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NASH EQUILIBRIUM POINTS AND THEIR FINDING FOR NONSMOOTH CASE

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Dedicated to Professor F. Clarke on the occasion of his 75-th birthday

Abstract. The purpose of this paper is to develop a numerical method for finding an equilibrium point in a model, in which the loss function of each object (subject) is described by a convex function with respect to one of its variables. Such models are found in medicine, economics, game theory, and biology. For the more complex case, with nonsmooth functions describing the state of each element of the system as damage, loss, or gain, the Steklov average integrals are used that turn nonsmooth functions into smooth ones. Numerical method for finding equilibrium points in the more general non-smooth case is constructed. In the process of optimization, the diameters of the sets, over which the averaging takes place, are decreased in accordance with the optimization steps. All limit points are proved to be equilibrium points. Under some conditions, the convergence rate can be estimated using the Kantorovich theorem. The necessity to develop new methods for finding Nash equilibrium points in the nonsmooth case is concluded.

Keywords. Clarke subdifferential; Convex functions; Non-cooperative Nash equilibrium point; Lebesgue integrals; Newton's optimization methods.

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1. Introduction

Let the physical or economic state of a system be described by m loss functions $f_1(x_1,x_2,\dots,x_m):\mathbb{R}^m\to\mathbb{R}$, $f_2(x_1,x_2,\dots,x_m):\mathbb{R}^m\to\mathbb{R}$, $\dots$, $f_m(x_1,x_2,\dots,x_m):\mathbb{R}^m\to\mathbb{R}$ depending on m variables $x_1,x_2,\dots,x_m$, where $\mathbb{R}^m$ is m-dimensional Euclidean space. Then an equilibrium point is a state $x_1^*,x_2^*,\dots,x_m^*$ for which changing any x_j^* leads to an increase in the corresponding function $f_j(\cdot)$, i.e.

$$f_j(x_1^*,x_2^*,\dots,x_j^*,\dots,x_m^*)\le f_j(x_1^*,x_2^*,\dots,x_{j-1}^*,x_j,x_{j+1}^*,\dots,x_m^*). \tag{1.1}$$

Equilibrium states were introduced into economics by J. Nash. In 1950-1953, his published many results on the existence of equilibrium points; see, e.g., [1]-[4]. The problem of equilibrium points finding in biology or economics is closely related to game theory and is of practical

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importance. Equilibrium points arise from interspecific competition in biology and intercompany competition in economics. Equilibrium points in medicine are homeostasis points [5]. These are points of balance between various states of the human body, e.g. blood pressure, temperature, blood cholesterol level, pulse rate. Some balance is achieved between different pills when we take medication. Here we consider non-cooperative games of m players, none of whom can influence other players' behavior (strategies). A player i chooses independently a pure strategy x_i from a compact convex set S_i, such that he minimizes his loss function $f_i(\cdot)$.

Consider a vector $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$, called a multistrategy and comprised of the pure strategies $x_i \in S_i$. We assume that the vector x belongs to the compact convex set $S = S_1 \times S_2 \times \dots S_m \in \mathbb{R}^m$, which is the Cartesian product of the compact sets $S_i, i \in 1:m$, and int $S \neq \emptyset$.

Definition 1.1. A multistrategy $x^* = (x_1^*, x_2^*, \dots, x_m^*) \in S$ of a non-cooperative game is called a non-cooperative equilibrium if inequality (1.1) is true for every $j \in 1:m$ and $x_j \in S_j$.

The definitions of equilibrium points in game theory, medicine, and economics are similar. In 1950, J. Nash proved the following theorem.

Theorem 1.1. [1]. Let S_i be a compact convex set for any $i \in 1:m$ and $f_i(\cdot)$ be convex with respect to $x_i \in S_i$. Then there is a non-cooperative equilibrium in a non-cooperative game with m players.

The aim of the paper is to develop numerical methods for finding equilibrium points in the nonsmooth case. To our knowledge, there are papers describing numerical methods for finding equilibrium points in special cases [6], however, there are no papers describing numerical methods for finding equilibrium points in the general case.

2. Discussion of the Problem

We describe a method of searching for an equilibrium state, provided that the functions $f_i(x_{-i}, x_i) : S \to R$, where $x_{-i} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$, are convex with respect to x_i. We assume that the inclusion $\{x \in \mathbb{R}^m \mid f_i(x) < f_i(x_0)\} \subset \text{int } S$ is true for any $i \in 1:m$, in which x_0 is a starting point. We can conclude from here that equilibrium points x^* belong to $\text{int} S$. Here $\text{int} S$ means the interior of the set S.

Let us denote the coordinate vectors by $e_1 = (1,0,0,\dots,0), e_2 = (0,1,0,\dots,0)$, $\dots, e_m = (0,0,0,\dots,1)$. It is known that coordinate descent method admits no convergence for nonsmooth functions [7]. Therefore, we will use the ideas from [8].

Equilibrium points finding algorithms for the smooth case

One way is to use gradient and second-order methods for the smooth case, i.e., $f_i(\cdot), i \in 1:m$, are differentiable functions with respect to the variables $x_j, j \in 1:m$. Denote the partial derivative of the function $f_i(\cdot)$ with respect to the variable x_i by $f'_{i,x_i}(\cdot)$.

Consider the vector function $\Theta(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$:

$$\Theta(x) = \begin{pmatrix} f'_{1,x_1}(x) \\ f'_{2,x_2}(x) \\ \dots \\ f'_{m,x_m}(x) \end{pmatrix} \tag{2.1}$$

We search a vector $x^* = (x_1^*, x_2^*, \dots, x_m^*)$, for which $\Theta(x^*) = 0$. It is clear that the vector x^* is an equilibrium point.

First, we give the general description of the Newton's method. From expansion accurate to the higher order terms $\Theta(x+\Delta x)=\Theta(x)+\Theta'(x)\Delta x+o(\Delta x)$, in which

$$\lim_{\Delta x\to 0}\frac{o(\Delta x)}{\|\Delta x\|}=0,$$

we obtain the value for the step Δx. Suppose

$$\Theta(x+\Delta x)\approx\Theta(x)+\Theta'(x)\Delta x=0.$$

It follows that $\Delta x=-(\Theta'(x))^{-1}\Theta(x)$, where

$$\Theta'(x)=\begin{pmatrix} f''_{1,x_1,x_1} & f''_{1,x_1,x_2} & \cdots & f''_{1,x_1,x_m}\\ f''_{2,x_2,x_1} & f''_{2,x_2,x_2} & \cdots & f''_{2,x_2,x_m}\\ \vdots & \vdots & \ddots & \vdots\\ f''_{m,x_m,x_1} & f''_{m,x_m,x_2} & \cdots & f''_{m,x_m,x_m}\end{pmatrix} \tag{2.2}$$

Algorithm 1 (The Newton's method for twice continuously differentiable functions $fi(\cdot)$)

At each step k, if the inverse matrix $(\Theta'(x_k))^{-1}$ exists, we find

$$\Delta x_k=-(\Theta'(x_k))^{-1}\Theta(x_k). \tag{2.3}$$

We set $x_{k+1}=x_k+2^{-l}\,\Delta x_k$, in which l is the smallest number from the set $M=\{0,1,2,\cdots\}$ for which the inequality $\|\Theta(x_{k+1})\|<\|\Theta(x_k)\|$ is correct and $x_{k+1}\in \operatorname{int} S$. It is easy to prove that there exists a number l for which the inequality $\|\Theta(x_{k+1})\|<\|\Theta(x_k)\|$ is true. Indeed,

$$\Theta(x_{k+1})=\Theta(x_k+2^{-l_s}\Delta x_k)=\Theta(x_k)+2^{-l_s}\Theta'(x_k)\Delta x_k+o(2^{-l_s}\Delta x_k)=$$

$$=\Theta(x_k)-2^{-l_s}\Theta(x_k)+o(2^{-l_s}\Delta x_k)=\Theta(x_k)(1-2^{-l_s})+o(2^{-l_s}\Delta x_k).$$

Since $\|o(2^{-l_s}\Delta x)\|\le 2^{-l_s}\varepsilon(l_s)\|\Theta(x_k)\|$, in which $\varepsilon(l_s)\to 0$ as $l_s\to\infty$, we have for big enough l_s

$$\|\Theta(x_{k+1})\|\le(1-2^{-l_s}+2^{-l_s}\varepsilon(l_s))\|\Theta(x_k)\|<\|\Theta(x_k)\|$$

what needed to be proven. Repeat the process as long as $\|\Theta(x_k)\|\le\varepsilon$, in which ε is a positive small number.

Let the inequality

$$L_1\|\Delta x\|\le\|\Theta'(x)\Delta x\|\le L_2\|\Delta x\| \tag{2.4}$$

hold true for some $L_1,L_2>0$ and any Δx. We assume that x belongs to a small neighborhood of an equilibrium point x^*, in which $\Theta(x^*)=0$ and the optimization process takes place with full step Δx_k i.e. $l=0$. Then it is possible to obtain an estimation of the convergence rate of the Newton's method. We have

$$\|\Theta(x_{k+1})\|=\|\Theta(x_k+\Delta x_k)-\Theta(x^*)\|=\|\Theta'(\xi)\Delta x_k\|,$$

in which ξ is a point on the line, connecting x_{k+1} and x^*, and between them. Let us substitute the expression for Δx_k from (2.3). Therefore, we obtain

$$\|\Theta'(\xi)\Delta x_k\|=\|\Theta'(\xi)(\Theta'(x_k))^{-1}\Theta(x_k)\|.$$

Due to the continuity of the matrix $\Theta'(\cdot)$ and the fact that at each step k according to the choice of the step $\Theta(x_k) = o(\Delta x_{k-1})$, and also the assumption (2.4), we obtain a chain of inequalities

$$L_1\|\Delta x_k\| \le \|\Theta'(\xi)\Delta x_k\| = \|\Theta'(\xi)(\Theta'(x_k))^{-1}\Theta(x_k)\| \le c_k\|\Theta(x_k)\|.$$

Here $c_k = \|\Theta'(\xi)(\Theta'(x_k))^{-1}\|$. We also have $\|o(\Delta x)\| \le \varepsilon(\Delta x)\|\Delta x\|$, in which $\varepsilon(\Delta x) \to 0$ as $\Delta x \to 0$. Let us define

$$N_k = N(\Delta x_k) = \frac{1}{\varepsilon(\Delta x_k)},$$

in which $N_k = N(\Delta x_k) \to_k \infty$ as $\|\Delta x_k\| \to 0$. Therefore,

$$\|o(\Delta x_{k-1})\| \le \frac{\|\Delta x_{k-1}\|}{N_{k-1}}.$$

We also have

$$\|\Theta(x_k)\| = \|o(\Delta x_{k-1})\| \le \frac{\|\Delta x_{k-1}\|}{N_{k-1}}.$$

As a result,

$$L_1\|\Delta x_k\| \le c_k\frac{\|\Delta x_{k-1}\|}{N_{k-1}}.$$

From the inequalities, we have

$$\|\Delta x_k\| \le c_k\frac{\|\Delta x_{k-1}\|}{L_1 N_{k-1}} = q_k\|\Delta x_{k-1}\|,$$

where $q_k = \frac{c_k}{L_1 N_{k-1}}$. Superlinear convergence follows directly due to $q_k \to_k 0$.

Finally, e obtain the following theorem.

Theorem 2.1. Let assumption (2.4) hold true for the twice continuously differentiable functions $f_i(\cdot), i \in 1:m$. Then the Newton method converges with superlinear rate in a small neighborhood of x^*.

This optimization process requires the existence of continuous second mixed derivatives with respect to the variables $x_i, x_j, i, j \in 1:m$, of the functions $f_i(\cdot), i \in 1:m$, and the existence of the inverse matrix $(\Theta'(x_k))^{-1}$ at any step k. Unfortunately, assumption (2.4) does not always hold. Moreover, Theorem 2.1 is true in a small neighborhood of the point x^* which is to be reached.

3. Solution of the Problem for the Nonsmooth Case

Let us use the ideas from [8]. We assume that $f_i(\cdot), i \in 1:m$, are Lipschitz functions with constants L_i, i.e., $\|f_i(u) - f_i(v)\| \le L_i\|u - v\|$ for all $u, v \in \mathbb{R}^m$. We construct functions

$$\varphi_i(x) = \frac{1}{\mu(D)}\int_D f_i(z+x)dz. \tag{3.1}$$

in which D is an arbitrary convex compact set, $0 \in \text{int}D, \mu(D) > 0$ is the Lebesgue measure of set D, and the integral is the Lebesgue integral. It is not difficult to verify that function $\varphi_i(\cdot)$ is convex with respect to its variable x_i.

The function $f_i(\cdot)$ has the partial derivative with respect to x_i almost everywhere (a.e.) on set S. In [8], it was proven that function $\varphi_i(\cdot)$ is continuously differentiable with respect to variable x_i. The partial derivative $\varphi_i(\cdot)$ with respect to x_i can be calculated by the formula [8]

$$\varphi'_{i,x_i}(x) = \frac{\partial \varphi_i(x)}{\partial x_i} = \frac{1}{\mu(D)} \int_D \frac{\partial f_i(x+y)}{\partial x_i} dy.$$

Functions $\varphi_i(\cdot), i \in 1:m$, have an equilibrium point according to Nash's theorem. Substitute $f_i(\cdot), i \in 1:m$, for $\Phi_i(\cdot): \mathbb{R}^m \to \mathbb{R}, i \in 1:m$, defined by

$$\Phi_i(x) = \frac{1}{\mu(D)} \int_D \varphi_i(x+y) dy,$$

in which the functions $\varphi_i(\cdot), i \in 1:m$, and the set D are defined above (see (3.1)). We take the integral of the integral, since in this way we obtain the twice continuously differentiable functions $\Phi_i(\cdot), i \in 1:m$, and the stationary points of $\Phi_i(\cdot)$ are $\varepsilon(D)-$stationary points of $f(\cdot)$ [8]. Since $\varphi_i(\cdot)$ is a Lipschitz [8], we have

$$\Phi'_i(x) = \frac{1}{\mu(D)} \int_D \varphi'_i(z+x) dz. \tag{3.2}$$

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We have proved that $\Phi_i(\cdot), i \in 1:m$, have Lipschitz second derivatives [8]. If D is a ball or a cube centered at zero with the diameter $d(D)$, then functions $\Phi_i(\cdot), i \in 1:m$, have Lipschitz second derivatives $\Phi''_i(\cdot)$ with constant [8]

$$L'_i = \frac{2L_i}{d^2(D)}.$$

We can apply the Newton's method to $\Phi_i(\cdot), i \in 1:m$, to find the equilibrium points. In the process of optimization, we will consistently decrease the step λ_k and the diameter $d(D_k)$ so that the inequality

$$\frac{\lambda_k}{d^2(D_k)} < \varepsilon_k, \tag{3.3}$$

was true for some sequence $\{\varepsilon_k\}$ in which $\varepsilon_k \to +0$ as $k \to \infty$. We will prove that the inequality (3.3) guarantees that any limit point of a sequence obtained by the Newton's method using the functions $\Phi'_i(\cdot), \Phi''_i(\cdot), i \in 1:m$, is an equilibrium point of the functions $f_i(\cdot), i \in 1:m$.

The Newton's method for finding equilibrium points for $f_i(\cdot)$, $i \in 1:m$, using the functions $\Phi_i(\cdot)$

Calculate $\Theta(z_k)$ and $\Theta'(z_k)$

$$\Theta(x) = \begin{pmatrix} \Phi'_{1,x_1}(x) \\ \Phi'_{2,x_2}(x) \\ \dots \\ \Phi'_{m,x_m}(x), \end{pmatrix}$$

$$\Theta'(x) = \begin{pmatrix} \Phi''_{1,x_1,x_1} & \Phi''_{1,x_1,x_2} & \cdots & \Phi''_{1,x_1,x_m} \\ \Phi''_{2,x_2,x_1} & \Phi''_{2,x_2,x_2} & \cdots & \Phi''_{2,x_2,x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi''_{m,x_m,x_1} & \Phi''_{m,x_m,x_2} & \cdots & \Phi''_{m,x_m,x_m} \end{pmatrix}$$

accordingly to (3.2) for the twice differentiable functions $\Phi_i(\cdot), i \in 1:m$, when D is a ball or a cube.

Take a sequence of sets $\{D_s\}, s = 1,2,\dots$ with non-empty interior, the diameters $d(D_s)$ of which tend to zero in $s \to \infty$. Let $D_s = B^m_{r_s}(0) = \{v \in \mathbb{R}^n \mid \|v\|\| \le r_s\}$ for $r_s \to +0$ and $s \to \infty$. Let us introduce for $i \in 1:m$ the following sequence of functions

$$\varphi_{i,s}(x) = \frac{1}{\mu(D_s)} \int_{D_s} f_i(x+y)dy$$

and

$$\Phi_{i,s}(x) = \frac{1}{\mu(D_s)} \int_{D_s} \varphi_{i,s}(x+y)dy. \tag{3.4}$$

The difference between (3.2) and (3.4) is that (3.2) is written for a constant D, while (3.4) is written for a set D_s depending on the parameter s.

Construct the functions $\Theta_s(\cdot)$ for the functions $\Phi_{i,s}(\cdot), i \in 1:m$, as written above. Let the inequality $\|\Phi''_{i,s}(\cdot)\| \le L_s$ hold true in which $\Phi''_{i,s}(\cdot)$ is the matrix of the second mixed derivatives. In [8], it was proved that $L_s = \frac{L}{d(D_s)}$, in which $L = \max_{i \in 1:m} L_i$.

It follows from here that, depending on the selected metric of the space $\mathbb{R}^m$, the norm $\|\Theta'_s(\cdot)\|$ is proportional to L_s. Suppose $\|\Theta'_s(\cdot)\| \le L_s$. Define the vector-function $\tilde{\Theta}_s(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ as a function of y:

$$\tilde{\Theta}_s(y,x) = \Theta_s(y) + 2L_s(y-x). \tag{3.5}$$

Then we have the inequality for the matrix $\tilde{\Theta}'_s(\cdot)$

$$L_s\|z\|^2 \le (\tilde{\Theta}'_s(x,x)z,z) \le 3L_s\|z\|^2 \ \ \forall z \in \mathbb{R}^n. \tag{3.6}$$

Let us construct the Newton's method for finding the roots of the equation $\Theta_s(x) = 0$ using the function $\tilde{\Theta}_s(\cdot)$. We will use the rule of consistent reduction of the length λ_k of k^{th} step and the diameter $d(D_k)$.

Description of the Newton's method for finding for the equilibrium points using $\Phi_{i,s}(\cdot)$.

Let a point x_k was constructed at the step k. Construct the point x_{k+1}. Take by definition $\tilde{\Theta}_{s,k}(\cdot) = \tilde{\Theta}_s(\cdot, x_k)$. The dependence of s on k is be written as $s = s(k)$. We calculate $\Delta x_k = -(\tilde{\Theta}'_{s,k}(x_k))^{-1}\tilde{\Theta}_{s,k}(x_k)$ at each step k. We set $x_{k+1} = x_k + 2^{-l}\Delta x_k$, in which l is the smallest number from the set $M = \{0,1,2,\cdots\}$ for which $\|\tilde{\Theta}_{s,k}(x_{k+1})\| < \|\tilde{\Theta}_{s,k}(x_k)\|$ and $x_{k+1} \in \operatorname{int} S$.

It is possible to prove that $\|\tilde{\Theta}_{s,k}(x_{k+1})\| < \|\tilde{\Theta}_{s,k}(x_k)\|$ for small $\|\Delta x_k\|$ and $\|\Delta x_k\| \to 0$ as $k \to \infty$ for fixed s in a small neighborhood of the equilibrium point x^*. We assume that we reach a small surrounding of the equilibrium point x^* for big k in which the process takes place with the full step Δx_k.

If the inequality

$$\frac{\|\Delta x_k\|}{d^2(D_{s(k)})} < \varepsilon_k \tag{3.7}$$

is fulfilled for a sequence $\{\varepsilon_k\}, \varepsilon_k \to +0$ as $k \to \infty$, then we decrease the diameter $d(D_{s(k)})$ of $D_{s(k)}$ and increase $k, s = s(k)$. Inequality (3.6) holds true for $\tilde{\Theta}'_{s(k),k}(\cdot)$ and all s,k. Firstly, we prove that

$$\lim_{k \to \infty} \Theta_{s(k)}(x_k) = 0 \tag{3.8}$$

and the sequence $\{x_k\}$ has a limit point x^*.

We have the expansion of the function $\tilde{\Theta}_{s(k),k}(\cdot)$ in the neighborhood of the point x_k

$$\tilde{\Theta}_{s(k),k}(x_{k+1}) = \tilde{\Theta}_{s(k),k}(x_k)) + \tilde{\Theta}'_{s(k),k}(x_k))\Delta x_k + o_{s(k),k}(\Delta x_k).$$

After substitution $\Delta x_k = -(\tilde{\Theta}'_{s(k),k}(x_k))^{-1}\tilde{\Theta}_{s,k}(x_k)$ in this expansion we obtain

$$\tilde{\Theta}_{s(k),k}(x_{k+1}) = o_{s(k),k}(\Delta x_k). \tag{3.9}$$

Let us prove that $o_{s(k),k}(\Delta x_k)$ is an infinitesimal function with respect to Δx_k as $k \to \infty$. Since $\tilde{\Theta}_{s(k),k}(\cdot)$ was obtained from $\Theta_s(\cdot)$ through adding the linear function, $o_{s(k),k}(\cdot)$ is the same infinitesimal function in the expansion of $\Theta_s(\cdot)$ in the surrounding of x_k.

Now we will obtain the upper bound for $o_{s(k),k}(\cdot)$. The following expansion takes place

$$\Theta_s(x_{k+1}) = \Theta_s(x_k) + \Theta'_s(x_k)\Delta x_k + o_{s,k}(\Delta x_k).$$

Since the function $\Theta_s(\cdot)$ is continuously differentiable for each s, according to the midpoint theorem, we have

$$\Theta_s(x_{k+1}) - \Theta_s(x_k) = \Theta'_s(\xi)(x_{k+1} - x_k) = \Theta'_s(\xi)\Delta x_k,$$

in which $\xi \in [x_k, x_{k+1}]$. Let us substitute this difference in the Taylor series and use the Lipschitzness of $\Theta'_s(\cdot)$ with the constant $\frac{2L}{d^2(D_s)}$. Therefore, since $\Theta'_s(\cdot)$ is Lipschitz with the constant $\frac{2L}{d^2(D_s)}$ [8], we obtain

$$\|o_{s,k}(\Delta x_k)\| \le \|(\Theta'_s(\xi) - \Theta'_s(x_k))\Delta x_k\| \le \frac{2L\|\Delta x_k\|}{d^2(D_s)}\|\Delta x_k\|.$$

It follows that

$$\frac{\|o_{s,k}(\Delta x_k)\|}{\|\Delta x_k\|} \le \frac{2L\|\Delta x_k\|}{d^2(D_s)}. \tag{3.10}$$

Hence, if

$$\lim_{k\to\infty} \frac{\|\Delta x_k\|}{d^2(D_{s(k)})} = 0 \tag{3.11}$$

holds true during optimization, then uniform infinitesimality of $o_{s(k),k}(\cdot)$ with respect to $s = s(k)$ and k follows from here. However, we organize our process in such a way that limit equality (3.11) was correct. The limit equality

$$\lim_{k\to\infty} L_{s(k)}\|x_{k+1} - x_k\| = \lim_{k\to\infty} L_{s(k)}\Delta x_k = \lim_{k\to\infty} \frac{L}{d(D_{s(k)})}\Delta x_k \le \lim_{k\to\infty} \frac{L}{d^2(D_{s(k)})}\Delta x_k = 0. \tag{3.12}$$

follows from the inequality (3.7) as we decrease $d(D_{s(k)})$ in the process of optimization. The equality (3.8) follows from (3.5), (3.9) and (3.12). It follows from the upper semicontinuity of the Clarke subdifferential and from equality (3.8) that the sequence $\{x_k\}$ converges to a limit point x^*, in which $0 \in \partial_{x_i} f_i(x^*)$ for all $i \in 1:m$, i.e. x^* is the equilibrium point.

All of the above stated is true if we reach a small neighborhood of the equilibrium point. In order to do this, we are to use the coordinate descent method with some modifications for the functions $\varphi_i(\cdot), i \in 1:m$. Thus, the following theorem is proved.

Theorem 3.1. Any limit points of the sequence obtained by Newton's method with starting points from small neighborhoods of the equilibrium points, are the equilibrium points if the equality (3.11) is satisfied in the process of optimization for the convex with respect to x_i, Lipschitzian functions $f_i(\cdot), i \in 1:m$, respectively.

The given Newton's method is also called the modified Newton's method. It is possible to show that there is a majorant Kantorovich function for any step k [9]. The step length and the convergence rate of the optimization method are estimated under the conditions of consistency (3.11) and some conditions indicated in the theorem given below. We can state the convergence of the whole sequence $\{x_k\}$ under the below given conditions in the theorem 3.1.

This is true for the reason that $\|\Delta x_k\|$ is compared with the step length of the majorant function. The conditions of the Kantorovich theorem [9, pp. 689-690], are fulfilled if we satisfy some requirements.

We will construct a sequence $\{x_k\}$ converging to the solution of the equation $\Theta(x^*) = 0$ for a ball $B^m_r(x_0) = \{y \in \mathbb{R}^m \mid \|y - x_0\| \leq r\}$. Suppose $Q_{s,0} = [\tilde{\Theta}'_{s,0}(x_0)]^{-1}$, $\|Q_{s,0}\Theta_s(x_0)\| \leq A_s$, and $\|Q_{s,0}\tilde{\Theta}''_{s,0}(x)\| \leq B_s$ for any $x \in B^m_r(x_0)$, $\triangle x_k = -[\tilde{\Theta}'_{s,0}(x_k)]^{-1}\tilde{\Theta}_{s,k}(x_k)$. We set $x_{k+1} = x_k + \triangle x_k$. During the optimization process, we change $s = s(k)$ and decrease the diameter $d(D_s)$ so that the requirements of Theorem 3.2 were satisfied.

Theorem 3.2. Assume that there exists a linear operator $Q_{s,0} = [\tilde{\Theta}'_{s,0}(x_0)]^{-1}$ for $s = s(0)$ and $k = 0$. If $q_s = A_sB_s \leq q < \frac{1}{2}$ is true for any s and the consistency condition $\lim_{k\to\infty} \frac{\Delta x_k}{d^2(D_{s(k)})} = 0$ is satisfied, then the equation $\Theta(x) = 0$ has a solution x^* to which the Newton's method converges with the rate

$$\|x^* - x_k\| \leq \frac{1}{2^k}[2q]^{2^k}C \tag{3.13}$$

for a constant C. The convergence rate of the modified Newton's method (for $q < \frac{1}{2}$) is estimated by the following inequality

$$\|x^* - x_k\| \leq C(1 - \sqrt{1 - 2q})^{k+1},\ k = 0, 1, 2, \dots \tag{3.14}$$

Remark 3.1. The proof of the convergence rate follows with some changes from [9, p. 690], since the convergence rate depends on the values q_k and A_k. The first value is limited by the value $\frac{1}{2}$. The second value tends to zero as $k \to \infty$.

Proof. It is easy to satisfy to the conditions of the theorem since $A_{s(k)} \to_k 0$ and we can decrease the diameters of the sets D_s when $q_s = A_sB_s < \frac{1}{2}$ and the point x_k can be considered as a new starting point. At each step k, there is a majorant function $\psi_{s,k}(\cdot)$ with $\psi_s(t) = B_st^2 - 2t + 2A_s$. Since $\|Q_{s,0}\Theta''_s(x_k)\| \leq \psi''_s(x_k)$, the step length $\triangle_k = \|x_{k+1} - x_k\|$ does not exceed the step length $t_{k+1} - t_k$ of the Newton's method for the equation $\psi_s(t) = 0$. It e solution is denoted by t_s. Thus the following can be written:

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k. \tag{3.15}$$

For the existence of the majorant equation $\psi_s(t) = 0, t \in \mathbb{R}$, for the operator equation $\Theta(x) = 0$, as it follows from the Taylor formula XVII.2.5 [9], it is sufficient that the following integral inequality holds true

$$\|\int_{x_k}^{x_{k+1}} Q_0\Theta''_s(x)(x_{k+1} - x, \cdot)dx\| \leq \int_{t_k}^{t_{k+1}} c_0\psi''_s(t)(t_{k+1} - t)dt$$

for big enough k, which is true if

$$\|Q_0\Theta''_s(x)\|\,\|x_{k+1} - x_k\| \leq c_0\psi''_s(t)(t_{k+1} - t_k).$$

Let us denote by $c_{s,k} = -\frac{1}{\psi_s'(t_k)}$, $A_{s,k} = c_{s,k}\psi_s(t_k)$, $B_{s,k} = c_{s,k}\psi_s''(t_k) = 2B_s c_{s,k}$, and $q_{s,k} = B_{s,k}A_{s,k}$. Let us note that

$$t_{k+1} - t_k = -\frac{\psi_s(t_k)}{\psi_s'(t_k)} = A_{s,k}, \ k = 0, 1, \ldots \tag{3.16}$$

According to the Taylor expansion for a second degree polynomial we have

$$A_{s,k} = c_{s,k}\psi_s(t_k) = c_s\psi_s(t_{k-1} + A_{s,k-1}) =$$
$$= c_{s,k}\left[\frac{1}{2}\psi_s''(t_{k-1})A_{s,k-1}^2 + \psi_s'(t_{k-1})A_{s,k-1} + \psi_s(t_{k-1})\right] =$$
$$= c_{s,k}\left[B_s A_{s,k-1}^2 - \frac{A_{s,k-1}}{c_{s,k-1}} + \frac{A_{s,k-1}}{c_{s,k-1}}\right] = c_{s,k}B_s A_{s,k-1}^2 =$$
$$= \frac{1}{2}\frac{c_{s,k}}{c_{s,k-1}}2B_s c_{s,k-1}A_{s,k-1}^2 = \frac{1}{2}\frac{c_{s,k}}{c_{s,k-1}}B_{s,k-1}A_{s,k-1}^2$$

However,

$$\frac{c_{s,k}}{c_{s,k-1}} = \frac{\psi_s'(t_k)}{\psi_s(t_{k-1})} = \frac{\psi_s(t_k-1) + \psi_s'(t_k-1)A_{s,k-1}}{\psi_s(t_{k-1})} = 1 - B_{s,k-1}A_{s,k-1} = 1 - q_{s,k-1}. \tag{3.17}$$

Therefore,

$$A_{s,k} = \frac{1}{2}\frac{B_{s,k}A_{s,k-1}^2}{1-q_{s,k-1}} = \frac{A_{s,k-1}}{2}\frac{q_{s,k-1}}{1-q_{s,k-1}}. \tag{3.18}$$

Similarly, from (3.17) we obtain

$$B_{s,k} = 2c_{s,k}B_s = 2B_s c_{s,k-1}\frac{c_{s,k}}{c_{s,k-1}} = \frac{B_{s,k-1}}{1-q_{s,k-1}}.$$

It follows that

$$q_{s,k} = B_{s,k}A_{s,k} = \frac{1}{2}\frac{B_{s,k-1}A_{s,k-1}q_{s,k-1}}{(1-q_{s,k-1})^2} = \frac{1}{2}\left[\frac{q_{s,k-1}}{1-q_{s,k-1}}\right]^2. \tag{3.19}$$

From (3.18) and (3.19), taking into account $q_{s,k} \leq \frac{1}{2}$, we obtain the following estimations

$$A_{s,k} \leq q_{s,k-1}A_{s,k}, \ q_{s,k} \leq 2q_{s,k-1}^2 \ n = 1, 2, \ldots \tag{3.20}$$

Consequently, $q_{s,k} \leq \frac{1}{2}[2q_{s,0}]^{2^k} = \frac{1}{2}[2q_s]^{2^k}$

$$A_{s,k} \leq q_{s,k-1}A_{s,k-1} \leq q_{s,k-1}q_{s,k-2}A_{s,k-2} \leq \ldots q_{s,k-1}q_{s,k-2}\ldots q_{s,0}A_{s,0},$$

in which $q_{s,0} = q_s$ and $A_{s,0} = A_s$. From (3.15) and (3.16), we obtain

$$\|x_{k+1} - x_k\| + \|x_{k+2} - x_{k+1}\| + \cdots \leq (t_{k+1} - t_k) + (t_{k+2} - t_{k+1}) + \cdots \leq$$
$$\leq \frac{1}{2^k}[2q_s]^{2^k-1}A_s \leq \frac{1}{2^k}[2q_s]^{2^k}\frac{A_s}{q_s} \leq \frac{1}{2^k}[2q]^{2^k}C, \tag{3.21}$$

since $\frac{A_s}{q_s} = \frac{1}{B_s} \leq C$ and B_s is an upper bound for the norm of the second derivatives and can only increase as $s \to \infty$. Passing to the limit on $k = k(s) \to \infty$ in (3.21), we obtain inequality (3.13).

We will use the modified Newton method $\Delta x_k = -(\tilde{\Theta}_{s,0}'(x_0))^{-1}\tilde{\Theta}_{s,k}(x_k)$ for solving the equality $\Theta_s(x) = 0$. We denote the obtained sequence as $\{x_k'\}$. Suppose $\varphi_s(t) = t + c_0\psi_s(t)$, where $c_0 = -\frac{1}{\psi_s'(t_0)} = \frac{1}{2}$ and $t_0 = 0$. Let us replace the modified Newton method for the equation

$\psi_s(t) = 0$ with the equation $t = \varphi_s(t)$, and we will solve it by using the successive approximations method. Suppose $t_s^* = \varphi_s(t_s^*)$. We can write

$$t_s^* - t_k' = \varphi_s(t_s^*) - \varphi_s(t_{k-1}') = \varphi_s'(\tilde{t}_k)(t^* - t_{k-1}'),\ \tilde{t}_k = \frac{t^* + t_{k-1}'}{2}.$$

However $\varphi_s'(t) = 1 + c_o\psi_s'(t) = B_s t$, so that

$$\varphi'(\tilde{t}_k) = B_s\tilde{t}_k \le B_s t^* = 1 - \sqrt{1-2q_s}.$$

Therefore, $t_s^* - t_k' \le [1-\sqrt{1-2q_s}](t_s^* - t_{k-1}')$. We can obtain the similar inequality for $t_s^* - t_{k-1}'$. Consequently,

$$t_s^* - t_k' \le [1-\sqrt{1-2q_s}]^k(t_s^* - t_0') = \frac{A_s}{q_s}[1-\sqrt{1-2q_s}]^{k+1}.$$

The inequality $\|x_s^* - x_k'\| \le t_s^* - t_k'$, similar to inequality (3.15), is correct for the modified Newton's method. Using this inequality, we obtain

$$\|x_s^* - x_k'\| \le \frac{A_s}{q_s}[1-\sqrt{1-2q_s}]^{k+1}.$$

Passing to the limit in s and considering

$$\frac{A_s}{q_s} = \frac{1}{B_s} \le C,\ q_s \le q < \frac{1}{2},$$

in which C is a constant for all s, we obtain inequality (3.14). □

4. Conclusion

We proposed a method for finding equilibrium points as the limit points of a sequence obtained by applying the numerical method described above. The coordinate descent method slowly converges to an equilibrium point in the general case, but by changing the initial points, one can obtain all equilibrium points with minimal intermediate calculations. A method for finding Nash equilibrium points using the matrices of second mixed derivatives (generalized matrices of second mixed derivatives) of the original functions was suggested. Such methods, under certain conditions, converge much faster than the coordinate descent method, but require more calculations at each step.

To speed up the convergence of the method, it was proposed to decrease consistently the diameter of the set D_m on which the integration is performed, and the step length of the optimization process. We gave the rules for successive decrease of the diameter of the set D_m and the step length. The Kantorovich theorem was used to estimate the convergence rate.

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