Commun. Optim. Theory 2024 (2024) 18 https://doi.org/10.23952/cot.2024.18

Communications in Optimization Theory
Available online at http://cot.mathres.org

A NOTE ON THE APPROXIMATION OF ZEROS OF PSEUDO-MONOTONE OPERATORS

BEHZAD DJAFARI ROUHANI*, VAHID MOHEBBI

Department of Mathematical Sciences, University of Texas at El Paso, El Paso, Texas 79968, USA

Dedicated to the memory of Professor Petr Zabreiko

Abstract. In this paper, we modify the Malitsky's algorithm to prove the strong convergence of the generated sequence to a zero of a pseudo-monotone operator without any knowledge of the Lipschitz constant of the operator. We also provide some examples of our main results.

Keywords. Coercive operator; Halpern's regularization; Pseudo-monotone operator; Variational inequality; Weakly upper semicontinuous.

2020 Mathematics Subject Classification. 47J20, 58E35, 65K15, 90C30.

1. Introduction

Finding zeros of operators plays a vital role in the field of nonlinear analysis and optimization. In fact, there are many problems that can be formulated in this setting (see [4, 7, 10] and the references therein). In this paper, we study the strong convergence of an iterative scheme for finding zeros of pseudo-monotone operators, which is a modification of Malitsky's algorithm. In [6], Malitsky proposed some projection methods for solving variational inequality problems with monotone and Lipschitz continuous mappings in Hilbert spaces. First, he introduced the projected reflected gradient algorithm with a constant stepsize using the Lipschitz constant of the operator, requiring only one projection per iteration, to prove that the generated sequence is weakly convergent to a solution of the monotone variational inequality. Also he showed that it has R-linear rate of convergence under the strong monotonicity assumption of the operator. Then he proposed another algorithm to avoid using the Lipschitz constant of the monotone operator and proved that the generated sequence is weakly convergent to a solution of the monotone variational inequality where the new algorithm needed at most two projections per iteration (see [6] and [11]).

*Corresponding author.
E-mail address: behzad@utep.edu (B. Djafari Rouhani).
Received: May 16, 2023; Accepted: September 25, 2023.

Subsequently, in [11] the authors modified the Malitsky's algorithm and proved that the generated sequence is weakly convergent to a solution of the monotone variational inequality, requiring only one projection per iteration. Also, they showed the R-linear convergence rate under the strong monotonicity assumption.

In this paper, inspired and motivated by the above papers, we modify the methods mentioned above to prove the strong convergence of the generated sequence to a zero of a pseudo-monotone operator without any knowledge of the Lipschitz constant of the operator.

This paper is organized as follows. In Section 2, we recall some definitions and preliminaries that will be needed in the sequel. In Section 3, we modify the Malitsky's algorithm to approximate zeros of pseudo-monotone operators in Hilbert spaces, and we prove the strong convergence of the generated sequence to a zero of the pseudo-monotone operator. In Section 4, we give some examples of applications of our main result.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote the weak convergence in H by $\rightharpoonup$ and the strong convergence by $\to$. Throughout the paper, C is a nonempty, closed and convex subset of H, $F : H \to H$ is an operator. An operator $F : H \to H$ is called:
(i) monotone if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \qquad \forall\, x, y \in H.$$

(ii) pseudo-monotone if

$$\langle F(x), y - x\rangle \geq 0 \text{ implies that } \langle F(y), x - y\rangle \leq 0, \qquad \forall\, x, y \in H.$$

(iii) asymptotically pseudo-monotone if, for each bounded sequence $\{x_k\}$ and any $y \in H$,

$$\limsup_{k\to\infty}\langle F(x_k), y - x_k\rangle \geq 0 \text{ implies that } \liminf_{k\to\infty}\langle F(y), x_k - y\rangle \leq 0.$$

(iv) Lipschitz continuous if there exists $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \qquad \forall\, x, y \in H.$$

(v) Coercive, if for any sequence $\{x_k\}$ in H such that $\|x_k\| \to \infty$, we have $\|F(x_k)\| \to \infty$.

It is easy to see that each monotone operator is pseudo-monotone as well as asymptotically pseudo-monotone. Moreover, every asymptotically pseudo-monotone operator is pseudo-monotone. A function $f : H \to \mathbb{R}$ is called weakly upper semicontinuous on H if for every $x \in H$,

$$f(x) \geq \limsup_{k\to\infty} f(x_k)$$

for each sequence $x_k \rightharpoonup x$ as $k \to +\infty$. It is well known that for any $x \in H$ there exists a unique $\bar{x} \in C$ such that

$$\|\bar{x} - x\| = \inf\Big\{\|y - x\| : y \in C\Big\}.$$

Now we define the mapping $P_C : H \to C$ by taking $P_C(x)$ to be this unique $\bar{x} \in C$. The mapping P_C is called the metric projection of H onto C (see [1]).

Lemma 2.1. ([1], Theorem 3.14) *Let $C \subset H$ be nonempty, closed and convex, $x \in H$ and $\bar{x} \in C$. Then $\bar{x} = P_C(x)$ if and only if*

$$\langle y - \bar{x}, x - \bar{x}\rangle \leq 0, \qquad \forall\, y \in C.$$

Lemma 2.2. ([1], Corollary 2.14) *Let $x, y \in H$ and $\alpha \in \mathbb{R}$. Then*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$$

Lemma 2.3. [9] *Consider sequences $\{s_k\} \subset [0,\infty)$, $\{t_k\} \subset \mathbb{R}$ and $\{\alpha_k\} \subset (0,1)$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$. Suppose that*

$$s_{k+1} \leq (1-\alpha_k)s_k + \alpha_k t_k, \qquad \forall\, k \geq 0. \tag{2.1}$$

If $\limsup_{n\to\infty} t_{k_n} \leq 0$ *for any subsequence $\{s_{k_n}\}$ of $\{s_k\}$ satisfying* $\liminf_{n\to\infty}(s_{k_n+1} - s_{k_n}) \geq 0$, *then* $\lim_{k\to\infty} s_k = 0$.

3. Strong Convergence

In this section, we assume that $F : H \to H$ is pseudo-monotone, Lipschitz continuous on H, satisfies a suitable condition and $F^{-1}(0) \neq \emptyset$. It is well known that $F^{-1}(0)$ is a closed and convex subset of H. By using the Halpern's regularization method, we prove the strong convergence of the sequence generated by Algorithm 1 to $P_{F^{-1}(0)}u$ where P is the metric projection and $u \in H$ is an arbitrary point.

Algorithm 1.

1. Initialization: Let x_0, y_0 and u belong to H, $\lambda_0 \in (0,+\infty)$, $\delta \in (1,+\infty)$, $\theta \in (0, \frac{\sqrt{2}-1}{\delta})$. Let $\alpha_k \in (0,1)$ be such that $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = +\infty$.

2. Iterative step: Given x_k and y_k, compute

$$z_{k+1} := x_k - \lambda_k F(y_k), \tag{3.1}$$

If $x_k = z_{k+1}$, then $F(y_k) = 0$, so y_k is a solution and we can stop. Otherwise let

$$x_{k+1} := \alpha_{k+1} u + (1-\alpha_{k+1}) z_{k+1}, \tag{3.2}$$

$$y_{k+1} := x_k - (1+\delta)\lambda_k F(y_k), \tag{3.3}$$

where

$$\lambda_{k+1} := \begin{cases} \min\left\{\frac{\theta\|y_{k+1}-y_k\|}{\|Fy_{k+1}-Fy_k\|}, \lambda_k\right\}, & \text{if } Fy_{k+1} - Fy_k \neq 0, \\ \lambda_k, & \text{otherwise.} \end{cases} \tag{3.4}$$

We divide the proof of our main theorem into several lemmas and propositions. Note that the Lipschitz continuity of F shows that $\frac{\theta}{L} = \frac{\theta\|y_{k+1}-y_k\|}{L\|y_{k+1}-y_k\|} \leq \frac{\theta\|y_{k+1}-y_k\|}{\|Fy_{k+1}-Fy_k\|}$ where L is the Lipschitz constant of F. It is clear that the sequence $\{\lambda_k\}$ is nonincreasing and bounded below by $\min\{\lambda_0, \frac{\theta}{L}\}$. Therefore $\lim_{k\to\infty} \lambda_k$ exists and is different from zero.

Lemma 3.1. *Let $\{y_k\}$ and $\{z_k\}$ be given by Algorithm 1. Then we have*

$$\langle F(y_{k-1}), y_k - z_{k+1}\rangle = \frac{1}{2\delta\lambda_{k-1}}\Big(\|z_{k+1}-z_k\|^2 - \|y_k - z_k\|^2 - \|z_{k+1}-y_k\|^2\Big).$$

Proof. Note that (3.1) and (3.3) imply that $z_k - y_k = \delta\lambda_{k-1}F(y_{k-1})$. Therefore we have

$$\begin{aligned}\langle F(y_{k-1}), y_k - z_{k+1}\rangle &= \frac{1}{\delta\lambda_{k-1}}\langle z_k - y_k, y_k - z_{k+1}\rangle \\ &= \frac{1}{2\delta\lambda_{k-1}}\Big(\|z_{k+1}-z_k\|^2 - \|y_k - z_k\|^2 - \|z_{k+1}-y_k\|^2\Big). \end{aligned} \tag{3.5}$$

□

Lemma 3.2. *Let $\{y_k\}$ and $\{z_k\}$ be given by Algorithm 1. Then we have*

$$2\lambda_k\langle F(y_k)-F(y_{k-1}),y_k-z_{k+1}\rangle \le \theta\Big((1+\sqrt{2})\|y_k-z_k\|^2+\|z_k-y_{k-1}\|^2+\sqrt{2}\|z_{k+1}-y_k\|^2\Big).$$

Proof. Using the Cauchy-Schwarz inequality and the Lipschitz continuity of F, we have

$$\begin{aligned}2\lambda_k\langle F(y_k)-F(y_{k-1}),y_k-z_{k+1}\rangle &\le 2\lambda_k\|F(y_k)-F(y_{k-1})\|\|y_k-z_{k+1}\|\\ &\le 2\theta\|y_k-y_{k-1}\|\|y_k-z_{k+1}\|\\ &\le \theta\Big(\frac{1}{\sqrt{2}}\|y_k-y_{k-1}\|^2+\sqrt{2}\|y_k-z_{k+1}\|^2\Big). \qquad (3.6)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|y_k-y_{k-1}\|^2 &= \|y_k-z_k\|^2+\|z_k-y_{k-1}\|^2+2\langle y_k-z_k,z_k-y_{k-1}\rangle\\ &\le \|y_k-z_k\|^2+\|z_k-y_{k-1}\|^2+2\|y_k-z_k\|\|z_k-y_{k-1}\|\\ &\le (2+\sqrt{2})\|y_k-z_k\|^2+\sqrt{2}\|z_k-y_{k-1}\|^2. \qquad (3.7)\end{aligned}$$

Now (3.6) and (3.7) imply that

$$2\lambda_k\langle F(y_k)-F(y_{k-1}),y_k-z_{k+1}\rangle \le \theta\Big((1+\sqrt{2})\|y_k-z_k\|^2+\|z_k-y_{k-1}\|^2+\sqrt{2}\|z_{k+1}-y_k\|^2\Big).$$

□

Lemma 3.3. *Let $\{y_k\}$ and $\{z_k\}$ be given by Algorithm 1 and let $x^*\in F^{-1}(0)$. Then we have*

$$\begin{aligned}\|x^*-z_{k+1}\|^2 &\le (1-\alpha_k)\|x^*-z_k\|^2+\alpha_k\|x^*-u\|^2-\alpha_k\|z_{k+1}-u\|^2\\ &\quad+\Big((1+\sqrt{2})\theta-\frac{\lambda_k}{\delta\lambda_{k-1}}\Big)\|y_k-z_k\|^2+\Big(\frac{\lambda_k}{\delta\lambda_{k-1}}-(1-\alpha_k)\Big)\|z_{k+1}-z_k\|^2\\ &\quad+\Big(\sqrt{2}\theta-\frac{\lambda_k}{\delta\lambda_{k-1}}\Big)\|z_{k+1}-y_k\|^2+\theta\|z_k-y_{k-1}\|^2.\end{aligned}$$

Proof. Let $x^*\in F^{-1}(0)$. By (3.1), we have $\lambda_kF(y_k)=x_k-z_{k+1}$. Therefore we get

$$\lambda_k\langle F(y_k),x^*-z_{k+1}\rangle=\langle x_k-z_{k+1},x^*-z_{k+1}\rangle.$$

Equivalently, we have

$$\|x^*-z_{k+1}\|^2=\|x^*-x_k\|^2-\|z_{k+1}-x_k\|^2+2\lambda_k\langle F(y_k),x^*-z_{k+1}\rangle. \qquad (3.8)$$

Since $\langle F(x^*),y_k-x^*\rangle=0$ and F is pseudo-monotone, we have $\langle F(y_k),y_k-x^*\rangle\ge 0$. Using (3.8), we get

$$\begin{aligned}\|x^*-z_{k+1}\|^2 &\le \|x^*-x_k\|^2-\|z_{k+1}-x_k\|^2+2\lambda_k\langle F(y_k),x^*-z_{k+1}\rangle\\ &\quad+2\lambda_k\langle F(y_k),y_k-x^*\rangle\\ &=\|x^*-x_k\|^2-\|z_{k+1}-x_k\|^2+2\lambda_k\langle F(y_k),y_k-z_{k+1}\rangle\\ &=\|x^*-x_k\|^2-\|z_{k+1}-x_k\|^2+2\lambda_k\langle F(y_k)-F(y_{k-1}),y_k-z_{k+1}\rangle\\ &\quad+2\lambda_k\langle F(y_{k-1}),y_k-z_{k+1}\rangle. \qquad (3.9)\end{aligned}$$

Using Lemmas 3.1 and 3.2 along with (3.9), we get

$$\begin{aligned}\|x^* - z_{k+1}\|^2 &\le \|x^* - x_k\|^2 - \|z_{k+1} - x_k\|^2 + 2\lambda_k\langle F(y_k) - F(y_{k-1}), y_k - z_{k+1}\rangle \\ &\quad + 2\lambda_k\langle F(y_{k-1}), y_k - z_{k+1}\rangle \\ &\le \|x^* - x_k\|^2 - \alpha_k\|z_{k+1} - u\|^2 - (1-\alpha_k)\|z_{k+1} - z_k\|^2 + \alpha_k(1-\alpha_k)\|u - z_k\|^2 \\ &\quad + \theta\Big((1+\sqrt{2})\|y_k - z_k\|^2 + \|z_k - y_{k-1}\|^2 + \sqrt{2}\|z_{k+1} - y_k\|^2\Big) \\ &\quad + \frac{\lambda_k}{\delta\lambda_{k-1}}\Big(\|z_{k+1} - z_k\|^2 - \|y_k - z_k\|^2 - \|z_{k+1} - y_k\|^2\Big) \qquad (3.10)\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|x^* - z_{k+1}\|^2 &\le \|x^* - x_k\|^2 - \alpha_k\|z_{k+1} - u\|^2 + \alpha_k(1-\alpha_k)\|u - z_k\|^2 \\ &\quad + \Big((1+\sqrt{2})\theta - \frac{\lambda_k}{\delta\lambda_{k-1}}\Big)\|y_k - z_k\|^2 + \Big(\frac{\lambda_k}{\delta\lambda_{k-1}} - (1-\alpha_k)\Big)\|z_{k+1} - z_k\|^2 \\ &\quad + \Big(\sqrt{2}\theta - \frac{\lambda_k}{\delta\lambda_{k-1}}\Big)\|z_{k+1} - y_k\|^2 + \theta\|z_k - y_{k-1}\|^2. \qquad (3.11)\end{aligned}$$

Now using (3.2) and Lemma 2.2, we get

$$\begin{aligned}\|x^* - z_{k+1}\|^2 &\le (1-\alpha_k)\|x^* - z_k\|^2 + \alpha_k\|x^* - u\|^2 - \alpha_k\|z_{k+1} - u\|^2 \\ &\quad + \Big((1+\sqrt{2})\theta - \frac{\lambda_k}{\delta\lambda_{k-1}}\Big)\|y_k - z_k\|^2 + \Big(\frac{\lambda_k}{\delta\lambda_{k-1}} - (1-\alpha_k)\Big)\|z_{k+1} - z_k\|^2 \\ &\quad + \Big(\sqrt{2}\theta - \frac{\lambda_k}{\delta\lambda_{k-1}}\Big)\|z_{k+1} - y_k\|^2 + \theta\|z_k - y_{k-1}\|^2. \qquad (3.12)\end{aligned}$$

□

In the following theorem, we show the strong convergence of the sequence $\{x_k\}$ generated by Algorithm 1 to an element of $F^{-1}(0)$.

Theorem 3.4. *Suppose that $F : H \to H$ is pseudo-monotone, coercive and Lipschitz continuous, and satisfies either one of the following conditions:*
(i) the operator F is sequentially weak-to-weak continuous,
(ii) F is asymptotically pseudo-monotone,
(iii) the function $\langle F(\cdot), y - \cdot\rangle : H \to \mathbb{R}$ is weakly upper semicontinuous for all $y \in H$.
(iv) for any arbitrary sequence $\{v_k\}$ such that $v_k \rightharpoonup v$ and $\limsup_{k\to\infty}\langle F(v_k), v_k - v\rangle \le 0$, it follows that, for all $y \in H$, $\liminf_{k\to\infty}\langle F(v_k), v_k - y\rangle \ge \langle F(v), v - y\rangle$.
If $F^{-1}(0) \neq \emptyset$, then the sequence $\{x_k\}$ generated by Algorithm 1 is strongly convergent to $P_{F^{-1}(0)}u$.

Proof. Let $x^* = P_{F^{-1}(0)}u$. For simplicity, denote $A_k := \Big((1+\sqrt{2})\theta - \frac{\lambda_k}{\delta\lambda_{k-1}}\Big)\|y_k - z_k\|^2$, $B_k := \Big(\frac{\lambda_k}{\delta\lambda_{k-1}} - (1-\alpha_k)\Big)\|z_{k+1} - z_k\|^2$ and $C_k := \Big((1+\sqrt{2})\theta - \frac{\lambda_k}{\delta\lambda_{k-1}}\Big)\|z_{k+1} - y_k\|^2$. Using Lemma 3.3, we have

$$\begin{aligned}\|x^* - z_{k+1}\|^2 &\le (1-\alpha_k)\|x^* - z_k\|^2 + \alpha_k\|x^* - u\|^2 - \alpha_k\|z_{k+1} - u\|^2 \\ &\quad + A_k + B_k + C_k - \theta\|z_{k+1} - y_k\|^2 + \theta\|z_k - y_{k-1}\|^2. \qquad (3.13)\end{aligned}$$

Note that the sequence $\{\lambda_k\}$ is nonincreasing and bounded away from zero, therefore $\lim_{k\to\infty}\lambda_k$ exists and is different from zero. Now by our assumptions on δ and θ, we have

$$\lim_{k\to\infty}\left((1+\sqrt{2})\theta-\frac{\lambda_k}{\delta\lambda_{k-1}}\right)=(1+\sqrt{2})\theta-\frac{1}{\delta}<0 \tag{3.14}$$

and

$$\lim_{k\to\infty}\left(\frac{\lambda_k}{\delta\lambda_{k-1}}-(1-\alpha_k)\right)<0. \tag{3.15}$$

Therefore it follows from (3.13), (3.14) and (3.15) that for large enough k, we have

$$\begin{aligned}\|x^*-z_{k+1}\|^2\leq(1-\alpha_k)\|x^*-z_k\|^2+\alpha_k\|x^*-u\|^2-\alpha_k\|z_{k+1}-u\|^2\\ -\theta\|z_{k+1}-y_k\|^2+\theta\|z_k-y_{k-1}\|^2.\end{aligned} \tag{3.16}$$

Denote $s_k:=\|x^*-z_k\|^2+\theta\|z_k-y_{k-1}\|^2$, then we can write the above inequality in the form

$$s_{k+1}\leq(1-\alpha_k)s_k+\alpha_k\Big(\|x^*-u\|^2-\|u-z_{k+1}\|^2+\theta\|z_k-y_{k-1}\|^2\Big). \tag{3.17}$$

Next we will show that $\lim_{k\to\infty}s_k=0$. In view of Lemma 2.3, it suffices to show that

$$\limsup_{n\to\infty}\Big(\|x^*-u\|^2-\|u-z_{k_n+1}\|^2+\theta\|z_{k_n}-y_{k_n-1}\|^2\Big)\leq 0 \tag{3.18}$$

for every subsequence $\{s_{k_n}\}$ of $\{s_k\}$ satisfying $\liminf_{n\to\infty}(s_{k_n+1}-s_{k_n})\geq 0$. Consider such a subsequence. By using (3.17), we get

$$\begin{aligned}0&\leq\liminf_{n\to\infty}(s_{k_n+1}-s_{k_n})\\ &\leq\limsup_{n\to\infty}(s_{k_n+1}-s_{k_n})\\ &\leq\limsup_{n\to\infty}\alpha_{k_n}\Big(-\|x^*-z_{k_n}\|^2+\|x^*-u\|^2-\|u-z_{k_n+1}\|^2\Big)\\ &\leq 0.\end{aligned}$$

Therefore, we have $\lim_{n\to\infty}(s_{k_n+1}-s_{k_n})=0$. Moreover, we get

$$\begin{aligned}0&\leq\liminf_{n\to\infty}(s_{k_n+1}-s_{k_n})\\ &\leq\liminf_{n\to\infty}\alpha_{k_n}\Big(-\|x^*-z_{k_n}\|^2+\|x^*-u\|^2-\|u-z_{k_n+1}\|^2\Big)\\ &\leq\limsup_{n\to\infty}\alpha_{k_n}\Big(-\|x^*-z_{k_n}\|^2+\|x^*-u\|^2-\|u-z_{k_n+1}\|^2\Big)\\ &\leq 0.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\alpha_{k_n}\Big(-\|x^*-z_{k_n}\|^2+\|x^*-u\|^2-\|u-z_{k_n+1}\|^2\Big)=0. \tag{3.19}$$

Now since $\lim_{n\to\infty}(s_{k_n+1}-s_{k_n})=0$, by using (3.13), we get

$$\begin{aligned}
0 &\le \liminf_{n\to\infty}\Big(-A_{k_n}-B_{k_n}-C_{k_n}\Big)\\
&\le \limsup_{n\to\infty}\Big(-A_{k_n}-B_{k_n}-C_{k_n}\Big)\\
&\le \limsup_{n\to\infty}\Big(-(s_{k_n+1}-s_{k_n})+\alpha_{k_n}\Big(-\|x^*-z_{k_n}\|^2+\|x^*-u\|^2-\|u-z_{k_n+1}\|^2\Big)\Big)\\
&\le \limsup_{n\to\infty}\Big(-(s_{k_n+1}-s_{k_n})\Big)+\limsup_{n\to\infty}\Big(\alpha_{k_n}\Big(-\|x^*-z_{k_n}\|^2+\|x^*-u\|^2-\|u-z_{k_n+1}\|^2\Big)\Big)\\
&=0
\end{aligned}$$

which implies that $\lim_{n\to\infty}\Big(-A_{k_n}-B_{k_n}-C_{k_n}\Big)=0$. Now the definitions of A_k, B_k and C_k along with (3.14) and (3.15) imply that

$$\lim_{n\to\infty}A_{k_n}=\lim_{n\to\infty}B_{k_n}=\lim_{n\to\infty}C_{k_n}=0. \tag{3.20}$$

Therefore we have

$$\lim_{n\to\infty}\|y_{k_n}-z_{k_n}\|^2=\lim_{n\to\infty}\|z_{k_n+1}-z_{k_n}\|^2=\lim_{n\to\infty}\|z_{k_n+1}-y_{k_n}\|^2=0. \tag{3.21}$$

On the other hand, using (3.2) in the following equality, we get

$$\begin{aligned}
\|x_{k_n}-z_{k_n+1}\| &\le \|x_{k_n}-z_{k_n}\|+\|z_{k_n+1}-z_{k_n}\|\\
&=\alpha_{k_n}\|u-z_{k_n}\|+\|z_{k_n+1}-z_{k_n}\|\\
&\le \alpha_{k_n}\|u-z_{k_n+1}\|+2\|z_{k_n+1}-z_{k_n}\|
\end{aligned} \tag{3.22}$$

Using (3.19) and (3.21), and taking the limit from (3.22) as $n\to\infty$, we get

$$\lim_{n\to\infty}\|x_{k_n}-z_{k_n+1}\|=0. \tag{3.23}$$

Using (3.1), we have

$$\lambda_{k_n}F(y_{k_n})=x_{k_n}-z_{k_n+1}. \tag{3.24}$$

Now since $\lim_{n\to\infty}\|x_{k_n}-z_{k_n+1}\|=0$, and $\lim_{k\to\infty}\lambda_k$ exists and is different from zero, taking the limit in (3.24), we get

$$\lim_{n\to\infty}F(y_{k_n})=0. \tag{3.25}$$

Now we claim that the subsequence $\{y_{k_n}\}$ is bounded, otherwise we have a contradiction with coercivity of the operator F. In the sequel, the boundedness of $\{y_{k_n}\}$ along with (3.21) imply that the subsequences $\{z_{k_n}\}$ and $\{z_{k_n+1}\}$ are bounded. Note that we have $\lim_{n\to\infty}(s_{k_n+1}-s_{k_n})=0$ and it means that

$$\lim_{n\to\infty}\Big(\|x^*-z_{k_n+1}\|^2+\theta\|z_{k_n+1}-y_{k_n}\|^2-\|x^*-z_{k_n}\|^2-\theta\|z_{k_n}-y_{k_n-1}\|^2\Big)=0. \tag{3.26}$$

Since by (3.21), $\lim_{n\to\infty}\|z_{k_n+1}-y_{k_n}\|^2=0$, from (3.26) we obtain that

$$\lim_{n\to\infty}\Big(\|x^*-z_{k_n+1}\|^2-\|x^*-z_{k_n}\|^2-\theta\|z_{k_n}-y_{k_n-1}\|^2\Big)=0,$$

which implies that

$$\lim_{n\to\infty}\Big(2\langle x^*-z_{k_n},z_{k_n}-z_{k_n+1}\rangle+\|z_{k_n+1}-z_{k_n}\|^2-\theta\|z_{k_n}-y_{k_n-1}\|^2\Big)=0.$$

Since $\{z_{k_n}\}$ is bounded and $\lim_{n\to\infty}\|z_{k_n+1}-z_{k_n}\|=0$, we deduce that

$$\lim_{n\to\infty}\|z_{k_n}-y_{k_n-1}\|^2=0. \tag{3.27}$$

Taking a subsequence of $\{z_{k_n+1}\}$ if needed, we may assume without loss of generality that it has a weak limit point $p\in H$ so that $z_{k_n+1}\rightharpoonup p$. It is clear that $y_{k_n}\rightharpoonup p$. Now we consider the following four cases:

(i) Assume that the operator F is sequentially weak-to-weak continuous. Then (3.25) implies that

$$\lim_{n\to\infty}F(y_{k_n})=F(p)=0, \tag{3.28}$$

that is $p\in F^{-1}(0)$.

(ii) If F is asymptotically pseudo-monotone, from (3.25) we obtain

$$\limsup_{n\to\infty}\langle F(y_{k_n}),y-y_{k_n}\rangle\geq 0, \qquad \forall\, y\in H$$

which implies that

$$\langle F(y),p-y\rangle=\liminf_{n\to\infty}\langle F(y),y_{k_n}-y\rangle\leq 0, \qquad \forall\, y\in H. \tag{3.29}$$

Let $z\in H$ be arbitrary and set $y=tp+(1-t)z$ where $t\in(0,1)$. Using (3.29), we have

$$(1-t)\langle F(tp+(1-t)z),z-p\rangle\geq 0, \tag{3.30}$$

Dividing (3.30) by $(1-t)$ and taking the limit as $t\to 1^-$, we get

$$\langle F(p),z-p\rangle\geq 0, \qquad \forall\, z\in H, \tag{3.31}$$

which shows $p\in F^{-1}(0)$.

(iii) If the function $\langle F(\cdot),y-\cdot\rangle : H\to\mathbb{R}$ is weakly upper semicontinuous, then it follows from (3.25) that

$$\limsup_{n\to\infty}\langle F(y_{k_n}),y-y_{k_n}\rangle\geq 0, \qquad \forall\, y\in H$$

which implies that

$$\langle F(p),y-p\rangle\geq 0, \qquad \forall\, y\in H. \tag{3.32}$$

Hence $p\in F^{-1}(0)$.

(iv) In this case, it follows from (3.25) that

$$\liminf_{n\to\infty}\langle F(y_{k_n}),y_{k_n}-p\rangle\leq 0, \qquad \forall\, y\in H$$

which implies that, for all $y\in H$, $\liminf_{k\to\infty}\langle F(y_{k_n}),y_{k_n}-y\rangle\geq\langle F(p),p-y\rangle$. Therefore

$$\langle F(p),p-y\rangle\leq 0, \qquad \forall\, y\in H.$$

Hence $p\in F^{-1}(0)$.

Therefore we have shown that $p\in F^{-1}(0)$ when the operator F satisfies either (i), (ii), (iii) or (iv). Now it follows from (3.27) that

$$\limsup_{n\to\infty}\left(\|x^*-u\|^2-\|u-z_{k_n+1}\|^2+\theta\|z_{k_n}-y_{k_n-1}\|^2\right)\leq\|x^*-u\|^2-\|u-p\|^2. \tag{3.33}$$

Since $x^* = P_{F^{-1}(0)}u$, we have $\|x^* - u\|^2 - \|u - p\|^2 \le 0$. Therefore we get

$$\limsup_{n\to\infty}\left(\|x^* - u\|^2 - \|u - z_{k_n+1}\|^2 + \theta\|z_{k_n} - y_{k_n-1}\|^2\right) \le 0, \tag{3.34}$$

which shows that (3.18) is satisfied. Therefore $z_k \to x^*$, and hence by (3.23), $x_k \to x^* = P_{F^{-1}(0)}u$. □

Corollary 3.5. *Suppose that $F : H \to H$ is coercive, monotone and Lipschitz continuous, and $F^{-1}(0) \neq \emptyset$. Then the sequence $\{x_k\}$ generated by Algorithm 1 is strongly convergent to $P_{F^{-1}(0)}u$.*

Proof. Since it is clear that every monotone operator is asymptotically pseudo-monotone, then the result follows from Theorem 3.4. □

Since from a practical point of view, it may not be easy to determine whether the zero set of the operator is nonempty, in the following remark, we give a sufficient condition for the zero set of the operator to be nonempty. In this case, the iterative sequence converges strongly to a zero of the operator. It is worth noting that this condition may prove useful when performing numerical experiments.

Remark 3.6. With the same assumptions as in Theorem 3.4, if there is a bounded subsequence $\{y_{k_n}\}$ of $\{y_k\}$ satisfying $\lim_{n\to\infty}\|x_{k_n} - z_{k_n+1}\| = 0$, then $F^{-1}(0) \neq \emptyset$. In fact, this condition implies that $\lim_{n\to\infty} F(y_{k_n}) = 0$, that is (3.25) holds. Then the same proof as in Theorem 3.4 shows that $F^{-1}(0) \neq \emptyset$.

4. Examples and Numerical Experiments

In this section, we provide some examples of applications of our main result. In fact, our results can be applied to find zeros of monotone and pseudo-monotone operators. Recently the authors studied the strong convergence of an inexact proximal point algorithm with possible unbounded errors for monotone operators in Banach and Hadamard spaces (see [2, 3]).

Karamardian introduced pseudo-monotone operators in 1976 (see [5]). The prototypical example of a pseudo-monotone operator is the gradient of a pseudo-convex function. Given an open convex set $C \subset H$, we recall that a differentiable function $f : C \to \mathbb{R}$ is said to be pseudo-convex if and only if the following statement holds:

$$\langle \nabla f(x), y - x\rangle \ge 0 \Rightarrow f(y) \ge f(x), \qquad \forall\, x, y \in C. \tag{4.1}$$

In [5], Karamardian showed that a differentiabe function f is pseudo-convex if and only if its gradient ∇f is pseudo-monotone. In the following, we give an example of application of our main theorem.

Example 4.1. Let H be a real Hilbert space and $f : H \to \mathbb{R}$ be a differentiabe and pseudo-convex function. Also suppose that ∇f is coercive and Lipschitz continuous. Assume that $\operatorname{Argmin} f \neq \emptyset$ and ∇f satisfies either one of the following conditions:
(i) the operator ∇f is sequentially weak-to-weak continuous,
(ii) ∇f is asymptotically pseudo-monotone,
(iii) the function $\langle \nabla f(\cdot), y - \cdot\rangle : H \to \mathbb{R}$ is weakly upper semicontinuous for all $y \in H$.
(iv) for any arbitrary sequence $\{v_k\}$ such that $v_k \rightharpoonup v$ and $\limsup_{k\to\infty}\langle \nabla f(v_k), v_k - v\rangle \le 0$, it follows that, for all $y \in H$, $\liminf_{k\to\infty}\langle \nabla f(v_k), v_k - y\rangle \ge \langle \nabla f(v), v - y\rangle$.

If we take $F = \nabla f$, then the sequence $\{x_k\}$ generated by Algorithm 1 is strongly convergent to an element of $\mathrm{Argmin} f = F^{-1}(0)$.

Example 4.2. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{2}x^2 - \sqrt{2}x + 5 & x \geq \sqrt{2}, \\ -x^4 + 4x^2 & -\sqrt{2} < x < \sqrt{2}, \\ \frac{1}{2}x^2 + \sqrt{2}x + 5 & x \leq -\sqrt{2}, \end{cases}$$

and $G : \mathbb{R}^2 \to \mathbb{R}$ be defined by $G(x,y) = f(\sqrt{x^2+y^2})$. Then $\nabla G : \mathbb{R}^2 \to \mathbb{R}^2$ is the operator $\nabla G(x,y) = \langle \frac{x}{\sqrt{x^2+y^2}} f'(\sqrt{x^2+y^2}), \frac{y}{\sqrt{x^2+y^2}} f'(\sqrt{x^2+y^2}) \rangle$ where

$$f'(x) = \begin{cases} x - \sqrt{2} & x \geq \sqrt{2}, \\ -4x^3 + 8x & -\sqrt{2} < x < \sqrt{2}, \\ x + \sqrt{2} & x \leq -\sqrt{2}. \end{cases}$$

It can be shown that ∇G is coercive, Lipschitz continuous and pseudo-monotone, but not monotone. Now in order to implement Algorithm 1 in Section 3 for this example, we take $\lambda_0 = 0.4$, $\delta = 1.01$, $\theta = \frac{0.4}{\delta}$, $\alpha_k = \frac{1}{k+1}$, $x_0 = y_0 = (1,-1)$ and $u = (-2,1)$. It is easy to see that $P_{F^{-1}(0)}(u) = (0,0)$. Note that the conditions of Theorem 3.4 are satisfied. Hence if the sequence $\{x_k\}$ is generated by Algorithm 1, then it converges strongly to $x^* = P_{F^{-1}(0)}(u) = (0,0)$.

We performed the numerical experiment for this example and the numerical results are displayed in Table 4.1. Also, for each starting point, the test was successful, meaning that the sequence $\{x_k\}$ converges to x^*. This problem was solved by the Optimization Toolbox in Matlab R2020a on a Laptop Intel(R) Core(TM) i7- 8665U CPU @ 1.90GHz RAM 8.00 GB.

Table 4.1

k	x_{k+1}	$\|x_{k+1} - x_k\|$	$\|x_{k+1} - x^*\|$
1	(-0.50000000, 0.00000000)	1.80277563	0.50000000
2	(-1.00000000, 0.33333333)	0.60092521	1.05409255
3	(-0.71452208, 0.50000000)	0.33056832	0.87209048
10	(-0.75214094, 0.37379939)	0.07165555	0.83990593
20	(-0.40010075, 0.20001144)	0.02065949	0.44730883
30	(-0.28010499, 0.14005143)	0.00967294	0.31316643
100	(-0.08912040, 0.04456020)	0.00098642	0.09963963
1000	(-0.00905571, 0.00452785)	0.00001011	0.01012460
10000	(-0.00090685, 0.00045342)	0.00000010	0.00101389

Example 4.3. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^{\frac{3}{2}} & x \geq 1, \\ \frac{3}{4}x^2 + \frac{1}{4} & -1 < x < 1, \\ (-x)^{\frac{3}{2}} & x \leq -1, \end{cases}$$

and $G:\mathbb{R}^2\to\mathbb{R}$ be defined by $G(x,y)=f(\sqrt{x^2+y^2})$. Then $\nabla G:\mathbb{R}^2\to\mathbb{R}^2$ is the operator $\nabla G(x,y)=\langle\frac{x}{\sqrt{x^2+y^2}}f'(\sqrt{x^2+y^2}),\frac{y}{\sqrt{x^2+y^2}}f'(\sqrt{x^2+y^2})\rangle$ where

$$f'(x)=\begin{cases}\frac{3}{2}\sqrt{x} & x\geq 1,\\ \frac{3}{2}x & -1<x<1,\\ \frac{-3}{2}\sqrt{-x} & x\leq -1.\end{cases}$$

It can be shown that ∇G is coercive, Lipschitz continuous and monotone. Now in order to implement Algorithm 1 in Section 3 for this example, we take $\lambda_0=0.4$, $\delta=1.1$, $\theta=\frac{0.4}{\delta}$, $\alpha_k=\frac{1}{k+1}$, $x_0=y_0=(6,-14)$ and $u=(-2,4)$. It is easy to see that $P_{F^{-1}(0)}(u)=(0,0)$. Note that the conditions of Theorem 3.4 are satisfied. Hence if the sequence $\{x_k\}$ is generated by Algorithm 1, then it converges strongly to $x^*=P_{F^{-1}(0)}(u)=(0,0)$.

We performed the numerical experiment for this example and the numerical results are displayed in Table 4.1. Also, for each starting point, the test was successful, meaning that the sequence $\{x_k\}$ converges to x^*. This problem was solved by the Optimization Toolbox in Matlab R2020a on a Laptop Intel(R) Core(TM) i7- 8665U CPU @ 1.90GHz RAM 8.00 GB.

Table 4.2

k	x_{k+1}	$\|x_{k+1}-x_k\|$	$\|x_{k+1}-x^*\|$
1	(1.53878809, -3.92383888)	11.01959320	4.21478118
2	(-0.14684487, -0.10180639)	4.17723483	0.17868396
3	(-0.58528093, 1.01565552)	1.20039465	1.17222434
10	(-0.76028376, 1.51165724)	0.09776704	1.69208126
20	(-0.44413475, 0.88800806)	0.04579848	0.99288166
30	(-0.30615016, 0.61228955)	0.02214052	0.68456294
100	(-0.09547839, 0.19095679)	0.00212174	0.21349618
1000	(-0.00968479, 0.01936959)	0.00002164	0.02165586
10000	(-0.00096984, 0.00193969)	0.00000021	0.00216864

Example 4.4. (Problem 1, [11]) This problem was also considered in [6, 8]. Let $H=\mathbb{R}^m$ and $A=[a_{ij}]$ be a square matrix of order $m\geq 2$ defined by

$$a_{ij}=\begin{cases}-1 & \text{if } j>i \text{ and } j=m+1-i,\\ 1 & \text{if } j<i \text{ and } j=m+1-i,\\ 0 & \text{otherwise.}\end{cases}$$

Define $F:H\to H$ by $F(x)=Ax$. If $m=2$, then F is a rotation with a $\pi/2$ angle. F is coercive, monotone and Lipschitz continuous. The unique zero of the operator is the origin, but the usual gradient method gives rise to a sequence satisfying $\|x_{k+1}\|>\|x_k\|$ for all k, that is the sequence is not convergent. We apply our method and show that the generated sequence converges to the solution of the problem. Since $F(0)=0$, then $F^{-1}(0)\neq\emptyset$.

Now in order to implement Algorithm 1 in Section 3 for this example, we take $\lambda_0=0.4$, $\delta=1.1$, $\theta=\frac{0.4}{\delta}$, $\alpha_k=\frac{1}{k+1}$, $x_0=y_0=(1,1,\cdots,1)$ and $u=(2,2,\cdots,2)$. Note that the conditions of Corollary 3.5 are satisfied. Hence the sequence $\{x_k\}$ generated by Algorithm 1 converges strongly to $x^*=(0,0,\cdots,0)$. Our stopping criterion is $\|x_{k-1}-x_k\|<\varepsilon$, and we take $\varepsilon=10^{-4}$.

We performed the numerical experiment for this example and the numerical results are displayed in Table 4.2. This problem was solved by the Optimization Toolbox in Matlab R2020a on a Laptop Intel(R) Core(TM) i7- 8665U CPU @ 1.90GHz RAM 8.00 GB.

Table 4.3		
m	Number of iterations	CPU time (Sec)
100	831	0.002655
200	989	0.004560
500	1243	0.015625
1000	1479	0.046875
2000	1759	0.093750
4000	2091	0.140625

5. CONCLUSIONS

In this paper, we studied the strong convergence of the generated sequence by a modified variant of the Malitsky's algorithm [6] to a zero of a pseudo-monotone operator without any knowledge of the Lipschitz constant of the operator. As a special case, we obtain the strong convergence of the generated sequence for a monotone operator, extending the results by Yang and Liu [11] who proved the weak convergence of the sequence in this case with their algorithm. We preformed some numerical experiments to show the efficiency of our algorithm. As a future direction for research, by using the ideas and methods in this paper, it might be interesting to study the possibility of extending the convergence results for monotone and pseudo-monotone operators in Banach spaces as well as for accretive operators.

Acknowledgement
This work aws done while the second author was visiting the University of Texas at El Paso. The second author would like to thank Professor Djafari Rouhani and the Department of Mathematical Sciences for their kind hospitality at the University of Texas at El Paso during his visit.

REFERENCES

[1] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, New York, 2011.
[2] B. Djafari Rouhani, Mohebbi, V. Proximal point methods with possible unbounded errors for monotone operators in Hadamard spaces, Optimization, 72 (2023) 2345-2366.
[3] B. Djafari Rouhani, V. Mohebbi, Strong convergence of an inexact proximal point algorithm in a Banach space, J. Optim. Theory Appl. 186 (2020) 134–147.
[4] A.N. Iusem, B.F. Svaiter, A variant of Korpelevich's method for variational inequalities with a new search strategy, Optimization 42 (1997) 309-321.
[5] S. Karamardian, Complementarity problems over cones with monotone and pseudomonotone maps. J. Optim. Theory Appl. 18 (1976), 445–454.
[6] Malitsky, Yu. Projected reflected gradient methods for monotone variational inequalities, SIAM J. Optim. 25 (2015), 502–520.
[7] Y.V. Malitsky, V.V. Semenov, An extragradient algorithm for monotone variational inequalities, . Cybernet. Systems Anal. 50 (2014), 271–277.
[8] P.E. Mainge, M.L. Gobinddass, Convergence of one-step projected gradient methods for variational inequalities, J. Optim. Theory Appl. 171 (2016) 146–168.
[9] S. Saejung, P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, Nonlinear Anal. 75 (2012) 742-750.
[10] M.V. Solodov, P. Tseng, Modified projection-type methods for monotone variational inequalities, SIAM J. Control Optim. 34 (1996) 1814–1830.
[11] J. Yang, H. Liu, A modified projected gradient method for monotone variational inequalities, J. Optim. Theory Appl. 179 (2018) 197–211.