitz real-valued function defined on a geodesic convex subset $\mathcal{G}$ of a Hadamard manifold $\mathscr{M}$. Then, $\Psi$ is referred to as a geodesic convex function at $y$, if for every $x \in \mathcal{G}$ and for every $\zeta \in \partial_c \Psi(y)$, we have

$$\Psi(x) - \Psi(y) \geq \left\langle \zeta, \exp_y^{-1} x \right\rangle_y.$$

Similarly, $\Psi$ is referred to as a strictly geodesic convex function at $y$, if for every $x \in \mathcal{G}$, $x \neq y$ and every $\zeta \in \partial_c \Psi(y)$, we have

$$\Psi(x) - \Psi(y) > \left\langle \zeta, \exp_y^{-1} x \right\rangle_y.$$

For more details on geodesic convex functions on Hadamard manifolds, we refer the reader to [30, 37, 38, 40, 41, 42, 43, 44, 45] and the references cited therein.

## 3. Pareto Efficiency Criteria for (NMFPPEC)

In this section, a nonsmooth multiobjective fractional programming problem with equilibrium constraints (NMFPPEC) is considered in the setting of Hadamard manifolds. We deduce KKT type necessary criteria of optimality for (NMFPPEC) by employing the generalized Guignard constraint qualification.

Let us consider the following (NMFPPEC) in the setting of Hadamard manifolds:

$$\text{(NMFPPEC)} \quad \text{Minimize} \quad \frac{\mathcal{A}(y)}{\mathcal{B}(y)} := \left( \frac{\mathcal{A}_1(y)}{\mathcal{B}_1(y)}, \frac{\mathcal{A}_2(y)}{\mathcal{B}_2(y)}, \ldots, \frac{\mathcal{A}_r(y)}{\mathcal{B}_r(y)} \right),$$

$$\begin{aligned} \text{subject to} \quad \Psi_j(y) &\leq 0, \quad \forall j \in \mathcal{I}^\Psi := \{1,2,\ldots,l\}, \\ \theta_j(y) &= 0, \quad \forall j \in \mathcal{I}^\theta := \{1,2,\ldots,p\}, \\ \mathcal{C}_j(y) &\geq 0, \quad \forall j \in \mathcal{T} := \{1,2,\ldots,m\}, \\ \mathcal{D}_j(y) &\geq 0, \quad \forall j \in \mathcal{T}, \\ \mathcal{D}_j(y)\mathcal{C}_j(y) &= 0, \quad \forall j \in \mathcal{T}. \end{aligned}$$

Here, each of the functions $\mathcal{A}_j$, $\mathcal{B}_j : \mathscr{M} \to \mathbb{R}$ $(j \in \mathcal{I} := \{1,2,\ldots,r\})$, $\Psi_j : \mathscr{M} \to \mathbb{R}$, $(j \in \mathcal{I}^\Psi)$, $\theta_j : \mathscr{M} \to \mathbb{R}$ $(j \in \mathcal{I}^\theta)$, $\mathcal{C}_j : \mathscr{M} \to \mathbb{R}$, $\mathcal{D}_j : \mathscr{M} \to \mathbb{R}$ $(j \in \mathcal{T})$ are assumed to be locally Lipschitz and are defined on some Hadamard manifold $\mathscr{M}$ having a dimension $n$, where $n \in \mathbb{N}$.

We use the symbol $\mathcal{F}$ to signify the set containing every feasible solution of the considered problem (NMFPPEC). Without any loss of generality, we assume that $\mathcal{A}_i(y) \geq 0$ and $\mathcal{B}_i(y) > 0$, for every $y \in \mathcal{F}$ and $i \in \mathcal{I}$. Throughout the rest of the paper, we shall use the following notation:

$$\Phi_i(y) := \frac{\mathcal{A}_i(y)}{\mathcal{B}_i(y)},\ \forall i \in \mathcal{I}, \text{ and},$$
$$\Pi_j(y) := \mathcal{D}_j(y)\mathcal{C}_j(y),\ \forall j \in \mathcal{T},$$

for every $y \in \mathscr{M}$.

We recall the concepts of Pareto efficiency as well as weak Pareto efficiency in the following definitions, which will be used in the paper (for instance, see, [21]).

**Definition 3.1.** Let $\hat{z} \in \mathcal{F}$ be arbitrary. Then $\hat{z}$ is termed as a Pareto efficient solution of (NMF-PPEC), provided that there does not exist any other feasible element $\widetilde{z} \in \mathcal{F}$, which satisfies the following inequality:

$$\Phi(\widetilde{z}) \preceq \Phi(\hat{z}),$$

that is,

$$\frac{\mathcal{A}(\widetilde{z})}{\mathcal{B}(\widetilde{z})} \preceq \frac{\mathcal{A}(\hat{z})}{\mathcal{B}(\hat{z})}.$$

**Definition 3.2.** Let $\hat{z} \in \mathcal{F}$ be arbitrary. Then $\hat{z}$ is termed as a weak Pareto efficient solution of (NMFPPEC), provided that there does not exist any other feasible element $\widetilde{z} \in \mathcal{F}$, which satisfies the following inequality:

$$\Phi(\widetilde{z}) \prec \Phi(\hat{z}),$$

that is,

$$\frac{\mathcal{A}(\widetilde{z})}{\mathcal{B}(\widetilde{z})} \prec \frac{\mathcal{A}(\hat{z})}{\mathcal{B}(\hat{z})}.$$

Suppose that $\hat{z} \in \mathcal{F}$ is any arbitrary feasible solution of (NMFPPEC). The index sets defined below will be crucial in the remaining portion of the article.

$$\mathscr{A}^{\Psi}(\hat{z}) := \{j \in \mathcal{I}^{\Psi} : \Psi_j(\hat{z}) = 0\},$$
$$\mathcal{R}_{+0}(\hat{z}) := \left\{j \in \mathcal{T} : \mathcal{C}_j(\hat{z}) > 0,\ \mathcal{D}_j(\hat{z}) = 0\right\},$$
$$\mathcal{R}_{0+}(\hat{z}) := \left\{j \in \mathcal{T} : \mathcal{C}_j(\hat{z}) = 0,\ \mathcal{D}_j(\hat{z}) > 0\right\},$$
$$\mathcal{R}_{00}(\hat{z}) := \left\{j \in \mathcal{T} : \mathcal{C}_j(\hat{z}) = 0,\ \mathcal{D}_j(\hat{z}) = 0\right\}.$$

**Remark 3.3.** (a) The index set $\mathscr{A}(\hat{z})$ is termed as the set of all active inequality indices for the function $\Psi$ at the point $\hat{z}$.

(b) The index set $\mathcal{R}_{00}(\hat{z})$ is termed as the degenerate index set at the point $\hat{z}$. The strict complementarity condition is said to be satisfied at $\hat{z}$ provided that $\mathcal{R}_{00}(\hat{z}) = \emptyset$.

(b) One can observe the fact that every index set that is defined above is dependent on the particular choice of $\hat{z} \in \mathcal{F}$. Nevertheless, in the remaining portion of the article, we shall not indicate such dependence explicitly when it will be easily perceivable from the context.

Let $\widetilde{y} \in \mathcal{F}$ be arbitrary. The sets $\mathscr{B}^k$ (for every $k \in \mathcal{I}$) and $\mathscr{B}$ as defined below will be crucial to discuss Guignard constraint qualification and optimality conditions for (NMFPPEC).

$$\mathscr{B}^k := \left\{ y \in \mathcal{F} : \Phi_j(y) \leq \Phi_j(\widetilde{y}), \ \forall j \in \mathcal{I}, \ j \neq k \right\},$$
$$\mathscr{B} := \left\{ y \in \mathcal{F} : \Phi_j(y) \leq \Phi_j(\widetilde{y}), \ \forall j \in \mathcal{I} \right\}.$$

**Remark 3.4.** (a) From the above definitions of the sets $\mathscr{B}^k$ and $\mathscr{B}$, it is clear that $\bigcap_{k\in\mathcal{I}} \mathscr{B}^k = \mathscr{B}$.

(b) In the case when $\mathcal{I} = \{1\}$, then (NMFPPEC) reduces to a single-objective fractional optimization problem with equilibrium constraints. In such a case, $\mathscr{B}^1 = \mathcal{F}$.

In the next definition, we recall the notion of the contingent cone for any subset of $\mathscr{M}$ (see, [20]).

**Definition 3.5.** Let $\mathcal{H} \subseteq \mathscr{M}$ and $\hat{y}$ be some arbitrary element in the closure of the set $\mathcal{H}$. Then the contingent cone (in other terms, Bouligand tangent cone) of the set $\mathcal{H}$ at the element $\hat{y}$ is symbolized by the notation $\mathscr{C}^{\text{Tan}}(\mathcal{H}, \hat{y})$, and is given by:

$$\mathscr{C}^{\text{Tan}}(\mathcal{H}, \hat{y}) := \{\xi \in T_{\hat{y}}\mathscr{M} : \exists \sigma_n \downarrow 0, \exists \{\xi_n\}_{n=1}^{\infty} \subset T_{\hat{y}}\mathscr{M}, \ \xi_n \to \xi, \ \exp_{\hat{y}}(\sigma_n \xi_n) \in \mathcal{H} \ \forall n \in \mathbb{N}\}.$$

The following notion of linearizing cone from [39] is an extension of Definition 6 from Treanţă et al. [35] for (NMFPPEC) in the setting of Hadamard manifolds.

**Definition 3.6.** Let $\hat{y} \in \mathcal{F}$ be arbitrary. The linearizing cone to the set $\mathscr{B}$ at the element $\hat{y}$ is the set defined as follows:

$$\begin{aligned} \mathscr{C}^{\text{Lin}}(\mathscr{B}, \hat{y}) := \Big\{ \overline{u} \in T_{\hat{y}}\mathscr{M} : & \langle \xi_j^{\Phi}(\hat{y}), \overline{u} \rangle \leq 0 \quad \forall \xi_j^{\Phi} \in \partial_c \Phi_j(\hat{y}), \ \forall j \in \mathcal{I}, \\ & \langle \xi_j^{\Psi}(\hat{y}), \overline{u} \rangle \leq 0, \ \ \forall \xi_j^{\Psi} \in \partial_c \Psi_j(\hat{y}), \ \forall j \in \mathscr{A}^{\Psi}, \\ & \langle \xi_j^{\theta}(\hat{y}), \overline{u} \rangle = 0, \ \ \forall \xi_j^{\theta} \in \partial_c \theta_j(\hat{y}), \ \forall j \in \mathcal{I}^{\theta}, \\ & \langle \xi_j^{\mathcal{C}}(\hat{y}), \overline{u} \rangle = 0, \ \ \forall \xi_j^{\mathcal{C}} \in \partial_c \mathcal{C}_j(\hat{y}), \ \forall j \in \mathcal{R}_{0+}, \\ & \langle \xi_j^{\mathcal{D}}(\hat{y}), \overline{u} \rangle = 0, \ \ \forall \xi_j^{\mathcal{D}} \in \partial_c \mathcal{D}_j(\hat{y}), \ \forall j \in \mathcal{R}_{+0}, \\ & \langle \xi_j^{\mathcal{C}}(\hat{y}), \overline{u} \rangle \geq 0, \ \ \forall \xi_j^{\mathcal{C}} \in \partial_c \mathcal{C}_j(\hat{y}), \ \forall j \in \mathcal{R}_{00}, \\ & \langle \xi_j^{\mathcal{D}}(\hat{y}), \overline{u} \rangle \geq 0, \ \ \forall \xi_j^{\mathcal{D}} \in \partial_c \mathcal{D}_j(\hat{y}), \ \forall j \in \mathcal{R}_{00} \Big\}. \end{aligned}$$

We now provide the following definition from [39], which is an extension of the notion of the modified linearizing cone from Singh and Mishra [31] from Euclidean space setting to Hadamard manifold setting.

**Definition 3.7.** Let $\hat{y} \in \mathcal{F}$ be arbitrary. The modified linearizing cone to the set $\mathscr{B}$ at the element $\hat{y}$ is the set defined as follows:

$$\begin{aligned}
\mathscr{C}^{\text{Lin}}_{\text{MFPPEC}}(\mathscr{B},\hat{y}) := \Big\{ \bar{u} \in T_{\hat{y}}\mathscr{M} : & \langle \xi^{\Phi}_j(\hat{y}), \bar{u}\rangle \leq 0 \quad \forall \xi^{\Phi}_j \in \partial_c \Phi_j(\hat{y}),\ \forall j \in \mathcal{I}, \\
& \langle \xi^{\Psi}_j(\hat{y}), \bar{u}\rangle \leq 0, \quad \forall \xi^{\Psi}_j \in \partial_c \Psi_j(\hat{y}),\ \forall j \in \mathscr{A}^{\Psi}, \\
& \langle \xi^{\theta}_j(\hat{y}), \bar{u}\rangle = 0, \quad \forall \xi^{\theta}_j \in \partial_c \theta_j(\hat{y}),\ \forall j \in \mathcal{I}^{\theta}, \\
& \langle \xi^{\mathcal{C}}_j(\hat{y}), \bar{u}\rangle = 0, \quad \forall \xi^{\mathcal{C}}_j \in \partial_c \mathcal{C}_j(\hat{y}),\ \forall j \in \mathcal{R}_{0+}, \\
& \langle \xi^{\mathcal{D}}_j(\hat{y}), \bar{u}\rangle = 0, \quad \forall \xi^{\mathcal{D}}_j \in \partial_c \mathcal{D}_j(\hat{y}),\ \forall j \in \mathcal{R}_{+0}, \\
& \langle \xi^{\mathcal{C}}_j(\hat{y}), \bar{u}\rangle \geq 0, \quad \forall \xi^{\mathcal{C}}_j \in \partial_c \mathcal{C}_j(\hat{y}),\ \forall j \in \mathcal{R}_{00}, \\
& \langle \xi^{\mathcal{D}}_j(\hat{y}), \bar{u}\rangle \geq 0, \quad \forall \xi^{\mathcal{D}}_j \in \partial_c \mathcal{D}_j(\hat{y}),\ \forall j \in \mathcal{R}_{00}, \\
\langle \xi^{\mathcal{C}}_j(\hat{y}), \bar{u}\rangle \langle \xi^{\mathcal{D}}_j(\hat{y}), \bar{u}\rangle = 0, & \quad \forall \xi^{\mathcal{C}}_j \in \partial_c \mathcal{C}_j(\hat{y}),\ \forall \xi^{\mathcal{D}}_j \in \partial_c \mathcal{D}_j(\hat{y}),\ \forall j \in \mathcal{R}_{00} \Big\}.
\end{aligned}$$

**Remark 3.8.**
(a) If $\mathscr{M}$ is considered to be the $n$-dimensional Euclidean space, and each of the components of the objective function and the constraints of (NMFPPEC) are assumed to be smooth, then the Definition 3.7 is a generalization of the notion of modified linearizing cone presented by Singh and Mishra [31] for a more wider category of optimization problems, that is, (NMFPPEC).
(b) It is significant to note that in view of Definition 3.6 and Definition 3.7, the following inclusion relation readily follows:

$$\mathscr{C}^{\text{Lin}}_{\text{MFPPEC}}(\mathscr{B},\hat{y}) \subseteq \mathscr{C}^{\text{Lin}}(\mathscr{B},\hat{y}).$$

We now define the (GGCQ) in the framework of Hadamard manifolds for our considered problem (NMFPPEC) (see, for instance, [39]).

**Definition 3.9.** Let $\bar{y} \in \mathcal{F}$ be any arbitrary feasible element. The generalized Guignard constraint qualification (GGCQ) holds at the point $\bar{y}$, provided that the following inclusion relation is satisfied:

$$\mathscr{C}^{\text{Lin}}_{\text{MFPPEC}}(\mathscr{B},\bar{y}) \subseteq \bigcap_{j \in \mathcal{I}} \text{cl}\,\text{co}\,\mathscr{C}^{\text{Tan}}\left(\mathscr{B}^j,\bar{y}\right).$$

The following theorem from [39] presents strong KKT type necessary criteria of optimality for (NMFPPEC).

**Theorem 3.10.** *Let $\bar{y} \in \mathcal{F}$ be any Pareto efficient solution of (NMFPPEC). Suppose that (GGCQ) holds at $\bar{y}$. Then we can obtain some real numbers $\alpha_j \in \mathbb{R}$ ($\alpha_j > 0$, $j \in \mathcal{I}$), $\sigma^{\Psi}_j \in \mathbb{R}$ ($j \in \mathcal{I}^{\Psi}$), $\sigma^{\theta}_j \in \mathbb{R}$ ($j \in \mathcal{I}^{\theta}$), $\sigma^{\mathcal{C}}_j \in \mathbb{R}$ ($j \in \mathcal{T}$) and $\sigma^{\mathcal{D}}_j \in \mathbb{R}$ ($j \in \mathcal{T}$), which satisfy the following:*

$$\begin{aligned}
0 \in \sum_{j \in \mathcal{I}} \alpha_j \Big[\partial_c \mathcal{A}_j(\bar{y}) - \chi_j \partial_c \mathcal{B}_j(\bar{y})\Big] + \sum_{j \in \mathcal{I}^{\Psi}} \sigma^{\Psi}_j \partial_c \Psi_j(\bar{y}) + \sum_{j \in \mathcal{I}^{\theta}} \sigma^{\theta}_j \partial_c \theta_j(\bar{y}) \\
- \sum_{j \in \mathcal{T}} \Big[\sigma^{\mathcal{C}}_j \partial_c \mathcal{C}_j(\bar{y}) + \sigma^{\mathcal{D}}_j \partial_c \mathcal{D}_j(\bar{y})\Big],
\end{aligned} \tag{3.1}$$

$$\mathcal{A}_j(\bar{y}) - \chi_j \mathcal{B}_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I},$$

$$\begin{aligned}
&\sigma_j^{\Psi} \geq 0, \quad \sigma_j^{\Psi}\Psi_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I}^{\Psi},\\
&\sigma_j^{\mathcal{C}} \textit{ free}, \forall j \in \mathcal{R}_{0+}, \quad \sigma_j^{\mathcal{C}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{C}} = 0, \ \forall j \in \mathcal{R}_{+0}, \qquad (3.2)\\
&\sigma_j^{\mathcal{D}} \textit{ free}, \forall j \in \mathcal{R}_{+0}, \quad \sigma_j^{\mathcal{D}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{D}} = 0, \forall j \in \mathcal{R}_{0+}.
\end{aligned}$$

We now furnish the following numerical example of (NMFPPEC) in the setting of a Hadamard manifold to illustrate the importance of Theorem 3.10.

**Example 3.11.** Consider the set $\mathscr{M} \subset \mathbb{R}^2$ as defined below:

$$\mathscr{M} := \{z = (z_1, z_2) \in \mathbb{R}^2, z_1, z_2 > 0\}.$$

One can verify that $\mathscr{M}$ is a Hadamard manifold (see, [26]). Let $y = (y_1, y_2) \in \mathscr{M}$ be arbitrary. The Riemannian metric associated to $\mathscr{M}$ is given by $\langle \hat{u}, \hat{w} \rangle_y = \langle \mathscr{G}(y)\hat{u}, \hat{w} \rangle, \quad \forall \hat{u}, \hat{w} \in T_y\mathscr{M} = \mathbb{R}^2$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^2$ and

$$\mathscr{G}(z) = \begin{pmatrix} \frac{1}{z_1^2} & 0 \\ 0 & \frac{1}{z_2^2} \end{pmatrix}.$$

Moreover, the exponential function denoted by $\exp_y : T_y\mathscr{M} \to \mathscr{M}$ for any arbitrary choice of $\hat{w} \in T_y\mathscr{M}$ is defined as $\exp_y(\hat{w}) = (x_1 e^{\frac{\hat{w}_1}{y_1}}, x_2 e^{\frac{\hat{w}_2}{y_2}})$. Similarly, $\exp_y^{-1} : \mathscr{M} \to T_y\mathscr{M}$ is the inverse of the exponential function for any $y \in \mathscr{M}$ and is defined as $\exp_y^{-1}(z) = \left( y_1 \ln \frac{z_1}{y_1}, y_2 \ln \frac{z_2}{y_2} \right)$. Consider the following (NMFPPEC) on the manifold $\mathscr{M}$:

$$(P) \quad \text{Minimize } \left( \frac{\mathcal{A}_1(z)}{\mathcal{B}_1(z)}, \frac{\mathcal{A}_1(z)}{\mathcal{B}_1(z)} \right) := \left( \frac{|z_1 - e|}{z_1}, \frac{\log z_2}{2} \right),$$

subject to

$$\begin{aligned}
\Psi(z) &:= 1 - \ln z_1 - \ln z_2 \leq 0,\\
\mathcal{C}(z) &:= \ln z_1 - 1 \geq 0,\\
\mathcal{D}(z) &:= \ln z_2 - 1 \geq 0,\\
\mathcal{C}(z)\mathcal{D}(z) &:= (\ln z_1 - 1)(\ln z_2 - 1) = 0.
\end{aligned}$$

Clearly, the functions $\mathcal{A}_j, \mathcal{B}_j : \mathscr{M} \to \mathbb{R}, (j = 1, 2)$, $\Psi : \mathscr{M} \to \mathbb{R}$, $\mathcal{C} : \mathscr{M} \to \mathbb{R}$, $\mathcal{D} : \mathscr{M} \to \mathbb{R}$ are locally Lipschitz functions defined on $\mathscr{M}$. We use the symbol $\mathcal{F}$ to signify the set containing every feasible solution of the problem (P). That is, we have:

$$\mathcal{F} = \{z \in \mathscr{M} : \ z_1 = e, \ z_2 \geq e, \text{ or, } z_1 \geq e, \ z_2 = e\}.$$

Choose the feasible solution $\hat{y} = (e,e) \in \mathcal{F}$. Consequently, we get the following:

$$\begin{aligned}
\partial_c\left(\mathcal{A}_1(\hat{y})\right) &= \left\{\left(e^2,0\right)^T\right\},\\
\partial_c\left(\mathcal{B}_1(\hat{y})\right) &= \left\{\left(e^2,0\right)^T\right\},\\
\partial_c\left(\mathcal{A}_2(\hat{y})\right) &= \left\{(0,e)^T\right\},\\
\partial_c\left(\mathcal{B}_2(\hat{y})\right) &= \left\{(0,0)^T\right\},\\
\partial_c\Psi(\hat{y}) &= \left\{(-e,-e)^T\right\},\\
\partial_c\mathcal{C}(\hat{y}) &= \left\{(e,0)^T\right\},\\
\partial_c\mathcal{D}(\hat{y}) &= \left\{(0,e)^T\right\}.
\end{aligned}$$

One can easily verify that the feasible solution $\hat{y} = (e,e) \in \mathcal{F}$ is indeed, a Pareto efficient solution of (P). Furthermore, (GGCQ) holds at $\hat{y}$. Let us now pick some real numbers $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$, $\sigma^{\Psi} = 0$, $\sigma^{\mathcal{C}} = \frac{e}{2}$, $\sigma^{\mathcal{D}} = \frac{1}{2}$. Further we choose $(e^2,0)^T \in \partial_c\mathcal{A}_1(\hat{y})$, $(0,e)^T \in \partial_c\mathcal{A}_2(\hat{y})$, $(e^2,0)^T \in \partial_c\mathcal{B}_1(\hat{y})$, $(0,0)^T \in \partial_c\mathcal{B}_2(\hat{y})$, $(-e,-e)^T \in \partial_c\Psi(\hat{y})$, $(e,0)^T \in \partial_c\mathcal{C}(\hat{y})$, $(0,e)^T \in \partial_c\mathcal{D}(\hat{y})$. Then, we can verify that the following relation is satisfied:

$$\begin{aligned}
&(0,0) \in \sum_{j=1}^{2}\alpha_j\left[\partial_c\mathcal{A}_j(\hat{y}) - \chi_j\partial_c\mathcal{B}_j(\hat{y})\right] + \sigma^{\Psi}\partial_c\Psi(\hat{y}) - \left[\sigma^{\mathcal{C}}\partial_c\mathcal{C}(\hat{y}) + \sigma^{\mathcal{D}}\partial_c\mathcal{D}(\hat{y})\right],\\
&\mathcal{A}_j(\hat{y}) - \chi_j\mathcal{B}_j(\hat{y}) = 0, \quad \forall j \in \{1,2\}.
\end{aligned}$$

Hence, every assumption and conclusion of Theorem 3.10 for the problem (P) is justified.

## 4. MOND-WEIR TYPE DUAL MODEL FOR FOR (NMFFPEC)

Let $w \in \mathscr{M}$ be arbitrary. Further, let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$. Then, related to the primal (NMFPPEC), the corresponding Mond-Weir type dual model (abbreviated as, (DP-MW)) is formulated as given below:

$$\text{(DP-MW)} \quad \text{Maximize } \mathscr{L}(w) := \left(\frac{\mathcal{A}_1(w)}{\mathcal{B}_1(w)}, \frac{\mathcal{A}_2(w)}{\mathcal{B}_2(w)}, \ldots, \frac{\mathcal{A}_l(w)}{\mathcal{B}_l(w)}\right), \tag{4.1}$$

subject to

$$\begin{aligned}
0 \in \sum_{j\in\mathcal{I}}\alpha_j\left[\partial_c\mathcal{A}_j(w) - \chi_j\partial_c\mathcal{B}_j(w)\right] + \sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\partial_c\Psi_j(w) +\\
\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\partial_c\theta_j(w) - \sum_{j\in\mathcal{T}}\left[\sigma_j^{\mathcal{C}}\partial_c\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\partial_c\mathcal{D}_j(w)\right],
\end{aligned}$$

$$\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(w) \geq 0, \ \sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(w) \geq 0, \ \sum_{j\in\mathcal{T}}\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) \leq 0, \ \sum_{j\in\mathcal{T}}\sigma_j^{\mathcal{D}}\mathcal{D}_j(w) \leq 0, \tag{4.2}$$

where

$$\begin{aligned}
&\mathcal{A}_j(w)-\chi_j\mathcal{B}_j(w)=0, \quad \forall j\in\mathcal{I},\\
&\sigma_j^{\Psi}\geq 0,\ \forall j\in\mathscr{A}^{\Psi},\\
&\sigma_j^{\mathcal{C}}=0,\ \forall j\in\mathcal{R}_{+0},\\
&\sigma_j^{\mathcal{D}}=0,\ \forall j\in\mathcal{R}_{0+},\\
&\forall j\in\mathcal{R}_{00},\ \text{either, } \sigma_j^{\mathcal{C}}>0,\ \sigma_j^{\mathcal{D}}>0 \text{ or, } \sigma_j^{\mathcal{C}}\sigma_j^{\mathcal{D}}=0.
\end{aligned} \tag{4.3}$$

The set containing every feasible element of (DP-MW) is signified by the symbol $\mathcal{F}_M$.

In the next theorem, a weak duality result relating our considered primal problem (MPPEC) and (DP-MW) is derived.

**Theorem 4.1.** *Let $\bar{z}\in\mathcal{F}$ and $(w,\alpha,\sigma)\in\mathcal{F}_M$ be arbitrary. Suppose that the function $\mathscr{P}$ as defined below*

$$\mathscr{P}(z):=\mathcal{A}_j(z)-\chi_j\mathcal{B}_j(z), \quad \textit{where,}$$
$$\chi_j:=\frac{\mathcal{A}_j(w)}{\mathcal{B}_j(w)}, \quad \forall j\in\mathcal{I},\ z\in\mathscr{M},$$

*is geodesic convex at the element $\bar{z}$. Further, suppose that each of the functions $\Psi_j\ (j\in\mathscr{A}^{\Psi})$, $\theta_j\ (j\in\mathcal{I}_+^{\theta})$, $-\theta_j\ (j\in\mathcal{I}_-^{\theta})$, $-\mathcal{C}_j\ (j\in\mathcal{R}_{0+}^{+}\cup\mathcal{R}_{00}^{+0}\cup\mathcal{R}_{00}^{++})$, $-\mathcal{D}_j\ (j\in\mathcal{R}_{+0}^{+}\cup\mathcal{R}_{00}^{0+}\cup\mathcal{R}_{00}^{++})$ are geodesic convex functions at $\bar{z}$. Moreover, we assume that*

$$\mathcal{R}_{0+}^{-}\cup\mathcal{R}_{+0}^{-}\cup\mathcal{R}_{00}^{-0}\cup\mathcal{R}_{00}^{0-}=\emptyset.$$

*Then*

$$\mathscr{L}(\bar{z})\nprec\mathscr{L}(w).$$

*Proof.* Given that $\bar{z}\in\mathcal{F}$ and $(w,\alpha,\sigma)\in\mathcal{F}_M$ are arbitrary feasible elements of (NMFPPEC) and (DP-MW), respectively. By *reductio ad absurdum*, we suppose that $\mathscr{L}(\bar{z})\prec\mathscr{L}(w)$. Consequently, the following inequalities can be obtained:

$$\frac{\mathcal{A}_j(\bar{z})}{\mathcal{B}_j(\bar{z})}<\frac{\mathcal{A}_j(w)}{\mathcal{B}_j(w)}, \quad \forall j\in\mathcal{I}. \tag{4.4}$$

From (4.4), we get the following inequalities:

$$\mathcal{A}_j(\bar{z})-\chi_j\mathcal{B}_j(\bar{z})<\mathcal{A}_j(w)-\chi_j\mathcal{B}_j(w), \quad \forall j\in\mathcal{I}.$$

In view of the definition of the function $\mathscr{P}$ in the hypothesis of the theorem, we infer that

$$\mathscr{P}_j(\bar{z})<\mathscr{P}_j(w), \quad \forall j\in\mathcal{I}.$$

Since for every $j\in\mathcal{I}$, the functions $\mathscr{P}_j$ are strictly geodesic convex at $\bar{y}$, we get the following inequalities:

$$\left\langle \xi_j^{\mathscr{P}},\ \exp_w^{-1}(\bar{z})\right\rangle<0, \quad \forall\xi_j^{\mathscr{P}}\in\partial_c\mathscr{P}_j,\ \forall j\in\mathcal{I}. \tag{4.5}$$

From the feasibility conditions of the problem (NMFPPEC), the following inequalities can be obtained:

$$\begin{aligned}
\Psi_j(\bar{z}) \leq 0 = \Psi_j(w), &\qquad \forall j \in \mathscr{A}^{\Psi}(w),\\
\theta_j(\bar{z}) \leq 0 = \theta_j(w), &\qquad \forall j \in \mathcal{I}_{+}^{\theta},\\
-\theta_j(\bar{z}) \leq 0 = -\theta_j(w), &\qquad \forall j \in \mathcal{I}_{-}^{\theta},\\
-\mathcal{C}_j(\bar{z}) \leq 0 = -\mathcal{C}_j(w), &\qquad \forall j \in \mathcal{R}_{0+} \cup \mathcal{R}_{00},\\
-\mathcal{D}_j(\bar{z}) \leq 0 = -\mathcal{D}_j(w), &\qquad \forall j \in \mathcal{R}_{+0} \cup \mathcal{R}_{00},
\end{aligned}$$

Since for every $j \in \mathscr{A}^{\Psi}(w)$, the functions $\Psi_j$ are geodesic convex at the element $w$, we get the following

$$\left\langle \xi_j^{\Psi}, \exp_w^{-1}(\bar{z}) \right\rangle \leq 0, \quad \forall \xi_j^{\Psi} \in \partial_c \Psi_j(w), \forall j \in \mathscr{A}^{\Psi}. \tag{4.6}$$

Similarly, in light of the geodesic convexity hypothesis on the functions $\theta_j$ $\left(j \in \mathcal{I}_{+}^{\theta}\right)$, $-\theta_j$ $\left(j \in \mathcal{I}_{-}^{\theta}\right)$, $-\mathcal{C}_j$ $(j \in \mathcal{R}_{0+}^{+} \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++})$, $-\mathcal{D}_j$ $(j \in \mathcal{R}_{+0}^{+} \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++})$, we have the following:

$$\begin{aligned}
\left\langle \xi_j^{\theta}, \exp_{\bar{z}}^{-1}(\bar{z}) \right\rangle \leq 0, &\qquad \forall \xi_j^{\theta} \in \partial_c \theta_j(w), \forall j \in \mathcal{I}_{+}^{\theta},\\
\left\langle \xi_j^{\theta}, \exp_w^{-1}(\bar{z}) \right\rangle \geq 0, &\qquad \forall \xi_j^{\theta} \in \partial_c \theta_j(w), \forall j \in \mathcal{I}_{-}^{\theta},\\
\left\langle \xi_j^{\mathcal{C}}, \exp_w^{-1}(\bar{z}) \right\rangle \geq 0, &\qquad \forall \xi_j^{\mathcal{C}} \in \partial_c \mathcal{C}_j(w), \forall j \in \mathcal{R}_{0+}^{+} \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++},\\
\left\langle \xi_j^{\mathcal{D}}, \exp_w^{-1}(\bar{z}) \right\rangle \geq 0, &\qquad \forall \xi_j^{\mathcal{D}} \in \partial_c \mathcal{D}_j(w), \forall j \in \mathcal{R}_{+0}^{+} \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++}.
\end{aligned}$$

Moreover, we have $\mathcal{R}_{0+}^{-} \cup \mathcal{R}_{+0}^{-} \cup \mathcal{R}_{00}^{-0} \cup \mathcal{R}_{00}^{0-} = \emptyset$. As a result, we arrive at the following inequalities:

$$\begin{aligned}
\left\langle \sum_{j \in \mathscr{A}^{\Psi}} \sigma_j^{\Psi} \xi_j^{\Psi}, \exp_w^{-1}(\bar{z}) \right\rangle &\leq 0,\\
\left\langle \sum_{j \in \mathcal{I}^{\theta}} \sigma_j^{\theta} \xi_j^{\theta}, \exp_w^{-1}(\bar{z}) \right\rangle &\leq 0,\\
\left\langle \sum_{j \in \mathcal{T}} \sigma_j^{\mathcal{C}} \xi_j^{\mathcal{C}}, \exp_w^{-1}(\bar{z}) \right\rangle &\geq 0,\\
\left\langle \sum_{j \in \mathcal{T}} \sigma_j^{\mathcal{D}} \xi_j^{\mathcal{D}}, \exp_w^{-1}(\bar{z}) \right\rangle &\geq 0.
\end{aligned} \tag{4.7}$$

We have $\alpha_j > 0$ for every $j \in \mathcal{I}$. Then adding each of the inequalities in (4.5) and (4.7), the following inequality can be obtained for every $\xi_j^{\mathcal{A}} \in \partial_c \mathcal{A}_j(w)$, $\xi_j^{\mathcal{B}} \in \partial_c \mathcal{B}_j(w)$, $\xi_j^{\Psi} \in \partial_c \Psi_j(\bar{z})$, $\xi_j^{\theta} \in \partial_c \theta_j(w)$, $\xi_j^{\mathcal{C}} \in \partial_c \mathcal{C}_j(w)$, $\xi_j^{\mathcal{D}} \in \partial_c \mathcal{D}_j(w)$:

$$\begin{aligned}
\Bigg\langle \sum_{j \in \mathcal{I}} \alpha_j \left[\xi_j^{\mathcal{A}} - \chi_j \xi_j^{\mathcal{B}}\right] + \sum_{j \in \mathscr{A}^{\Psi}} \sigma_j^{\Psi} \xi_j^{\Psi} + \sum_{j \in \mathcal{I}^{\theta}} \sigma_j^{\theta} \xi_j^{\theta}&\\
- \sum_{j \in \mathcal{T}} \left[\sigma_j^{\mathcal{C}} \xi_j^{\mathcal{C}}(w) + \sigma_j^{\mathcal{D}} \xi_j^{\mathcal{D}}\right], \exp_w^{-1}(\bar{z}) \Bigg\rangle &< 0,
\end{aligned}$$

which contradicts the fact that $w$ is a feasible element of (MWP). Hence, the proof is complete. □

In the next corollary, we derive another weak duality relation relating our primal problem (NMFPPEC) and (DP-MW). The proof of the corollary follows in similar lines as the proof of Theorem 4.1.

**Corollary 4.2.** *Let $\bar{z} \in \mathcal{F}$ and $(w,\alpha,\sigma) \in \mathcal{F}_M$ be arbitrary. Suppose that the function $\mathscr{P}$ as defined below*

$$\mathscr{P}(z) := \mathcal{A}_j(z) - \chi_j \mathcal{B}_j(z), \quad \text{where,}$$
$$\chi_j := \frac{\mathcal{A}_j(w)}{\mathcal{B}_j(w)}, \quad \forall j \in \mathcal{I},\, z \in \mathscr{M},$$

*is strictly geodesic convex at the element $w$. Further, suppose that each of the functions $\Psi_j$ $\left(j \in \mathscr{A}^{\Psi}\right)$, $\theta_j$ $\left(j \in \mathcal{I}_+^{\theta}\right)$, $-\theta_j$ $\left(j \in \mathcal{I}_-^{\theta}\right)$, $-\mathcal{C}_j$ $(j \in \mathcal{R}_{0+}^{+} \cup \mathcal{R}_{00}^{+0} \cup \mathcal{R}_{00}^{++})$, $-\mathcal{D}_j$ $(j \in \mathcal{R}_{+0}^{+} \cup \mathcal{R}_{00}^{0+} \cup \mathcal{R}_{00}^{++})$ are geodesic convex functions at $w$. Moreover, we assume that*

$$\mathcal{R}_{0+}^{-} \cup \mathcal{R}_{+0}^{-} \cup \mathcal{R}_{00}^{-0} \cup \mathcal{R}_{00}^{0-} = \emptyset.$$

*Then*

$$\mathscr{L}(\bar{z}) \not\preceq \mathscr{L}(w).$$

In the next theorem, a strong duality relation relating our considered primal problem (MPPEC) and the Mond-Weir dual problem (DP-MW) is deduced.

**Theorem 4.3.** *Let $\bar{z} \in \mathcal{F}$ be any arbitrary Pareto efficient solution of (NMFPPEC) at which (GGCQ) holds. Then there exist some $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$ such that $(\bar{z},\alpha,\sigma) \in \mathcal{F}_M$. Moreover, we have*

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} = \mathscr{L}(\bar{z},\alpha,\sigma).$$

*Then the following statements hold true:*

*(a) Suppose that each of the hypothesis stated in Theorem 4.1 holds. Then $(\bar{z},\alpha,\sigma)$ is a weak Pareto efficient solution of (DP-MW).*
*(b) Suppose that each of the hypotheses stated in the Corollary 4.2 holds. Then $(\bar{z},\alpha,\sigma)$ is a Pareto efficient solution of (DP-MW).*

*Proof.* According to the provided hypothesis, we have that $\bar{z} \in \mathcal{F}$ is any arbitrary Pareto efficient solution of (NMFPPEC) and (GGCQ) holds at $\bar{z}$. In light of Theorem 3.10, we obtain some real multipliers $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$ such that

$$\begin{aligned} 0 \in \sum_{j\in\mathcal{I}} \alpha_j \left[\partial_c \mathcal{A}_j(\bar{z}) - \chi_j \partial_c \mathcal{B}_j(\bar{z})\right] + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \partial_c \Psi_j(\bar{z}) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \partial_c \theta_j(\bar{z}) \\ - \sum_{j\in\mathcal{T}} \left[\sigma_j^{\mathcal{C}} \partial_c \mathcal{C}_j(\bar{y}) + \sigma_j^{\mathcal{D}} \partial_c \mathcal{D}_j(\bar{z})\right], \end{aligned} \tag{4.8}$$
$$\mathcal{A}_j(\bar{z}) - \chi_j \mathcal{B}_j(\bar{z}) = 0, \quad \forall j \in \mathcal{I},$$

$$\begin{aligned}&\sigma_j^{\Psi} \geq 0, \quad \sigma_j^{\Psi}\Psi_j(\bar{y}) = 0, \quad \forall j \in \mathcal{I}^{\Psi},\\ &\sigma_j^{\mathcal{C}} \text{ free}, \forall j \in \mathcal{R}_{0+}, \quad \sigma_j^{\mathcal{C}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{C}} = 0, \ \forall j \in \mathcal{R}_{+0},\\ &\sigma_j^{\mathcal{D}} \text{ free}, \forall j \in \mathcal{R}_{+0}, \quad \sigma_j^{\mathcal{D}} \geq 0, \forall j \in \mathcal{R}_{00}, \quad \sigma_j^{\mathcal{D}} = 0, \forall j \in \mathcal{R}_{0+}.\end{aligned} \tag{4.9}$$

Consequently, it follows that

$$(\bar{z}, \alpha, \sigma) \in \mathcal{F}_M,$$

and

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} = \mathscr{L}(\bar{z}, \alpha, \sigma).$$

(a) By *reductio ad absurdum*, we suppose that $(\bar{z}, \alpha, \sigma)$ is not a weak Pareto efficient solution of (DP-MW). As a result, one can find some $(\bar{u}, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_M$, which satisfies the following:

$$\mathscr{L}(\bar{z}) \prec \mathscr{L}(\bar{u}),$$

which contradicts Theorem 4.1. Hence, the proof is complete.

(b) By *reductio ad absurdum*, we suppose that $(\bar{z}, \alpha, \sigma)$ is not a Pareto efficient solution of (DP-MW). As a result, one can find some $(\bar{u}, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_M$, which satisfies the following:

$$\mathscr{L}(\bar{z}) \preceq \mathscr{L}(\bar{u}),$$

which contradicts Corollary 4.2. Hence, the proof is complete.

□

In the next theorem, a strict converse duality relation relating our considered primal problem (NMFPPEC) and (DP-MW) is established.

**Theorem 4.4.** *Suppose that $\bar{y} \in \mathcal{F}$ be any Pareto efficient solution of (NMFPPEC) at which (GGCQ) holds. Let $(\overline{w}, \overline{\alpha}, \overline{\sigma})$ be a feasible element of (DP-MW), such that $\mathscr{L}(\bar{y}) \preceq \mathscr{L}(\overline{w})$. Suppose that each of the hypotheses stated in Corollary 4.2 holds. then $\bar{y} = \overline{w}$.*

*Proof.* According to the provided hypothesis, we have that $\bar{y} \in \mathcal{F}$ be any weak Pareto efficient solution of (NMFPPEC) at which (GGCQ) holds.
By *reductio ad absurdum*, we suppose that $\bar{y} \neq \overline{w}$. As a result, in light of Theorem 4.3, we can get $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$ such that $(\bar{y}, \alpha, \sigma) \in \mathcal{F}_M$. Moreover, we have

$$\frac{\mathcal{A}(\bar{y})}{\mathcal{B}(\bar{y})} = \mathscr{L}(\bar{y}, \alpha, \sigma).$$

On the other hand, in view of the conclusions of the strong duality theorem (Theorem 4.3), we can infer that $(\bar{y}, \alpha, \sigma)$ is a Pareto efficient solution for (NMFPPEC). Since $\bar{y} \in \mathcal{F}$ and $(\overline{w}, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_{MW}$, then from Theorem 4.3, we get

$$\mathscr{L}(\bar{y}) \npreceq \mathscr{L}(\overline{w}),$$

which is a contradiction. Hence, the proof is complete. □

**Remark 4.5.** (1) If $\mathscr{M} = \mathbb{R}^n$, then Theorem 4.1 and Theorem 4.3 generalize Theorem 6 and Theorem 7 deduced in [31] from multiobjective (MPEC) to (NMFPPEC).

(2) The weak, strong as well as strict converse duality relations (Theorem 4.1, Theorem 4.3 and Theorem 4.4) extend Theorem 4.1, Theorem 4.2 and Theorem 4.3, respectively, deduced in [41] for a wider category of optimization problems, that is, (NMFPPEC).

**Example 4.6.** Consider the (NMFPPEC) (Problem (P)) defined in Example 3.11 on the manifold $\mathscr{M}$. We use the symbol $\mathcal{F}$ to signify the set containing every feasible solution of the problem (P). That is, we have:

$$\mathcal{F} = \{z \in \mathscr{M}, z_1 = e, z_2 \geq e, \text{ or}, z_1 \geq e, z_2 = e\}.$$

Let $w \in \mathscr{M}$ be arbitrary. Further, let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \{1,2\})$, $\sigma^{\Psi} \in \mathbb{R}$, $\sigma^{\mathcal{C}} \in \mathbb{R}$, $\sigma^{\mathcal{D}} \in \mathbb{R}$. Then, related to the primal (P), the corresponding Mond-Weir type dual model (abbreviated as, (DP-MW)) is formulated as given below:

$$\text{(DP-MW)} \quad \text{Maximize } \mathscr{L}(w) = \left(\frac{\mathcal{A}_1(w)}{\mathcal{B}_1(w)}, \frac{\mathcal{A}_2(w)}{\mathcal{B}_2(w)}\right) := \left(\frac{|w_1 - e|}{e}, \frac{\log w_2}{2}\right),$$

subject to

$$0 \in \sum_{j=1}^{2} \alpha_j \left[\partial_c \mathcal{A}_j(w) - \chi_j \partial_c \mathcal{B}_j(w)\right] + \sigma^{\Psi} \partial_c \Psi(w) - \left[\sigma^{\mathcal{C}} \partial_c \mathcal{C}(w) + \sigma^{\mathcal{D}} \partial_c \mathcal{D}(w)\right],$$

$$\sigma^{\Psi}\Psi(w) \geq 0,\ \sigma^{\mathcal{C}}\mathcal{C}(w) \leq 0,\ \sigma^{\mathcal{D}}\mathcal{D}(w) \leq 0,$$

where

$$\mathcal{A}_j(w) - \chi_j \mathcal{B}_j(w) = 0, \quad \forall j \in \{1,2\}. \tag{4.10}$$

Choose the feasible solution $\hat{y} = (e,e) \in \mathcal{F}$. Consequently, we get the following:

$$\begin{aligned}
\partial_c \mathcal{A}_1(\hat{y}) &= \left\{\left(e^2, 0\right)^T\right\},\\
\partial_c \mathcal{B}_1(\hat{y}) &= \left\{\left(e^2, 0\right)^T\right\},\\
\partial_c \mathcal{A}_2(\hat{y}) &= \left\{(0, e)^T\right\},\\
\partial_c \mathcal{B}_2(\hat{y}) &= \left\{(0, 0)^T\right\},\\
\partial_c \Psi(\hat{y}) &= \left\{(-e, -e)^T\right\},\\
\partial_c \mathcal{C}(\hat{y}) &= \left\{(e, 0)^T\right\},\\
\partial_c \mathcal{D}(\hat{y}) &= \left\{(0, e)^T\right\}.
\end{aligned}$$

One can easily verify that the feasible solution $\hat{y} = (e,e) \in \mathcal{F}$ is indeed, a Pareto efficient solution of (P). Furthermore, (GGCQ) holds at $\hat{y}$. Let us now pick some real numbers $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$, $\sigma^{\Psi} = 0$, $\sigma^{\mathcal{C}} = \frac{e}{2}$, $\sigma^{\mathcal{D}} = \frac{1}{2}$. Further we choose $(e^2,0)^T \in \partial_c \mathcal{A}_1(\hat{y})$, $(0,e)^T \in \partial_c \mathcal{A}_2(\hat{y})$, $(e^2,0)^T \in \partial_c \mathcal{B}_1(\hat{y})$, $(0,0)^T \in \partial_c \mathcal{B}_2(\hat{y})$, $(-e,-e)^T \in \partial_c \Psi(\hat{y})$, $(e,0)^T \in \partial_c \mathcal{C}(\hat{y})$, $(0,e)^T \in \partial_c \mathcal{D}(\hat{y})$. Then, we can verify that the following relation is satisfied:

$$(0,0) \in \sum_{j=1}^{2} \alpha_j \left[\partial_c \mathcal{A}_j(\hat{y}) - \chi_j \partial_c \mathcal{B}_j(\hat{y})\right] + \sigma^{\Psi} \partial_c \Psi(\hat{y}) - \left[\sigma^{\mathcal{C}} \partial_c \mathcal{C}(\hat{y}) + \sigma^{\mathcal{D}} \partial_c \mathcal{D}(\hat{y})\right],$$

$$\mathcal{A}_j(\hat{y}) - \chi_j \mathcal{B}_j(\hat{y}) = 0, \quad \forall j \in \{1,2\}.$$

Thus, we see that $\hat{y}$ is a feasible element of (DP-MW). Further, every assumption of strong duality theorem is satisfied. One can verify that $(\hat{y},\alpha,\sigma)$ is a Pareto efficient solution of (DP-MW).

## 5. WOLFE TYPE DUAL MODEL FOR (NMFFPEC)

In the following lemma, we establish a different variant of the necessary criteria of Pareto efficiency for (NMFPPEC) derived in Theorem 3.10, that will be helpful to formulate the Wolfe type dual model for (NMFPPEC).

**Theorem 5.1.** *Let $\bar{y}\in\mathcal{F}$ be any Pareto efficient solution of (NMFPPEC). Suppose that (GGCQ) holds at $\bar{y}$. Then we can obtain some real Lagrange multipliers $\varepsilon_j\in\mathbb{R}$ ($\varepsilon_j>0$, $j\in\mathcal{I}$), $\overline{\sigma}_j^{\Psi}\in\mathbb{R}$ ($j\in\mathcal{I}^{\Psi}$), $\overline{\sigma}_j^{\theta}\in\mathbb{R}$ ($j\in\mathcal{I}^{\theta}$), $\overline{\sigma}_j^{\mathcal{C}}\in\mathbb{R}$ ($j\in\mathcal{T}$) and $\overline{\sigma}_j^{\mathcal{D}}\in\mathbb{R}$ ($j\in\mathcal{T}$), which satisfy the following:*

$$
\begin{aligned}
0\in\sum_{i\in\mathcal{I}}\varepsilon_i\mathcal{B}_i(\bar{y})\Bigg[&\partial_c\mathcal{A}_i(\bar{y})+\sum_{j\in\mathcal{I}^{\Psi}}\overline{\sigma}_j^{\Psi}\partial_c\Psi_j(\bar{y})+\sum_{j\in\mathcal{I}^{\theta}}\overline{\sigma}_j^{\theta}\partial_c\theta_j(\bar{y})\\
&-\sum_{j\in\mathcal{T}}\left(\overline{\sigma}_j^{\mathcal{C}}\partial_c\mathcal{C}_j(\bar{y})+\overline{\sigma}_j^{\mathcal{D}}\partial_c\mathcal{D}_j(\bar{y})\right)\Bigg]-\sum_{i\in\mathcal{I}}\varepsilon_i\operatorname{grad}\mathcal{B}_i(\bar{y})\Bigg[\mathcal{A}_i(\bar{y})\\
&+\sum_{j\in\mathcal{I}^{\Psi}}\overline{\sigma}_j^{\Psi}\Psi_j(\bar{y})+\sum_{j\in\mathcal{I}^{\theta}}\overline{\sigma}_j^{\theta}\theta_j(\bar{y})-\sum_{j\in\mathcal{T}}\left(\overline{\sigma}_j^{\mathcal{C}}\mathcal{C}_j(\bar{y})+\overline{\sigma}_j^{\mathcal{D}}\mathcal{D}_j(\bar{y})\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\sigma}_j^{\Psi}\geq 0,\quad \sigma_j^{\Psi}\Psi_j(\bar{y})=0,\quad \forall j\in\mathcal{I}^{\Psi},\\
&\overline{\sigma}_j^{\mathcal{C}}\ \textit{free},\ \forall j\in\mathcal{R}_{0+},\quad \overline{\sigma}_j^{\mathcal{C}}\geq 0,\ \forall j\in\mathcal{R}_{00},\quad \overline{\sigma}_j^{\mathcal{C}}=0,\ \forall j\in\mathcal{R}_{+0},\\
&\overline{\sigma}_j^{\mathcal{D}}\ \textit{free},\ \forall j\in\mathcal{R}_{+0},\quad \overline{\sigma}_j^{\mathcal{D}}\geq 0,\ \forall j\in\mathcal{R}_{00},\quad \overline{\sigma}_j^{\mathcal{D}}=0,\ \forall j\in\mathcal{R}_{0+}.
\end{aligned}
$$

*Proof.* According to the provided hypotheses, $\bar{y}$ is a Pareto efficient solution for (MFPPEC) at which (GGCQ) holds. In view of Theorem 3.10, one can obtain some real numbers $\alpha_j\in\mathbb{R}$ ($\alpha_j>0$, $j\in\mathcal{I}$), $\sigma_j^{\Psi}\in\mathbb{R}$ ($j\in\mathcal{I}^{\Psi}$), $\sigma_j^{\theta}\in\mathbb{R}$ ($j\in\mathcal{I}^{\theta}$), $\sigma_j^{\mathcal{C}}\in\mathbb{R}$ ($j\in\mathcal{T}$) and $\sigma_j^{\mathcal{D}}\in\mathbb{R}$ ($j\in\mathcal{T}$), which satisfy the following:

$$
\begin{aligned}
0\in\sum_{j\in\mathcal{I}}\alpha_j\Big[\partial_c\mathcal{A}_j(\bar{y})-\chi_j\partial_c\mathcal{B}_j(\bar{y})\Big]&+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\partial_c\Psi_j(\bar{y})+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\partial_c\theta_j(\bar{y})\\
&-\sum_{j\in\mathcal{T}}\left[\sigma_j^{\mathcal{C}}\partial_c\mathcal{C}_j(\bar{y})+\sigma_j^{\mathcal{D}}\partial_c\mathcal{D}_j(\bar{y})\right],
\end{aligned}\tag{5.1}
$$

$$
\mathcal{A}_j(\bar{y})-\chi_j\mathcal{B}_j(\bar{y})=0,\quad \forall j\in\mathcal{I},
$$

$$
\begin{aligned}
&\sigma_j^{\Psi}\geq 0,\quad \sigma_j^{\Psi}\Psi_j(\bar{y})=0,\quad \forall j\in\mathcal{I}^{\Psi},\\
&\sigma_j^{\mathcal{C}}\ \text{free},\ \forall j\in\mathcal{R}_{0+},\quad \sigma_j^{\mathcal{C}}\geq 0,\ \forall j\in\mathcal{R}_{00},\quad \sigma_j^{\mathcal{C}}=0,\ \forall j\in\mathcal{R}_{+0},\\
&\sigma_j^{\mathcal{D}}\ \text{free},\ \forall j\in\mathcal{R}_{+0},\quad \sigma_j^{\mathcal{D}}\geq 0,\ \forall j\in\mathcal{R}_{00},\quad \sigma_j^{\mathcal{D}}=0,\ \forall j\in\mathcal{R}_{0+}.
\end{aligned}
$$

Let us now define $\varepsilon_j$ ($j\in\mathcal{I}$) in the following manner:

$$
\varepsilon_j:=\frac{\alpha_j}{\mathcal{B}_j(\bar{y})},\quad \forall j\in\mathcal{I}.
$$

Consequently, from (5.1), we can get the following:

$$\begin{aligned} 0 \in \sum_{j\in\mathcal{I}} \varepsilon_j \mathcal{B}_j(\bar{y}) \left[ \partial_c \mathcal{A}_j(\bar{y}) - \frac{\mathcal{A}_j(\bar{y})}{\mathcal{B}_j(\bar{y})} \partial_c \mathcal{B}_j(\bar{y}) \right] + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \partial_c \Psi_j(\bar{y}) \\ + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \partial_c \theta_j(\bar{y}) - \sum_{j\in\mathcal{T}} \left[ \sigma_j^{\mathcal{C}} \partial_c \mathcal{C}_j(\bar{y}) + \sigma_j^{\mathcal{D}} \partial_c \mathcal{D}_j(\bar{y}) \right]. \end{aligned} \tag{5.2}$$

From (5.2), the following equation can be obtained:

$$\begin{aligned} 0 \in \sum_{j\in\mathcal{I}} \varepsilon_j \mathcal{B}_j(\bar{y}) \partial_c \mathcal{A}_j(\bar{y}) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \partial_c \Psi_j(\bar{y}) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \partial_c \theta_j(\bar{y}) \\ - \sum_{j\in\mathcal{T}} \left( \sigma_j^{\mathcal{C}} \partial_c \mathcal{C}_j(\bar{y}) + \sigma_j^{\mathcal{D}} \partial_c \mathcal{D}_j(\bar{y}) \right) - \sum_{j\in\mathcal{I}} \varepsilon_j \mathcal{A}_j(\bar{y}) \partial_c \mathcal{B}_j(\bar{y}). \end{aligned} \tag{5.3}$$

Further, let us define the following:

$$\begin{aligned} \overline{\sigma}_j^{\Psi} &:= \frac{\sigma_j^{\Psi}}{\sum_{i\in\mathcal{I}} \varepsilon_i \mathcal{B}_i(\bar{y})}, \quad \forall j \in \mathcal{I}^{\Psi}, \\ \overline{\sigma}_j^{\theta} &:= \frac{\sigma_j^{\theta}}{\sum_{i\in\mathcal{I}} \varepsilon_i \mathcal{B}_i(\bar{y})}, \quad \forall j \in \mathcal{I}^{\theta}, \\ \overline{\sigma}_j^{\mathcal{C}} &:= \frac{\sigma_j^{\mathcal{C}}}{\sum_{i\in\mathcal{I}} \varepsilon_i \mathcal{B}_i(\bar{y})}, \quad \forall j \in \mathcal{T}, \\ \overline{\sigma}_j^{\mathcal{D}} &:= \frac{\sigma_j^{\mathcal{D}}}{\sum_{i\in\mathcal{I}} \varepsilon_i \mathcal{B}_i(\bar{y})}, \quad \forall j \in \mathcal{T}. \end{aligned}$$

Then from the feasibility conditions of (NMFPPEC) and the definition of indices, the required relations follow. □

Suppose that $w \in \mathscr{M}$ is any arbitrary element. Further, let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$. Let $e = (1,1,\ldots,1) \in \mathbb{R}^r$ be the unit vector having $r$ components. Then, related to the primal (NMFPPEC), the corresponding Wolfe-type dual model (abbreviated as, (DP-W)) is formulated as given below:

$$\text{(DP-W)} \quad \text{Maximize } \mathcal{L}(w,\alpha,\sigma) := (\mathcal{L}_1(w,\alpha,\sigma), \mathcal{L}_2(w,\alpha,\sigma), \ldots, \mathcal{L}_r(w,\alpha,\sigma)),$$

subject to

$$\begin{aligned} 0 \in \sum_{i\in\mathcal{I}} \alpha_i \mathcal{B}_i(w) \Bigg[ \partial_c \mathcal{A}_i(w) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \partial_c \Psi_j(w) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \partial_c \theta_j(w) \\ - \sum_{j\in\mathcal{T}} \left( \sigma_j^{\mathcal{C}} \partial_c \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \partial_c \mathcal{D}_j(w) \right) \Bigg] - \sum_{i\in\mathcal{I}} \alpha_i \operatorname{grad} \mathcal{B}_i(w) \Bigg[ \mathcal{A}_i(w) \\ + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \Psi_j(w) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \theta_j(w) - \sum_{j\in\mathcal{T}} \left( \sigma_j^{\mathcal{C}} \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \mathcal{D}_j(w) \right) \Bigg], \end{aligned}$$

where, for every $j \in \mathcal{I}$, the function $\mathcal{L}_j(w,\alpha,\sigma)$ is defined as:

$$\mathcal{L}_j(w,\alpha,\sigma) := \frac{\mathcal{A}_j(w) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi} \Psi_j(w) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta} \theta_j(w) - \sum_{j\in\mathcal{T}} \left[ \sigma_j^{\mathcal{C}} \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \mathcal{D}_j(w) \right]}{\mathcal{B}_j(w)},$$

and

$$\sigma_j^{\Psi} \geq 0,\ \forall j \in \mathscr{A}^{\Psi},$$
$$\sigma_j^{\mathcal{C}} = 0,\ \forall j \in \mathcal{R}_{+0},$$
$$\sigma_j^{\mathcal{D}} = 0,\ \forall j \in \mathcal{R}_{0+},$$
$$\forall j \in \mathcal{R}_{00},\ \text{either},\ \sigma_j^{\mathcal{C}} > 0,\ \sigma_j^{\mathcal{D}} > 0 \text{ or},\ \sigma_j^{\mathcal{C}}\sigma_j^{\mathcal{D}} = 0.$$

We use the notation $\mathcal{F}_W$ to signify the set containing every feasible solution of the problem (DP-W). For the sake of convenience, we now construct an auxiliary function $\Omega : \mathscr{M} \to \mathbb{R}$ in the following manner:

$$\begin{aligned}\Omega(\cdot) := &\sum_{i\in\mathcal{I}} \alpha_i \mathcal{B}_i(w)\left[\mathcal{A}_i(\cdot) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi}\Psi_j(\cdot) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta}\theta_j(\cdot) - \sum_{j\in\mathcal{T}}\left(\sigma_j^{\mathcal{C}}\mathcal{C}_j(\cdot) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(\cdot)\right)\right] \\ &- \sum_{i\in\mathcal{I}} \alpha_i \mathcal{B}_i(\cdot)\left[\mathcal{A}_i(w) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi}\Psi_j(w) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta}\theta_j(w) - \sum_{j\in\mathcal{T}}\left(\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\right)\right],\end{aligned}$$

where, $w \in \mathcal{F}_w$. Throughout the remaining portion of the section, we shall always assume that

$$\begin{aligned}\mathcal{A}_j(w) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi}\Psi_j(w) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta}\theta_j(w) - \sum_{j\in\mathcal{T}}\left[\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\right] &\geq 0, \\ \mathcal{B}_j(w) &> 0,\end{aligned}$$

for every $j \in \mathcal{I}$.

The weak duality relation relating our considered primal problem (NMFPPEC) and Wolfe dual model (DP-W) is established in the following theorem.

**Theorem 5.2.** *Suppose that $\bar{z} \in \mathcal{F}$ and $(w, \alpha, \sigma) \in \mathcal{F}_W$ are arbitrary elements. Further, assume that the function $\Omega$ is geodesic convex at $w$. Then we have the following*

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} \not\prec \mathcal{L}(w, \alpha, \sigma).$$

*Proof.* From the feasibility conditions of the problem (DP-W), we have that:

$$\begin{aligned}0 \in &\sum_{i\in\mathcal{I}} \alpha_i \mathcal{B}_i(w)\left[\partial_c \mathcal{A}_i(w) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi}\partial_c\Psi_j(w) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta}\partial_c\theta_j(w)\right. \\ &\left. - \sum_{j\in\mathcal{T}}\left(\sigma_j^{\mathcal{C}}\partial_c\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\partial_c\mathcal{D}_j(w)\right)\right] - \sum_{i\in\mathcal{I}} \alpha_i \operatorname{grad}\mathcal{B}_i(w)\left[\mathcal{A}_i(w)\right. \\ &\left. + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi}\Psi_j(w) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta}\theta_j(w) - \sum_{j\in\mathcal{T}}\left(\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\right)\right].\end{aligned}$$

By *reductio ad absurdum*, we suppose that

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} \prec \mathcal{L}(w, \alpha, \sigma).$$

Consequently, we have the following inequality for every $j \in \mathcal{I}$:

$$\frac{\mathcal{A}_j(\bar{z})}{\mathcal{B}_j(\bar{z})} < \frac{\mathcal{A}_j(w) + \sum_{j\in\mathcal{I}^{\Psi}} \sigma_j^{\Psi}\Psi_j(w) + \sum_{j\in\mathcal{I}^{\theta}} \sigma_j^{\theta}\theta_j(w) - \sum_{j\in\mathcal{T}}\left[\sigma_j^{\mathcal{C}}\mathcal{C}_j(w) + \sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\right]}{\mathcal{B}_j(w)}.$$

Given that $\alpha_i > 0$ for every $i \in \mathcal{I}$. Then, we get the following:

$$\sum_{j\in\mathcal{I}} \alpha_j \big(\mathcal{A}_j(\bar{z})\mathcal{B}_j(w)\big) < \sum_{j\in\mathcal{I}} \alpha_j \mathcal{B}_j(\bar{z}) \Bigg( \mathcal{A}_j(w) + \sum_{j\in\mathcal{I}^\Psi} \sigma_j^\Psi \Psi_j(w) + \sum_{j\in\mathcal{I}^\theta} \sigma_j^\theta \theta_j(w) - \sum_{j\in\mathcal{T}} \left(\sigma_j^{\mathcal{C}} \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \mathcal{D}_j(w)\right) \Bigg).$$

Equivalently, we can rewrite the above inequality in the following manner:

$$\begin{aligned} &\sum_{j\in\mathcal{I}} \alpha_j \mathcal{B}_j(w) \left[ \mathcal{A}_j(\bar{z}) + \sum_{j\in\mathcal{I}^\Psi} \sigma_j^\Psi \Psi_j(\bar{z}) + \sum_{j\in\mathcal{I}^\theta} \sigma_j^\theta \theta_j(\bar{z}) - \sum_{j\in\mathcal{T}} \left(\sigma_j^{\mathcal{C}} \mathcal{C}_j(\bar{z}) + \sigma_j^{\mathcal{D}} \mathcal{D}_j(\bar{z})\right) \right] \\ &- \sum_{j\in\mathcal{I}} \alpha_j \mathcal{B}_j(\bar{z}) \left[ \mathcal{A}_j(w) + \sum_{j\in\mathcal{I}^\Psi} \sigma_j^\Psi \Psi_j(w) + \sum_{j\in\mathcal{I}^\theta} \sigma_j^\theta \theta_j(w) - \sum_{j\in\mathcal{T}} \left(\sigma_j^{\mathcal{C}} \mathcal{C}_j(w) + \sigma_j^{\mathcal{D}} \mathcal{D}_j(w)\right) \right] \\ &< \sum_{j\in\mathcal{I}} \alpha_j \mathcal{B}_j(w) \left[ \sum_{j\in\mathcal{I}^\Psi} \sigma_j^\Psi \Psi_j(\bar{z}) + \sum_{j\in\mathcal{I}^\theta} \sigma_j^\theta \theta_j(\bar{z}) - \sum_{j\in\mathcal{T}} \left(\sigma_j^{\mathcal{C}} \mathcal{C}_j(\bar{z}) + \sigma_j^{\mathcal{D}} \mathcal{D}_j(\bar{z})\right) \right]. \end{aligned} \tag{5.4}$$

We now note that $\alpha_j > 0$ and $\mathcal{B}_j > 0$ for every $j \in \mathcal{I}$. Combining these with the feasibility conditions of (NMFPPEC), we infer that

$$\sum_{j\in\mathcal{I}} \alpha_j \mathcal{B}_j(w) \left[ \sum_{j\in\mathcal{I}^\Psi} \sigma_j^\Psi \Psi_j(\bar{z}) + \sum_{j\in\mathcal{I}^\theta} \sigma_j^\theta \theta_j(\bar{z}) - \sum_{j\in\mathcal{T}} \left(\sigma_j^{\mathcal{C}} \mathcal{C}_j(\bar{z}) + \sigma_j^{\mathcal{D}} \mathcal{D}_j(\bar{z})\right) \right] \leq 0. \tag{5.5}$$

From inequalities (5.4), (5.5) and in view of the definition of function $\Omega$, we have the following:

$$\Omega(\bar{z}) < 0 = \Omega(w).$$

By invoking the geodesic convexity assumption on $\Omega$ at $w$, we get

$$\langle \xi^\Omega, \exp_w^{-1}(\bar{z}) \rangle_w < 0, \quad \forall \xi^\Omega \in \partial_c \Omega(w).$$

which contradicts the fact that $w \in \mathcal{F}_W$. Thus, the proof is complete. □

In the next corollary, we present another weak duality relation relating our considered primal problem (NMFPPEC) and (DP-W). The proof of the corollary follows in similar lines as the proof of Theorem 5.2.

**Corollary 5.3.** *Suppose that $\bar{z} \in \mathcal{F}$ and $(w,\alpha,\sigma) \in \mathcal{F}_W$ are arbitrary elements. Further, assume that the function $\Omega$ is strictly geodesic convex at z. Then we have the following*

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} \npreceq \mathcal{L}(w,\alpha,\sigma).$$

In the next theorem, a strong duality relation relating our considered primal problem (NMF-PPEC) and Wolfe dual problem (DP-W) is established.

**Theorem 5.4.** *Let $\bar{z} \in \mathcal{F}$ be any arbitrary Pareto efficient solution of (NMFPPEC) at which (GGCQ) holds. Then, some real numbers $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^\Psi \in \mathbb{R}$ $(j \in \mathcal{I}^\Psi)$, $\sigma_j^\theta \in \mathbb{R}$ $(j \in \mathcal{I}^\theta)$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$ exist, such that $(\bar{z},\alpha,\sigma) \in \mathcal{F}_W$. Furthermore, the corresponding values of the objective functions of (NMFPPEC) and (DP-W) are equal, that is:*

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})} = \mathcal{L}(\bar{z},\alpha,\sigma).$$

*The following assertions hold true.*

(a) *Suppose that each of the hypothesis stated in Theorem 5.2 hold true. Then $(\bar{z},\alpha,\sigma)$ is a weak Pareto efficient solution of (DP-W).*

(b) *Suppose that each of the hypothesis stated in the Corollary 5.3 hold true. Then $(\bar{z},\alpha,\sigma)$ is a Pareto efficient solution of (DP-W).*

*Proof.* According to the provided hypothesis, we have that $\bar{z}\in\mathcal{F}$ is any arbitrary Pareto efficient solution of (NMFPPEC) and (GGCQ) holds at $\bar{z}$.
As a result, in light of Theorem 3.10, we obtain some real multipliers $\alpha_j\in\mathbb{R}$, $\alpha_j>0$ $(j\in\mathcal{I})$, $\sigma_j^{\Psi}\in\mathbb{R}$ $(j\in\mathcal{I}^{\Psi})$, $\sigma_j^{\theta}\in\mathbb{R}$ $(j\in\mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}}\in\mathbb{R}$ $(j\in\mathcal{T})$, $\sigma_j^{\mathcal{D}}\in\mathbb{R}$ $(j\in\mathcal{T})$, satisfying the following

$$
\begin{aligned}
0\in\sum_{i\in\mathcal{I}}\alpha_i\mathcal{B}_i(w)\Big[\partial_c\mathcal{A}_i(w)&+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\partial_c\Psi_j(w)+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\partial_c\theta_j(w)\\
&-\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\partial_c\mathcal{C}_j(w)+\sigma_j^{\mathcal{D}}\partial_c\mathcal{D}_j(w)\Big)\Big]-\sum_{i\in\mathcal{I}}\alpha_i\,\mathrm{grad}\,\mathcal{B}_i(w)\Big[\mathcal{A}_i(w)\\
&+\sum_{j\in\mathcal{I}^{\Psi}}\sigma_j^{\Psi}\Psi_j(w)+\sum_{j\in\mathcal{I}^{\theta}}\sigma_j^{\theta}\theta_j(w)-\sum_{j\in\mathcal{T}}\Big(\sigma_j^{\mathcal{C}}\mathcal{C}_j(w)+\sigma_j^{\mathcal{D}}\mathcal{D}_j(w)\Big)\Big],
\end{aligned}
$$

and,

$$
\begin{aligned}
&\sigma_j^{\Psi}\geq 0,\quad \sigma_j^{\Psi}\Psi_j(\bar{y})=0,\quad \forall j\in\mathcal{I}^{\Psi},\\
&\sigma_j^{\mathcal{C}}\ \text{free},\forall j\in\mathcal{R}_{0+},\quad \sigma_j^{\mathcal{C}}\geq 0,\forall j\in\mathcal{R}_{00},\quad \sigma_j^{\mathcal{C}}=0,\ \forall j\in\mathcal{R}_{+0},\\
&\sigma_j^{\mathcal{D}}\ \text{free},\forall j\in\mathcal{R}_{+0},\quad \sigma_j^{\mathcal{D}}\geq 0,\forall j\in\mathcal{R}_{00},\quad \sigma_j^{\mathcal{D}}=0,\forall j\in\mathcal{R}_{0+}.
\end{aligned}
$$

Consequently, it follows that

$$(\bar{z},\alpha,\sigma)\in\mathcal{F}_W,$$

and

$$\frac{\mathcal{A}(\bar{z})}{\mathcal{B}(\bar{z})}=\mathcal{L}(\bar{z},\alpha,\sigma).$$

(a) By *reductio ad absurdum*, we suppose that $(\bar{z},\alpha,\sigma)$ is not a weak Pareto efficient solution of (DP-W). As a result, one can find some $(\overline{u},\overline{\alpha},\overline{\sigma})\in\mathcal{F}_W$, which satisfies the following:

$$\mathcal{L}(\bar{z})\prec\mathcal{L}(\overline{u}).$$

This contradicts the consequences of Theorem 5.2.

(b) By *reductio ad absurdum*, we suppose that $(\bar{z},\alpha,\sigma)$ is not a Pareto efficient solution of (DP-W). As a result, one can find some $(\overline{u},\overline{\alpha},\overline{\sigma})\in\mathcal{F}_W$, which satisfies the following:

$$\mathcal{L}(\bar{z})\preceq\mathcal{L}(\overline{u}).$$

This contradicts the consequences of Corollary 5.3.

Hence, the proof is complete. □

In the next theorem, a strict converse duality relation relating our considered primal problem (MPPEC) and Wolfe dual problem (DP-W) is established.

**Theorem 5.5.** *Suppose that $\bar{y}\in\mathcal{F}$ is any Pareto efficient solution of (NMFPPEC) at which (GGCQ) holds. Let $(\overline{w},\overline{\alpha},\overline{\sigma})$ be a feasible element of (DP-W), such that $\mathcal{L}(\bar{y})\preceq\mathcal{L}(\overline{w})$. Suppose that each of the hypotheses stated in Corollary 4.2 holds. then $\bar{y}=\overline{w}$.*

*Proof.* According to the provided hypothesis, we have that $\overline{y} \in \mathcal{F}$ be any weak Pareto efficient solution of (NMFPPEC) at which (GGCQ) holds. By *reductio ad absurdum*, we suppose that $\overline{y} \neq \overline{w}$. As a result, in light of Theorem 4.3, we can get $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \mathcal{I})$, $\sigma_j^{\Psi} \in \mathbb{R}$ $(j \in \mathcal{I}^{\Psi})$, $\sigma_j^{\theta} \in \mathbb{R}$ $(j \in \mathcal{I}^{\theta})$, $\sigma_j^{\mathcal{C}} \in \mathbb{R}$ $(j \in \mathcal{T})$, $\sigma_j^{\mathcal{D}} \in \mathbb{R}$ $(j \in \mathcal{T})$ such that $(\overline{y}, \alpha, \sigma) \in \mathcal{F}_M$. Moreover, we have $\frac{\mathcal{A}(\overline{y})}{\mathcal{B}(\overline{y})} = \mathcal{L}(\overline{y}, \alpha, \sigma)$. On the other hand, in view of the conclusions of the strong duality theorem (Theorem 4.3), we can infer that $(\overline{y}, \alpha, \sigma)$ is a Pareto efficient solution for (NMFPPEC). Since $\overline{y} \in \mathcal{F}$ and $(\overline{w}, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_{MW}$, then we have from Theorem 4.3 $\mathcal{L}(\overline{y}) \nleq \mathcal{L}(\overline{w})$, which is a contradiction. Hence, the proof is complete. □

**Remark 5.6.**
(1) If $\mathscr{M} = \mathbb{R}^n$, then Theorems 5.2 and 5.4 generalize Theorem 4 and Theorem 5 deduced in [31] from multiobjective (MPEC) to (NMFPPEC).
(2) Theorems 5.2, 5.4 and 5.5 extend Theorem 3.1, Theorem 3.2 and Theorem 3.3, respectively, deduced in [41] for a wider category of optimization problems, that is, (NMFPPEC).

**Example 5.7.** Consider the (NMFPPEC) (Problem (P)) defined in Example 3.11 on the manifold $\mathscr{M}$. We use the symbol $\mathcal{F}$ to signify the set containing every feasible solution of the problem (P). That is,

$$\mathcal{F} = \{z \in \mathscr{M}, z_1 = e, z_2 \geq e, \text{ or}, z_1 \geq e, z_2 = e\}.$$

Let $w \in \mathscr{M}$ be arbitrary. Further, let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ $(j \in \{1,2\})$, $\sigma^{\Psi} \in \mathbb{R}$, $\sigma^{\mathcal{C}} \in \mathbb{R}$, $\sigma^{\mathcal{D}} \in \mathbb{R}$. Then, related to primal (P), the corresponding Wolfe type dual model (abbreviated as, (DP-W)) is formulated as given below:

$$\text{(DP-W)} \quad \text{Maximize } \mathcal{L}(w) = (\mathcal{L}_1(w), \mathcal{L}_2(w)),$$

subject to

$$0 \in \sum_{i \in \{1,2\}} \alpha_i \mathcal{B}_i(w) \left[ \partial_c \mathcal{A}_i(w) + \sigma^{\Psi} \partial_c \Psi(w) - \left( \sigma^{\mathcal{C}} \partial_c \mathcal{C}(w) + \sigma^{\mathcal{D}} \partial_c \mathcal{D}(w) \right) \right]$$
$$- \sum_{i \in \{1,2\}} \alpha_i \operatorname{grad} \mathcal{B}_i(w) \left[ \mathcal{A}_i(w) + \sigma^{\Psi} \Psi_j(w) - \left( \sigma^{\mathcal{C}} \mathcal{C}(w) + \sigma^{\mathcal{D}} \mathcal{D}(w) \right) \right].$$

Choose the feasible solution $\hat{y} = (e, e) \in \mathcal{F}$. Consequently, we get the following:

$$\begin{aligned}
\partial_c \mathcal{A}_1(\hat{y}) &= \left\{ \left(e^2, 0\right)^T \right\}, \\
\partial_c \mathcal{B}_1(\hat{y}) &= \left\{ \left(e^2, 0\right)^T \right\}, \\
\partial_c \mathcal{A}_2(\hat{y}) &= \left\{ (0, e)^T \right\}, \\
\partial_c \mathcal{B}_2(\hat{y}) &= \left\{ (0, 0)^T \right\}, \\
\partial_c \Psi(\hat{y}) &= \left\{ (-e, -e)^T \right\}, \\
\partial_c \mathcal{C}(\hat{y}) &= \left\{ (e, 0)^T \right\}, \\
\partial_c \mathcal{D}(\hat{y}) &= \left\{ (0, e)^T \right\}.
\end{aligned}$$

One can easily verify that the feasible solution $\hat{y} = (e,e) \in \mathcal{F}$ is indeed, a Pareto efficient solution of (P). Furthermore, (GGCQ) holds at $\hat{y}$. Let us now pick some real numbers $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$, $\sigma^{\Psi} = 0$, $\sigma^{\mathcal{C}} = \frac{e}{2}$, $\sigma^{\mathcal{D}} = \frac{1}{2}$. Further we choose $(e^2,0)^T \in \partial_c \mathcal{A}_1(\hat{y})$, $(0,e)^T \in \partial_c \mathcal{A}_2(\hat{y})$, $(e^2,0)^T \in \partial_c \mathcal{B}_1(\hat{y})$, $(0,0)^T \in \partial_c \mathcal{B}_2(\hat{y})$, $(-e,-e)^T \in \partial_c \Psi(\hat{y})$, $(e,0)^T \in \partial_c \mathcal{C}(\hat{y})$, $(0,e)^T \in \partial_c \mathcal{D}(\hat{y})$. Then, we can verify that $\hat{y}$ is a feasible element of (DP-MW).
Further, every assumption of the strong duality theorem is satisfied. One can verify that $(\hat{y}, \alpha, \sigma)$ is a Pareto efficient solution of (DP-W).

## 6. Conclusions and Future Research Directions

In this article, we have explored a category of (NMFPPEC) in the setting of Hadamard manifolds. (GGCQ) for (NMFPPEC) and KKT type necessary criteria of Pareto efficiency for (NMFPPEC) are presented. Mond-Weir as well as Wolfe type dual models related to (NMFPPEC) are formulated. Weak, strong, and strict converse duality results are derived relating (NMFPPEC) and the respective dual models. Suitable non-trivial examples have been furnished to demonstrate the significance of the results established in this article.

The various results that are derived in this article extend as well as generalize some well-known results from the literature. In particular, we have extended the corresponding results presented by Ghosh et al. [12] smooth multiobjective fractional programming problems with equilibrium constraints to (NMFPPEC). Moreover, the duality results derived in the present article extend and generalize similar results derived by Singh and Mishra [31] for a more general class of optimization problems in the setting of Hadamard manifolds. Further, the results of this paper extend corresponding results of [41] for a wider category of problems, namely, (NMFPPEC).

For future research, investigating optimality conditions for nonsmooth multiobjective fractional programming problems with vanishing constraints would be an interesting problem. This would be our future course of study.

**Acknowledgement**
The second author was supported by the Council of Scientific and Industrial Research (CSIR), New Delhi, India, Grant Number 09/1023(0044)/2021-EMR-I.

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