Commun. Optim. Theory 2024 (2024) 19 https://doi.org/10.23952/cot.2024.19

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

ON THE RELAXATION COMPLEXITY OF NONCONVEX QUADRATIC GLOBAL OPTIMIZATION

TONGLI ZHANG[1,2], YONG XIA[2*]

[1]School of Mathematics and Physics, Nanjing Institute of Technology, Nanjing 211167, China
[2]School of Mathematical Sciences, Beihang University, Beijing 100191, China

In memory of Professor Naum Z. Shor on the occasion of his 85th birthday

Abstract. We study the relaxation complexity for nonconvex quadratic global optimization, which is defined as the number of convex relaxation subproblems to be solved. The relaxation complexity for quadratic programming with fixed nonconvex-rank is known to be a polynomial function of the dimension. In this paper, we show that the relaxation complexity for nonconvex quadratic optimization with convex quadratic constraints may not depend on the dimension, as long as the objective function has a fixed nonconvex rank.

Keywords. Dikin ellipsoid; Global optimization; Löwner-John ellipsoid; Quadratic constrained quadratic optimization.

2020 Mathematics Subject Classification. 90C20, 90C26.

1. Introduction

We study the following nonconvex quadratic optimization problem with convex quadratic constraints:

$$\begin{aligned}\text{(QCQP)}\quad \min\quad & f(x) = x^T Gx + 2g^T x \\ \text{s.t.}\quad & x \in \Omega := \left\{x \in \mathbb{R}^n :\ x^T A_i x + 2a_i^T x + \varsigma_i \le 0,\ i = 1,\cdots,m\right\},\end{aligned}$$

where $G \in \mathbb{R}^{n\times n}$ is symmetric and not positive semidefinite, $A_i \in \mathbb{R}^{n\times n}$ are symmetric and positive semidefinite, $g, a_i \in \mathbb{R}^n$ and $\varsigma_i \in \mathbb{R}$ for $i = 1,\cdots,m$. We assume that Slater condition holds, i.e., Ω contains an interior point. We also assume that Ω is bounded.

Problem (QCQP) contains many special cases such as the trust-region subproblem ($m = 1$, $A_1 = I$) with key applications in nonlinear programming [18], the Celis-Dennis-Tapia subproblem ($m = 2$, $A_1 = I$) [2, 12], the multi-ball constrained quadratic optimization ($A_i \equiv I$) [17]

*Corresponding author.
E-mail address: yxia@buaa.edu.cn (Y. Xia).
Received July 2, 2023; Accepted October 5, 2023.

with applications in finding Chebyshev center of the intersection of balls [16], ellipsoid constrained quadratic optimization [7] and references therein, and box constrained quadratic optimization ($A_i \equiv e_i e_i^T$, $a_i \equiv 0$) [6], where e_i is the i-th column of I. In general, (QCQP) is NP-hard, even when G has a negative eigenvalue [13]. For recent global optimization methods based on branch-and-bound/branch-and-cut for solving (QCQP), we refer to [3, 9, 10, 11].

For any $\varepsilon \in (0,1)$, we call $\hat{x} \in \Omega$ an ε-approximate solution if

$$f(\hat{x}) - f_{\min} \leq \varepsilon \left(f_{\max} - f_{\min}\right),$$

where $f_{\min}$ and $f_{\max}$ denote the minimum and maximum of $f(x)$ over Ω, respectively. When $m = 1$ and $A_1 = I$ (i.e., the trust-region subproblem), an ε-approximate solution of (QCQP) can be found in polynomial time [4]. We refer to [15] for more references. For general (QCQP) with $m \geq 2$, Fu et al. [4] presented a polynomial-time algorithm to find an $\varepsilon(\tau)$-approximate solution, where $\varepsilon(\tau) = 1-(1-\tau)/(m(1+\tau))^2$ for all $\tau \in (0, 1-1/\sqrt{2})$. Until recently, the approximate ratio ε is corrected to be $(1-(1-\tau)/(\sqrt{m^2+m}(1+\tau))^2$ [19]. Moreover, when $g = a_i \equiv 0$ for $i = 1,\cdots,m$, the ratio $\varepsilon(\tau)$ is sharpened $(1-(1-\tau)/(\sqrt{m}(1+\tau))^2$ [5]. Note that $\varepsilon(\tau)$ is an increase function of τ. It is unknown whether there is an efficient algorithm to find an ε-approximate solution of (QCQP) with $\varepsilon < \varepsilon(0) = 1-1/(m^2+m)$ for $m \geq 2$.

When $A_i \equiv 0$, (QCQP) reduces to the classical quadratic programming (QP). Vavasis [14] introduced the weak Löwner-John ellipsoid approach to find an ε-approximate solution of nonconvex (QP). The relaxation complexity, i.e., the number of convex relaxation subproblems to be solved, is polynomial in n, if the nonconvex rank of G, denoted by q, is fixed.

In this paper, we first extend Vavasis's algorithm [14] to solve (QCQP), and show that the relaxation complexity is polynomial in the dimension n for fixed q. As a main contribution, the relaxation complexity of a modified version of Vavasis's algorithm by replacing Löwner-John ellipsoid pair with Dikin ellipsoid pair, can substantially reduce the relaxation complexity to be independent of the dimension n.

The remainder of this paper is organized as follows. In Section 2, we extend Vavasis's algorithm from (QP) to (QCQP). In Section 3, we improve Vavasis's algorithm for (QCQP) to achieve a much lower relaxation complexity. Conclusions are made in Section 4.

Throughout this paper, $A \succ (\succeq) 0$ represent positive (semi)definite. Denote by A^T and A^{-1} the transpose and inverse of A, respectively. In addition, $A^{-T} = (A^{-1})^T$. Let $\mathrm{diag}(A)$ be the vector formed by the diagonal elements of A and $\det(A)$ be the determinant of A. $I \in \mathbb{R}^{n\times n}$ is the identity matrix. $I_i \in \mathbb{R}^{n\times n}$ is a matrix where all the elements are 0 except the i-th row and the i-th column is 1. Denote by $\nabla^2 L(x)$ the Hessian matrix of $L(x)$. $\lceil\cdot\rceil$ denotes the smallest integer larger than or equal to "$\cdot$".

2. A LÖWNER-JOHN ELLIPSOID BASED APPROXIMATION ALGORITHM

We extend Vavasis's algorithm presented in [14] for (QP) to solve (QCQP).

First, as shown in [8], one can find in polynomial time a weak Löwner-John ellipsoid pair satisfying

$$E^{wlj}(c_{wlj};1) \subseteq \Omega \subseteq E^{wlj}(c_{wlj};(n+1)\sqrt{n}), \tag{2.1}$$

where

$$E^{wlj}(c_{wlj};r) := \left\{x \in \mathbb{R}^n:\ (x-c_{wlj})^T M (x-c_{wlj}) \leq r^2\right\},$$

with $c_{wlj} \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$, and $M \succ 0$. For convenience, we denote the radius ratio of the inner and outer ellipsoids as ρ. It is clear that $\rho = 1/((n+1)\sqrt{n})$ for the weak Löwner-John ellipsoid pair. Then there is a nonsingular and symmetric matrix P such that $M = PP^T$. Let the eigenvalue decomposition of $P^{-1}GP^{-T}$ be $P^{-1}GP^{-T} = Q^T\Lambda Q$, where Q is orthogonal and Λ is diagonal. Then, for $U = P^{-T}Q^T$, we have

$$U^TMU = I,\ \Lambda = U^TGU.$$

By introducing

$$y = U^{-1}\left(x - c_{wlj}\right), \tag{2.2}$$

we can rewrite (QCQP) as

$$\begin{aligned} \text{(EP)}\quad & \min_{y\in\mathbb{R}^n}\ y^T\Lambda y + 2d^Ty + \zeta \\ & \text{s.t.}\quad y \in \Omega_1 := \left\{ y\in\mathbb{R}^n:\ y^TB_iy + 2b_i^Ty + \kappa_i \le 0,\ i = 1,\cdots,m \right\}, \end{aligned}$$

where $d = U^TGc_{wlj} + U^Tg$, $\zeta = c_{wlj}^TGc_{wlj} + 2g^Tc_{wlj}$, $B_i = U^TA_iU$, $b_i = U^TA_ic_{wlj} + U^Ta_i$, $\kappa_i = c_{wlj}^TA_ic_{wlj} + 2a_i^Tc_{wlj} + \varsigma_i$. Similarly, $E^{wlj}(c_{wlj};1)$ and $E^{wlj}(c_{wlj};(n+1)\sqrt{n})$ are reformulated as

$$E_1 := \left\{ y\in\mathbb{R}^n:\ y^Ty \le 1 \right\},\ E_2 := \left\{ y\in\mathbb{R}^n:\ y^Ty \le n(n+1)^2 \right\},$$

respectively. As the mapping from x to y is linear, we have

$$E_1 \subseteq \Omega_1 \subseteq E_2.$$

Without loss of generality, let $\mathrm{diag}(\Lambda) = [\lambda_1,\cdots,\lambda_n]$ with

$$\lambda_1 \le \cdots \le \lambda_q < 0 \le \lambda_{q+1} \le \cdots \le \lambda_n.$$

Let $y = (z^T,\ w^T)^T$, where $z \in \mathbb{R}^q$ and $w \in \mathbb{R}^{n-q}$. The projections of Ω_1, E_1, and E_2 in z-space are

$$\begin{aligned} &\Omega_2 := \left\{ z\in\mathbb{R}^q:\ \exists w\in\mathbb{R}^{n-q} \text{ s.t. } (z^T,w^T)^T \in \Omega_1 \right\}, \\ &E_3 := \left\{ z\in\mathbb{R}^q:\ z^Tz \le 1 \right\},\ E_4 := \left\{ z\in\mathbb{R}^q:\ z^Tz \le n(n+1)^2 \right\}, \end{aligned}$$

respectively. Trivially, we have

$$E_3 \subseteq \Omega_2 \subseteq E_4.$$

Moreover, it holds that

$$E_4 \subseteq L := \left[-(n+1)\sqrt{n},\ (n+1)\sqrt{n}\right]^q.$$

We equally divide each dimension of L into t parts. So L is divided into t^q sub-rectangles

$$L_k = \left\{ z\in\mathbb{R}^q:\ l_j^k \le z_j \le u_j^k,\ j = 1,\cdots,q \right\},\ k = 1,\cdots,t^q$$

with $u_j^k - l_j^k = 2(n+1)\sqrt{n}/t$. Then, solving (EP) amounts to solve t^q subproblems over the set $\{(z^T,w^T)^T \in \Omega_1:\ z \in L_k\}$. Since

$$\lambda_j(z_j - l_j^k)(z_j - u_j^k) \ge 0,\ j = 1,\cdots,q,$$

each subproblem can be lower bounded by its convex relaxation:

$$(\mathrm{REP}_k) \quad \min_{(z,w)\in\mathbb{R}^n} \quad \varphi_k(z,w) := \phi_k(z) + \sum_{j=q+1}^{n} \left(\lambda_j w_j^2 + 2d_j w_j\right) + \zeta$$
$$\text{s.t.} \quad z \in L_k, (z^T, w^T)^T \in \Omega_1,$$

where d_j is the j-th component of d, and

$$\phi_k(z) = \sum_{j=1}^{q} \left[\lambda_j(u_j^k + l_j^k) + 2d_j\right] z_j - \sum_{j=1}^{q} \lambda_j u_j^k l_j^k,$$

The optimal solution of (REP_k) remains a feasible candidate solution for (EP). The candidate with the minimal objective function value of (EP) is selected as the best approximate solution of (EP). It makes sense that the larger t, the better the approximation. Precisely, according to the proof of Theorem 2 in [14], for any given precision $\varepsilon \in (0,1)$, setting $t = \left\lceil (n+1)\sqrt{qn/\varepsilon} \right\rceil$ outputs an ε-approximate solution of (EP). As a summary, we have the following relaxation complexity result.

Theorem 2.1. *For any $\varepsilon \in (0,1)$, if we approximate Ω with the weak Löwner-John ellipsoid pair* (2.1)*, an ε-approximate solution of* (QCQP) *can be obtained by solving at most $\left\lceil (n+1)\sqrt{nq/\varepsilon} \right\rceil^q$ convex quadratic constrained quadratic optimization subproblems* (REP_k).

Theorem 2.1 is a natural extension of Theorem 2 in [14] as (QCQP) reduces to (QP) when $A_i \equiv 0$ for $i = 1, \cdots, m$.

Remark 2.2. Since the number of subproblems that need to be solved by Vavasis's algorithm is $\left\lceil \sqrt{q/\varepsilon\rho^2} \right\rceil^q$, which is a polynomial about ρ, it is clear that the larger ρ, the fewer the number of subproblems that need to be solved.

When $A_i \succ 0$ for $i = 1, \cdots, m$, the Löwner-John ellipsoid pair [8] for Ω is given by

$$E^{lj}(c;1) \subseteq \Omega \subseteq E^{lj}(c;n), \tag{2.3}$$

where

$$E^{lj}(c;r) = \left\{x \in \mathbb{R}^n : (x-c)^T C^{-2}(x-c) \le r^2\right\},$$

and (C,c) is the optimal solution of the following semidefinite programming problem [1]:

$$\min_{C\in\mathbb{R}^{n\times n}, c,\mu\in\mathbb{R}^n} \quad \log\det\left(C^{-1}\right)$$
$$\text{s.t.} \quad \begin{pmatrix} -\mu_i - \varsigma_i + a_i^T A_i^{-1} a_i & 0 & (c + A_i^{-1}a_i)^T \\ 0 & \mu_i I & C \\ c + A_i^{-1}a_i & C & A_i^{-1} \end{pmatrix} \succeq 0, \ i = 1, \cdots, m.$$

Replacing the weak Löwner-John ellipsoid pair (2.1) with the Löwner-John ellipsoid pair (2.3), we can strengthen Theorem 2.1 as follows.

Corollary 2.3. *Suppose $A_i \succ 0$ for $i = 1, \cdots, m$. For any $\varepsilon \in (0,1)$, with the Löwner-John ellipsoid pair* (2.3)*, solving at most $\left\lceil n\sqrt{q/\varepsilon} \right\rceil^q$ convex quadratic constrained quadratic optimization subproblems yields an ε-approximate solution of* (QCQP).

Moreover, when Ω is central symmetric, [1] shows that (2.3) can be strengthened to be

$$E^{lj}(c;1) \subseteq \Omega \subseteq E^{lj}(c;\sqrt{n}),$$

so that the number of subproblems in Corollary 2.3 can reduce to $\left\lceil \sqrt{nq/\varepsilon} \right\rceil^q$.

3. A DIKIN ELLIPSOID BASED APPROXIMATION ALGORITHM

In this section, we improve Vavasis's algorithm for solving (QCQP) by replacing the Löwner-John ellipsoid pair with the Dikin ellipsoid pair.

Consider the logarithmic barrier function

$$L(x) = -\sum_{i=1}^{m} \log\left(-\left(x^T A_i x + 2a_i^T x + \varsigma_i\right)\right).$$

The unique minimizer of $L(x)$, denoted by c_d is called the analytic center of Ω. Define the Dikin ellipsoid as

$$E^d(c_d;r) = \left\{x \in \mathbb{R}^n :\ (x-c_d)^T \nabla^2 L(c_d)(x-c_d) \le r^2\right\}.$$

It is shown in [4] and then corrected in [19] that

$$E^d(c_d;1) \subseteq \Omega \subseteq E^d(c_d;\sqrt{m^2+m}). \tag{3.1}$$

Replacing the Löwner-John ellipsoid pair (2.3) with the Dikin ellipsoid pair (3.1), we can further strengthen Corollary 2.3 as follows.

Corollary 3.1. *For a given $\varepsilon \in (0,1)$, with the Dikin ellipsoid pair (3.1), solving at most*

$$\left\lceil \sqrt{(m^2+m)q/\varepsilon} \right\rceil^q \tag{3.2}$$

convex quadratic constrained quadratic optimization subproblems yields an ε-approximate solution of (QCQP).

Remark 3.2. Corollary 3.1 can be further improved when all quadratic constraints of (QCQP) are homogeneous, i.e., $a_i \equiv 0$ for $i = 1,\cdots,m$. It is not difficult to verify that $c_d = 0$ and

$$\nabla^2 L(c_d) = 2\sum_{i=1}^{m} \frac{A_i}{-\varsigma_i}.$$

Consequently, (3.1) is improved to be

$$E^d(c_d;\sqrt{2}) \subseteq \Omega \subseteq E^d(c_d;\sqrt{2m}), \tag{3.3}$$

see [5]. Therefore, the upper bound of the number of subproblems (3.2) further reduces to

$$\left\lceil \sqrt{mq/\varepsilon} \right\rceil^q \tag{3.4}$$

Remark 3.3. Comparing with Theorems 2.1 and Corollary 2.3, the upper bound of the number of subproblems (3.2) in Corollary 3.1 is dimensional-free. This is advantageous for optimization problems with low nonconvex-rank and small number of convex quadratic constraints.

Remark 3.4. Remark 3.3 cannot be applied for (QP). Actually, when $A_i \equiv 0$ for $i = 1, \cdots, m$, for compact Ω, we have the following Dikin ellipsoid pair [1]:

$$E^d(c_d; 1) \subseteq \Omega \subseteq E^d(c_d; \sqrt{m^2 - m}). \tag{3.5}$$

Since Ω must be compact, under Slater condition, it holds that $m \geq n+1$. It follows that

$$\sqrt{m^2 - m} \geq \sqrt{(n+1)^2 - (n+1)} \geq \sqrt{(n+1)n} > n.$$

This implies that the number of subproblems for solving (QP) by the Dikin ellipsoid pair is greater than that by the Löwner-John ellipsoid pair.

4. Conclusion

We study the relaxation complexity (i.e., the number of convex optimization subproblems to be solved) for globally solving nonconvex (QCQP). We first extend Vavasis's algorithm from (QP) to (QCQP), and then replace the (weak) Löwner-John ellipsoid pair with the Dikin ellipsoid pair as a further improvement. It turns out that the relaxation complexity is no longer dependent of the dimension if the nonconvexity-rank is fixed. Thus the presented new algorithm seems to be more suitable for very large-scale (QCQP) with a low nonconvexixty-rank. The relaxation complexity for (QCQP) with a high nonconvexity-rank remains unknown. Future works also include further extensions for general convex constrained nonconvex optimization problems.

Acknowledgements
This research was supported by the National Natural Science Foundation of China under Grants 12171021 and 11822103, the Beijing Natural Science Foundation under Grant Z180005, and the Fundamental Research Funds for the Central Universities.

References

[1] S. Boyd, L. Vandenberghe, Convex optimization, Cambridge University press, Cambridge, 2004.
[2] M.R. Celis, J.E. Dennis, R.A. Tapia, A trust region strategy for nonlinear equality constrained optimization. In Numerical Optimization, 1984 (Boulder, Colo., 1984), pp. 71-82, SIAM, Philadelphia, 1985.
[3] X. Cen, Y. Xia, A new global optimization scheme for quadratic programs with low-rank nonconvexity, INFORMS J. Comput. 33 (2021) 1368-1383.
[4] M. Fu, Z.-Q. Luo, Y. Ye, Approximation algorithms for quadratic programming, J. Comb. Optim. 2 (1998) 29-50.
[5] D. Henrion, S. Tarbouriech, D. Arzelier, LMI approximations for the radius of the intersection of ellipsoids: Survey, J. Optimiz. Theory App. 108 (2001) 1-28.
[6] M. Hladík, M. Černỳ and M. Rada, A new polynomially solvable class of quadratic optimization problems with box constraints, Optim. Lett. 15 (2021) 2331-2341.
[7] Y. Hsia, S. Wang, Z. Xu, Improved semidefinite approximation bounds for nonconvex nonhomogeneous quadratic optimization with ellipsoid constraints, Oper. Res. Lett. 43 (2015) 378-383.
[8] L. Lovász, An Algorithmic Theory of Numbers, Graphs and Convexity, SIAM, 1986.
[9] C. Lu, Z. Deng, Q. Jin, An eigenvalue decomposition based branch-and-bound algorithm for nonconvex quadratic programming problems with convex quadratic constraints, J. Global Optim. 67 (2017) 475-493.
[10] H. Luo, X. Bao, G. Lim, J. Peng, New global algorithms for quadratic programming with a few negative eigenvalues based on alternative direction method and convex relaxation, Math. Program. Comput. 11 (2019) 119-171.

[11] H. Luo, S. Chen, H. Wu, A new branch-and-cut algorithm for non-convex quadratic programming via alternative direction method and semidefinite relaxation, Numer. Algor. 88 (2021) 993-1024.
[12] T. Montanher, A. Neumaier, F. Domes, A computational study of global optimization solvers on two trust region subproblems, J. Global Optim. 71 (2018) 915-934.
[13] P.M. Pardalos, S.A. Vavasis, Quadratic programming with one negative eigenvalue is NP-hard, J. Global Optim. 1 (1991) 15-22.
[14] S.A. Vavasis, Approximation algorithms for indefinite quadratic programming, Math. Program. 57 (1992) 279-311.
[15] Y. Xia, A survey of hidden convex optimization. J. Oper. Res. Soc. China. 8 (2020) 1-28.
[16] Y. Xia, M. Yang, S. Wang, Chebyshev center of the intersection of balls: complexity, relaxation and approximation, Math. Program. 187 (2021) 287-315.
[17] Z. Xu, Y. Xia, J. Wang, Cheaper relaxation and better approximation for multi-ball constrained quadratic optimization and extension, J. Global Optim. 80 (2021) 341-356.
[18] Y.-X. Yuan, Recent advances in trust region algorithms, Math. Program. 151 (2015) 249-281.
[19] T. Zhang, Y. Xia, Comment on "Approximation algorithms for quadratic programming", J. Comb. Optim. 44 (2022) 1099-1103.