Commun. Optim. Theory 2024 (2024) 2 https://doi.org/10.23952/cot.2024.2

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

CLOSED CONVEX SETS OF MOTZKIN, GENERALIZED MINKOWSKI, AND PARETO BORDERED TYPES

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Dedicated to Francis Clarke on the occasion of his 75th birthday

Abstract. In this paper the generalized Minkowski sets are defined and characterized. On the other hand, the Motzkin decomposable sets, along with their epigraphic versions are considered and characterized in new ways. Among them, the closed convex sets with one single minimal face, i.e. translated closed convex cones, along with their epigraphic counterparts are particularly studied. Finally, the generalized Minkowski sets along with the class of closed convex sets with full Pareto like relative boundary are considered and studied. The latter ones are called *Pareto bordered* sets and their epigraphic counterparts are also considered and studied. It turns out that the Pareto bordered sets are generalized Minkowski.

Keywords. Closed convex sets; Generalized Minkowski sets; Motzkin decomposable sets; Pareto bordered sets.

2020 Mathematics Subject Classification. 52A20, 26B25.

1. Introduction

Every compact convex subset of the Euclidean space is, according to the Minkowski theorem [8], the convex hull of the set of its extreme points (see also [2, p.52] and [3, Theorem 2.7.2]). However, the class of all closed convex sets that are representable as their own sets of extreme points is significantly much larger and we called them *Minkowski sets*. Indeed, there are many unbounded Minkowski sets and we characterized them in [6]. For such subsets the extreme points are obviously minimal and lowest dimensional faces as well. In this paper we first prove that a face of a closed convex subset of the Euclidean space is minimal if and only if it is lowest dimensional. Such faces are also shown to be lineality invariant, and several other properties of them are proved. We next enlarge the class of Minkowski sets to the one of *generalized Minkowski sets*, which are intensively studied here. This latter class consists of

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Received: December 3, 2022; Accepted: May 4, 2023.

all closed convex subsets of the Euclidean space that are representable as the convex hull of the union of their minimal (for the inclusion) faces (or, equivalently, the union of their lowest dimensional faces, according to Corollary 2.4 below). The Minkowski sets (see [6]) are obviously particular examples of such closed convex sets. On the other hand, every Minkowski set C can produce extra sets with the above property simply by considering $C+U$, where U is a suitable subspace. We will prove that, in fact, every set with the above property is the sum of a Minkwoski set with a subspace. We will also consider the lower semicontinuous (l.s.c., in brief) proper convex functions from $\mathbb{R}^n$ to $\mathbb{R}\cup\{+\infty\}$ whose epigraphs are Motzkin decomposable or generalized Minkowski sets and obtain characterizations of such classes of functions from those of the corresponding classes of sets.

The paper is organized as follows: In the second section we prove several characterizations of the lowest dimensional faces of a closed convex subset of the Euclidean space. This is done by proving several helpful statements before. The equivalence between the minimality of a face of a closed convex set and its quality to be lowest dimensional appears as a corollary afterwards. The third section is devoted to new characterizations of Motzkin decomposable sets and their epigraphic versions. We recall that a set $C\subseteq\mathbb{R}^n$ is said to be Motzkin decomposable [4] if there exist a compact convex set K and a closed convex cone D such that $C=K+D$. Note that several characterizations of Motzkin decomposable sets were given earlier in [5]. The particular case of translated closed convex cones is also studied by highlighting their property to have one single (i.e., unique) minimal face. Actually, this property characterizes closed convex cones. Finally, in the last section the generalized Minkowski sets along with the class of the closed convex sets whose Pareto like sets cover their relative boundaries are considered and studied. Such sets are called *Pareto bordered* sets and they are showed to be generalized Minkowski sets whenever their lin-orthogonal sections are at least two dimensional.

2. Definitions and preliminary results

Throughout the whole paper, $0^+C:=\{y\in\mathbb{R}^n : y+C\subseteq C\}$ stands for the *recession cone* of a closed convex set C. Recall that 0^+C is a convex cone [10, Theorem 18.1] and $\operatorname{lin} C:=0^+C\cap(-0^+C)$ is a subspace of $\mathbb{R}^n$ called *the lineality* of C.

Recall that the *polar cone* of a set $S\subseteq\mathbb{R}^n$ is

$$S^0:=\{x^*\in\mathbb{R}^n : \langle x^*,x\rangle\le 0 \qquad \forall x\in S\},$$

and the *barrier cone* $\operatorname{barr}(C)$ of C consists of those vectors $x^*\in\mathbb{R}^n$ for which there exists $\alpha_{x^*}\in\mathbb{R}$ with the property $\langle c,x^*\rangle\le\alpha_{x^*}$, for every $c\in C$. In other words,

$$\operatorname{barr}(C):=\left\{x^*\in\mathbb{R}^n \mid \sup_{c\in C}\langle c,x^*\rangle<+\infty\right\}.$$

It is well known that $\operatorname{barr}(C)$ is a convex cone and $[\operatorname{barr}(C)]^0=0^+C$.

Definition 2.1. A vector x^* is said to be *normal* to a convex set $C\subseteq\mathbb{R}^n$ at a point $x\in C$ if $\langle c-x,x^*\rangle\le 0$, for all $c\in C$. The set of vectors normal to C at x is a closed convex cone denoted by $N_C(x)$ and is called the *normal cone* to C at x.

Note that the normal cone to C at every interior point of C reduces to $\{0\}$. We extend the, possibly multivalued, mapping $N_C : C\rightrightarrows\mathbb{R}^n$ to the whole space $\mathbb{R}^n$ by setting $N_C(x):=\emptyset$ whenever $x\in\mathbb{R}^n\setminus C$. For $A\subseteq\mathbb{R}^n$, we consider the set $N_C(A):=\cup_{x\in C}N_C(x)$. We call $N_C(\mathbb{R}^n)$ the *total normal cone* of C; clearly, it is the union of the normal cones to C at all points of C. Note that the total normal cone need not be convex.

For a l.s.c. proper convex function $f:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, one has the equality $\partial f^* = (\partial f)^{-1}$, i.e.

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*), \tag{2.1}$$

where f^* stands for the conjugate function of f and $\partial f(x)$ for the subdifferential of f at x. Recall that f^* and ∂f are defined by

$$f^*(x^*) := \sup_x \{\langle x, x^*\rangle - f(x)\},$$
$$\partial f(x) := \{x^* \mid f(y) \geq f(x) + \langle x^*, y-x\rangle, \forall y\}.$$

For example, the support function σ_C of C is the conjugate δ_C^* of the indicator function δ_C of C. Recall that these two functions are defined by

$$\sigma_C : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\},\ \sigma_C(x^*) := \sup_{c\in C}\langle c, x^*\rangle,$$
$$\delta_C : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}, \delta_C(x) := \begin{cases} 0 & \text{if } c \in C \\ +\infty & \text{if } c \notin C. \end{cases}$$

We also observe that

$$\operatorname{dom}(\sigma_C) := \{x^* \in \mathbb{R}^n \mid \sigma_C(x^*) < +\infty\} = \operatorname{barr}(C).$$

and

$$\operatorname{dom}(\partial\sigma_C) := \{x^* \in \mathbb{R}^n \mid \partial\sigma_C \neq \emptyset\} = N_C(\mathbb{R}^n),$$

as

$$\partial\delta_C = N_C;$$

moreover, as a consequence of (2.1), if C is convex and closed, then

$$x^* \in N_C(x) \Leftrightarrow x \in \partial\sigma_C(x^*).$$

We denote by $\operatorname{ext}(A)$ the set of extreme points of $A \subseteq \mathbb{R}^n$.

Theorem 2.2. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set, $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* $\operatorname{lin} C$ *(i.e., a linear subspace such that $U \oplus \operatorname{lin} C = \mathbb{R}^n$), and F be a nonempty face of C. The following statements are equivalent:*

(1) F is lowest dimensional.
(2) F is an affine variety.
(3) $F = x + \operatorname{lin} C$ for some $x \in C$.
(4) $F \cap U$ is a singleton.
(5) $F \cap U$ is the singleton of an extreme point of $C \cap U$.
(6) $F = x + \operatorname{lin} C$ for some $x \in \operatorname{ext}(C \cap U)$.

We postpone the proof of Theorem 2.2 to the end of this section.

Remark 2.3. In fact, 6 implies that F is a face of C. To see this, we consider two points $p, q \in C$ such that $(1-t)p + tq \in x + \operatorname{lin} C$ for some $t \in (0,1)$ and the decomposition $p = p_U + p_l$ and $q = q_U + q_l$, where $p_U, q_U \in U$ and $p_l, q_l \in \operatorname{lin} C$, due to the direct sum decomposition

$U \oplus \text{lin } C = \mathbb{R}^n$. Note that $p_U = p - p_l$, $q_U = q - q_l \in C + \text{lin } C = C$, namely $p_U, q_U \in C \cap U$, and therefore

$$(1-t)p_U + tq_U = (1-t)p + tq - ((1-t)p_l + tq_l) \in x + \text{lin } C - \text{lin } C = x + \text{lin } C.$$

Consequently,

$$(1-t)p_U + tq_U \in C \cap U \cap (x + \text{lin } C) \subseteq U \cap (x + \text{lin } C) = \{x\}$$

(the latter equality following from the fact that $x \in U$), which shows that $p_U, q_U = x$, as x is an extreme point of $C \cap U$. Thus

$$p = x + p_l, \, q = x + q_l \in x + \text{lin } C,$$

which completes the proof that F is a face of C.

Corollary 2.4. *A face F of the closed convex set $C \subseteq \mathbb{R}^n$ is lowest dimensional if and only if F is minimal.*

Proof. If F is lowest dimensional, then F is obviously minimal, as a proper face of F would have dimension strictly smaller than dim F. Conversely, assume that F is minimal and not lowest dimensional, i.e. F is not an affine variety, according to Theorem 2.2. Since F is minimal, it is not a closed half of any affine variety either, as every halfflat has a proper face. According to [13, Theorem 2.6.12], $F = \text{conv rbd } F$ and the inequality dim $F_a <$ dim F holds for every $a \in \text{rbd } F$, where F_a is the smallest face of F containing a, i.e. the intersection of all faces of F that contain a (see [13, Corollary 2.6.11]). This shows that F is not minimal, as it has a proper face F_a. We have thus obtained a contradiction. □

We will denote by $MF(C)$ the union of the minimal (equivalently, lowest dimensional) faces of a closed convex set $C \subseteq \mathbb{R}^n$.

Corollary 2.5. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin C *. Then*

$$MF(C) = \text{ext}(C \cap U) + \text{lin } C.$$

In particular,

$$MF(C) = \text{ext}(C \cap (\text{lin } C)^{\perp}) + \text{lin } C.$$

In order to prove Theorem 2.2, we need some preliminary results.

Proposition 2.6. *If F is a nonempty face of a closed convex set $C \subseteq \mathbb{R}^n$, then*

$$\text{lin } F = \text{lin } C.$$

Proof. The inclusion $\subseteq$ is obvious, as one has $0^+(F) \subseteq 0^+(C)$ (see [10, Theorem 8.3, p. 63]). To prove the opposite inclusion, let $d \in \text{lin } C$ and take $x \in F$. Since $x = \frac{1}{2}(x + d + x - d)$ and $x + d, x - d \in C$, we have $x + d, x - d \in F$, which shows that both d and $-d$ belong to $0^+(F)$, that is, $d \in \text{lin } F$, and the equality lin $F =$ lin C is completely proved. □

Corollary 2.7. *If F is a nonempty face of a closed convex set $C \subseteq \mathbb{R}^n$, then F contains l_C-dimensional affine varieties, where $l_C = \text{dim}(\text{lin } C)$. In particular* $\dim F \geq \dim(\text{lin } C)$.

Lemma 2.8. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin C. *A nonempty face F of C is lowest dimensional if and only if it is an l_C-dimensional affine variety of $\mathbb{R}^n$ parallel with* lin C. *In such a case the intersection $F \cap U$ is a singleton, say $\{x_F\}$, with x_F being an extreme point of $C \cap U$, and $F = x_F + \operatorname{lin} C$.*

Proof. Assume that F is a lowest dimensional face of the closed convex set C. Then F has no proper faces and empty relative boundary therefore. In other words the boundary of F in $\operatorname{aff}(F)$ is empty, which shows, due to the connectedness of $\operatorname{aff}(F)$, that $F = \operatorname{aff}(F)$ (see e.g [12, p. 86]). Thus $F = x + V$ for some $x \in F$ and some subspace V of $\mathbb{R}^n$. Since

$$F + V = x + V + V = x + V = F,$$

we have $V \subseteq 0^+C$ as well as $V = -V \subseteq -0^+C$, namely

$$V \subseteq 0^+C \cap (-0^+C) = \operatorname{lin} C.$$

To prove the converse inclusion, let $d \in \operatorname{lin} C$. Then we have $x + d, x - d \in C$; hence, since $\frac{1}{2}(x+d) + \frac{1}{2}(x-d) \in F$ and F is a face, we have $x + d \in F = x + V$, that is, $d \in V$. This shows that $V = \operatorname{lin} C$.

Conversely, if the face F of C is an l_C-dimensional affine variety of $\mathbb{R}^n$ parallel with lin C, then it is obviously lowest dimensional due to Corollary 2.7.

The intersection $F \cap U$ is obviously a singleton, say $\{x_F\}$, and one has $F = x_F + \operatorname{lin} C$. In order to justify the extreme property of x_F, assume that $x_F = (1-t)p + tq$ for some $p, q \in C \cap U$ and some $t \in]0,1[$. Since $x_F \in F$ and F is a face of C, it follows that $p, q \in F = x_F + \operatorname{lin} C$. Thus $p = x_F + p_l$ and $q = x_F + q_l$ for some $p_l, q_l \in \operatorname{lin} C$. On the other hand,

$$p_l = p - x_F,\ q_l = q - x_F \in U.$$

as $p, q, x_F \in U$. Thus $p_l, q_l \in U \cap \operatorname{lin} C$, which shows that $p_l = q_l = 0$ and therefore $p = x_F = q$. □

Proof of Theorem 2.2. The equivalence [1 ⇔ 3] and the implication [1 ⟹ 5] follow from Lemma 2.8.

[2 ⟹ 3]. Assume that $F = x + V$ for some $x \in F$ and a subspace V of $\mathbb{R}^n$. Since $V = \operatorname{lin} F$, statement 3 follows via Proposition 2.6.

The implications [3 ⟹ 2], [5 ⟹ 4] and [6 ⟹ 3] are obvious.

The implications [4 ⟹ 3] and [5 ⟹ 6] follow from the equality

$$F = (F \cap U) + \operatorname{lin} F.$$

□

3. Closed convex sets with a single minimal face and Motzkin decomposable sets

In this section we first characterize the closed convex translated cones along with their epigraphic counterpats in terms of their minimal faces. The Motzkin decomposable sets along with the Motzkin decomposable functions are characterized afterwards.

Remark 3.1. A closed convex cone K has one single minimal face. Its single minimal face is lin K. Indeed we first observe that 0 is the only extreme point of $K \cap (\text{lin } K)^{\perp}$. Therefore lin $K = 0 + \text{lin } K$ is the only minimal face of K, due to Theorem 2.2(6). Note that a translated closed convex cone has the same property.

Proposition 3.2. *If a closed convex set $C \subseteq \mathbb{R}^n$ has one single minimal face F, then this face is contained in any face of C. In other words, F is the intersection of all faces of C.*

Proof. It is an immediate consequence of the fact that every face of C contains a minimal face. $\square$

Proposition 3.3. *A nonempty closed convex set $C \subseteq \mathbb{R}^n$ has one single minimal face if and only if it is a closed convex cone or a translated closed convex cone.*

Proof. The obvious implication is Remark 3.1. We now assume that the closed convex set C has one single minimal face. We assume that the origin of $\mathbb{R}^n$ belongs to the minimal face F of C, as otherwise we translate the set with the opposite of a vector in F. We will prove that, under this assumption, C is actually a cone. In what follows we will use an inductive argument on the dimension of C to show that the positive multiples of the vectors in C remain in C. Indeed, if C is 0-dimensional, i.e. $C = \{0\}$, then it is obviously a cone. We now assume that $\dim C \geq 1$ and the proper faces of C, which have a strictly smaller dimension, are cones. If $x \in C$ would be a point such that $tx \notin C$ for some $t > 0$, then $sx \in \text{rbd}\, C$, where $s = \sup\{\lambda > 0 \mid \lambda x \in C\} \geq 1$, i.e sx belongs to a proper face of C, which is, by the inductive hypothesis, a cone. In particular, $tx = ts^{-1}(sx)$ belongs to that proper face and therefore to C, which is absurd. $\square$

Remark 3.4. The epigraph of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a nonempty closed convex cone if and only if f is in the orbit of a l.s.c. proper sublinear function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ with respect to the action

$$(\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^{\mathbb{R}^n} \longrightarrow \mathbb{R}^{\mathbb{R}^n}, ((u,v),\phi) \mapsto (u,v) \oplus \phi,$$

where $((u,v) \oplus \phi)(x) := \phi(x+u) - v$. Note that

$$\text{epi}\,((u,v) \oplus \phi) = \text{epi}\ \phi - (u,v),$$

for all $((u,v),\phi) \in (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}^{\mathbb{R}^n}$. For the notions of group actions and orbits, we refer to [11].

From Remark 3.4 and Proposition 3.3, one obtains the following result.

Corollary 3.5. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. proper convex function. Then there exist $u \in \mathbb{R}^n$ and $v \in \mathbb{R}$ such that $f(\cdot + u) - v$ is sublinear if and only if* epi f *has one single minimal face.*

We say that a nonempty closed convex set $C \subseteq \mathbb{R}^n$ is Motzkin decomposable (M-decomposable in short) if there exist a decomposition $C = C_0 + K$, with the set $C_0 \subseteq \mathbb{R}^n$ being convex and compact and K being a closed convex cone. Then we say that C_0 and K are the compact and conic components, respectively, of that decomposition. The conic component is uniquely determined, namely, one has $K = 0^+(C)$. The original Motzkin Theorem [9] asserts that every polyhedral convex set is the sum of a polytope and a polyhedral convex cone, namely it is M-decomposable. From the following characterization of Motzkin decomposable sets, we will obtain a new characterization of closed convex cones.

Proposition 3.6. *For a nonempty closed convex set $C \subseteq \mathbb{R}^n$ and a compact convex set $C_0 \subseteq C$, the following statements are equivalent:*

(1) C is Motzkin decomposable and C_0 is a compact component of C.
(2) $N_C(\mathbb{R}^n) = N_C(C_0)$.

Proof. [1 $\Longrightarrow$ 2] Let $x^* \in N_C(\mathbb{R}^n)$. Then $x^* \in N_C(x)$ for some $x \in C$. By 1, there exist $c_0 \in C_0$ and $d \in 0^+(C)$ such that $x = c_0 + d$. For every $c \in C$, since $c + d \in C$ we have $\langle x^*, c - c_0\rangle = \langle x^*, c + d - x\rangle \leq 0$. This shows that $x^* \in N_C(c_0)$, which proves that $N_C(\mathbb{R}^n) \subseteq N_C(c_0) \subseteq N_C(C_0)$. Since the inclusion $N_C(C_0) \subseteq N_C(\mathbb{R}^n)$ is obvious, we obtain 2.

[2 $\Longrightarrow$ 1] From the well known equality

$$C = \{x \in \mathbb{R}^n : \langle x^*, x\rangle \leq \delta_C^*(x^*) \qquad \forall x^* \in N_C(\mathbb{R}^n)\}$$

(see [10, Theorem 13.1]), using that $\delta_C^*(x^*) = \delta_{C_0}^*(x^*)$ for every $x^* \in N_C(C_0)$, by 2 we obtain

$$C = \left\{x \in \mathbb{R}^n : \langle x^*, x\rangle \leq \delta_{C_0}^*(x^*) \qquad \forall x^* \in N_C(C_0)\right\}. \tag{3.1}$$

Since $\delta_{C_0}^*$ is continuous, we can replace $N_C(C_0)$ with its closure in (3.1); therefore, by $\operatorname{cl} N_C(C_0) = \operatorname{cl} N_C(\mathbb{R}^n) = \operatorname{cl} \operatorname{barr}(C) = (\operatorname{barr}(C))^{00} = (0^+(C))^0$, we obtain

$$C = \left\{x \in \mathbb{R}^n : \langle x^*, x\rangle \leq \delta_{C_0}^*(x^*) \qquad \forall x^* \in \left(0^+(C)\right)^0\right\}.$$

Observe that this equality can be equivalently written

$$C = \left\{x \in \mathbb{R}^n : \langle x^*, x\rangle \leq \delta_{C_0}^*(x^*) + \delta_{(0^+(C))^0}(x^*) \qquad \forall x^* \in \mathbb{R}^n\right\}. \tag{3.2}$$

Hence, since $\delta_{(0^+(C))^0} = \delta_{0^+(C)}^*$ and $\delta_{C_0}^* + \delta_{0^+(C)}^* = \delta_{C_0+0^+(C)}^*$, equality (3.2) yields

$$C = \left\{x \in \mathbb{R}^n : \langle x^*, x\rangle \leq \delta_{C_0+0^+(C)}^*(x^*) \qquad \forall x^* \in \mathbb{R}^n\right\} = C_0 + 0^+(C),$$

which proves 1. □

Corollary 3.7. *A closed convex set $C \subseteq \mathbb{R}^n$ is a cone with vertex $x_0 \in C$ (that is, $C - x_0$ is a cone) if and only if $N_C(\mathbb{R}^n) = N_C(x_0)$.*

Proof. Apply Proposition 3.6 with $C_0 := \{x_0\}$. □

The following corollary is a counterpart to Corollary 3.7 for functions.

Corollary 3.8. *For a l.s.c. proper convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ and a point $u \in \operatorname{dom} f$ such that $\operatorname{dom} f - u$ is a cone, there exists $v \in \mathbb{R}$ such that $f(\cdot + u) - v$ is sublinear if and only if $\partial f(\mathbb{R}^n) = \partial f(u)$.*

Proof. Let $g := f(\cdot + u) - v$. Since $\operatorname{dom} g = \operatorname{dom} f - u$ and $\partial g(x) = \partial f(x + u)$ for every $x \in \mathbb{R}^n$, we assume, without loss of generality, that $u = 0$ and $v = 0$, so that $\operatorname{dom} f$ is a cone and $f(0) = 0$. If f is sublinear, from the well known (and easy to prove) equality

$$\partial f(x) = \partial f(0) \cap \{x^* \in \mathbb{R}^n : \langle x^*, x\rangle = f(x)\},$$

which holds for every $x \in \mathbb{R}^n$, it immediately follows that $\partial f(\mathbb{R}^n) = \partial f(0)$.

Conversely, assume that the latter equality holds. Using the well known and easy to prove formula

$$N_{\operatorname{epi} f}(x, f(x)) = \left(N_{\operatorname{dom} f}(x) \times \{0\}\right) \cup \mathbb{R}_+ \left(\partial f(x) \times \{-1\}\right),$$

which holds for every $x \in \text{dom } f$, and Corollary 3.7, we obtain

$$\begin{aligned} N_{\text{epi } f}\left(\mathbb{R}^{n+1}\right) &= \bigcup_{x \in \text{dom } f} \left(\left(N_{\text{dom } f}(x) \times \{0\} \right) \cup \mathbb{R}_+ \left(\partial f(x) \times \{-1\} \right) \right) \\ &= \left(N_{\text{dom } f}(\mathbb{R}^n) \times \{0\} \right) \cup \mathbb{R}_+ \left(\partial f(\mathbb{R}^n) \times \{-1\} \right) \\ &= \left(N_{\text{dom } f}(0) \times \{0\} \right) \cup \mathbb{R}_+ \left(\partial f(0) \times \{-1\} \right) = N_{\text{epi } f}(0,0). \end{aligned}$$

Hence, by Corollary 3.7, the set epi f is a cone, and therefore f is sublinear. □

A necessary condition for a nonempty closed convex set to be Motzkin decomposable is provided next.

Proposition 3.9. *If a nonempty closed convex set $C \subseteq \mathbb{R}^n$ is Motzkin decomposable, then*

$$N_C(\mathbb{R}^n) = \left(0^+ (C)\right)^0. \tag{3.3}$$

Proof. We first observe that the inclusion $\subseteq$ in (3.3) holds for every closed convex set C. Assume now that C is Motzkin decomposable and C_0 is a compact component of C, and let $x^* \in \left(0^+ (C)\right)^0$. Since C_0 is compact, we have $x^* \in N_{C_0}(x)$ for some $x \in C_0$. Then, for every $c \in C_0$ and $d \in 0^+ (C)$, we have

$$\langle x^*, c+d \rangle = \langle x^*, c \rangle + \langle x^*, d \rangle \leq \langle x^*, x \rangle,$$

which shows that $x^* \in N_C(x) \subseteq N_C(\mathbb{R}^n)$, thus completing the proof of (3.3). □

The converse of Proposition 3.9 does not hold true; indeed, using Corollary 3.11 below, one can see that [4, Example 14] provides a counterexample.

We recall that a nonempty closed convex set is said to be hyperbolic [1] if it is contained in some Motzkin decomposable set with the same recession cone.

Proposition 3.10. *For a nonempty closed convex set $C \subseteq \mathbb{R}^n$, the following statements are equivalent:*

(1) $N_C(\mathbb{R}^n) = (0^+ (C))^0$.
(2) $N_C(\mathbb{R}^n)$ *is closed.*

Proof. Implication $[1 \Longrightarrow 2]$ is obvious.

$[2 \Longrightarrow 1]$ By 2, the Gauss range of C, that is, the intersection of $N_C(\mathbb{R}^n)$ with the unit sphere, is closed. Hence, by [7, Theorem 18], the set C is hyperbolic, which, in view of [1, Proposition 5], means that $\text{barr}(C) = \left(0^+ (C)\right)^0$; on the other hand, the inclusions $N_C(\mathbb{R}^n) \subseteq \text{barr}(C) \subseteq \text{cl } N_C(\mathbb{R}^n)$ combined with 2 yield $\text{barr}(C) = N_C(\mathbb{R}^n)$. From these two equalities, 1 immediately follows. □

Combining Proposition 3.10 with [7, Lemma 19 and Theorem 18], one obtains the following corollary. Recall that a hyperplane H is said to be asymptotic to the closed convex set $C \subseteq \mathbb{R}^n$ if $H \cap C = \emptyset$ and the gap between H and C is 0 (that is, $\inf\{h - c : h \in H, c \in C\} = 0$).

Corollary 3.11. *For a nonempty closed convex set $C \subseteq \mathbb{R}^n$, the following statements are equivalent:*

(1) $N_C(\mathbb{R}^n) = (0^+ (C))^0$.

(2) C is hyperbolic and has no asymptotic hyperplane.

As defined in [10], the recession function $f0^+ : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ of a l.s.c. proper convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is the l.s.c. proper sublinear function defined by epi $f0^+ = 0^+ (\text{epi } f)$. We omit the easy proof of the following Proposition.

Proposition 3.12. *If $s : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a l.s.c. proper sublinear function, then*

$$(\text{epi } s)^0 = \left((\text{dom } s)^0 \times \{0\}\right) \cup \mathbb{R}_+ (\partial s(0) \times \{-1\}).$$

Applying Proposition 3.9 to the epigraph of a l.s.c. proper convex function yields the following corollary.

Corollary 3.13. *If a l.s.c. proper convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is Motzkin decomposable, then $(\text{dom } f0^+)^0 = N_{\text{dom } f}(\mathbb{R}^n)$ and $\partial f0^+(0) = \partial f(\mathbb{R}^n)$.*

Proof. According to [7, Lemma 27], we have

$$N_{\text{epi } f}\left(\mathbb{R}^{n+1}\right) = \left(N_{\text{dom } f}(\mathbb{R}^n) \times \{0\}\right) \cup \mathbb{R}_+ (\partial f(\mathbb{R}^n) \times \{-1\}).$$

Hence, the conclusion easily follows from combining Proposition 3.9, applied to $C :=$ epi f, with Proposition 3.12, applied to $s := f0^+$. □

4. Generalized Minkowski sets

In this section we introduce the notions of generalized Minkowski set and generalized Minkowski function and provide several characterizations of such new notions. In this respect, several auxiliary results are also proved.

Definition 4.1. A nonempty closed convex set $C \subseteq \mathbb{R}^n$ which is the convex hull of the union of its minimal faces is called a *generalized Minkowski set*, that is, if

$$C = \text{conv } MF(C). \tag{4.1}$$

Definition 4.2. A l.s.c. proper convex function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ which has a generalized Minkowski epigraph is called a *generalized Minkowski function.*

Theorem 4.3. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin C*. Then, C is a generalized Minkowski set if and only if $C \cap U$ is a Minkowski set.*

In particular, C is a generalized Minkowski set if and only if the orthogonal slice $C \cap (\text{lin } C)^\perp$ is a Minkowski set.

Proof. Assume that $C \cap U$ is a Minkowski set. This quality of $C \cap U$ combined with [10, p. 65] and the additivity of the conv operator and Corollary 2.5 leads us to

$$\begin{aligned} C = (C \cap U) + \text{lin } C &= \text{conv ext}(C \cap U) + \text{lin } C \\ &= \text{conv ext}(C \cap U) + \text{conv lin } C \\ &= \text{conv}(\text{ext}(C \cap U) + \text{lin } C) \\ &= \text{conv } MF(C). \end{aligned}$$

Conversely, assume that $C = \text{conv}\, MF(C)$ and let $x \in C \cap U$. Since $x \in C$, it follows that $x = \lambda_1 x_1 + \cdots + \lambda_m x_m$ for some $\lambda_1, \ldots, \lambda_m \in [0,1]$ such that $\lambda_1 + \cdots + \lambda_m = 1$ and some $x_1 \in F_1, \ldots, x_m \in F_m$, where

$$F_1 = x_{F_1} + \text{lin}\, C, \ldots, F_m = x_{F_m} + \text{lin}\, C$$

are lowest dimensional faces of C and $x_{F_1}, \ldots, x_{F_m} \in \text{ext}(C \cap U)$ (see Theorem 2.2). Consequently, we have $x_1 = x_{F_1} + p_{l_1}, \ldots, x_m = x_{F_m} + p_{l_m}$ for some $p_{l_1}, \ldots, p_{l_m} \in \text{lin}\, C$ and therefore $x = \lambda_1 x_{F1} + \cdots + \lambda_m x_{F_m} + \lambda_1 p_{l_1} + \cdots + \lambda_m p_{l_m}$, namely

$$U \ni x - \lambda_1 x_{F_1} - \cdots - \lambda_m x_{F_m} = \lambda_1 p_{l_1} + \cdots + \lambda_m p_{l_m} \in \text{lin}\, C,$$

which shows that $\lambda_1 p_{l_1} + \cdots + \lambda_m p_{l_m} \in U \cap (\text{lin}\, C) = \{0\}$ and therefore

$$x = \lambda_1 x_{F1} + \cdots + \lambda_m x_{F_m} \in \text{conv ext}(C \cap U).$$

Thus, the inclusion $C \cap U \subseteq \text{conv ext}(C \cap U)$ is now completely done, and the opposite inclusion is obvious. □

We now recall that the linearity space of a l.s.c. proper convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is

$$\text{lin}\, f := \left\{ d \in \mathbb{R}^n : f0^+(-d) = -f0^+(d) \right\}.$$

The subspace $\text{lin}\, f$ consists of those directions in which f is affine. By [10, Theorem 4.8], the recession function $f0^+$ is linear on $\text{lin}\, f$ and, in view of [10, Theorem 8.8], one has

$$\text{graph} \left(f0^+ \right)_{|\, \text{lin}\, f} = \text{lin epi}\, f. \tag{4.2}$$

Taking all this into account, one easily obtains the following corollary.

Corollary 4.4. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. proper convex function and U be a supplementary subspace to* lin f*. Then, f is a generalized Minkowski function if and only if $f_{|U}$ is a Minkowski function.*

In particular, f is a generalized Minkowski function if and only if $f_{|(\text{lin}\, f)^\perp}$ is a Minkowski function.

Proof. Observe first that $U \times \mathbb{R}$ is a supplementary subspace to graph $\left(f0^+ \right)_{|\, \text{lin}\, f}$. Hence, by (4.2) and the equality $(\text{epi}\, f) \cap (U \times \mathbb{R}) = \text{epi}\, f_{|U}$, it suffices to apply Theorem 4.3. □

According to the following result, the set of minimizers of a generalized Minkowski function is (if nonempty) a generalized Minkowski set.

Proposition 4.5. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is generalized Minkowski, then* arg min f *is either empty or a generalized Minkowski set.*

Proof. We assume that $\arg\min f \neq \emptyset$. Let $x \in \arg\min f$. Since epi f is a generalized Minkowski set and containd the point $(x, f(x))$, there exist minimal faces F_i, $i = 1, ..k$, of epi f, $(x_i, \alpha_i) \in F_i$ and $\lambda_i > 0$ such that

$$\sum_{i=1}^{k} \lambda_i = 1 \text{ and } (x, f(x)) = \sum_{i=1}^{k} \lambda_i (x_i, \alpha_i).$$

Hence, using the inequalities $f(x) \leq \alpha_i$, we can easily deduce that $\alpha_i = f(x)$, so that $x_i \in \operatorname{arg\,min} f$. Since $\operatorname{arg\,min} f \times \{f(x)\}$ is a face of epi f and, as we have just seen,

$$F_i \cap (\operatorname{arg\,min} f \times \{f(x)\}) \neq \emptyset$$

(because $(x_i, f(x)) \in F_i \cap (\operatorname{arg\,min} f \times \{f(x)\})$), from the minimality of F_i it follows that $F_i \subseteq \operatorname{arg\,min} f \times \{f(x)\}$. Using this inclusion, one can easily prove that $\pi_{\mathbb{R}^n}(F_i) \cap \operatorname{arg\,min} f$ (with $\pi_{\mathbb{R}^n}$ denoting the projection mapping from $\mathbb{R}^n \times \mathbb{R}$ onto $\mathbb{R}^n$) is a minimal face of $\operatorname{arg\,min} f$ (for the minimality, observe that if $G \subseteq \pi_{\mathbb{R}^n}(F_i) \cap \operatorname{arg\,min} f$ is a face of $\operatorname{arg\,min} f$, then $G \times \{f(x)\}$ is a face of epi f and $G \times \{f(x)\} \subseteq F_i$, so that the minimality of F_i yields $G \times \{f(x)\} = F_i$, from which one gets

$$\pi_{\mathbb{R}^n}(F_i) \cap \operatorname{arg\,min} f = G \cap \operatorname{arg\,min} f = G,$$

thus proving the minimality of $\pi_{\mathbb{R}^n}(F_i) \cap \operatorname{arg\,min} f$). Therefore, the equality

$$x = \sum_{i=1}^{k} \lambda_i x_i$$

shows that x belongs to the convex hull of the union of the minimal faces of $\operatorname{arg\,min} f$, which concludes the proof. ☐

Corollary 4.6. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is Minkowski, then* $\operatorname{arg\,min} f$ *is either empty or a Minkowski set.*

Proof. If $\operatorname{arg\,min} f \neq \emptyset$, then, by Proposition 4.5, the set $\operatorname{arg\,min} f$ is generalized Minkowski, so it suffices to observe that the minimal faces of $\operatorname{arg\,min} f$ are singletons, which is equivalent to saying that lin $\operatorname{arg\,min} f = \{0\}$. But this follows from the inclusion lin $\operatorname{arg\,min} f \times \{0\} \subseteq$ lin epi $f = \{0\}$, the latter equality being a consequence of the fact that epi f contains no lines, as it is a Minkowski set. ☐

Another consequence of Theorem 4.3 is the next result.

Theorem 4.7. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then, C is a generalized Minkowski set if and only if there exist a Minkowski set $C_0 \subseteq \mathbb{R}^n$, a supplementary subspace $V \subseteq \mathbb{R}^n$ to* aff C_0 − aff C_0 *and a linear subspace $L \subseteq V$ such that*

$$C = C_0 + L. \tag{4.3}$$

Corollary 4.8. *In particular, C is a generalized Minkowski set if and only if there exist a Minkowski set $C_0 \subseteq \mathbb{R}^n$ and a linear subspace $L \subseteq (\text{aff } C_0 - \text{aff } C_0)^{\perp}$ such that*

$$C = C_0 + L.$$

Proof. If C is a generalized Minkwoski set, then (4.3) holds with

$$C_0 := C \cap (\text{lin } C)^{\perp},\ V = (\text{aff } C_0 - \text{aff } C_0)^{\perp} \text{ and } L := \text{lin } C.$$

Indeed, by Theorem 4.3, the set $C \cap (\text{lin } C)^{\perp}$ is Minkowski and, on the other hand, we have

$$\text{aff}\left(C \cap (\text{lin } C)^{\perp}\right) - \text{aff}\left(C \cap (\text{lin } C)^{\perp}\right) \subseteq (\text{lin } C)^{\perp} - (\text{lin } C)^{\perp} = (\text{lin } C)^{\perp},$$

and hence $\text{lin } C = (\text{lin } C)^{\perp\perp} \subseteq (\text{aff } C_0 - \text{aff } C_0)^{\perp}$.

Conversely, assume that (4.3) holds for some Minkowski set $C_0 \subseteq \mathbb{R}^n$, some supplementary subspace $V \subseteq \mathbb{R}^n$ to $\operatorname{aff} C_0 - \operatorname{aff} C_0$ and some subspace $L \subseteq V$. By applying a translation if necessary, we assume, without loss of generality, that $0 \in C_0$. Then, $\operatorname{aff} C_0 - \operatorname{aff} C_0 = \operatorname{aff} C_0$. We will now prove that

$$L \subseteq \operatorname{lin} C \subseteq V. \tag{4.4}$$

The first inclusion being obvious, we will only prove the second one. To this aim, let $d \in \operatorname{lin} C$. Denoting by π the projection mapping onto $\operatorname{aff} C_0$ corresponding to the direct sum $V \oplus \operatorname{aff} C_0 = \mathbb{R}^n$, we have

$$\pi(d) \in \pi(\operatorname{lin} C) \subseteq \operatorname{lin} \pi(C) = \operatorname{lin} C_0 = \{0\};$$

we have here used the equality $\pi(C) = C_0$. Thus, $\pi(d) = 0$, that is, $d \in V$, which completes the proof of (4.4). We deduce that $\operatorname{lin} C \cap \operatorname{aff} C_0 \subseteq V \cap \operatorname{aff} C_0 = \{0\}$; consequently, $\operatorname{aff} C_0 \subseteq U$ for some supplementary subspace U to $\operatorname{lin} C$. We have $C \cap U = (C_0 + L) \cap U = C_0$, the latter equality being an easy consequence of the inclusions $C_0 \subseteq \operatorname{aff} C_0 \subseteq U$ and $U \cap L \subseteq U \cap \operatorname{aff} C_0 \subseteq U = \{0\}$. To complete the proof, it suffices to apply Theorem 4.3. □

Remark 4.9. The condition that L is contained in a supplementary space to $\operatorname{aff} C_0 - \operatorname{aff} C_0$ is essential for the validity of the "if" statement of Theorem 4.7, as can be seen by considering, e.g., the convex hull C_0 of a parabola in $\mathbb{R}^2$ and the line L through the origin orthogonal to the axis of the parabola.

Let V be a subspace of $\mathbb{R}^n$. Recall that a set $S \subseteq \mathbb{R}^n$ is said to be V-*invariant* if $S + V = S$.

Remark 4.10. Let U be a supplementary subspace to V. One can easily prove the equivalence

$$S \text{ is } V\text{-invariant} \ \Leftrightarrow \ (S \cap U) + V = S.$$

Indeed, if S is V-invariant, then $(S \cap U) + V \subseteq S + V = S$, and, for the opposite inclusion, consider an element $s \in S$, use the direct sum decomposition

$$U \oplus V = \mathbb{R}^n$$

to write

$$s = s^U + s^V, \qquad \text{with } s^U \in U \text{ and } s^V \in V, \tag{4.5}$$

and observe that

$$s^U = s - s^V \in S - V = S + V = S.$$

Conversely, if $(S \cap U) + V = S$ and $s \in S$ is decomposed as in (4.5), by the uniqueness of this decomposition one has $s^U \in S \cap U$, and hence

$$s + V = s^U + s^V + V = s^U + V \subseteq (S \cap U) + V = S,$$

which proves that $S + V = S$, as the inclusion $\supseteq$ is obvious.

In particular, one has

$$S \text{ is } V\text{-invariant} \ \Leftrightarrow \ \left(S \cap V^{\perp}\right) + V = S.$$

Lemma 4.11. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* $\operatorname{lin} C$. *If $S \subseteq C$ is a* $(\operatorname{lin} C)$*-invariant set such that* $\operatorname{conv} S = C$, *then*

$$\operatorname{conv}(S \cap U) = C \cap U. \tag{4.6}$$

In particular,

$$\text{conv}\left(S\cap(\text{lin}\, C)^{\perp}\right)=C\cap(\text{lin}\, C)^{\perp}.$$

Proof. By Remark 4.10, we have

$$\begin{aligned}(C\cap U)+\text{lin}\, C &= C=\text{conv}\, S=\text{conv}\,((S\cap U)+\text{lin}\, C)\\ &= \text{conv}\,(S\cap U)+\text{conv}\,(\text{lin}\, C)\\ &= \text{conv}\,(S\cap U)+\text{lin}\, C,\end{aligned}$$

and hence we obtain (4.6). □

Lemma 4.12. *Let $C\subseteq\mathbb{R}^n$ be a nonempty closed convex set and $U\subseteq\mathbb{R}^n$ be a supplementary subspace to* lin C*. If $C\cap U$ is a Minkowski set, then*

$$\text{conv}\, MF\,(C)=C.$$

Proof. By Corollary 2.5, we have

$$\begin{aligned}\text{conv}\, MF\,(C) &= \text{conv}\,(\text{ext}\,(C\cap U)+\text{lin}\, C)\\ &= \text{conv}\,(\text{ext}\,(C\cap U))+\text{conv lin}\, C\\ &= (C\cap U)+\text{lin}\, C=C.\end{aligned}$$

□

Proposition 4.13. *For a nonempty closed convex set $C\subseteq\mathbb{R}^n$, the following statements are equivalent:*

(1) C is generalized Minkowski.
(2) There exists a smallest $(\text{lin}\, C)$*-invariant set $S\subseteq C$ such that* conv $S=C$.
(3) There exists a minimal $(\text{lin}\, C)$*-invariant set $S\subseteq C$ such that* conv $S=C$.

In 2 and 3, one has $S=MF\,(C)$.

Proof. $1\Rightarrow 2$. Since the closed convex set $C\cap(\text{lin}\, C)^{\perp}$ is Minkowski, according to Theorem 4.3, it follows that $\text{ext}\left(C\cap(\text{lin}\, C)^{\perp}\right)$ is, according to [6, Proposition 3.1], the smallest set whose convex hull equals $C\cap(\text{lin}\, C)^{\perp}$. By Corollary 2.5, the set $MF\,(C)$ is $(\text{lin}\, C)$-invariant. Let $S\subseteq C$ be a $(\text{lin}\, C)$-invariant set such that conv $S=C$. From Lemma 4.11, it follows that

$$\text{ext}\left(C\cap(\text{lin}\, C)^{\perp}\right)\subseteq S\cap(\text{lin}\, C)^{\perp},$$

which, by Remark 4.10, implies that

$$MF\,(C)=\text{ext}\left(C\cap(\text{lin}\, C)^{\perp}\right)+\text{lin}\, C\subseteq\left(S\cap(\text{lin}\, C)^{\perp}\right)+\text{lin}\, C=S.$$

Since, by Lemma 4.12, we have conv $MF\,(C)=C$, in view of (4.1) the set $MF\,(C)$ is the smallest $(\text{lin}\, C)$-invariant subset of C whose convex hull equals C.
$2\Rightarrow 3$. Obvious.
$3\Rightarrow 1$. By Lemma 4.11, we have $\text{conv}(S\cap(\text{lin}\, C)^{\perp})=C\cap(\text{lin}\, C)^{\perp}$. We will actually prove that $S\cap(\text{lin}\, C)^{\perp}$ is minimal among the $(\text{lin}\, C)$-invariant sets whose convex hulls equal $C\cap(\text{lin}\, C)^{\perp}$. Indeed, if conv $S_1=C\cap(\text{lin}\, C)^{\perp}$ for some $(\text{lin}\, C)$-invariant set $S_1\subseteq S\cap(\text{lin}\, C)^{\perp}$, then

$$\begin{aligned}\text{conv}(S_1+\text{lin}\, C) &= \text{conv}\, S_1+\text{conv lin}\, C=\left(C\cap(\text{lin}\, C)^{\perp}\right)+\text{conv lin}\, C\\ &= \left(C\cap(\text{lin}\, C)^{\perp}\right)+\text{lin}\, C=C\end{aligned}$$

and $S_1 + \text{lin}\, C \subseteq \left(S \cap (\text{lin}\, C)^{\perp}\right) + \text{lin}\, C = S$. But since $S_1 + \text{lin}\, C$ is $(\text{lin}\, C)$-invariant and S is a minimal $(\text{lin}\, C)$-invariant set with conv $S = C$, it follows that $S_1 = S_1 + \text{lin}\, C = S$. Thus

$$S \cap (\text{lin}\, C)^{\perp} = S_1 \cap (\text{lin}\, C)^{\perp} = S_1,$$

which proves the minimality of $S \cap (\text{lin}\, C)^{\perp}$. According to [6, Proposition 3.1], it follows that the set $C \cap (\text{lin}\, C)^{\perp}$ is Minkowski; therefore, by Lemma 4.12, we have conv $MF(C) = C$. □

5. The role of Pareto bordered sets in this setting

For a nonempty closed convex set $C \subseteq \mathbb{R}^n$, we will denote by $M(C)$ the *Pareto like set* of C :

$$\begin{aligned} M(C) &:= \{x \in C : (x - 0^+C) \cap C \subseteq x + 0^+C\} \\ &= \{x \in C : (x - 0^+C) \cap C = x + \text{lin}\, C\}. \end{aligned}$$

Definition 5.1. A nonempty closed convex set $C \subseteq \mathbb{R}^n$ with the properrty $M(C) = \text{rbd}(C)$ will be called a *Pareto bordered set*.

In [6], we paid some special attention to Pareto bordered sets, but we did not use the term ”Pareto bordered” there.

”is not an affine variety, then since aff C

Remark 5.2. As already observed in the proof of Lemma 2.8, since every nonempty closed convex set C is connected, one has $\text{rbd}(C) \neq \emptyset$ if and only if $C \neq \text{aff}\, C$, that is, if and only if C is not an affine variety. On the other hand, if C is an affine variety, then $M(C) = C$; consequently, if C is Pareto bordered, then $\text{rbd}(C) \neq \emptyset$.

Remark 5.3. ([6, Proposition 4.4]) If $C \subseteq \mathbb{R}^n$ is a closed convex set, then $M(C)$ is $(\text{lin}\, C)$-invariant.

Definition 5.4. A l.s.c. proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ which has a Pareto bordered epigraph is called a *Pareto bordered function.*

According to the following result, the set of minimizers of a Pareto bordered function is (if nonempty) a Pareto bordered set.

Proposition 5.5. *If* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is Pareto bordered, then* $\arg\min f$ *is either empty or a Pareto bordered set.*

Proof. We assume that $\arg\min f \neq \emptyset$. Let

$$x \in \text{rbd}(\arg\min f) \text{ and } d \in 0^+ (\arg\min f) \text{ be such that } x - d \in \arg\min f.$$

The latter condition means that $f(x-d) = f(x)$, so that

$$(x, f(x)) - (d, 0) = (x - d, f(x)) \in \text{epi}\, f.$$

Since $(x, f(x)) \in \text{rbd}(\text{epi}\, f)$ and $(d, 0) \in 0^+ (\text{epi}\, f)$, given the Pareto bordered character of epi f we obtain

$$(x, f(x)) - (d, 0) \in \left((x, f(x)) - 0^+ (\text{epi}\, f)\right) \cap \text{epi}\, f \subseteq (x, f(x)) + 0^+ (\text{epi}\, f),$$

from which we deduce that $-(d, 0) \in 0^+ (\text{epi}\, f)$, which is equivalent to saying that $-d \in 0^+ (\arg\min f)$. Therefore $x - d \in x + 0^+ (\arg\min f)$, showing that $x \in M(\arg\min f)$. This proves the inclusion $\text{rbd}(\arg\min f) \subseteq M(\arg\min f)$, which says that $\arg\min f$ is a Pareto bordered set. □

Theorem 5.6. *If $C \subseteq \mathbb{R}^n$ is a Pareto bordered set and* $\dim(C \cap (\operatorname{lin} C)^\perp) \geq 2$*, then C is a generalized Minkowski set.*

Remark 5.7. The dimension assumption in Theorem 5.6 is not superfluous: consider, e.g., the case when C is a halfspace. On the other hand, the converse of Theorem 5.6 does not hold, as shown by [6, Example 4.9], which exhibits a Minkowski set that is not Pareto bordered.

In order to prove Theorem 5.6, we need the following lemmas. We will denote by π the projection mapping onto $(\operatorname{lin} C)^\perp$.

Lemma 5.8. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* $\operatorname{lin} C$*. If π_U denotes the projection mapping onto U corresponding to the direct sum decomposition $U \oplus \operatorname{lin} C = \mathbb{R}^n$, then*

$$\pi_U(C) = C \cap U \tag{5.1}$$

and

$$C = \pi_U(C) + \operatorname{lin} C. \tag{5.2}$$

In particular, if π denotes the orthogonal projection mapping onto $(\operatorname{lin} C)^\perp$*, then*

$$\pi(C) = C \cap (\operatorname{lin} C)^\perp$$

and

$$C = \pi(C) + \operatorname{lin} C.$$

Proof. Equality (5.1) is an immediate consequence of the decomposition

$$C = (C \cap U) + \operatorname{lin} C, \tag{5.3}$$

since π_U is linear. Equality (5.2) follows from (5.3) and (5.1). □

Corollary 5.9. *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* $\operatorname{lin} C$*. If π_U denotes the projection mapping onto U corresponding to the direct sum decomposition $U \oplus \operatorname{lin} C = \mathbb{R}^n$, then*

$$\pi_U^{-1}(\pi_U(C)) = C.$$

In particular, if π denotes the orthogonal projection mapping onto $(\operatorname{lin} C)^\perp$*, then*

$$\pi^{-1}(\pi(C)) = C.$$

Proof. The inclusion $\supseteq$ being obvious, we will only prove the opposite one. Let $x \in \pi_U^{-1}(\pi_U(C))$. Then, in view of (5.2), we have

$$x = \pi_U(x) + x - \pi_U(x) \in \pi_U(C) + \operatorname{lin} C = C,$$

which proves the inclusion $\subseteq$. □

Lemma 5.10. *If $C \subseteq \mathbb{R}^n$ is a nonempty closed convex set, then*

$$\operatorname{aff}\left(C \cap (\operatorname{lin} C)^\perp\right) = (\operatorname{aff} C) \cap (\operatorname{lin} C)^\perp \tag{5.4}$$

and

$$\operatorname{rbd}(C \cap (\operatorname{lin} C)^\perp) + \operatorname{lin} C = \operatorname{rbd} C. \tag{5.5}$$

Proof. The inclusion $\subseteq$ in (5.4) being obvious, we will only prove the opposite one. To this aim, let $x \in (\text{aff } C) \cap (\text{lin } C)^{\perp}$. Since $x \in \text{aff } C$, it follows that

$$x = \lambda_1 x_1 + \cdots + \lambda_m x_m \tag{5.6}$$

for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that $\lambda_1 + \cdots + \lambda_m = 1$ and some $x_1, \ldots, x_m \in C$. Applying π to both sides of (5.6), we obtain

$$x = \lambda_1 \pi(x_1) + \cdots + \lambda_m \pi(x_m).$$

Thus, by Lemma 5.8, we have $x \in \text{aff}\left(C \cap (\text{lin } C)^{\perp}\right)$, which completes the proof of the inclusion $\supseteq$ in (5.4).

To prove the inclusion $\supseteq$ in (5.5), let $x \in \text{rbd } C$ and, for $r > 0$, take

$$x_r \in B(x, r) \cap (\text{aff } C) \cap (\mathbb{R}^n \setminus C).$$

By Corollary 5.9, we have $\pi(x_r) \notin C$; hence, since π is nonexpansive, using (5.4) we get

$$\begin{aligned} \pi(x_r) &\in B(\pi(x), r) \cap (\text{aff } C) \cap (\text{lin } C)^{\perp} \cap (\mathbb{R}^n \setminus C) \\ &= B(\pi(x), r) \cap (\text{aff } C) \cap (\text{lin } C)^{\perp} \cap \left(\mathbb{R}^n \setminus \left(C \cap (\text{lin } C)^{\perp}\right)\right) \\ &= B(\pi(x), r) \cap \text{aff}\left(C \cap (\text{lin } C)^{\perp}\right) \cap \left(\mathbb{R}^n \setminus \left(C \cap (\text{lin } C)^{\perp}\right)\right). \end{aligned}$$

This shows that $\pi(x) \in \text{rbd}(C \cap (\text{lin } C)^{\perp})$, from which we deduce that

$$x = \pi(x) + x - \pi(x) \in \text{rbd}(C \cap (\text{lin } C)^{\perp}) + \text{lin } C,$$

as was to be proved.

To prove the inclusion $\subseteq$ in (5.5), it suffices to prove that

$$\text{rbd}(C \cap (\text{lin } C)^{\perp}) \subseteq \text{rbd } C, \tag{5.7}$$

since $\text{rbd } C + \text{lin } C = \text{rbd } C$ in view of [6, Proposition 4.6]. To see that (5.7) holds, just observe that, for every $x \in \text{rbd}(C \cap (\text{lin } C)^{\perp})$ and $r > 0$, using (5.4) one obtains

$$\begin{aligned} &B(x, r) \cap (\text{aff } C) \cap (\mathbb{R}^n \setminus C) \\ &\quad \supseteq B(x, r) \cap (\text{aff } C) \cap (\text{lin } C)^{\perp} \cap (\mathbb{R}^n \setminus C) \\ &\quad = B(x, r) \cap (\text{aff } C) \cap (\text{lin } C)^{\perp} \cap \left(\left(\mathbb{R}^n \setminus \left(C \cap (\text{lin } C)^{\perp}\right)\right)\right) \\ &\quad = B(x, r) \cap \text{aff}\left(C \cap (\text{lin } C)^{\perp}\right) \cap \left(\left(\mathbb{R}^n \setminus \left(C \cap (\text{lin } C)^{\perp}\right)\right)\right) \neq \emptyset. \end{aligned}$$

Now, from the inclusion (5.7), we deduce that

$$\text{rbd}(C \cap (\text{lin } C)^{\perp}) + \text{lin } C \subseteq \text{rbd } C + \text{lin } C = \text{rbd } C,$$

the latter equality being due to [6, Proposition 4.6]. □

The proof of Theorem 5.6. By using (5.5) and [6, Proposition 4.5], the equality $M(C) = \text{rbd } C$ can be rewritten as

$$M(C \cap (\text{lin } C)^{\perp}) + \text{lin } C = \text{rbd } (C \cap (\text{lin } C)^{\perp}) + \text{lin } C.$$

From this equation, taking into account that the sets $M(C\cap(\text{lin}\, C)^{\perp})$ and rbd $(C\cap(\text{lin}\, C)^{\perp})$ are contained in $(\text{lin}\, C)^{\perp}$, we obtain

$$\begin{aligned} M(C\cap(\text{lin}\, C)^{\perp}) &= \Big(M(C\cap(\text{lin}\, C)^{\perp})+\text{lin}\, C\Big)\cap \text{lin}\, C)^{\perp} \\ &= \Big(\text{rbd}(C\cap(\text{lin}\, C)^{\perp})+\text{lin}\, C\Big)\cap \text{lin}\, C)^{\perp} \\ &= \text{rbd}\,(C\cap(\text{lin}\, C)^{\perp}). \end{aligned}$$

Then, in view of Remark 5.2, by using [6, Proposition 4.8] and Theorem 4.3, the statement follows immediately. □

Acknowledgments
The authors are very grateful to an anonymous referee for his/her extremely careful review, which included numerous suggestions for improvements and some corrections. In particular, thanks to his/her detection of two wrong proofs, we have been able to delete a wrong result (replacing it with a weaker version) and to correct the wrong proof of a right result. Juan Enrique Martínez-Legaz was supported by the Ministerio de Ciencia, Innovación y Universidades [PGC2018-097960-B-C21] and the Severo Ochoa Programme for Centres of Excellence in R&D [SEV-2015-0563]. He is affiliated with MOVE (Markets, Organizations and Votes in Economics). Cornel Pintea was supported by Ministerio de Ciencia, Innovación y Universidades [PGC2018-097960-B-C21] and by AGC31333/23.03.2022 and AGC31334/23.03.2022.

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