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OPTIMALITY CONDITIONS AND LAGRANGE DUALITY FOR NONSMOOTH FRACTIONAL SEMI-INFINITE PROGRAMMING WITH VANISHING CONSTRAINTS

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Dedicated to Professor Francis Clarke on the occasion of his 75th birthday

Abstract. This paper considers the nonsmooth fractional semi-infinite programming problems with vanishing constraints. Using Clarke subdifferentials, we first obtain both necessary and sufficient Karush-Kuhn-Tucker optimality conditions for nonsmooth fractional semi-infinite programming problems with vanishing constraints. Then, the duality relations of types of Lagrange dual problems and saddle point optimality criteria are explored under generalized convexity assumptions.

Keywords. Clarke subdifferentials; Karush-Kuhn-Tucker optimality conditions; Nonsmooth fractional semi-infinite programming problems; Saddle point; Vanishing constraints;

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1. Introduction

Some problems in structural topology optimization in certain engineering applications could be reformulated by mathematical programming problems with vanishing constraints, see [1, 13]. For some recent results in this direction, see, e.g., the papers [14, 15, 17, 20]. In [11], the mathematical programming problem with vanishing constraints with an infinite number of constraints, or semi-infinite programming [10], was proposed and investigated the optimality conditions by employing the Clarke subdifferentials. The Karush-Kuhn-Tucker optimality conditions and Mond-Weir and Wolfe type duality for semi-infinite programming with vanishing constraints were also considered in [2, 23, 24, 25]. In addition, to deal with some optimization problems that involve ratio terms, fractional optimization problems were investigated, see e.g. [6, 7, 16, 22] and references therein. However, to the best of our knowledge, there is no paper dealing with the fractional semi-infinite programming problems with vanishing constraints. Moreover, Lagrange duality and saddle point criteria for nonsmooth semi-infinite programming problems with vanishing constraints have also not been investigated yet. It is well known that

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Lagrange duality may be easier to deal from algorithmic point of view rather than other dualities; see, e.g., [5, 21].

Motivated by the above observations, in this paper, we establish Karush-Kuhn-Tucker optimality conditions and investigate Lagrange duality problems for the nonsmooth fractional semi-infinite programming problems with vanishing constraints. The paper is organized as follows. The basic concepts and some preliminaries are recalled in Section 2. The KKT necessary and sufficient optimality conditions for the nonsmooth fractional semi-infinite programming problems with vanishing constraints in terms of Clarke subdifferentials are discussed in Section 3. Section 4 is devoted to delving into Lagrange dual problems and saddle point criteria of the nonsmooth fractional semi-infinite programming problems with vanishing constraints.

2. PRELIMINARIES

In this paper, the notation $\langle\cdot,\cdot\rangle$ is used to denote the inner product in the Euclidean space $\mathbb{R}^n$. By $B(x,\delta)$ we designate the open ball centered at x with radius $\delta>0$. For $A\subseteq\mathbb{R}^n$, int$A$, cl$A$, ∂A, span$A$ and co$A$ stand for its interior, closure, boundary, linear hull, convex hull of A, respectively. The cone and the convex cone (containing the origin) generated by A are denoted resp by cone$A$, pos$A$. It should be mentioned that, for the given sets A_1,A_2 in $\mathbb{R}^n$,

$$\mathrm{span}(A_1\cup A_2)=\mathrm{span}A_1+\mathrm{span}A_2 \text{ and } \mathrm{pos}(A_1\cup A_2)=\mathrm{pos}A_1+\mathrm{pos}A_2.$$

The negative polar cone, the strictly negative polar cone and the orthogonal complement of A are defined respectively by

$$A^-:=\{x^*\in\mathbb{R}^n|\langle x^*,x\rangle\le 0,\ \forall x\in A\},$$

$$A^s:=\{x^*\in\mathbb{R}^n|\langle x^*,x\rangle< 0,\ \forall x\in A\},$$

$$A^\perp:=\{x^*\in\mathbb{R}^n|\langle x^*,x\rangle= 0,\ \forall x\in A\}.$$

It is easy to check that $A^s\subset A^-$ and if $A^s\neq\emptyset$ then cl$A^s=A^-$. Moreover, the bipolar theorem (see, e.g., [3]) states that $A^{--}=$ cl pos$A$. For a given nonempty subset A of $\mathbb{R}^n$, the contingent cone [3] of A at $\bar{x}\in$ cl$A$ is

$$\mathscr{T}(A,\bar{x}):=\{x\in\mathbb{R}^n\mid \exists\tau_k\downarrow 0,\exists x_k\to x,\ \forall k\in\mathbb{N},\bar{x}+\tau_kx_k\in A\}.$$

Note that if A is a convex set then $\mathscr{T}(A,\bar{x})=\mathrm{clcone}(A-\bar{x})$. If $\langle x^*,x\rangle\ge 0(\langle x^*,x\rangle=0)$ for all $x^*\in A^*$, where A^* is a subset of the dual space of $\mathbb{R}^n$, we write $\langle A^*,x\rangle\ge 0$ ($\langle A^*,x\rangle=0$, resp.). The cardinality of the index set I is denoted by $|I|$. For an index subset $I\subset\{1,...,n\}$, $x_I=0(x_I\ge 0)$ stands for $x_i=0$ ($x_i\ge 0$, respectively) for all $i\in I$.

Definition 2.1. [8] Let $\bar{x}\in\mathbb{R}^n$ and $\phi:\mathbb{R}^n\to\mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of $\phi:\mathbb{R}^n\to\mathbb{R}$ at $\bar{x}$ in direction u is defined by

$$\phi^o(\bar{x},u):=\limsup_{\tau\downarrow 0,x\to\bar{x}}\frac{\phi(x+\tau u)-\phi(x)}{\tau}.$$

The Clarke subdifferential of ϕ at $\bar{x}$ is

$$\partial^C\phi(\bar{x}):=\{x^*\in\mathbb{R}^n|\langle x^*,d\rangle\le\phi^o(\bar{x},d),\ \forall d\in\mathbb{R}^n\}.$$

We recall the following properties from [8].

Lemma 2.2. *Let ϕ, ψ be the functions from $\mathbb{R}^n$ to $\mathbb{R}$, which are Lipschitz near $\bar{x}$. Then, the following assertions hold:*

(i) *The function $v \to \phi^o(\bar{x}, v)$ is finite, positively homogenous, subadditive on $\mathbb{R}^n$, $\phi^o(\bar{x}, 0) = 0$, $\phi^o(\bar{x}, v) = \max_{x^* \in \partial^C \phi(\bar{x})} \langle x^*, v\rangle$ and $\partial(\phi^o(\bar{x}, .))(0) = \partial^C \phi(\bar{x})$, where ∂ denotes the subdifferential in sense of convex analysis.*

(ii) *$\partial^C \phi(\bar{x})$ is a nonempty, convex and compact subset of $\mathbb{R}^n$.*

(iii) *$\partial^C(\lambda\phi)(\bar{x}) = \lambda \partial^C \phi(\bar{x}), \forall \lambda \in \mathbb{R}$ and $\partial^C(\phi + \psi)(\bar{x}) \subseteq \partial^C \phi(\bar{x}) + \partial^C \psi(\bar{x})$.*

(iv) *If ϕ is convex on $\mathbb{R}^n$ then $\partial^C \phi(\bar{x}) = \partial \phi(\bar{x})$. If ϕ is continously differentiable at $\bar{x}$, then $\partial^C \phi(\bar{x}) = \{\nabla \phi(\bar{x})\}$.*

(v) *$\partial^C \phi(\bar{x}) = \mathrm{co}\{x^* \in \mathbb{R}^n \mid \exists x_k \to \bar{x}, x_k \notin S \cup \Omega_\phi, \nabla \phi(x_k) \to x^*\}$, where S is any set of Lebesgue measure 0 in $\mathbb{R}^n$ and Ω_ϕ is the set of points at which a given function ϕ fails to be differentiable.*

(vi) *If ϕ is locally Lipschitz on an open set containing $[x, y]$, then $\phi(x) - \phi(y) = \langle x^*, y - x\rangle$, for some $c \in [x, y)$ and $x^* \in \partial^C \phi(c)$.*

In this paper, we consider the following nonsmooth fractional semi-infinite programming with vanishing constraints (P):

$$
\begin{aligned}
\min \quad & f(x) = \tfrac{u(x)}{v(x)} \\
\text{s.t.} \quad & g_t(x) \leq 0, t \in T, \\
& h_i(x) = 0, i = 1, ..., q, \\
& H_i(x) \geq 0, i = 1, ..., l, \\
& G_i(x) H_i(x) \leq 0, i = 1, ..., l,
\end{aligned}
$$

where u, v, $g_t(t \in T)$, $h_i(i = 1, ..., q)$ and $G_i, H_i(i = 1, ..., l)$ are Lipschitzian functions from $\mathbb{R}^n$ to $\mathbb{R}$. The index set T is an arbitrary nonempty set, not necessary finite. Let us denote $I_h := \{1, ..., q\}$ and $I_l := \{1, ..., l\}$. The feasible solution set of (P) is

$$\Omega := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0 (t \in T), h_i(x) = 0 (i \in I_h), H_i(x) \geq 0, G_i(x) H_i(x) \leq 0 (i \in I_l)\}.$$

We further assume that $v_i(x) > 0, i \in I$ for all $x \in \mathbb{R}^n$, and that $u_i(\bar{x}) \leq 0$ for the reference point $\bar{x} \in \Omega$. The point $\bar{x}$ is a locally solution of (P), denoted by $\bar{x} \in \mathrm{locS(P)}$, if there exists a neighborhood $U \in \mathscr{U}(\bar{x})$ such that $\frac{u(x)}{v(x)} \geq \frac{u(\bar{x})}{v(\bar{x})}, \forall x \in \Omega \cap U$. If $U = \mathbb{R}^n$, the word "locally" is omitted.

The notation $\mathbb{R}^{|T|}_+$ signifies the collection of all the functions $\lambda : T \to \mathbb{R}$ taking values λ_t's positive only at finitely many points of T, and equal to zero at the other points. For a given $\bar{x} \in \Omega$, $I_g(\bar{x}) := \{t \in T | g_t(\bar{x}) = 0\}$ indicates the index set of all active constraints at $\bar{x}$. The set of active constraint multipliers at $\bar{x} \in \Omega$ is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}^{|T|}_+ | \lambda_t g_t(\bar{x}) = 0, \forall t \in T\}.$$

Notice that $\lambda \in \Lambda(\bar{x})$ if there exists a finite index set $J \subset I_g(\bar{x})$ such that $\lambda_t > 0$ for all $t \in J$ and $\lambda_t = 0$ for all $t \in T \setminus J$. For each $\bar{x} \in \Omega$, define

$$I_+(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) > 0\}, I_0(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) = 0\},$$

$$I_{+0}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) > 0, G_i(\bar{x}) = 0\},$$

$$I_{+-}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) > 0, G_i(\bar{x}) < 0\},$$

$$I_{0+}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) = 0, G_i(\bar{x}) > 0\},$$

$$I_{00}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) = 0, G_i(\bar{x}) = 0\},$$
$$I_{0-}(\bar{x}) := \{i \in I_l \mid H_i(\bar{x}) = 0, G_i(\bar{x}) < 0\}.$$

Definition 2.3. Let $\bar{x} \in \Omega$.
(i) The point $\bar{x}$ is called a strong stationary point of (P) iff there exists $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \times\Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\lambda^H_{I_+(\bar{x})} = 0$, $\lambda^H_{I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} \geq 0$, $\lambda^G_{I_{+-}(\bar{x}) \cup I_{0+}(\bar{x}) \cup I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} = 0$ and $\lambda^G_{I_{+0}(\bar{x})} \geq 0$ such that

$$0 \in \partial^C u(\bar{x}) - \frac{u(\bar{x})}{v(\bar{x})} \partial^C v(\bar{x}) + \sum_{t \in T} \lambda^g_t \partial^C g_t(\bar{x}) + \sum_{i \in I_h} \lambda^h_i \partial^C h_i(\bar{x}) - \sum_{i \in I_l} \lambda^H_i \partial^C H_i(\bar{x}) + \sum_{i \in I_l} \lambda^G_i \partial^C G_i(\bar{x}).$$

(ii) The point $\bar{x}$ is said to be a VC-stationary point of (P) iff there exists $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^m_+ \times \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\lambda^H_{I_+(\bar{x})} = 0$, $\lambda^H_{I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} \geq 0$, $\lambda^G_{I_{+-}(\bar{x}) \cup I_{0+}(\bar{x}) \cup I_{0-}(\bar{x})} = 0$ and $\lambda^G_{I_{+0}(\bar{x}) \cup I_{00}(\bar{x})} \geq 0$ satisfying

$$0 \in \partial^C u(\bar{x}) - \frac{u(\bar{x})}{v(\bar{x})} \partial^C v(\bar{x}) + \sum_{t \in T} \lambda^g_t \partial^C g_t(\bar{x}) + \sum_{i \in I_h} \lambda^h_i \partial^C h_i(\bar{x}) - \sum_{i \in I_l} \lambda^H_i \partial^C H_i(\bar{x}) + \sum_{i \in I_l} \lambda^G_i \partial^C G_i(\bar{x}).$$

It is easy to see that if $\bar{x} \in \Omega$ is a strong stationary point of (P) then $\bar{x}$ is a VC-stationary point of (P).

For $\bar{x} \in \Omega$ and $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{|T|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$, let us define

$$I^+_g(\bar{x}) := \{t \in I_g(\bar{x}) \mid \lambda^g_t > 0\},$$
$$I^+_h(\bar{x}) := \{i \in I_h(\bar{x}) \mid \lambda^h_i > 0\}, I^-_h(\bar{x}) := \{i \in I_h(\bar{x}) \mid \lambda^h_i < 0\},$$

$$\hat{I}^+_+(\bar{x}) := \{i \in I_+(\bar{x}) \mid \lambda^H_i > 0\},$$
$$\hat{I}^+_0(\bar{x}) := \{i \in I_0(\bar{x}) \mid \lambda^H_i > 0\}, \hat{I}^-_0(\bar{x}) := \{i \in I_0(\bar{x}) \mid \lambda^H_i < 0\},$$
$$\hat{I}^+_{0+}(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda^H_i > 0\}, \hat{I}^-_{0+}(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda^H_i < 0\},$$
$$\hat{I}^+_{00}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda^H_i > 0\}, \hat{I}^-_{00}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda^H_i < 0\},$$
$$\hat{I}^+_{0-}(\bar{x}) := \{i \in I_{0-}(\bar{x}) \mid \lambda^H_i > 0\},$$

$$I^+_{+0}(\bar{x}) := \{i \in I_{+0}(\bar{x}) \mid \lambda^G_i > 0\}, I^-_{+0}(\bar{x}) := \{i \in I_{+0}(\bar{x}) \mid \lambda^G_i < 0\},$$
$$I^+_{+-}(\bar{x}) := \{i \in I_{+-}(\bar{x}) \mid \lambda^G_i > 0\},$$
$$I^+_{0+}(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda^G_i > 0\}, I^-_{0+}(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda^G_i < 0\},$$
$$I^+_{00}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda^G_i > 0\}, I^-_{00}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda^G_i < 0\},$$
$$I^+_{0-}(\bar{x}) := \{i \in I_{0-}(\bar{x}) \mid \lambda^G_i > 0\}.$$

Definition 2.4. (see [4, 18]) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, $\Omega \subset \mathbb{R}^n$ and $\bar{x} \in \Omega$.

(i) ϕ is said to be ∂^C-convex at $\bar{x}$ on Ω if, for all $x \in \Omega$, $\phi(x) - \phi(\bar{x}) \geq \langle \partial^C \phi(\bar{x}), x - \bar{x} \rangle$.

(ii) ϕ is called strictly ∂^C-convex at $\bar{x}$ on Ω if, for all $x \in \Omega \setminus \{\bar{x}\}$,
$$\phi(x) - \phi(\bar{x}) > \langle \partial^C \phi(\bar{x}), x - \bar{x} \rangle.$$

(iii) ϕ is termed ∂^C-pseudoconvex at $\bar{x}$ on Ω if for any $x \in \Omega$,
$$\phi(x) - \phi(\bar{x}) < 0 \Rightarrow \langle \partial^C \phi(\bar{x}), x - \bar{x} \rangle < 0.$$

(iv) ϕ is declared strictly ∂^C-pseudoconvex at $\bar{x}$ on Ω if for all $x \in \Omega \setminus \{\bar{x}\}$,

$$\phi(x) - \phi(\bar{x}) \leq 0 \Rightarrow \langle \partial^C \phi(\bar{x}), x - \bar{x} \rangle < 0.$$

(v) ϕ is called ∂^C-quasiconvex at $\bar{x}$ on Ω if for any $x \in \Omega$,

$$\phi(x) - \phi(\bar{x}) \leq 0 \Rightarrow \langle \partial^C \phi(\bar{x}), x - \bar{x} \rangle \leq 0.$$

Lemma 2.5. [19] *Let $\{C_t | t \in \Gamma\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and $K = \operatorname{pos}\left(\bigcup_{t\in\Gamma} C_t\right)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different C_t.*

Lemma 2.6. [10] *Suppose that S,T,P are arbitrary (possibly infinite) index sets, $a_s = a(s) = (a_1(s),\ldots,a_n(s))$ maps S onto $\mathbb{R}^n$, and so do a_t and a_p. Suppose that the set $\operatorname{co}\{a_s, s \in S\} + \operatorname{pos}\{a_t, t \in T\} + \operatorname{span}\{a_p, p \in P\}$ is closed. Then the following statements are equivalent:*

$$I: \quad \begin{cases} \langle a_s, x\rangle < 0, s \in S, S \neq \emptyset \\ \langle a_t, x\rangle \leq 0, t \in T \\ \langle a_p, x\rangle = 0, p \in P \end{cases} \quad \textit{has no solution } x \in \mathbb{R}^n;$$

$$II: \quad 0 \in \operatorname{co}\{a_s, s \in S\} + \operatorname{pos}\{a_t, t \in T\} + \operatorname{span}\{a_p, p \in P\}.$$

Lemma 2.7. [12] *If A is a nonempty compact subset of $\mathbb{R}^n$, then*

(i) $\operatorname{co}A$ *is a compact set;*
(ii) *if* $0 \notin \operatorname{co}A$, *then* $\operatorname{pos}A$ *is a closed cone.*

3. Karush-Kuhn-Tucker optimality conditions

In this section, both KKT necessary and sufficient optimality conditions for the nonsmooth fractional semi-infinite programming with vanishing constraints are established. We write the index set I_g instead of $I_g(\bar{x})$ for the convenience. The other index sets are wrote similarly. Firstly, we present the following constraint qualifications, which are similar to Abadie constraint qualification in the literature:
(i) (ACQ) holds at $\bar{x} \in \Omega$ if

$$(\bigcup_{t\in I_g} \partial^C g_t(\bar{x}))^- \cap (\bigcup_{i\in I_h} \partial^C h_i(\bar{x}))^\perp \cap (\bigcup_{i\in I_{0+}} \partial^C H_i(\bar{x}))^\perp \cap (\bigcup_{i\in I_{00}\cup I_{0-}} -\partial^C H_i(\bar{x}))^- \cap (\bigcup_{i\in I_{+0}} \partial^C G_i(\bar{x}))^-$$
$$\subseteq \mathscr{T}(\Omega,\bar{x}),$$

(ii) (VC-ACQ) holds at $\bar{x} \in \Omega$ if

$$(\bigcup_{t\in I_g} \partial^C g_t(\bar{x}))^- \cap (\bigcup_{i\in I_h} \partial^C h_i(\bar{x}))^\perp \cap (\bigcup_{i\in I_{0+}} \partial^C H_i(\bar{x}))^\perp \cap (\bigcup_{i\in I_{00}\cup I_{0-}} -\partial^C H_i(\bar{x}))^- \cap (\bigcup_{i\in I_{+0}\cup I_{00}} \partial^C G_i(\bar{x}))^-$$
$$\subseteq \mathscr{T}(\Omega,\bar{x}).$$

It is straightforward that (ACQ) implies (VC-ACQ).

Proposition 3.1. *Let $\bar{x} \in \operatorname{locS(P)}$.*

(i) If (ACQ) holds at $\bar{x}$ and the set

$$\Delta := \operatorname{pos}\left(\bigcup_{t\in I_g}\partial^C g_t(\bar{x})\cup\bigcup_{i\in I_{00}\cup I_{0-}}(-\partial^C H_i(\bar{x}))\cup\bigcup_{i\in I_{+0}}\partial^C G_i(\bar{x})\right)$$
$$+\operatorname{span}\left(\bigcup_{i\in I_h}\partial^C h_i(\bar{x})\cup\bigcup_{i\in I_{0+}}\partial^C H_i(\bar{x})\right)$$

is closed, then $\bar{x}$ is a strong stationary point of (P).
(ii) If (VC-ACQ) holds at $\bar{x}$ and the set

$$\Delta_1 := \operatorname{pos}\left(\bigcup_{t\in I_g}\partial^C g_t(\bar{x})\cup\bigcup_{i\in I_{00}\cup I_{0-}}(-\partial^C H_i(\bar{x}))\cup\bigcup_{i\in I_{+0}\cup I_{00}}\partial^C G_i(\bar{x})\right)$$
$$+\operatorname{span}\left(\bigcup_{i\in I_h}\partial^C h_i(\bar{x})\cup\bigcup_{i\in I_{0+}}\partial^C H_i(\bar{x})\right)$$

is closed, then $\bar{x}$ is a VC-stationary point of (P).

Proof. Owing to the similarity, we only prove (ii). As $\bar{x}\in \operatorname{locS(P)}$, there exists $U\in\mathscr{U}(\bar{x})$ such that

$$\frac{u(x)}{v(x)}\geq\frac{u(\bar{x})}{v(\bar{x})},\forall x\in\Omega\cap U. \tag{3.1}$$

This together with $v(x)>0$ implies that

$$u(x)-\frac{u(\bar{x})}{v(\bar{x})}v(x)\geq 0=u(\bar{x})-\frac{u(\bar{x})}{v(\bar{x})}v(\bar{x}),\forall x\in\Omega\cap U. \tag{3.2}$$

Hence, $\bar{x}$ is the solution of the following nonsmooth semi-infinite programming with vanishing constraints

$$\begin{array}{lll}(P_1):\min & u(x)-\frac{u(\bar{x})}{v(\bar{x})}v(x)\\ \quad\text{s.t.} & x\in\Omega.\end{array}$$

Applying Theorem 3.1 in [25], there exists $(\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\Lambda(\bar{x})\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\lambda^H_{I_+(\bar{x})}=0$, $\lambda^H_{I_{00}(\bar{x})\cup I_{0-}(\bar{x})}\geq 0$, $\lambda^G_{I_{+-}(\bar{x})\cup I_{0+}(\bar{x})\cup I_{0-}(\bar{x})}=0$ and $\lambda^G_{I_{+0}(\bar{x})\cup I_{00}(\bar{x})}\geq 0$ satisfying

$$0\in\partial^C\left(u-\frac{u(\bar{x})}{v(\bar{x})}.v\right)(\bar{x})+\sum_{t\in T}\lambda^g_t\partial^C g_t(\bar{x})+\sum_{i\in I_h}\lambda^h_i\partial^C h_i(\bar{x})-\sum_{i\in I_l}\lambda^H_i\partial^C H_i(\bar{x})+\sum_{i\in I_l}\lambda^G_i\partial^C G_i(\bar{x}).$$

It follows from Lemma 2.6 (iii) that

$$\partial^C\left(u-\frac{u(\bar{x})}{v(\bar{x})}.v\right)(\bar{x})\subset\partial^C u(\bar{x})-\frac{u(\bar{x})}{v(\bar{x})}.\partial^C v(\bar{x}),$$

and hence, the conclusion is obtained. □

Proposition 3.2. *Let $\bar{x}$ be a strong stationary point of (P). Suppose that $\hat{I}^-_{0+}\cup I^+_{+0}=\emptyset$ and $g_t(t\in I_g), h_i(i\in I^+_h), -h_i(i\in I^-_h), -H_i(i\in\hat{I}^+_{0+}\cup\hat{I}^+_{00}\cup\hat{I}^+_{0-})$ are ∂^C-quasiconvex at $\bar{x}$ on Ω. If $u-\frac{u(\bar{x})}{v(\bar{x})}v$ are ∂^C-pseudoconvex at $\bar{x}$ on Ω, then $\bar{x}$ is a solution of (P).*

Proof. Since $\bar{x}$ is a strong stationary point of (P), there exists $(\lambda_J^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^{|J|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$, where J is a finite subset of I_g, with $\lambda_{I_+}^H = 0$, $\lambda_{I_{00} \cup I_{0-}}^H \geq 0$, $\lambda_{I_{+-} \cup I_{0+} \cup I_{00} \cup I_{0-}}^G = 0$, $\lambda_{I_{+0}}^G \geq 0$ and $\xi^u \in \partial^C u(\bar{x}), \xi^v \in \partial^C v(\bar{x}), \xi_t^g \in \partial^C g_i(\bar{x})(t \in J), \xi_i^h \in \partial^C h_i(\bar{x})(i \in I_h), \xi_i^H \in \partial^C H_i(\bar{x})(i \in I_{0+} \cup I_{00} \cup I_{0-}), \xi_i^G \in \partial^C G_i(\bar{x})(i \in I_{+0})$ such that

$$\xi^u - \frac{u(\bar{x})}{v(\bar{x})} \xi^v + \sum_{t \in J} \lambda_t^g \xi_t^g + \sum_{i \in I_h} \lambda_i^h \xi_i^h - \sum_{i \in I_{0+} \cup I_{00} \cup I_{0-}} \lambda_i^H \xi_i^H + \sum_{i \in I_{+0}} \lambda_i^G \xi_i^G = 0. \tag{3.3}$$

For an arbitrary $x \in \Omega$, one gets that $g_t(x) \leq 0 = g_t(\bar{x})$ for each $t \in I_g$. Thus, the ∂^C-quasiconvexity at $\bar{x}$ on Ω of $g_t(t \in I_g)$ give us that $\langle \xi_t^g, x - \bar{x} \rangle \leq 0$ for all $t \in J$, which in turn together with $\lambda_J^g \in \mathbb{R}_+^{|J|}$ leads that

$$\left\langle \sum_{t \in J} \lambda_t^g \xi_t^g, x - \bar{x} \right\rangle \leq 0. \tag{3.4}$$

We deduce from $x, \bar{x} \in \Omega$ that $h_i(x) = h_i(\bar{x}) = 0, \forall i \in I_h$, and hence,

$$h_i(x) \leq h_i(\bar{x}), \forall i \in I_h^+ \text{ and } -h_i(x) \leq -h(\bar{x}), \forall i \in I_h^-.$$

We deduce from the above inequalities, the ∂^C-quasiconvexity at $\bar{x}$ on Ω of $h_i(i \in I_h^+)$ and $-h_i(i \in I_h^-)$ and $\partial^C(-h_i)(\bar{x}) = -\partial^C h_i(\bar{x})(i \in I_h^-)$ that

$$\langle \xi_i^h, x - \bar{x} \rangle \leq 0, \forall i \in I_h^+ \quad \text{and} \quad \langle -\xi_i^h, x - \bar{x} \rangle \leq 0, \forall i \in I_h^-.$$

This, taking into account the definitions of I_h^+, I_h^-, results in

$$\left\langle \sum_{i \in I_h} \lambda_i^h \xi_i^h, x - \bar{x} \right\rangle \leq 0. \tag{3.5}$$

Again, we derive from $x \in \Omega$ that $-H_i(x) \leq 0, \forall i \in I_l$, and thus, $-H_i(x) \leq -H_i(\bar{x}), i \in \hat{I}_{0+}^+ \cup \hat{I}_{00}^+ \cup \hat{I}_{0-}^+$. Therefore, by the ∂^C-quasiconvexity of $-H_i, i \in \hat{I}_{0+}^+ \cup \hat{I}_{00}^+ \cup \hat{I}_{0-}^+$ at $\bar{x}$ on Ω, one yields that

$$\langle -\xi_i^H, x - \bar{x} \rangle \leq 0, \forall i \in \hat{I}_{0+}^+ \cup \hat{I}_{00}^+ \cup \hat{I}_{0-}^+. \tag{3.6}$$

As $I_{+0}^+ \cup \hat{I}_{0+}^- = \emptyset$, we infer from (3.3) - (3.6) that

$$\begin{aligned} &\left\langle \xi^u - \frac{u(\bar{x})}{v(\bar{x})} \xi^v, x - \bar{x} \right\rangle \\ &= -\left\langle \sum_{t \in T} \lambda_t^g \xi_t^g + \sum_{i \in I_h} \lambda_i^h \xi_i^h - \sum_{i \in I_{0+} \cup I_{00} \cup I_{0-}} \lambda_i^H \xi_i^H + \sum_{i \in I_{+0}} \lambda_i^G \xi_i^G, x - \bar{x} \right\rangle \geq 0, \end{aligned} \tag{3.7}$$

for all $x \in \Omega$.

Suppose to the contrary that $\bar{x}$ is not a solution of (P). This amounts to the existence of a feasible point $\tilde{x} \in \Omega$ such that

$$\frac{u(\tilde{x})}{v(\tilde{x})} < \frac{u(\bar{x})}{v(\bar{x})},$$

or equivalently,

$$u(\tilde{x}) - \frac{u(\bar{x})}{v(\bar{x})} v(\tilde{x}) < u(\bar{x}) - \frac{u(\bar{x})}{v(\bar{x})} v(\bar{x}).$$

This together with the ∂^C-pseudoconvexity at $\bar{x}$ on Ω of $u-\frac{u(\bar{x})}{v(\bar{x})}v$ give us the conclusion

$$\left\langle \xi^u - \frac{u(\bar{x})}{v(\bar{x})}\xi^v, x-\bar{x}\right\rangle < 0,$$

contradicting with (3.7). □

Proposition 3.3. *Let $\bar{x}$ be a VC-stationary point of (P). Assume that $\hat{I}^-_{0+}\cup I^+_{+0}\cup I^+_{00}=\emptyset$ and $g_t(t\in I_g), h_i(i\in I^+_h), -h_i(i\in I^-_h), -H_i(i\in \hat{I}^+_{0+}\cup\hat{I}^+_{00}\cup\hat{I}^+_{0-})$ are ∂^C-quasiconvex at $\bar{x}$ on Ω. If $u-\frac{u(\bar{x})}{v(\bar{x})}v$ are ∂^C-pseudoconvex at $\bar{x}$ on Ω, then $\bar{x}$ is a solution of (P).*

Proof. The proof is analogous to those in Proposition 3.2. □

Example 3.4. Consider the following (P):

$$\begin{aligned} \min \quad & f(x)=\tfrac{|x_1|+x_2^2}{x_2^2+1},\\ \text{s.t.} \quad & g_t(x)=-tx_1\le 0, t\in T=\mathbb{N},\\ & H_1(x)=x_1^3+3x_2\ge 0,\\ & G_1(x)H_1(x)=|x_1|(x_1^3+3x_2)\le 0.\end{aligned}$$

Then, $\Omega=\{x\in\mathbb{R}^2 \mid x_1>0, x_1^3+3x_2=0\}\cup\{x\in\mathbb{R}^2\mid x_1=0,x_2\ge 0\}$. For $\bar{x}=(0,0)\in\Omega$, direct calculations give that

$$\mathscr{T}(\Omega,\bar{x})=\{x\in\mathbb{R}^2\mid x_1\ge 0, x_2=0\}\cup\{x\in\mathbb{R}^2\mid x_1=0,x_2\ge 0\},$$

$$u(\bar{x})=0, v(\bar{x})=1, u(x)-\frac{u(\bar{x})}{v(\bar{x})}.v(x)=u(x),$$

$$\partial^C u(\bar{x})=[-1,1]\times\{0\}, \partial^C v(\bar{x})=\{(0,0)\},$$

$$I_g=\mathbb{N}, \partial^C g_t(\bar{x})=\{(-t,0)\}, t\in T,$$

$$I_+=I_{0+}=I_{0-}=\emptyset, I_{00}=\{1\}, \partial^C G_1(\bar{x})=[-1,1]\times\{0\}, \partial^C H_1(\bar{x})=\{(0,3)\},$$

$$(\bigcup_{t\in I_g}\partial^C g_t(\bar{x}))^-=\{x\in\mathbb{R}^2\mid x_1\ge 0\},$$

$$(\bigcup_{i\in I_{00}}(-\partial^C H_i(\bar{x}))^-=(-\partial^C H_1(\bar{x}))^-=\{x\in\mathbb{R}^2\mid x_2\ge 0\},$$

$$(\bigcup_{i\in I_{00}}\partial^C G_i(\bar{x}))^-=(\partial^C G_1(\bar{x}))^-=\{x\in\mathbb{R}^2\mid x_1=0\},$$

$$(\bigcup_{t\in I_g}\partial^C g_t(\bar{x}))^-\cap(\bigcup_{i\in I_{00}}(-\partial^C H_i(\bar{x})))^-\cap(\bigcup_{i\in I_{00}}\partial^C G_i(\bar{x}))^-=\{x\in\mathbb{R}^2\mid x_1=0,x_2\ge 0\}.$$

Hence,

$$(\bigcup_{t\in I_g}\partial^C g_t(\bar{x}))^-\cap(\bigcup_{i\in I_{00}}(-\partial^C H_i(\bar{x})))^-\cap(\bigcup_{i\in I_{00}}\partial^C G_i(\bar{x}))^-\subset\mathscr{T}(\Omega,\bar{x}).$$

Thus, (VC-ACQ) holds at $\bar{x}$. Moreover,

$$\Delta_1=\text{pos}\left(\bigcup_{t\in I_g}\partial^C g_t(\bar{x})\cup\bigcup_{i\in I_{00}}(-\partial^C H_i(\bar{x}))\cup\bigcup_{i\in I_{00}}\partial^C G_i(\bar{x})\right)=\{x\in\mathbb{R}^2\mid x_2\le 0\}$$

is closed. Because of the fact $\frac{u(x)}{v(x)} = \frac{|x_1|+x_2^2}{x_2^2+1} \geq 0 = \frac{u(\bar{x})}{v(\bar{x})}$, $\forall x \in \Omega$, we assert that $\bar{x} \in S(P)$. Thus, all assumptions in Proposition 3.1 (ii) are fulfilled. Hence, we deduce from the conclusion f Proposition 3.1 (ii) that $\bar{x}$ is a VC-stationary point of (P).

Now, we can check directly that $\bar{x}$ is a VC-stationary point of (P) as follows. Let $\lambda_1^H = 0$, $\lambda_1^G = 0$ and $\lambda^g : T \to \mathbb{R}$ be defined by

$$\lambda^g(t) = \begin{cases} \frac{1}{2}, & \text{if } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} (0,0) &= \left(\frac{1}{2}(0,0) - \frac{0}{1}.(0,0)\right) + \frac{1}{2}(1,0) + \frac{1}{2}.(-1,0) - 0.(0,3) + 0.(1,0) \\ &\in \left([-1,1]\times\{0\} - \frac{0}{1}.(0,0)\right) + \sum_{t\in T} \lambda_t^g(-t,0) - \lambda_1^H(0,1) + \lambda_1^G.[-1,1]\times\{0\}, \end{aligned}$$

which means that $\bar{x}$ is a VC-stationary point of (P).

Furthermore, $\hat{I}_{00}^+ = \hat{I}_{00}^- = I_{00}^- = I_{00}^+ = \emptyset$. We can check that $u - \frac{u(\bar{x})}{v(\bar{x})}v = u, g_t(t \in I_g)$ are ∂^C-convex at $\bar{x}$ on Ω. For instance, since

$$u(x) - u(\bar{x}) = |x_1| + x_2^2 \geq \{\beta x_1, \beta \in [-1,1]\} = \langle \partial^C u(\bar{x}), x - \bar{x}\rangle, \forall x \in \Omega,$$

$u = u - \frac{u(\bar{x})}{v(\bar{x})}v$ is ∂^C-convex at $\bar{x}$ on Ω. Hence, all assumptions in Proposition 3.3 (i) are satisfied. Then, it follows that $\bar{x}$ is a solution of (P).

4. Lagrange Duality and Saddle Point Criteria

In this section, we consider the Lagrange duality schemes and saddle point optimality criteria for (P). For a fixed $s \geq 0$, in the line of [9], we could associate (P) with the semi-infinite programming with vanishing constraints $(P)_s$ as follows:

$$\begin{aligned} (P)_s : \min \quad & u(x) + sv(x) \\ \text{s.t.} \quad & x \in \Omega. \end{aligned}$$

Proposition 4.1. *Let $\bar{x} \in \Omega$.*

(i) [9] $\bar{s} = \frac{u(\bar{x})}{v(\bar{x})} = \min\{\frac{u(x)}{v(x)}, x \in \Omega\}$ *if and only if* $F_{\bar{s}}(\bar{x}) = \min\{u(x) - \bar{s}v(x), x \in \Omega\} = 0$.

(ii) The point $\bar{x}$ is a (local) solution of $(P)_s$ if and only if $\bar{x}$ is a (local) solution of (P).

Proof. (ii) Let $\bar{x} \in \Omega$ be a local solution of $(P)_s$. Then, there exists $U \in \mathscr{U}(\bar{x})$ such that

$$u(x) - \frac{u(\bar{x})}{v(\bar{x})}v(x) \geq u(\bar{x}) - \frac{u(\bar{x})}{v(\bar{x})}v(\bar{x}), \forall x \in \Omega \cap U.$$

This leads that

$$\frac{u(x)}{v(x)} \geq \frac{u(\bar{x})}{v(\bar{x})}, \forall x \in \Omega \cap U,$$

i.e. $\bar{x}$ is a local solution of $(P)_s$. The conversion is proved analogously. ∎

4.1. **Lagrange duality.** For a fixed $s \geq 0$, $x \in \Omega$ and $\lambda = (\lambda^g, \lambda^h, \lambda^H, \lambda^G) \in \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$, we define

$$L_s(x,\lambda) = u(x) + sv(x) + \sum_{t \in T} \lambda_t^g g_t(x) + \sum_{i \in I_h} \lambda_i^h h_i(x) - \sum_{i \in I_l} \lambda_i^H H_i(x) + \sum_{i \in I_l} \lambda_i^G G_i(x),$$

and $\varphi_s(\lambda) = \min_{x \in \Omega} L_s(x,\lambda)$. We propose the following Lagrange; see, e.g., [5], type dual model depending on $x \in \Omega$ for $(P)_s$:

$$\begin{aligned} D_s(x): \quad & \max \varphi_s(\lambda) \\ \text{s.t.} \quad & \lambda^g_{T \setminus I_g(x)} \geq 0, \lambda^H_{I_+(x)} \geq 0, \\ & \lambda^G_{I_{+-}(x) \cup I_{0-}(x)} \geq 0, \lambda^G_{I_{0+}(x)} \leq 0. \end{aligned}$$

Denote $\Omega_{D_s(x)}$ the feasible region of the $D_s(x)$. Now, we consider the Lagrange type duality which is independent on a feasible point of $(P)_s$ as follows:

$$\begin{aligned} (D_s): \quad & \max \varphi_s(\lambda) \\ \text{s.t.} \quad & \lambda \in \Omega_{D_s} := \cap_{x \in \Omega} \Omega_{D_s(x)}, \end{aligned}$$

where $\Omega_{D_s} = \cap_{x \in \Omega} \Omega_{D_s(x)} \neq \emptyset$ is the feasible set of the (D_s).

Proposition 4.2. *(Weak duality)*

(i) If x is a feasible point of $(P)_s$ and λ is a feasible point of $D_s(x)$, then $\varphi_s(\lambda) \leq u(x) + sv(x)$.

(ii) If x is a feasible point of $(P)_s$ and λ is a feasible point of D_s, then $\varphi_s(\lambda) \leq u(x) + sv(x)$.

Proof. (i) Since $\varphi_s(\lambda) = \min_{x \in \Omega} L_s(x,\lambda)$, one has, for all $x \in \Omega$,

$$\varphi_s(\lambda) \leq u(x) + sv(x) + \sum_{t \in T} \lambda_i^g g_t(x) + \sum_{i \in I_h} \lambda_i^h h_i(x) - \sum_{i \in I_l} \lambda_i^H H_i(x) + \sum_{i \in I_l} \lambda_i^G G_i(x). \tag{4.1}$$

We deduce from $x \in \Omega$ that $g_t(x) \leq 0 (t \in T)$, $h_i(x) = 0 (i \in I_h)$, $-H_i(x) \leq 0 (i \in I_l)$, $G_i(x) H_i(x) \leq 0 (i \in I_l)$. Hence,

$$\sum_{t \in I_g(x)} \lambda_t^g g_t(x) = 0 \text{ and } g_t(x) < 0, \lambda_t^g \geq 0, \forall t \in T \setminus I_g(x),$$

$$\sum_{i \in I_0(x)} \lambda_i^H H_i(x) = 0 \text{ and } -H_i(x) < 0, \lambda_i^H \geq 0, \forall i \in I_+(x),$$

$$\sum_{i \in I_{+0}(x) \cup I_{00}(x)} \lambda_i^G G_i(x) = 0 \text{ and } G_i(x) > 0, \lambda_i^G \leq 0, \forall i \in I_{0+}(x),$$

$$G_i(x) < 0, \lambda_i^G \geq 0, \forall i \in I_{+-}(x) \cup I_{0-}(x).$$

The above inequalities imply that

$$\sum_{t \in T} \lambda_t^g g_t(x) + \sum_{i \in I_h} \lambda_i^h h_i(x) - \sum_{i \in I_l} \lambda_i^H H_i(x) + \sum_{i \in I_l} \lambda_i^G G_i(x) \leq 0. \tag{4.2}$$

This, together with (4.1), leads that $\varphi_s(\lambda) \leq u(x) + sv(x)$, which completes the proof.
(ii) The conclusion follows from $\Omega_{D_s} = \cap_{x \in \Omega} \Omega_{D_s(x)}$. $\square$

Corollary 4.3. *Let $\bar{s} = -\frac{u(\bar{x})}{v(\bar{x})}$. If $\bar{x}$ and $\bar{\lambda}$ are feasible points for $(P)_{\bar{s}}$ and $D_{\bar{s}}(\bar{x})$, resp, and $u(\bar{x}) + \bar{s}v(\bar{x}) = \varphi_{\bar{s}}(\bar{\lambda})$, then $\bar{x}$ and $\bar{\lambda}$ are optimal solutions for $(P)_{\bar{s}}$ and $D_{\bar{s}}(\bar{x})$, resp. Thus, $\bar{x}$ is also an optimal solution for (P).*

Proposition 4.4. *(Strong duality) Let $\bar{x}$ be a local optimal solution of (P) such that the (VC-ACQ) holds at $\bar{x}$ and Δ_1 is closed. If $u, v, g_t(t \in I_g^+(\bar{x})), h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x})), H_i(i \in \hat{I}_{0+}^-(\bar{x})), -H_i(i \in \hat{I}_{0+}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x})) \cup \hat{I}_{0-}^+(\bar{x})), G_i(i \in I_{+0}^+(\bar{x}) \cup I_{00}^+(\bar{x}))$ are ∂^C-convex at $\bar{x}$, then there exists $\bar{\lambda}$ such that $\bar{\lambda}$ is a solution of $D_{\bar{s}}(\bar{x})$ and $u(\bar{x}) + \bar{s}v(\bar{x}) = \varphi_{\bar{s}}(\bar{\lambda})$, where $\bar{s} = -\frac{u(\bar{x})}{v(\bar{x})}$.*

Proof. It follows from Proposition 4.1 that $\bar{x}$ is also a solution of $(P)_{\bar{s}}$, where $\bar{s} = -\frac{u(\bar{x})}{v(\bar{x})}$. We derive from the assumptions and Proposition 3.1 that there exists $(\bar{\lambda}_J^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}_+^{|J|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$, where J is a finite subset of $I_g(\bar{x})$, with $\bar{\lambda}^H_{I_+(\bar{x})} = 0$, $\bar{\lambda}^H_{I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} \geq 0$, $\bar{\lambda}^G_{I_{+-}(\bar{x}) \cup I_{0+}(\bar{x}) \cup I_{0-}(\bar{x})} = 0$, $\bar{\lambda}^G_{I_{+0}(\bar{x}) \cup I_{00}(\bar{x})} \geq 0$ and $\xi^u \in \partial^C u(\bar{x}), \xi^v \in \partial^C v(\bar{x}), \xi_t^g \in \partial^C g_i(\bar{x})(t \in J), \xi_i^h \in \partial^C h_i(\bar{x})(i \in I_h), \xi_i^H \in \partial^C H_i(\bar{x})(i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x}) \cup I_{0-}(\bar{x})), \xi_i^G \in \partial^C G_i(\bar{x})(i \in I_{+0}(\bar{x}) \cup I_{00}(\bar{x}))$ such that

$$\xi^u - \frac{u(\bar{x})}{v(\bar{x})}\xi^v + \sum_{t \in J} \bar{\lambda}_t^g \xi_t^g + \sum_{i \in I_h} \bar{\lambda}_i^h \xi_i^h - \sum_{i \in I_{0+} \cup I_{00} \cup I_{0-}} \bar{\lambda}_i^H \xi_i^H + \sum_{i \in I_{+0} \cup I_{00}} \bar{\lambda}_i^G \xi_i^G = 0. \tag{4.3}$$

$$\bar{\lambda}^H_{I_+(\bar{x})} = 0, \bar{\lambda}^H_{I_{00}(\bar{x}) \cup I_{0-}(\bar{x})} \geq 0, \tag{4.4}$$

$$\bar{\lambda}^G_{I_{+-}(\bar{x}) \cup I_{0+}(\bar{x}) \cup I_{0-}(\bar{x})} = 0, \bar{\lambda}^G_{I_{+0}(\bar{x}) \cup I_{00}(\bar{x})} \geq 0. \tag{4.5}$$

Since $\bar{\lambda}^g \in \Lambda(\bar{x})$, one has $\bar{\lambda}_t^g g_t(\bar{x}) = 0$ for all $t \in T$, and thus, $\sum_{t \in T} \bar{\lambda}_t^g g_t(\bar{x}) = 0$. As $g_t(\bar{x}) < 0 (t \in T \setminus I_g(\bar{x}))$, one has $\lambda^g_{T \setminus I_g(x)} = 0$, leading that $\bar{\lambda} \in \Omega_{D_{\bar{s}}(\bar{x})}$. The fact that $\bar{x} \in \Omega$ asserts that $\sum_{i \in I_h} \bar{\lambda}_i^h h_i(\bar{x}) = 0$. Moreover, we deduce from $\bar{\lambda}^H_{I_+(\bar{x})} = 0$ and $H_i(\bar{x}) = 0$ for all $i \in I_0(\bar{x})$ that $\sum_{i \in I_l} \bar{\lambda}_i^H H_i(\bar{x}) = 0$. Analogously, since $\bar{\lambda}^G_{I_{+-}(\bar{x}) \cup I_{0+}(\bar{x}) \cup I_{0-}}(\bar{x}) = 0$ and $G_i(\bar{x}) = 0$ for all $i \in I_{+0}(\bar{x}) \cup I_{00}(\bar{x})$, one has $\sum_{i \in I_l} \bar{\lambda}_i^G G_i(\bar{x}) = 0$. Therefore,

$$\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) + \sum_{i \in I_h} \bar{\lambda}_i^h h_i(\bar{x}) - \sum_{i \in I_l} \bar{\lambda}_i^H H_i(\bar{x}) + \sum_{i \in I_l} \bar{\lambda}_i^G G_i(\bar{x}) = 0,$$

which implies

$$u(\bar{x}) + \bar{s}v(\bar{x}) = L_{\bar{s}}(\bar{x}, \bar{\lambda}). \tag{4.6}$$

Moreover, we infer from the ∂^C-convexity of $u, v, g_t(t \in I_g^+(\bar{x})), h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x}))$, $H_i(i \in \hat{I}_{0+}^-(\bar{x})), -H_i(i \in \hat{I}_{0+}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x})) \cup \hat{I}_{0-}^+(\bar{x})), G_i(i \in I_{+0}^+(\bar{x}) \cup I_{00}^+(\bar{x}))$ at $\bar{x}$ and the definitions of the index sets that, for any $x \in \Omega$,

$$\begin{gathered}
u(x) - u(\bar{x}) \geq \langle \xi^u, x - \bar{x} \rangle, \\
v(x) - v(\bar{x}) \geq \langle \xi^v, x - \bar{x} \rangle, \\
g_t(x) - g_t(\bar{x}) \geq \langle \xi_t^g, x - \bar{x} \rangle, \bar{\lambda}_t^g > 0, \forall t \in I_g^+(\bar{x}), \\
h_i(x) - h_i(\bar{x}) \geq \langle \xi_i^h, x - y \rangle, \bar{\lambda}_i^h > 0, \forall i \in I_h^+(\bar{x}), \\
-h_i(x) - (-h_i(\bar{x})) \geq \langle -\xi_i^h, x - \bar{x} \rangle, \bar{\lambda}_i^h < 0, \forall i \in I_h^-(\bar{x}), \\
H_i(x) - H_i(\bar{x}) \geq \langle \xi_i^H, x - \bar{x} \rangle, \bar{\lambda}_i^H < 0, \forall i \in \hat{I}_{0+}^-(\bar{x}),
\end{gathered}$$

$$-H_i(x)-(-H_i(\bar{x}))\geq\langle -\xi_i^H,x-\bar{x}\rangle,\bar{\lambda}_i^H>0,\forall i\in\hat{I}_{0+}^+(\bar{x})\cup\hat{I}_{00}^+(\bar{x})\cup\hat{I}_{0-}^+(\bar{x}),$$
$$G_i(x)-G_i(\bar{x})\geq\langle\xi_i^G,x-\bar{x}\rangle,\bar{\lambda}_i^G>0,\forall i\in I_{+0}^+(\bar{x})\cup I_{00}^+(\bar{x}).$$

The above inequalities together with (4.4) and (4.5) entail that

$$u(x)+\bar{s}v(x)+\sum_{t\in T}\bar{\lambda}_t g_t(x)+\sum_{i\in I_h}\bar{\lambda}_i^h h_i(x)-\sum_{i\in I_l}\bar{\lambda}_i^H H_i(x)+\sum_{i\in I_l}\bar{\lambda}_i^G G_i(x)$$
$$-\left(u(\bar{x})+\bar{s}v(\bar{x})+\sum_{t\in T}\bar{\lambda}_t g_t(\bar{x})+\sum_{i\in I_h}\bar{\lambda}_i^h h_i(\bar{x})-\sum_{i\in I_l}\bar{\lambda}_i^H H_i(\bar{x})+\sum_{i\in I_l}\bar{\lambda}_i^G G_i(\bar{x})\right)$$
$$\left\langle\xi^u+\bar{s}\xi^v+\sum_{t\in T}\bar{\lambda}_t^g\xi_t^g(\bar{x})+\sum_{i\in I_h}\bar{\lambda}_i^h\xi_i^h(\bar{x})-\sum_{i\in I_l}\bar{\lambda}_i^H\xi_i^H(\bar{x})+\sum_{i\in I_l}\bar{\lambda}_i^G\xi_i^G(\bar{x}),x-\bar{x}\right\rangle\geq 0.$$

Granting this, taking into account (4.3), we get

$$L_{\bar{s}}(x,\bar{\lambda})\geq L_{\bar{s}}(\bar{x},\bar{\lambda}),\forall x\in\Omega. \tag{4.7}$$

This, together with (4.6), shows that

$$u(\bar{x})+\bar{s}v(\bar{x})=L_{\bar{s}}(\bar{x},\bar{\lambda})=\min_{x\in\Omega}L_{\bar{s}}(x,\bar{\lambda})=\varphi_{\bar{s}}(\bar{\lambda}). \tag{4.8}$$

Moreover, by invoking Proposition 4.2, we have $\varphi_{\bar{s}}(\lambda)\leq u(\bar{x})+\bar{s}v(\bar{x}),\forall\lambda\in\Omega_{D_{\bar{s}}(\bar{x})}$. Combining this and (4.8), one yields $\varphi_{\bar{s}}(\lambda)\leq\varphi_{\bar{s}}(\bar{\lambda}),\forall\lambda\in\Omega_{D_{\bar{s}}(\bar{x})}$. □

Example 4.5. Consider the following (P):

$$\begin{aligned}\min\quad & f(x)=\frac{x_1^2+x_2^2+x_1}{|x_2|+1}\\ \text{s.t.}\quad & g_t(x)=-x_1+t-1\leq 0,\ \forall t\in T:=[0,1],\\ & H_1(x)=x_1-x_2\geq 0, G_1(x)H_1(x)=|x_1|(x_1-x_2)\leq 0.\end{aligned}$$

Then, $\Omega=\cup_{i=1}^3\Omega^i$, where $\Omega^1=\left\{x\in\mathbb{R}^2\mid x_1>0,x_1-x_2=0\right\}$, $\Omega^2=\left\{x\in\mathbb{R}^2\mid x_1=0,x_2=0\right\}$ and $\Omega^3=\left\{x\in\mathbb{R}^2\mid x_1=0,x_2<0\right\}$. For a fixed $s\geq 0$, the $(P)_s$ is

$$\begin{aligned}\min\quad & u(x)+sv(x)=x_1^2+x_2^2+x_1+s(|x_2|+1)\\ & \text{s.t. } x\in\Omega.\end{aligned}$$

As $g_t(x)=-x_1+t-1\leq 0(\forall t\in T=[0,1])$ is equivalent to $g_1(x)=-x_1\leq 0$, we could consider the Lagrange function for $(P)_s$ as follows

$$\begin{aligned}L_s(x,\lambda)&=x_1^2+x_2^2+x_1+s(|x_2|+1)+\lambda_1^g(-x_1)-\lambda_1^H(x_1-x_2)+\lambda_1^G|x_1|\\ &=\frac{1}{2}(x_1^2+2\lambda_1^G|x_1|)+\frac{1}{2}(x_1^2-2(\lambda_1^g+\lambda_1^H-1)x_1)+\frac{1}{2}(x_2^2+2s|x_2|)+\frac{1}{2}(x_2^2+2\lambda_1^H x_2)+s.\end{aligned}$$

Therefore, we have $\lambda=(\lambda_1^g,\lambda_1^H,\lambda_1^G)$ and

$$\varphi_s(\lambda)=\min_{x\in\Omega}L_s(x,\lambda)=-\frac{(\lambda_1^G)^2+(\lambda_1^g+\lambda_1^H-1)^2+s^2+(\lambda_1^H)^2}{2}+s.$$

Since $\Omega=\cup_{i=1}^3\Omega^i$, one has the three Lagrange type dual problems as follows. For any $x\in\Omega^1$, $I_g(x)=\emptyset,I_{0+}(x)=\{1\},I_+(x)=I_{00}(x)=I_{0-}(x)=\emptyset$,

$$D_s^1(x):\quad \max\{\varphi_s(\lambda)\mid\lambda_1^g\geq 0,\lambda_1^H\in\mathbb{R},\lambda_1^G\leq 0\}.$$

For any $x \in \Omega^2$, $I_g(\bar{x}) = \{1\}, I_+(x) = I_{0+}(x) = I_{0-}(x) = \emptyset, I_{00}(x) = \{1\}$,

$$D_s^2(x): \quad \max\{\varphi_s(\lambda) \mid \lambda_1^g \in \mathbb{R}, \lambda_1^H \in \mathbb{R}, \lambda_1^G \in \mathbb{R}\}.$$

For any $x \in \Omega^3$, $I_g(\bar{x}) = \emptyset, I_{+-}(x) = I_0(x) = \emptyset, I_{+0}(x) = \{1\}$,

$$D_s^3(x): \quad \max\{\varphi_s(\lambda) \mid \lambda_1^g \geq 0, \lambda_1^H \geq 0, \lambda_1^G \in \mathbb{R}\}.$$

Denote $\Omega_{D_s^i(x)}$ the feasible region of $D_s^i(x)$ for $i = 1,2,3$. Then, one gets

$$(D_s): \quad \max\{\varphi_s(\lambda) \mid \lambda \in \Omega_{D_s} := \bigcap_{x \in \Omega_{D_s^i(x)}, i=1,2,3} \Omega_{D_s^i(x)}\},$$

and thus, $\Omega_{D_s} = \{\lambda \mid \lambda^g \geq 0, \lambda^H \geq 0, \lambda^G \leq 0\}$. Hence, it follows from Proposition 4.2 that for all $x \in \Omega$ and $\lambda \in \Omega_{D_s}(\Omega_{D_s^i(x)}, i = 1,2,3)$, $u(x) + sv(x) \geq \varphi_s(\lambda)$. This conclusion could be checked directly as follows. For all $x \in \Omega$ and $\lambda \in \Omega_{D_s}(\Omega_{D_s^i(x)}, i = 1,2,3)$,

$$\begin{aligned} u(x) + sv(x) &= x_1^2 + x_2^2 + x_1 + s(|x_2| + 1) \\ &\geq s \\ &\geq -\frac{(\lambda_1^G)^2 + (\lambda_1^g + \lambda_1^H - 1)^2 + s^2 + (\lambda_1^H)^2}{2} + s = \varphi_s(\lambda). \end{aligned}$$

Now, taking $\bar{x} = (0,0)$, one can justify that $\bar{s} = 0$ and $\bar{x}$ is a local solution of (P). By some calculations, we have

$$T(\Omega, \bar{x}) = \Omega, \partial^C u(\bar{x}) = (1,0), \partial^C v(\bar{x}) = \{0\} \times [-1,1], I_g(\bar{x}) = \{1\},$$

$$\partial^C g_1(\bar{x}) = \{(-1,0)\}, (\bigcup_{t \in I_g(\bar{x})} \partial^C g_t(\bar{x}))^- = \{x \in \mathbb{R}^2 \mid x_1 \geq 0\},$$

$$I_+(\bar{x}) = I_{0+}(\bar{x}) = I_{0-}(\bar{x}) = \emptyset, I_{00}(\bar{x}) = \{1\}, \partial^C G_1(\bar{x}) = [-1,1] \times \{0\}, \partial^C H_1(\bar{x}) = \{(1,-1)\},$$

$$(\bigcup_{i \in I_{00}(\bar{x})} (-\partial^C H_i(\bar{x})))^- = \{x \in \mathbb{R}^2 \mid x_1 - x_2 \geq 0\}, (\bigcup_{i \in I_{00}(\bar{x})} \partial^C G_i(\bar{x}))^- = \{x \in \mathbb{R}^2 \mid x_1 = 0\},$$

$$(\bigcup_{t \in I_g} \partial^C g_t(\bar{x}))^- \cap (\bigcup_{i \in I_{00}} (-\partial^C H_i(\bar{x})))^- \cap (\bigcup_{i \in I_{00}} \partial^C G_i(\bar{x}))^- = \{x \in \mathbb{R}^2 \mid x_1 = 0\}.$$

Hence, (VC-ACQ) holds at $\bar{x}$. Moreover,

$$\Delta_1 = \operatorname{pos}\left(\bigcup_{t \in I_g(\bar{x})} \partial^C g_t(\bar{x}) \cup \bigcup_{i \in I_{00}(\bar{x})} (-\partial^C H_i(\bar{x})) \cup \bigcup_{i \in I_{00}(\bar{x})} \partial^C G_i(\bar{x})\right) = \{x \in \mathbb{R}^2 \mid x_2 \geq 0\}$$

is closed. We can check that $u, v, g_t(t \in I_g^+(\bar{x})), -H_1, G_1$ are ∂^C-convex at $\bar{x}$ on Ω. Thus, all assumptions in Proposition 4.4 holds. It is easy to see that, $\forall x \in \Omega, \forall \lambda \in \Omega_{D_{\bar{s}}(x)}$,

$$x_1^2 + x_2^2 + x_1 + \bar{s}(|x_2| + 1) = -\frac{(\lambda_1^G)^2 + (\lambda_1^g + \lambda_1^H - 1)^2 + \bar{s}^2 + (\lambda_1^H)^2}{2} + \bar{s}$$

is only possible for $x_1 = x_2 = 0$ and $\lambda_1^g = 1, \lambda_1^G = \lambda_1^H = 0$. Hence, there exists $\bar{\lambda} = (1,0,0)$ such that $u(\bar{x}) + \bar{s}v(\bar{x}) = \varphi_{\bar{s}}(\bar{\lambda})$ and

$$\varphi_{\bar{s}}(\bar{\lambda}) = 0 \geq \varphi_{\bar{s}}(\lambda), \forall \lambda \in \Omega_{D_{\bar{s}}(x)},$$

i.e., the conclusion of Proposition 4.4 holds.

4.2. **Saddle point optimality criteria.** In this section, we propose saddle point conditions for $(P)_s$ and investigate the relationships between strong duality and saddle point conditions.

Definition 4.6. A point $(\bar{x},\bar{\lambda})$ with $\bar{x}\in\Omega$ and $\bar{\lambda}\in\Omega_{D_s(\bar{x})}$ is said to be a saddle point of the Lagrange function L_s, if

$$L_s(\bar{x},\lambda)\le L_s(\bar{x},\bar{\lambda})\le L_s(x,\bar{\lambda}),\forall x\in\Omega,\forall\lambda\in\Omega_{D_s(\bar{x})}.$$

Proposition 4.7. *Let $\bar{x}$ be a local optimal solution of the (P) and the assumptions of Proposition 4.4 hold. Then, there exists $\bar{\lambda}\in\Lambda(\bar{x})\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$, such that $(\bar{x},\bar{\lambda})$ is a saddle point of $L_{\bar{s}}$, where $\bar{s}=-\frac{u(\bar{x})}{v(\bar{x})}$. Conversely, if $(\bar{x},\bar{\lambda})\in\Omega\times\Omega_{D_{\bar{s}}(\bar{x})}$ is a Lagrange saddle point of $L_{\bar{s}}$, then $\varphi(\bar{\lambda})=u(\bar{x})+\bar{s}v(\bar{x})$ and $\bar{x}$ and $\bar{\lambda}$ are optimal solutions to $(P)_{\bar{s}}$ and $D_{\bar{s}}(\bar{x})$, resp.*

Proof. From (4.7), we have for all $x\in\Omega$,

$$L_{\bar{s}}(\bar{x},\bar{\lambda})\le L_{\bar{s}}(x,\bar{\lambda}).\tag{4.9}$$

From (4.8) and (4.2), one gets that, for all $\lambda\in\Omega_{D_{\bar{s}}(\bar{x})}$, we have

$$\begin{aligned}L_{\bar{s}}(\bar{x},\bar{\lambda})&=u(\bar{x})+\bar{s}v(\bar{x})\\&\ge u(\bar{x})+\bar{s}v(\bar{x})+\sum_{t\in T}\lambda_t^g g_t(\bar{x})+\sum_{i\in I_h}\lambda_i^h h_i(\bar{x})-\sum_{i\in I_l}\lambda_i^H H_i(\bar{x})+\sum_{i\in I_l}\lambda_i^G G_i(\bar{x})\\&=L_{\bar{s}}(\bar{x},\lambda).\end{aligned}$$

This together with (4.9) implies that $(\bar{x},\bar{\lambda})$ is a saddle point of $L_{\bar{s}}$.

Now, let $(\bar{x},\bar{\lambda})\in\Omega\times\Omega_{D_{\bar{s}}(\bar{x})}$ be a saddle point of $L_{\bar{s}}$. Then, for all $\lambda\in\Omega_{D_{\bar{s}}(\bar{x})}$,

$$\begin{aligned}&u(\bar{x})+\bar{s}v(\bar{x})+\sum_{t\in T}\lambda_t^g g_t(\bar{x})+\sum_{i\in I_h}\lambda_i^h h_i(\bar{x})-\sum_{i\in I_l}\lambda_i^H H_i(\bar{x})+\sum_{i\in I_l}\lambda_i^G G_i(\bar{x})\\&\le u(\bar{x})+\bar{s}v(\bar{x})+\sum_{t\in T}\bar{\lambda}_t^g g_t(\bar{x})+\sum_{i\in I_h}\bar{\lambda}_i^h h_i(\bar{x})-\sum_{i\in I_l}\bar{\lambda}_i^H H_i(\bar{x})+\sum_{i\in I_l}\bar{\lambda}_i^G G_i(\bar{x}).\end{aligned}\tag{4.10}$$

Letting $\lambda=0$ in (4.10), we obtain

$$\sum_{t\in T}\bar{\lambda}_t^g g_t(\bar{x})+\sum_{i\in I_q}\bar{\lambda}_i^h h_i(\bar{x})-\sum_{i\in I_l}\bar{\lambda}_i^H H_i(\bar{x})+\sum_{i\in I_l}\bar{\lambda}_i^G G_i(\bar{x})\ge 0.\tag{4.11}$$

As $(\bar{x},\bar{\lambda})\in\Omega\times\Omega_{D_{\bar{s}}(\bar{x})}$, we derive from (4.2) that

$$\sum_{t\in T}\bar{\lambda}_t^g g_t(\bar{x})+\sum_{i\in I_q}\bar{\lambda}_i^h h_i(\bar{x})-\sum_{i\in I_l}\bar{\lambda}_i^H H_i(\bar{x})+\sum_{i\in I_l}\bar{\lambda}_i^G G_i(\bar{x})\le 0.\tag{4.12}$$

From (4.11) and (4.12), we get

$$\sum_{t\in T}\bar{\lambda}_t^g g_t(\bar{x})+\sum_{i\in I_q}\bar{\lambda}_i^h h_i(\bar{x})-\sum_{i\in I_l}\bar{\lambda}_i^H H_i(\bar{x})+\sum_{i\in I_l}\bar{\lambda}_i^G G_i(\bar{x})=0,\tag{4.13}$$

which, along with $L_{\bar{s}}(\bar{x},\bar{\lambda})\le L_{\bar{s}}(x,\bar{\lambda})$ for all $x\in\Omega$, brings us that

$$u(\bar{x})+\bar{s}v(\bar{x})=L_{\bar{s}}(\bar{x},\bar{\lambda})=\min_{x\in\Omega}L_{\bar{s}}(x,\bar{\lambda})=\varphi_{\bar{s}}(\bar{\lambda}).$$

By Corollary 4.3, $\bar{x}$ and $\bar{\lambda}$ are optimal solutions to $(P)_{\bar{s}}$ and $D_{\bar{s}}(\bar{x})$, resp. $\square$

Proposition 4.8. *Let $\bar{x} \in \Omega$ be a VC-stationary point of (P). Suppose that $u, v, g_t(t \in I_g^+(\bar{x}))$, $h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x})), H_i(i \in \hat{I}_{0+}^-(\bar{x}))$, $-H_i(i \in \hat{I}_{0+}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x})) \cup \hat{I}_{0-}^+(\bar{x}))$, $G_i(i \in I_{+0}^+(\bar{x}) \cup I_{00}^+(\bar{x}))$ are ∂^C-convex at $\bar{x}$. Then $(\bar{x}, \bar{\lambda})$ is a saddle point of $L_{\bar{s}}$, where $\bar{s} = -\frac{u(\bar{x})}{v(\bar{x})}$.*

Proof. For an arbitrary $x \in \Omega$, we have

$$L_{\bar{s}}(x, \bar{\lambda}) - L_{\bar{s}}(\bar{x}, \bar{\lambda}) = u(x) + \bar{s}v(x) + \sum_{t \in T} \bar{\lambda}_t g_t(x) + \sum_{i \in I_h} \bar{\lambda}_i^h h_i(x) - \sum_{i \in I_l} \bar{\lambda}_i^H H_i(x) + \sum_{i \in I_l} \bar{\lambda}_i^G G_i(x)$$

$$- \left(u(\bar{x}) + \bar{s}v(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t^g g_t(\bar{x}) + \sum_{i \in I_q} \bar{\lambda}_i^h h_i(\bar{x}) - \sum_{i \in I_l} \bar{\lambda}_i^H H_i(\bar{x}) + \sum_{i \in I_l} \bar{\lambda}_i^G G_i(\bar{x}) \right).$$

By analyzing similarly to the proof in Proposition 4.4, we derive from the fact $\bar{x}$ is a VC-stationary point of (P) that

$$L_{\bar{s}}(x, \bar{\lambda}) - L_{\bar{s}}(\bar{x}, \bar{\lambda}) \geq 0, \forall x \in \Omega. \tag{4.14}$$

It follows from (4.2) and (4.13) that, for all $\lambda \in \Omega_{D_{\bar{s}}(\bar{x})}$,

$$\begin{aligned} L_{\bar{s}}(\bar{x}, \lambda) &= u(\bar{x}) + \bar{s}v(\bar{x}) + \sum_{t \in T} \lambda_t^g g_t(\bar{x}) + \sum_{i \in I_q} \lambda_i^h h_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^H H_i(\bar{x}) + \sum_{i \in I_l} \lambda_i^G G_i(\bar{x}) \\ &\leq u(\bar{x}) + \bar{s}v(\bar{x}) = u(\bar{x}) + \bar{s}v(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t^g g_t(\bar{x}) + \sum_{i \in I_q} \bar{\lambda}_i^h h_i(\bar{x}) - \sum_{i \in I_l} \bar{\lambda}_i^H H_i(\bar{x}) + \sum_{i \in I_l} \bar{\lambda}_i^G G_i(\bar{x}) \\ &= L_{\bar{s}}(\bar{x}, \bar{\lambda}). \end{aligned}$$

This along with (4.14) leads that $(\bar{x}, \bar{\lambda})$ is a saddle point of $L_{\bar{s}}$. $\square$

Example 4.9. Consider the problem (P)

$$\begin{aligned} \min \quad & f(x) = \frac{x_1 + |x_2|}{x_2^2 + 1} \\ \text{s.t.} \quad & g_t(x) = x_1 - x_2 + 1 - t \leq 0, \forall t \in T = [0, 1], \\ & H_1(x) = -x_1^2 - x_2^2 + 1 \geq 0, G_1(x)H_1(x) = x_2(-x_1^2 - x_2^2 + 1) \leq 0. \end{aligned}$$

The feasible region is $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_2 \leq 1, x_1 = -\sqrt{1 - x_2^2}\}$. Choosing $\bar{x} = (-1, 0) \in \Omega$ and $\bar{\lambda} = (\bar{\lambda}_0^g, \bar{\lambda}_1^H, \bar{\lambda}_1^G) = (1, 1, 0)$, one obtains that $\bar{x}$ is a VC-stationary point of (P), where $I_g(\bar{x}) = \{0\}$, $I_+(\bar{x}) = I_{0+}(\bar{x}) = I_{0-}(\bar{x}) = \emptyset$ and $I_{00}(\bar{x}) = \{1\}$.

Moreover, we can check that $u, v, g_t(t \in I_g^+(\bar{x})), -H_1, G_1$ are ∂^C-convex at $\bar{x}$ on Ω, i.e., the provisos of Proposition 4.8 are fulfilled at $\bar{x}$. Furthermore, as $u(\bar{x}) = 0, v(\bar{x}) = 1, \bar{s} = -\frac{u(\bar{x})}{v(\bar{x})} = 0$ and

$$L_{\bar{s}}(x, \lambda) = x_1 + |x_2| + \lambda_0^g(x_1 - x_2 + 1) - \lambda_1^H(-x_1^2 - x_2^2 + 1) + \lambda_1^G x_2,$$

one has

$$L_{\bar{s}}(\bar{x}, \lambda) = L_{\bar{s}}(\bar{x}, \bar{\lambda}) = -1, L(x, \bar{\lambda}) = x_1^2 + x_2^2 + 2x_1 + |x_2| - x_2.$$

For $x \in \Omega$, we get that $0 \leq x_2 \leq 1$ and

$$L(x, \bar{\lambda}) = (-\sqrt{1 - x_2^2})^2 + x_2^2 + 2(-\sqrt{1 - x_2^2}) + x_2 - x_2 = 1 - 2\sqrt{1 - x_2^2} \geq -1,$$

and hence,

$$L_{\bar{s}}(\bar{x}, \lambda) \leq L_{\bar{s}}(\bar{x}, \bar{\lambda}) \leq L_{\bar{s}}(x, \bar{\lambda}), \forall x \in \Omega, \forall \lambda \in \Omega_{D_{\bar{s}}(\bar{x})}.$$

Therefore, the conclusion of Proposition 4.8 is verified.

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