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NEW PREDICTOR-CORRECTOR INTERIOR-POINT ALGORITHM WITH AET FUNCTION HAVING INFLECTION POINTS

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Dedicated to the memory of Naum Shor on the occasion of his 85th birthday

Abstract. In this paper, we introduce a new predictor-corrector interior-point algorithm (PC IPA) for solving $P_*(\kappa)$-linear complementarity problems. For the determination of search directions, we use the algebraically equivalent transformation (AET) technique. In this method, we apply the function $\varphi(t) = t^2 - t + \sqrt{t}$ which has inflection point. It is interesting that the kernel corresponding to this AET function is neither self-regular, nor eligible. We present the complexity analysis of the proposed interior-point algorithm and we show that its iteration bound matches the best known iteration bound for this type of PC IPAs given in the literature. It should be mentioned that usually the iteration bound is given for a fixed update and proximity parameter. In this paper, we provide a set of parameters for which the PC IPA is well defined. Moreover, we also show the efficiency of the algorithm by providing numerical results.

Keywords. Complexity analysis; Interior-point algorithm; Linear complementarity problems; Predictor-corrector.

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1. Introduction

Linear complementarity problems (LCPs) are intensively studied due to the fact that they have several applications in different fields, such as economics, optimization theory and engineering, see [4, 15]. The Karush-Kuhn-Tucker optimality conditions of linear programming (LP) and quadratic optimization problems lead to LCPs. The Arrow-Debreu competitive market equilibrium problem with linear and Leontief utility functions can also be formulated as LCP [42]. Bras et al. [2] showed that LCP can be used to test copositivity of matrices. Wang and Sun analysed sparse Markowitz portfolio selection by using stochastic LCP approach, see [41].

The LCPs can be solved by using several methods, such as different pivot and criss-cross algorithms [5, 6, 16, 17]. Fukuda and Terlaky gave a general form of the dual of the LCPs for

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oriented matroids [17]. Later, Csizmadia and Illés analysed it for LCPs related to the criss-cross algorithm [5]. Several interior-point algorithms (IPAs) have been also introduced for $P_*(\kappa)$-LCPs, see [4, 22, 23, 29, 30, 36]. Illés et al. gave existentially polytime (EP) theorem for the dual LCP with arbitrary matrix and they proposed IPAs to solve LCPs with general matrices in EP-sense. This means that their generalized IPAs either solve the problems with rational coefficient matrix in polynomial time or give a polynomial size certificate that the matrix is not $P_*(\bar{\kappa})$ with apriori fixed, positive $\bar{\kappa}$, see [22–24]. These results have been summarized in [33].

For detailed study on LCPs, we refer the reader to the books of Cottle et al. [4] and Kojima et al. [29]. In the book of Kojima et al. [29] the theory of interior-point algorithms for solving LCPs is highlighted. The LCPs belong to the class of NP-complete problems, see [3, 29]. However, Kojima et al. [29] proved that assuming that the problem's matrix has a special property, i.e. it is $P_*(\kappa)$-matrix, the IPAs give approximate solution for LCPs in polynomial time in the size of the problem, the handicap of the problem's matrix and the starting point's duality gap. Illés [25] et al. proposed a rounding procedure that computes a maximally complementary (exact) solution in polynomial time from an ε-optimal solution produced by IPAs for $P_*(\kappa)$-LCPs with appropriate $\varepsilon > 0$.

Predictor-corrector (PC) IPAs turned out to be efficient in practice. They perform in a major iteration a predictor and one or several corrector steps. The predictor step is a greedy step and aims to approach the optimal solution set of the corresponding problem. After a predictor step the obtained strictly feasible solution may not be in a small neighbourhood of the central path. Hence, the corrector step is responsible for returning the iterate in the designated small neighbourhood. Mizuno, Todd and Ye [32] provided the first PC IPA for LP problems, which uses only one corrector step in a main iteration. Miao [31] extended this IPA to $P_*(\kappa)$-LCPs. When only one step is needed to return into the small neighbourhood of the central path, then we call these methods Mizuno-Todd-Ye-type (MTY-type) of PC IPAs. Potra and Sheng [36] introduced a MTY-type PC IPA for $P_*(\kappa)$-LCPs. Illés and Nagy [21], Kheirfam [27], Darvay et al. [8, 9] also introduced MTY-type PC IPAs.

Defining a proper search direction plays an important role in the analysis of the IPAs. As mentioned before, Peng et al. [34] provided the notion of self-regular barrier and they determined new long-step IPAs. Later on, Darvay [7] introduced the *algebraically equivalent transformation* (AET) technique to determine search directions in case of LP problems. He applied a continuously differentiable and invertible function on the nonlinear equation of the system characterizing the central path. In his work, Darvay used the square root function in the AET technique. Following his result, a large amount of papers appeared, where IPAs based on the AET technique were proposed, see [8, 9, 26, 28, 37]. In [26], the authors defined a new class of AET functions in order to define primal-dual IPAs for solving $P_*(\kappa)$-LCPs and the function $\varphi(t) = t^2 - t + \sqrt{t}$ is member of this class. In this paper, we introduce a new PC IPA for solving $P_*(\kappa)$-LCPs which is based on the AET function $\varphi(t) = t^2 - t + \sqrt{t}$.

The paper is organized as follows. In Section 2, some basic concepts and results related to the theory of $P_*(\kappa)$-LCPs are presented. Section 3 is devoted to define the new PC IPA which is based on a new search direction. Section 4 contains the complexity analysis of the new PC IPA. In Section 5, we provide a set of parameters for which the PC IPA is well defined. Section 6 summarizes numerical results that show the efficiency of the algorithm. In Section 7, some concluding remarks and further research plans are enumerated.

2. Preliminaries

2.1. Linear complementarity problems and $P_*(\kappa)$-matrices.

In the linear complementarity problems (LCPs), we search for vectors $\mathbf{x},\mathbf{s} \in \mathbb{R}^n$ that satisfy the following constraints:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q}, \quad \mathbf{x},\mathbf{s}\geq \mathbf{0}, \quad \mathbf{x}\mathbf{s}=\mathbf{0}, \tag{LCP}$$

where $M \in \mathbb{R}^{n\times n}$, $\mathbf{q} \in \mathbb{R}^n$ and $\mathbf{x}\mathbf{s}$ is the componentwise product of vectors $\mathbf{x}$ and $\mathbf{s}$. We use the following notations to denote feasible region, the interior and the solutions set of LCPs:

$$\begin{aligned}
\mathscr{F} &:= \{(\mathbf{x},\mathbf{s}) \in \mathbb{R}^n_{\oplus} \times \mathbb{R}^n_{\oplus} : -M\mathbf{x}+\mathbf{s}=\mathbf{q}\},\\
\mathscr{F}^+ &:= \{(\mathbf{x},\mathbf{s}) \in \mathbb{R}^n_{+} \times \mathbb{R}^n_{+} : -M\mathbf{x}+\mathbf{s}=\mathbf{q}\},\\
\mathscr{F}^* &:= \{(\mathbf{x},\mathbf{s}) \in \mathscr{F} : \mathbf{x}\mathbf{s}=\mathbf{0}\}.
\end{aligned}$$

Note that $\mathbb{R}^n_{\oplus}$ denotes the n-dimentional nonnegative orthant and $\mathbb{R}^n_{+}$ the positive orthant.

Thorughout the paper, we assume that $\mathscr{F}^+ \neq \emptyset$ and M is a $P_*(\kappa)$-matrix, see [29]. We present the central path problem for (LCP) as follows:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q} \qquad \mathbf{x},\mathbf{s}> \mathbf{0}, \qquad \mathbf{x}\mathbf{s}=\mu\mathbf{e}. \tag{CPP}$$

where $\mathbf{e}$ denotes the n-dimensional all-one vector and $\mu > 0$. Kojima et al. [29] proved the uniqueness of the central path for $P_*(\kappa)$-LCPs. They also showed that the sequence $\{(\mathbf{x}(\mu),\mathbf{s}(\mu)) \mid \mu > 0\}$ of solutions lying on the central path parameterised by $\mu > 0$ approaches the solution $(\mathbf{x},\mathbf{s})$ of the (LCP).

2.2. Algebraic equivalent transformation technique.

We present the AET technique in case of $P_*(\kappa)$-LCPs. Note that the AET method was introduced by Darvay [7] for LP problem. Let $\varphi : (\bar{\xi},\infty) \to \mathbb{R}$, with $0 \leq \bar{\xi} < 1$, be a continuously differentiable function, such that $\varphi'(t) > 0,\ \forall t > \bar{\xi}$. We also use the notation $\varphi(\mathbf{x}) = [\varphi(x_1),\varphi(x_2)\ldots,\varphi(x_n)]^T$. In this way, system ($CPP$) can be written in the following form:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q} \qquad \mathbf{x},\mathbf{s}> \mathbf{0}, \qquad \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\varphi(\mathbf{e}), \tag{CPP_φ}$$

The Newton-system is

$$\begin{aligned}
-M\Delta\mathbf{x}+\Delta\mathbf{s} &= \mathbf{0},\\
S\Delta\mathbf{x}+X\Delta\mathbf{s} &= \mathbf{a}_\varphi,
\end{aligned} \tag{2.3}$$

where $X=\operatorname{diag}(\mathbf{x})$, $S=\operatorname{diag}(\mathbf{s})$ and

$$\mathbf{a}_\varphi = \mu\frac{\varphi(\mathbf{e})-\varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)}. \tag{2.4}$$

It can be observed that for different functions φ we obtain different search directions. Let

$$\mathbf{v}=\sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}, \quad \mathbf{d} = \sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{d}_x=\frac{\mathbf{d}^{-1}\Delta\mathbf{x}}{\sqrt{\mu}}=\frac{\mathbf{v}\Delta\mathbf{x}}{\mathbf{x}}, \quad \mathbf{d}_s=\frac{\mathbf{d}\Delta\mathbf{s}}{\sqrt{\mu}}=\frac{\mathbf{v}\Delta\mathbf{s}}{\mathbf{s}}. \tag{2.5}$$

Using (2.5) we get

$$\Delta\mathbf{x}=\frac{\mathbf{x}\mathbf{d}_x}{\mathbf{v}} \quad \text{and} \quad \Delta\mathbf{s}=\frac{\mathbf{s}\mathbf{d}_s}{\mathbf{v}}.$$

Substituting these in the second equation of system (2.3) we obtain:

$$\frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_x}{\mathbf{v}} + \frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_s}{\mathbf{v}} = \mu \frac{\varphi(\mathbf{e}) - \varphi\left(\frac{\mathbf{xs}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{xs}}{\mu}\right)}. \tag{2.6}$$

The scaled system of the transformed Newton system (2.3) is

$$\begin{aligned} -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{d}_x + \mathbf{d}_s &= \mathbf{p}_\varphi, \end{aligned} \tag{2.7}$$

where $\bar{M} = DMD$, $D = \operatorname{diag}(\mathbf{d})$ and

$$\mathbf{p}_\varphi = \frac{\varphi(\mathbf{e}) - \varphi(\mathbf{v}^2)}{\mathbf{v}\varphi'(\mathbf{v}^2)}. \tag{2.8}$$

It can be shown that the scaled transformed Newton system (2.7) has a unique solution, see [8].

2.3. **Special case of the new class of AET.** In this paper, we deal with the AET function $\varphi : (0,\infty) \to \mathbb{R}$

$$\varphi(t) = t^2 - t + \sqrt{t}. \tag{2.9}$$

This function was firstly used in [26], where the authors introduced a whole new class of AET functions for primal-dual IPAs. Up to our best knowledge, this is the first AET function analysed in the literature of IPAs which has inflection point.
Using (2.4) and (2.8), we can calculate the corresponding $\mathbf{a}_\varphi$ and $\mathbf{p}_\varphi$ in this case:

$$\mathbf{a}_\varphi = \mu \frac{\mathbf{e} - \frac{\mathbf{x}^2\mathbf{s}^2}{\mu^2} + \frac{\mathbf{xs}}{\mu} - \sqrt{\frac{\mathbf{xs}}{\mu}}}{2\frac{\mathbf{xs}}{\mu} - \mathbf{e} + \frac{1}{2}\sqrt{\frac{\mu}{\mathbf{xs}}}} = \frac{4\mu^2\sqrt{\mathbf{xs}} + 2\mu\mathbf{xs}\sqrt{\mathbf{xs}} - 3\mu\mathbf{xs}\sqrt{\mu}}{2(4\mathbf{xs}\sqrt{\mathbf{xs}} - 2\mu\sqrt{\mathbf{xs}} + \mu\sqrt{\mu})} - \frac{\mathbf{xs}}{2} \tag{2.10}$$

and

$$\mathbf{p}_\varphi = \frac{\mathbf{e} - \mathbf{v}^4 + \mathbf{v}^2 - \mathbf{v}}{2\mathbf{v}^3 - \mathbf{v} + \frac{1}{2}\mathbf{e}} = \frac{2(\mathbf{e} - \mathbf{v})(\mathbf{e} + \mathbf{v}^2 + \mathbf{v}^3)}{4\mathbf{v}^3 - 2\mathbf{v} + \mathbf{e}}. \tag{2.11}$$

Hence, in our case the coordinate function c_f of $\mathbf{p}_\varphi$ is

$$c_f(t) = \frac{2(1 - t^4 + t^2 - t)}{4t^3 - 2t + 1}. \tag{2.12}$$

In the following section, we present the new PC IPA for solving $P_*(\kappa)$-LCPs.

3. Predictor-corrector interior point algorithm based on a new AET function

Several IPAs use firstly corrector steps and after that predictor step, these algorithms are called corrector-predictor IPAs, see Potra [35]. The reason for using these methods is that after a corrector step we reach a proper neighbourhood of the central path. Our IPA also performs firstly a corrector step and after that a predictor one.

3.1. **Corrector step.** The scaled corrector system coincides with system (2.7) with $\mathbf{p}_\varphi$ given in (2.11).

This system has the following solution:

$$\mathbf{d}_x^c = (I+\bar{M})^{-1}\mathbf{p}_\varphi, \quad \mathbf{d}_s^c = \bar{M}(I+\bar{M})^{-1}\mathbf{p}_\varphi.$$

Using this and (2.5) the search directions $\Delta^c\mathbf{x}$ and $\Delta^c\mathbf{s}$ can be calculated. The point after a corrector step is

$$\mathbf{x}^+ = \mathbf{x} + \Delta^c\mathbf{x}, \quad \mathbf{s}^+ = \mathbf{s} + \Delta^c\mathbf{s}.$$

In the next subsection, we deal with the predictor step.

3.2. **Predictor step.** Based on (2.10), vector $\mathbf{a}_\varphi$ can be represented as

$$\mathbf{a}_\varphi = f(\mathbf{x},\mathbf{s},\mu) + g(\mathbf{x},\mathbf{s}),$$

where $f : \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_\oplus \to \mathbb{R}^n$ with $f(\mathbf{x},\mathbf{s},0) = \mathbf{0}$ and $g : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}^n$. Since, we would like to make as greedy predictor step as possible, we set $\mu = 0$ in this decomposition.

Then, we obtain:

$$\begin{aligned} -M\Delta^p\mathbf{x} + \Delta^p\mathbf{s} &= \mathbf{0}, \\ S\Delta^p\mathbf{x} + X\Delta^p\mathbf{s} &= g(\mathbf{x},\mathbf{s}). \end{aligned} \tag{3.1}$$

In the specific case, when we set $\mu = 0$ in $\mathbf{a}_\varphi$ then $g(\mathbf{x},\mathbf{s}) = -\frac{1}{2}\mathbf{x}\mathbf{s}$ is computed. Using (2.5) and (3.1), we get the scaled predictor system

$$\begin{aligned} -\bar{M}_+\mathbf{d}_x^p + \mathbf{d}_s^p &= \mathbf{0}, \\ \mathbf{d}_x^p + \mathbf{d}_s^p &= -\frac{1}{2}\mathbf{v}^+, \end{aligned} \tag{3.2}$$

where $\mathbf{v}^+ = \sqrt{\frac{\mathbf{x}^+\mathbf{s}^+}{\mu}}$, $\mathbf{d}^+ = \sqrt{\frac{\mathbf{x}^+}{\mathbf{s}^+}}$, $D^+ = \operatorname{diag}(\mathbf{d}^+)$, $\bar{M}_+ = D^+MD^+$ and which has the solution:

$$\mathbf{d}_x^p = -\frac{1}{2}(I+\bar{M}_+)^{-1}\mathbf{v}^+ \quad \text{and} \quad \mathbf{d}_s^p = -\frac{1}{2}\bar{M}_+(I+\bar{M}_+)^{-1}\mathbf{v}^+.$$

From this and (2.5), we can calculate the predictor search directions $\Delta^p\mathbf{x}$ and $\Delta^p\mathbf{s}$. The point after a predictor step is

$$\mathbf{x}^p = \mathbf{x}^+ + \theta\Delta^p\mathbf{x}, \quad \mathbf{s}^p = \mathbf{s}^+ + \theta\Delta^p\mathbf{s}, \quad \mu^p = \left(1-\frac{\theta}{2}\right)\mu,$$

where $\theta \in (0,1)$ is the update parameter.

Let us define the proximity measure, which is used to measure the distance of the iterates $(\mathbf{x},\mathbf{s})$ from the central path:

$$\delta(\mathbf{x},\mathbf{s},\mu) := \delta(\mathbf{v}) := \frac{\|\mathbf{p}_\varphi\|}{2} = \left\| \frac{(\mathbf{e}-\mathbf{v})(\mathbf{e}+\mathbf{v}^2+\mathbf{v}^3)}{4\mathbf{v}^3 - 2\mathbf{v} + \mathbf{e}} \right\|. \tag{3.3}$$

The neighbourhood is given in the following way:

$$\mathcal{N}_2(\tau,\mu) := \{(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \delta(\mathbf{x},\mathbf{s},\mu) \le \tau\}, \tag{3.4}$$

where δ is given in (3.3), τ is a threshold parameter and $\mu > 0$.

The PC IPA for solving $P_*(\kappa)$-LCPs is given in Algorithm 1.

In the next section, we present the complexity analysis of the proposed PC IPA.

Algorithm 1: New PC IPA for $P_*(\kappa)$-LCPs

Let $\varepsilon > 0$ *be the accuracy parameter,* $0 < \theta < 1$ *the update parameter and* τ *the proximity parameter. Furthermore, a known upper bound* κ *of the handicap* $\hat{\kappa}(M)$ *is given. Assume that for* $(\mathbf{x}^0, \mathbf{s}^0)$ *the* $\left(\mathbf{x}^0\right)^T \mathbf{s}^0 = n\mu^0$, $\mu^0 > 0$ *holds such that* $\delta(\mathbf{x}^0, \mathbf{s}^0, \mu^0) \le \tau$.

begin

$k := 0$;

while $\left(\mathbf{x}^k\right)^T \mathbf{s}^k > \varepsilon$ **do begin**

(corrector step)
compute $(\Delta^c x^k, \Delta^c s^k)$ *from system (2.7) using (2.5);*
let $(\mathbf{x}^+)^k := \mathbf{x}^k + \Delta^c x^k$ *and* $(\mathbf{s}^+)^k := \mathbf{s}^k + \Delta^c s^k$;

(predictor step)
compute $(\Delta^p x^k, \Delta^p s^k)$ *from system (3.2) using (2.5);*
let $(\mathbf{x}^p)^k := (\mathbf{x}^+)^k + \theta \Delta^p \mathbf{x}^k$ *and* $(\mathbf{s}^p)^k := (\mathbf{s}^+)^k + \theta \Delta^p \mathbf{s}^k$;

(update of the parameters and the iterates)
$(\mu^p)^k = \left(1 - \frac{\theta}{2}\right) \mu^k$;
$\mathbf{x}^{k+1} := (\mathbf{x}^p)^k, \quad \mathbf{s}^{k+1} := (\mathbf{s}^p)^k, \quad \mu^{k+1} := (\mu^p)^k$;
$k := k+1$;

end

end.

4. Analysis of the predictor-corrector algorithm

In this section, we present the complexity analysis of the proposed PC IPA. In the first part, we deal with the corrector step. This is a full-Newton step, so the analysis of this part is very similar to that of small-update IPA presented in [26] for the special case when $\varphi(t) = t^2 - t + \sqrt{t}$. We summarize the lemmas and corollaries used in this case.

The first lemma refers to the strict feasibility of the corrector step.

Lemma 4.1. *[Corollary 3.1 in [26]] Let* $(\boldsymbol{x}, \boldsymbol{s}) \in \mathscr{F}^+$ *be given, such that* $\delta(\boldsymbol{x}, \boldsymbol{s}; \mu) < \frac{1}{\sqrt{1+4\kappa}}$. *In case of* $\varphi(t) = t^2 - t + \sqrt{t}$ *we have that* $(\mathbf{x}^+, \mathbf{s}^+) \in \mathscr{F}^+$.

The next lemma shows the quadratic convergence of the corrector step.

Lemma 4.2. *[Corollary 3.2 in [26]] Let* $(\boldsymbol{x}, \boldsymbol{s}) \in \mathscr{F}^+$ *and* $\bar{\mathbf{v}} = \sqrt{\frac{\boldsymbol{x}^+ \boldsymbol{s}^+}{\mu}}$ *be given, such that* $\delta(\boldsymbol{x}, s; \mu) \le \frac{1}{\sqrt{1+4\kappa}}$. *In case of* $\varphi(t) = t^2 - t + \sqrt{t}$ *we have that*

$$\delta(\boldsymbol{x}^+, \boldsymbol{s}^+; \mu) \le 2(10 + 4\kappa)\delta(\boldsymbol{x}, s; \mu)^2.$$

In the following lemma, an upper bound is given on the duality gap.

Lemma 4.3. *[Corollary 3.3 in [26]] Let* $\delta := \delta(\boldsymbol{x}, s; \mu)$ *and suppose that the vectors* $\boldsymbol{x}^+$ *and* $\boldsymbol{s}^+$ *are obtained using a full-Newton step, thus* $\boldsymbol{x}^+ = \boldsymbol{x} + \Delta\boldsymbol{x}$ *and* $\boldsymbol{s}^+ = \boldsymbol{s} + \Delta\boldsymbol{s}$. *In case of* $\varphi(t) =$

$t^2 - t + \sqrt{t}$ *we have*

$$\left(\boldsymbol{x}^+\right)^T \boldsymbol{s}^+ \le \mu(n + 9\delta^2).$$

We present technical lemmas that will be used later in the analysis. The following result is independent from the selection of the AET function. Let

$$\mathbf{q}_\varphi = \mathbf{d}_x^c - \mathbf{d}_s^c. \tag{4.1}$$

Lemma 4.4. *[Lemma 5.4 in [8]] The following inequality holds:*

$$\|\mathbf{q}_\varphi\| \le 2\sqrt{1+4\kappa}\,\delta,$$

where $\delta = \delta(\mathbf{x}, \mathbf{s}, \mu)$.

Throughout this section, we assume that M is a $P_*(\kappa)$-matrix for a given $\kappa \ge \hat{\kappa}(M) \ge 0$ upper bound.

Easy computations lead (for details see [8], page 2641) to the following inequality

$$(1+4\kappa)\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p + \sum_{i\in I_-} d_{x_i}^p d_{s_i}^p \quad \ge \quad 0, \tag{4.2}$$

The next lemma is a technical one which will be used later in the analysis.

Lemma 4.5. *Let* $\bar{f} : (\bar{d}, +\infty) \to \mathbb{R}_+$ *be a function, where* $\bar{d} > 0$ *and* $|\bar{f}(t)| \ge \bar{k}\,|1-t|$, *for* $t > \bar{d}$, *where* $\bar{k} > 0$. *Then,*

$$\|\bar{f}(\mathbf{v})\| \ge \bar{k}\|\mathbf{e} - \mathbf{v}\|.$$

Proof. We have

$$\|\bar{f}(\mathbf{v})\| \quad = \quad \sqrt{\sum_{i=1}^n \left(\bar{f}(v_i)\right)^2} \ge \bar{k}\sqrt{\sum_{i=1}^n (1-v_i)^2} = \bar{k}\|\mathbf{e} - \mathbf{v}\|,$$

which leads to the result. □

The following result depends on the current AET function $\varphi(t) = t^2 - t + \sqrt{t}$, therefore we need to prove a similar result to that of Lemma 5.3. in [8]. Since many steps of the proof of our new lemma are very similar to the proof of the earlier result, therefore we only focus on the differences caused by the new AET function.

Lemma 4.6. *Let* $\varphi(t) = t^2 - t + \sqrt{t}$. *Then,*

$$\|\mathbf{d}_x^p \mathbf{d}_s^p\| \le \frac{n\left(1+\frac{\kappa}{2}\right)(1+4\delta^+)^2}{2},$$

where $\delta^+ = \delta(\mathbf{x}^+, \mathbf{s}^+, \mu) = \left\| \dfrac{(\mathbf{e} - \mathbf{v}^+)(\mathbf{e} + (\mathbf{v}^+)^2 + (\mathbf{v}^+)^3)}{4(\mathbf{v}^+)^3 - 2\mathbf{v}^+ + \mathbf{e}} \right\|.$

Proof. We use the second equation of the scaled predictor system (3.2), and we obtain the following:

$$\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p \le \frac{1}{4}\|\mathbf{d}_x^p + \mathbf{d}_s^p\|^2 = \frac{\|\mathbf{v}^+\|^2}{16}.$$

Using the previous bound and (4.2), we have

$$\begin{aligned}\|\mathbf{v}^+\|^2 \geq \frac{1}{4}\|\mathbf{v}^+\|^2 = \|\mathbf{d}_x^p + \mathbf{d}_s^p\|^2 &= \|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 + 2\left(\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p + \sum_{i\in I_-} d_{x_i}^p d_{s_i}^p\right) \\ &\geq \|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 - 8\kappa \sum_{i\in I_+} d_{x_i}^p d_{s_i}^p \\ &\geq \|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 - \frac{1}{2}\kappa\|\mathbf{v}^+\|^2.\end{aligned}$$

This means that $\|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 \leq \left(1+\frac{\kappa}{2}\right)\|\mathbf{v}^+\|^2$. Exactly on the same way as in [8] (page 2642), we have

$$\|\mathbf{v}^+\| \leq \sqrt{n}(\sigma^+ + 1), \tag{4.3}$$

where $\sigma^+ = \|\mathbf{e} - \mathbf{v}^+\|$. Now we need to compute a lower bound on δ^+. Using Lemma 4.5, we derive

$$\delta^+ = \frac{1}{2}\left\|\frac{2(\mathbf{e}-\mathbf{v}^+)(\mathbf{e}+(\mathbf{v}^+)^2+(\mathbf{v}^+)^3)}{4(\mathbf{v}^+)^3 - 2\mathbf{v}^+ + \mathbf{e}}\right\| = \|\bar{f}(\mathbf{v})\| \geq \frac{1}{4}\left\|\mathbf{e}-\mathbf{v}^+\right\| = \frac{\sigma^+}{4}, \tag{4.4}$$

where $\bar{f}(t) = \frac{(1-t)(1+t^2+t^3)}{4t^3-2t+1}$ and it can be shown that $|\bar{f}(t)| \geq \frac{1}{4}|1-t|$. Using (4.3) and (4.4), we obtain

$$\|\mathbf{v}^+\| \leq \sqrt{n}(1+4\delta^+). \tag{4.5}$$

From the previous steps,

$$\|\mathbf{d}_x^p\mathbf{d}_s^p\| \leq \frac{1}{2}\left(\|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2\right) \leq \frac{1}{2}\left(1+\frac{\kappa}{2}\right)\|\mathbf{v}^+\|^2 \leq \frac{n(1+\frac{\kappa}{2})(1+4\delta^+)^2}{2},$$

which yields the result. □

In the next part, we deal with the predictor step. Consider the following notations:

$$\mathbf{x}^p(\alpha) = \mathbf{x}^+ + \alpha\,\theta\,\Delta^p\mathbf{x}, \quad \mathbf{s}^p(\alpha) = \mathbf{s}^+ + \alpha\,\theta\,\Delta^p\mathbf{s},$$

for $0 \leq \alpha \leq 1$ and $\theta \in (0,1)$. We have

$$\mathbf{x}^p(\alpha) = \frac{\mathbf{x}^+}{\mathbf{v}^+}(\mathbf{v}^+ + \alpha\,\theta\,\mathbf{d}_x^p), \quad \mathbf{s}^p(\alpha) = \frac{\mathbf{s}^+}{\mathbf{v}^+}(\mathbf{v}^+ + \alpha\,\theta\,\mathbf{d}_s^p).$$

Using the second equation of the scaled predictor system (3.2) and the computations given in (5.17), page 2643 in [8], we get the following:

$$\begin{aligned}\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha) &= \mu\left(\left(\mathbf{v}^+\right)^2 + \alpha\theta\mathbf{v}^+\left(-\frac{1}{2}\mathbf{v}^+\right) + \alpha^2\,\theta^2\,\mathbf{d}_x^p\mathbf{d}_s^p\right) \\ &= \mu\left(\left(1-\frac{1}{2}\alpha\theta\right)\left(\mathbf{v}^+\right)^2 + \alpha^2\,\theta^2\,\mathbf{d}_x^p\mathbf{d}_s^p\right). \end{aligned} \tag{4.6}$$

From (4.6) and following the same ideas as in [8], we have the following:

$$
\begin{aligned}
\min\left(\frac{\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)}{\mu\left(1-\frac{1}{2}\alpha\theta\right)}\right) &= \min\left(\left(\mathbf{v}^+\right)^2+\frac{\alpha^2\theta^2}{1-\frac{1}{2}\alpha\theta}\mathbf{d}_x^p\mathbf{d}_s^p\right)\\
&\geq \min\left(\left(\mathbf{v}^+\right)^2-\frac{\alpha^2\theta^2}{1-\frac{1}{2}\alpha\theta}\|\mathbf{d}_x^p\mathbf{d}_s^p\|_\infty\mathbf{e}\right)\\
&\geq \min\left(\left(\mathbf{v}^+\right)^2-\frac{\theta^2}{1-\frac{1}{2}\theta}\|\mathbf{d}_x^p\mathbf{d}_s^p\|_\infty\mathbf{e}\right) \qquad (4.7)
\end{aligned}
$$

The last inequality follows from the fact that

$$h(\alpha)=\frac{\alpha^2\theta^2}{1-\frac{1}{2}\alpha\theta}$$

is strictly increasing for $0\leq\alpha\leq 1$ and each fixed $0<\theta<1$. Moreover, using

$$|1-v_i^+|\leq\|\mathbf{e}-\mathbf{v}^+\|,\ \ \forall i=1,\ldots,n$$

we have

$$1-\sigma^+\leq v_i^+\leq 1+\sigma^+,\ \ \forall i=1,\ldots,n.$$

Using this and (4.4) we have

$$\min\left(\mathbf{v}^+\right)^2\geq(1-\sigma^+)^2\geq(1-4\,\delta^+)^2 \qquad (4.8)$$

From Lemma 4.6 and (4.8) and using that $\|\cdot\|_\infty\leq\|\cdot\|$, we obtain

$$\min\left(\frac{\mathbf{x}^p(\alpha)\,\mathbf{s}^p(\alpha)}{\mu\,(1-\frac{1}{2}\,\alpha\,\theta)}\right) \ \geq\ (1-4\,\delta^+)^2-\frac{n\left(1+\frac{\kappa}{2}\right)\,(1+4\,\delta^+)^2\,\theta^2}{2-\theta} =: u(\delta^+,\theta,n). \qquad (4.9)$$

Now, it is clear that $u(\delta^+,\theta,n)>0$ is sufficient for strict feasibility, namely $\mathbf{x}^p(\alpha)\,\mathbf{s}^p(\alpha)>0$ for $0\leq\alpha\leq 1$. Therefore, $\mathbf{x}^p(\alpha)$ and $\mathbf{s}^p(\alpha)$ do not change sign on $0\leq\alpha\leq 1$. Since $\mathbf{x}^p(0)=\mathbf{x}^+>\mathbf{0}$ and $\mathbf{s}^p(0)=\mathbf{s}^+>\mathbf{0}$, we can conclude that $\mathbf{x}^p(1)=\mathbf{x}^p>\mathbf{0}$ and $\mathbf{s}^p(1)=\mathbf{s}^p>\mathbf{0}$.

We are ready to summarize the previous analysis in the following lemma.

Lemma 4.7. *Let* $(\mathbf{x}^+,\mathbf{s}^+)>\mathbf{0}$ *and* $\mu>0$ *such that* $\delta^+:=\delta(\mathbf{x}^+,\mathbf{s}^+,\mu)<\frac{1}{4}$. *Furthermore, let* $0<\theta<1$. *Let* $\mathbf{x}^p=\mathbf{x}^++\theta\Delta^p\mathbf{x},\quad \mathbf{s}^p=\mathbf{s}^++\theta\Delta^p\mathbf{s}$ *be the iterates after a predictor step. Then, in case of* $\varphi(t)=t^2-t+\sqrt{t}$, *we have* $\mathbf{x}^p,\mathbf{s}^p>\mathbf{0}$ *if* $u(\delta^+,\theta,n)>0$, *where*

$$u(\delta^+,\theta,n):=(1-4\,\delta^+)^2-\frac{n\left(1+\frac{\kappa}{2}\right)\,(1+4\,\delta^+)^2\,\theta^2}{2-\theta}.$$

This result is very similar to Lemma 5.5. [8] and the proof differs only in those parts that are affected by the AET function φ. Later, we will present how could we fix the parameters to get $u(\delta^+,\theta,n)>0$.

Let us introduce

$$\mathbf{v}^p=\sqrt{\frac{\mathbf{x}^p\mathbf{s}^p}{\mu^p}}, \qquad (4.10)$$

where $\mu^p=\left(1-\frac{\theta}{2}\right)\mu$.

The next lemma analyses the effect of a predictor step and the update of μ on the proximity measure.

Lemma 4.8. *Let $\varphi(t) = t^2 - t + \sqrt{t}$, $\delta^+ := \delta(\mathbf{x}^+, \mathbf{s}^+, \mu) < \frac{1}{4}$, $\mu^p = \left(1 - \frac{\theta}{2}\right)\mu$, where $0 < \theta < 1$, and let $\mathbf{x}^p$ and $\mathbf{s}^p$ denote the iterates after a predictor step. Then,*

$$\delta^p := \delta(\mathbf{x}^p, \mathbf{s}^p, \mu^p) \leq 2(10 + 4\kappa)\delta^2 + \frac{2\theta^2}{2-\theta} n \left(1 + \frac{\kappa}{2}\right)(1 + 4\delta^+)^2,$$

where $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu)$.

Proof. Let us compute δ^p by using (2.11)

$$\begin{aligned}
\delta^p &:= \left\| \frac{(\mathbf{e} - \mathbf{v}^p)\left(\mathbf{e} + (\mathbf{v}^p)^2 + (\mathbf{v}^p)^3\right)}{4(\mathbf{v}^p)^3 - 2\mathbf{v}^p + \mathbf{e}} \right\| \\
&= \left\| (\mathbf{e} - (\mathbf{v}^p)^2) \frac{\left(\mathbf{e} + (\mathbf{v}^p)^2 + (\mathbf{v}^p)^3\right)}{(\mathbf{e} + \mathbf{v}^p)\left(4(\mathbf{v}^p)^3 - 2\mathbf{v}^p + \mathbf{e}\right)} \right\| \\
&= \left\| (\mathbf{e} - (\mathbf{v}^p)^2)\, g(\mathbf{v}) \right\| \leq 2 \left\| \mathbf{e} - (\mathbf{v}^p)^2 \right\|, \qquad (4.11)
\end{aligned}$$

where we used that for the function $g : \mathbb{R}_+ \to \mathbb{R}$, $g(t) = \frac{1+t^2+t^3}{(1+t)(4t^3-2t+1)}$ we have $|g(t)| \leq 2$, for all $t > 0$. It should be mentioned that the last inequality of (4.11) can be obtained by using a modified version of Lemma 4.5. From (4.6), we have

$$(\mathbf{v}^p)^2 = \left(\mathbf{v}^+\right)^2 + \frac{\theta^2}{1 - \frac{1}{2}\theta} \mathbf{d}_x^p \mathbf{d}_s^p, \qquad (4.12)$$

which implies

$$\delta^p \leq 2 \left\| \mathbf{e} - \left(\mathbf{v}^+\right)^2 - \frac{\theta^2}{1 - \frac{1}{2}\theta} \mathbf{d}_x^p \mathbf{d}_s^p \right\| \leq 2 \left(\left\| \mathbf{e} - \left(\mathbf{v}^+\right)^2 \right\| + \frac{\theta^2}{1 - \frac{1}{2}\theta} \|\mathbf{d}_x^p \mathbf{d}_s^p\| \right). \qquad (4.13)$$

Using (4.12) and inequality (5.25) given in [8], we get

$$\left\| \mathbf{e} - \left(\mathbf{v}^+\right)^2 \right\| \leq \|\mathbf{e} - \mathbf{v}^2 - \mathbf{v}\mathbf{p}_\varphi\| + \left\| \frac{\mathbf{p}_\varphi^2}{4} \right\| + \left\| \frac{\mathbf{q}_\varphi^2}{4} \right\|, \qquad (4.14)$$

where $\mathbf{q}_\varphi$ is as given in (4.1). Using (2.11), after some calculations, we obtain

$$\begin{aligned}
\|\mathbf{e} - \mathbf{v}^2 - \mathbf{v}\mathbf{p}_\varphi\| &= \left\| \frac{-2\mathbf{v}^5 + 4\mathbf{v}^3 + \mathbf{v}^2 - 4\mathbf{v} + \mathbf{e}}{4\mathbf{v}^3 - 2\mathbf{v} + \mathbf{e}} \right\| \\
&= \left\| \frac{(4\mathbf{v}^3 - 2\mathbf{v} + \mathbf{e})(-2\mathbf{v}^5 + 4\mathbf{v}^3 + \mathbf{v}^2 - 4\mathbf{v} + \mathbf{e})}{(\mathbf{e} - \mathbf{v})^2(\mathbf{e} + \mathbf{v}^2 + \mathbf{v}^3)^2} \cdot \frac{\mathbf{p}_\varphi^2}{4} \right\| \\
&= \left\| h(\mathbf{v}) \, \frac{\mathbf{p}_\varphi^2}{4} \right\| \leq 8 \left\| \frac{\mathbf{p}_\varphi^2}{4} \right\|, \qquad (4.15)
\end{aligned}$$

where in the last inequality we used that $|h(t)| \leq 8$, for all $t > 0$, where

$$h(t) = \frac{(4t^3-2t+1)(-2t^5+4t^3+t^2-4t+1)}{(1-t)^2(1+t^2+t^3)^2} = \frac{(-4t^3+2t-1)(2t^3+4t^2+2t-1)}{(1+t^2+t^3)^2}.$$

Note that the last inequality of (4.15) can be derived by using a modified version of Lemma 4.5. Using Lemma 4.4, (3.3), (4.14) and (4.15), we have

$$\left\| \mathbf{e} - \left(\mathbf{v}^{+}\right)^2 \right\| \leq 9 \left\| \frac{\mathbf{p}_{\varphi}^2}{4} \right\| + \left\| \frac{\mathbf{q}_{\varphi}^2}{4} \right\| \leq 9 \left(\frac{\|\mathbf{p}_{\varphi}\|}{2} \right)^2 + \left(\frac{\|\mathbf{q}_{\varphi}\|}{2} \right)^2 \leq (10+4\kappa)\delta^2. \tag{4.16}$$

From Lemma 4.6, (4.11) and (4.16), we have

$$\delta^p \leq 2(10+4\kappa)\delta^2 + \frac{2\theta^2}{2-\theta} n \left(1+\frac{\kappa}{2}\right)(1+4\delta^{+})^2, \tag{4.17}$$

which gives the result. □

The next lemma provides an upper bound for the duality gap after a main iteration.

Lemma 4.9. *Let $\varphi(t) = t^2 - t + \sqrt{t}$ and $0 < \theta < 1$. If $\mathbf{x}^p$ and $\mathbf{s}^p$ are the iterates obtained after the predictor step of the algorithm, then*

$$\left(\mathbf{x}^p\right)^T \mathbf{s}^p \leq \left(1 - \frac{\theta}{2} + \frac{\theta^2}{8}\right) \left(\mathbf{x}^{+}\right)^T \mathbf{s}^{+} < \frac{2(n+9\delta^2)\mu^p}{2-\theta}.$$

Proof. Using (4.6) with $\alpha = 1$ and (4.10), we have

$$\begin{aligned} \left(\mathbf{x}^p\right)^T \mathbf{s}^p &= \mu^p \mathbf{e}^T \left(\mathbf{v}^p\right)^2 = \mu \mathbf{e}^T \left(\left(1-\frac{\theta}{2}\right) \left(\mathbf{v}^{+}\right)^2 + \theta^2 \mathbf{d}_x^p \mathbf{d}_s^p \right) \\ &= \left(1-\frac{\theta}{2}\right) \left(\mathbf{x}^{+}\right)^T \mathbf{s}^{+} + \mu\theta^2 \left(\mathbf{d}_x^p\right)^T \mathbf{d}_s^p. \end{aligned} \tag{4.18}$$

Using the technique given in the proof of Lemma 5.7 given in [8], we obtain

$$\left(\mathbf{x}^p\right)^T \mathbf{s}^p \leq \left(1 - \frac{\theta}{2} + \frac{\theta^2}{8}\right) \left(\mathbf{x}^{+}\right)^T \mathbf{s}^{+}.$$

If $0 < \theta < 1$, then

$$1 - \frac{\theta}{2} + \frac{\theta^2}{8} < 1. \tag{4.19}$$

Using this, $\mu^p = \left(1 - \frac{\theta}{2}\right)\mu$, (4.19) and Lemma 4.3, we have

$$\begin{aligned} \left(\mathbf{x}^p\right)^T \mathbf{s}^p &\leq \left(1 - \frac{\theta}{2} + \frac{\theta^2}{8}\right) \left(\mathbf{x}^{+}\right)^T \mathbf{s}^{+} \\ &< \left(\mathbf{x}^{+}\right)^T \mathbf{s}^{+} < \mu(n+9\delta^2) = \frac{2(n+9\delta^2)\mu^p}{2-\theta}, \end{aligned} \tag{4.20}$$

which yields the result. □

In the following section, we provide a set of parameters for which the PC IPA is well defined. This means that if we have a feasible solution for which $\delta(\mathbf{x}, \mathbf{s}, \mu) \leq \tau$, then after a corrector and predictor step, we get that $\delta^p(\mathbf{x}^p, \mathbf{s}^p, \mu^p) \leq \tau$. We use some ideas given in [11].

5. Complexity bound

We introduce the following notation:

$$L(\tau) = \frac{(1-2\tau)^2}{(1+2\tau)^2}. \tag{5.1}$$

The next lemma gives a condition related to the parameters τ and θ for which the PC IPA is well defined.

Lemma 5.1. *Let $\varphi(t) = t^2 - t + \sqrt{t}$, $\tau = \frac{1}{2r(10+4\kappa)}$, where $r \geq 2$ and $0 < \theta \leq \frac{1}{q(10+4\kappa)\sqrt{n}}$, where $q \geq 2$. If*

i. $r \leq \frac{2}{5}q$,

ii. $\frac{n\left(1+\frac{\kappa}{2}\right)\theta^2}{2-\theta} < L(\tau)$,

then the PC IPA proposed in Algorithm 1 is well defined.

Proof. Let $(\mathbf{x},\mathbf{s}) \in \mathscr{F}^+$ such that $\delta(\mathbf{x},\mathbf{s},\mu) \leq \tau$. After a corrector step, applying Lemma 4.2 we have

$$\delta^+ := \delta(\mathbf{x}^+,\mathbf{s}^+,\mu) \leq 2(10+4\kappa)\delta^2,$$

which is monotonically increasing with respect to δ. Using this and $r \geq 2$ we get

$$\delta^+ \leq 2(10+4\kappa)\tau^2 = \frac{1}{2r^2(10+4\kappa)} = \frac{1}{r}\tau \leq \frac{1}{2}\tau. \tag{5.2}$$

Using $\frac{n\left(1+\frac{\kappa}{2}\right)\theta^2}{2-\theta} < L(\tau)$ and (5.2), we obtain

$$(1-4\delta^+)^2 - \frac{n\left(1+\frac{\kappa}{2}\right)\theta^2(1+4\delta^+)^2}{2-\theta} > (1-2\tau)^2 - \frac{n\left(1+\frac{\kappa}{2}\right)\theta^2}{2-\theta}(1+2\tau)^2 > 0, \tag{5.3}$$

where we used that the function appearing on the left hand side is increasing with respect to δ^+. Condition $\bar{u}(\delta^c,\theta,r) > 0$ from Lemma 4.7 is satisfied. Furthermore, using $\tau = \frac{1}{2r(10+4\kappa)}$, $r \geq 2$ and (5.2) we have $\delta^+ \leq \frac{1}{4r(10+4\kappa)} < \frac{1}{80} < \frac{1}{4}$. From Lemma 4.8, after a predictor step and a μ-update we have

$$\delta^p \leq 2(10+4\kappa)\delta^2 + \frac{2\theta^2}{2-\theta}n\left(1+\frac{\kappa}{2}\right)(1+4\delta^+)^2, \tag{5.4}$$

where $\delta := \delta(x,s,\mu)$. Using $\theta \leq \frac{1}{q(10+4\kappa)\sqrt{n}}$ and $\kappa \geq 0$, we get

$$\frac{2}{2-\theta} \leq \frac{20q}{20q-1}. \tag{5.5}$$

We also consider

$$\theta \leq \frac{1}{q(10+4\kappa)\sqrt{n}} \leq \frac{1}{q\left(1+\frac{\kappa}{2}\right)\sqrt{n}}. \tag{5.6}$$

Using (5.5) and (5.6), we get

$$\begin{aligned} \frac{2n\left(1+\frac{\kappa}{2}\right)\theta^2}{2-\theta} &\leq n\left(1+\frac{\kappa}{2}\right)\frac{20q}{20q-1}\,\frac{1}{q(10+4\kappa)\sqrt{n}}\,\frac{1}{q\left(1+\frac{\kappa}{2}\right)\sqrt{n}} \\ &= \frac{20}{q(20q-1)}\,\frac{1}{10+4\kappa} = \frac{40r}{(20q-1)q}\tau. \end{aligned} \tag{5.7}$$

Using (5.2) and $\kappa \geq 0$, we obtain

$$\delta^+ \leq 2(10+4\kappa)\,\delta^2 \leq \frac{1}{r}\tau = \frac{1}{2r^2(10+4\kappa)} \leq \frac{1}{20r^2}. \tag{5.8}$$

From (5.7) and (5.8) we get

$$\frac{2n\left(1+\frac{\kappa}{2}\right)\,\theta^2\,(1+4\delta^+)^2}{2-\theta} \quad \leq \quad \frac{40r}{(20q-1)q}\left(1+\frac{1}{5r^2}\right)^2\tau. \tag{5.9}$$

From $r \geq 2$, $q \geq 2$ and $r \leq \frac{2}{5}q$ we obtain

$$\begin{aligned} \frac{40r}{(20q-1)q}\left(1+\frac{1}{5r^2}\right)^2 &= 40\frac{r}{q}\frac{1}{20q-1}\left(1+\frac{1}{5r^2}\right)^2 \\ &\leq \frac{40}{39}\cdot\frac{2}{5}\cdot\frac{441}{400} < \frac{1}{2}. \end{aligned} \tag{5.10}$$

From (5.2), (5.4), (5.9) and (5.10) we get

$$\delta^p < \left(\frac{1}{2}+\frac{1}{2}\right)\tau = \tau, \tag{5.11}$$

hence the PC IPA is well defined, hence if we have a feasible solution for which $\delta(\mathbf{x},\mathbf{s},\mu) \leq \tau$, then after a corrector and predictor step we will get that $\delta^p(\mathbf{x}^p,\mathbf{s}^p,\mu^p) \leq \tau$. □

The following lemma gives a sufficient condition for satisfying Condition ii. from Lemma 5.1.

Lemma 5.2. *Let $\varphi(t) = t^2 - t + \sqrt{t}$, $\tau = \frac{1}{2r(10+4\kappa)}$, where $r \geq 2$ and $0 < \theta \leq \frac{1}{q(10+4\kappa)\sqrt{n}}$, where $q \geq 2$. Consider L given in (5.1). If*

$$\frac{1}{q^2} < L\left(\frac{1}{20r}\right), \tag{5.12}$$

then condition ii. of Lemma 5.1 is satisfied.

Proof. From $0 < \theta \leq \frac{1}{q(10+4\kappa)\sqrt{n}}$ and $q \geq 2$ we have

$$\frac{1}{2-\theta} \leq 1. \tag{5.13}$$

Furthermore, using the properties of the function φ, (5.6) and $\kappa \geq 0$ we get

$$\frac{n\left(1+\frac{\kappa}{2}\right)\theta^2}{2-\theta} \leq n\left(1+\frac{\kappa}{2}\right)\frac{1}{q^2\left(1+\frac{\kappa}{2}\right)^2 n} = \frac{1}{q^2\left(1+\frac{\kappa}{2}\right)} \leq \frac{1}{q^2}. \tag{5.14}$$

Besides this, from $\tau = \frac{1}{2r(10+4\kappa)}$ and $\kappa \geq 0$ we obtain

$$\tau \leq \frac{1}{20r}. \tag{5.15}$$

It should be mentioned that the function $L(\tau)$ is strictly decreasing with respect to τ, hence using (5.15) we obtain

$$L(\tau) \geq L\left(\frac{1}{20r}\right). \tag{5.16}$$

In this way, using (5.12), (5.13), (5.14) and (5.16) we obtain

$$\frac{n\left(1+\frac{\kappa}{2}\right)\theta^2}{2-\theta} \leq \frac{1}{q^2} < L\left(\frac{1}{20r}\right) \leq L(\tau),$$

which yields the result. ☐

Lemma 5.3. *Let $\varphi(t)=t^2-t+\sqrt{t}$, $\tau=\frac{1}{2r(10+4\kappa)}$, where $r\geq 2$ and $0<\theta\leq\frac{1}{q(10+4\kappa)\sqrt{n}}$, where $q\geq 2$. If Condition i. from Lemma 5.1 is satisfied, then Condition ii. from Lemma 5.1 also holds.*

Proof. Consider the following function

$$z(r)=\frac{2\left(1+\frac{1}{10r}\right)}{5r\,\left(1-\frac{1}{10r}\right)},$$

which is decreasing with respect to r. Thus, for $r\geq 2$ we have

$$z(r)\leq z(2)<1. \tag{5.17}$$

Using (5.17) and Condition i. of Lemma 5.1 we obtain that

$$q\geq\frac{5}{2}r>\frac{1}{\sqrt{L\left(\frac{1}{20r}\right)}}.$$

Hence, if Condition i. of Lemma 5.1 holds, then (5.12) is satisfied. Using Lemma 5.2, we obtain that Condition ii. from Lemma 5.1 also holds. ☐

Corollary 5.4. *Let $\varphi(t)=t^2-t+\sqrt{t}$, $\tau=\frac{1}{2r(10+4\kappa)}$, where $r\geq 2$ and $0<\theta\leq\frac{1}{q(10+4\kappa)\sqrt{n}}$. If $q\geq\frac{5}{2}r$, then the PC IPA proposed in Algorithm 1 is well defined.*

Lemma 5.5. *Let $\varphi(t)=t^2-t+\sqrt{t}$, $(\mathbf{x}^0,\mathbf{s}^0)\in\mathscr{F}^+$, $\tau=\frac{1}{2r(10+4\kappa)}$, where $r\geq 2$ and $0<\theta\leq\frac{1}{q(10+4\kappa)\sqrt{n}}$, where $q\geq\frac{5}{2}r$. Furthermore, let $\mu^0=\frac{\left(\mathbf{x}^0\right)^T\mathbf{s}^0}{n}$ and $\delta(\mathbf{x}^0,\mathbf{s}^0,\mu^0)\leq\tau$. Moreover, let $\mathbf{x}^k$ and $\mathbf{s}^k$ be the iterates obtained after k iterations of Algorithm 1. Then, $\left(\mathbf{x}^k\right)^T\mathbf{s}^k\leq\varepsilon$ for*

$$k\geq 1+\left\lceil\frac{2}{\theta}\log\frac{3\left(\mathbf{x}^0\right)^T\mathbf{s}^0}{2\varepsilon}\right\rceil.$$

Proof. Using $\tau=\frac{1}{2r(10+4\kappa)}$, $r\geq 2$, $n\geq 1$ and $\kappa\geq 0$, we get

$$\delta^2\leq\frac{1}{16(10+4\kappa)^2}<\frac{n}{1600}. \tag{5.18}$$

Using Lemma 4.9 we have

$$\left(\mathbf{x}^k\right)^T\mathbf{s}^k<\frac{2(n+9\delta^2)\mu^k}{2-\theta}<\frac{1609n\mu^k}{1600\left(1-\frac{\theta}{2}\right)}<\frac{3}{2}\left(1-\frac{\theta}{2}\right)^{k-1}\left(\mathbf{x}^0\right)^T\mathbf{s}^0.$$

The inequality $\left(\mathbf{x}^k\right)^T\mathbf{s}^k\leq\varepsilon$ holds if $\frac{3}{2}\left(1-\frac{\theta}{2}\right)^{k-1}\left(\mathbf{x}^0\right)^T\mathbf{s}^0\leq\varepsilon$. We take logarithms, hence

$$(k-1)\log\left(1-\frac{\theta}{2}\right)+\log\frac{3\left(\mathbf{x}^0\right)^T\mathbf{s}^0}{2}\leq\log\varepsilon.$$

From $\log(1+\theta) \leq \theta$, $\theta \geq -1$, it follows that the above inequality holds if

$$-\frac{\theta}{2}(k-1) + \log \frac{3\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{2} \leq \log \varepsilon.$$

This yields the desired result. □

Theorem 5.6. *Let $\varphi(t) = t^2 - t + \sqrt{t}$, $\tau = \frac{1}{2r(10+4\kappa)}$, where $r \geq 2$ and $0 < \theta \leq \frac{1}{q(10+4\kappa)\sqrt{n}}$, where $q \geq \frac{5}{2}n$. Then, the PC IPA proposed in Algorithm 1 is well defined and it performs at most*

$$\mathscr{O}\left((10+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{2\varepsilon}\right)$$

iterations. The output is a pair $(\mathbf{x}, \mathbf{s})$ satisfying $\mathbf{x}^T\mathbf{s} \leq \varepsilon$.

Proof. The result follows from Corollary 5.4 and Lemma 5.5. □

Corollary 5.7. *Let $\varphi(t) = t^2 - t + \sqrt{t}$ and consider $0 < \tau \leq \frac{1}{40+16\kappa}$ and $0 < \theta \leq \frac{4}{5\sqrt{n}}\tau$. Then, the PC IPA proposed in Algorithm 1 is well defined and it performes at most*

$$\mathscr{O}\left((10+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{2\varepsilon}\right)$$

iterations. The output is a pair $(\mathbf{x}, \mathbf{s})$ satisfying $\mathbf{x}^T\mathbf{s} \leq \varepsilon$.

Proof. If $\tau \leq \frac{1}{40+16\kappa}$, then we can find $r \geq 2$ such that

$$\tau = \frac{1}{2r(10+4\kappa)}. \tag{5.19}$$

Using $\theta \leq \frac{4}{5\sqrt{n}}\tau$ and $\tau \leq \frac{1}{40+16\kappa}$ we have

$$\theta \leq \frac{4}{5\sqrt{n}}\tau \leq \frac{1}{5(10+4\kappa)\sqrt{n}}.$$

Hence, we can find $q \geq 5$ such that

$$\theta = \frac{1}{q(10+4\kappa)\sqrt{n}}. \tag{5.20}$$

Moreover, from (5.19), (5.20) and $\theta \leq \frac{4}{5\sqrt{n}}\tau$ we have

$$\theta = \frac{1}{q(10+4\kappa)\sqrt{n}} = \frac{2\tau r}{q\sqrt{n}} \leq \frac{4}{5\sqrt{n}}\tau,$$

therefore $q \geq \frac{5}{2}r$ holds. All conditions from Lemma 5.1 are satisfied, hence from Corollary 5.4 and Lemma 5.5 we obtain the desired result. □

6. NUMERICAL RESULTS

We implemented a variant of the proposed PC IPA in the C++ programming language. There is a detailed explanation about the implementation in [38]. We did all computations on a desktop computer with Intel quad-core 3.3 GHz processor and 16 GB RAM.

We used Algorithm 2 in our implementation. We set the values $\theta = 0.999$ and $\varepsilon = 10^{-5}$ and the maximum of α to 3. We did not give upper bound for the proximity parameter τ and we observed that in some cases the value of the proximity measure might become very large. However, we ensured that feasibility holds in all cases.

Algorithm 2: Implemented PC IPA for $P_*(\kappa)$-LCPs

Let $\varepsilon = 10^{-5}$ *be the accuracy parameter,* $\theta = 0.999$ *the update parameter and* $\tau > 0$ *the proximity parameter,* $\sigma = 0.9999$ *decrease parameter,* $0 < \alpha < 3$ *the step length.*
Assume that for $(\mathbf{x}^0, \mathbf{s}^0) \in \mathscr{F}^+$*, the* $\left(\mathbf{x}^0\right)^T \mathbf{s}^0 = n\mu^0$*,* $\mu^0 > 0$ *holds such that* $-M\boldsymbol{x} + \boldsymbol{s} = \boldsymbol{q}$ *for a qiven matrix M and vector* $\boldsymbol{q}$*.*

begin

$k := 0$;

while $\left(\mathbf{x}^k\right)^T \mathbf{s}^k > \varepsilon$ **do begin**

$(\mu^p)^k = (1-\theta)\,\mu^k$;

(predictor step)

compute $(\Delta^p x^k, \Delta^p s^k)$ *from system (3.2) using (2.5);*

calculate local κ *with* $\kappa(\Delta^p x^k) = -\dfrac{1}{4}\dfrac{\Delta^p (x^k)^T M \Delta^p x^k}{\sum_{i \in \mathscr{I}_+(\Delta^p(x^k))} \Delta^p x_i^k (M\Delta^p x^k)_i}$

compute α *with*

$\alpha_{x_h} := \{-\frac{x_i}{\Delta x_i} : \Delta x_i < 0\}$ *and* $\alpha_{s_h} := \{-\frac{s_i}{\Delta s_i} : \Delta s_i < 0\}$

$\alpha_h := \sigma \cdot \min\{\alpha_{x_h}, \alpha_{s_h}\}$

$\alpha := \min\{\alpha_h, 3\}$

let $(\mathbf{x}^p)^k := \mathbf{x}^k + \alpha\Delta^p \mathbf{x}^k$ *and* $(\mathbf{s}^p)^k := \mathbf{s}^k + \alpha\Delta^p \mathbf{s}^k$;

(corrector step)

compute $(\Delta^c x^k, \Delta^c s^k)$ *from system (2.7) using (2.5);*

compute α *with*

$\alpha_{x_h} := \{-\frac{x_i}{\Delta x_i} : \Delta x_i < 0\}$ *and* $\alpha_{s_h} := \{-\frac{s_i}{\Delta s_i} : \Delta s_i < 0\}$

$\alpha_h := \sigma \cdot \min\{\alpha_{x_h}, \alpha_{s_h}\}$

$\alpha := \min\{\alpha_h, 3.0\}$

let $(\mathbf{x}^+)^k := (\mathbf{x}^p)^k + \alpha\Delta^c x^k$ *and* $(\mathbf{s}^+)^k := (\mathbf{s}^p)^k + \alpha\Delta^c s^k$;

(update of the parameters and the iterates)

$\mathbf{x}^{k+1} := (\mathbf{x}^+)^k, \quad \mathbf{s}^{k+1} := (\mathbf{s}^+)^k, \quad \mu^{k+1} := \left(\frac{(\mathbf{x}^{k+1})^T \mathbf{s}^{k+1}}{n}\right)^k$;

$k := k+1$;

end

end.

In the literature there are only a few numerical results about LCPs, where the problem's matrix has positive handicap. Gurtuna et al. [19] and Asadi et al. [1] provided numerical results based on 2×2 or 3×3 matrices that are related to $P_*(\kappa)$-LCPs having positive handicap. Darvay et al. [8] presented numerical results on $P_*(\kappa)$-problems that were generated by Illés and Morapitiye [20] and on $P_*(\kappa)$-LPCs with matrix having exponential handicap, see (6.1). They also used their PC IPA to solve special, non-sufficient LCPs. They tested copositivity of matrices and they obtained promising results in spite of the fact that they did not have $P_*(\kappa)$-matrices in those cases.

It is proven by De Klerk and E.-Nagy [12] that the handicap of the matrix can be exponential in the size of the problem. They considered a matrix which was proposed by Csizmadia:

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}, \tag{6.1}$$

and E.-Nagy proved that $\hat{\kappa}(M) = 2^{2n-8} - 0.25$, see [13].

We tested our implementation on problems generated by using matrix (6.1) in order to show promising results on problems with matrix having exponential handicap. We generated test examples based on the results of E.-Nagy and Varga showed in [14].

We considered 10-10 problems by using the matrix (6.1) with generating random starting point pairs $(\mathbf{x},\mathbf{s})$ of sizes of 0, 30, 50, 70, 100.

With these starting points we calculated the right hand side vector by $\mathbf{q} = -M\mathbf{x} + \mathbf{s}$. The number of the generated examples from the corresponding intervals can be found in Table 1. These examples can be found on the website [39].

values of $\mathbf{s}$values of $\mathbf{x}$	$[0,1]^n$	$[0,10]^n$	$[0,100]^n$	$[9,11]^n$	$[9900,11000]^n$
$[0,1]^n$	50	0	0	50	0
$[0,10]^n$	0	50	0	0	0
$[0,100]^n$	0	0	50	0	50
$[9,11]^n$	50	0	0	0	0
$[9900,11000]^n$	0	0	50	0	0

Table 1. Testing data generated from (6.1).

We compared our algorithm with other PC IPA that was presented in [8]. We also used another versions of our implemented IPA by changing the right hand side of the Newton system and scaled system and using other AET functions and a kernel function. In this way we compared these versions with our new PC IPA using the AET function $\varphi_1(t) = t^2 - t + \sqrt{t}$. The chosen AET functions are $\varphi_2(t) = t$ and $\varphi_3(t) = \sqrt{t}$. The parametric kernel function used here is $\Psi_{p,\sigma} = \frac{t^{p+1}-1}{p+1} + \frac{e^{\sigma(1-t)}-1}{\sigma}$, $p \in [0,1]$, $\sigma \geq 1$, see [18, 40]. We set $p = 1$ and $\sigma = 5$ in our computations based on the kernel function $\Psi_{p,\sigma}$.

Table 2 contains the average iteration numbers for each IPA on the problems with size 100x100. We can see in Table 2 that if $\mathbf{x}$ and $\mathbf{s}$ are from the same interval, the size of the interval does not significantly change the number of iterations. Furthermore, it can be seen in these cases that the number of iterations seems to be slightly increasing if the size of the problem is increasing in spite of the exponential handicap. We show this observation on Figure 1.

IPAProblems	$\mathbf{x},\mathbf{s} \in [0,1]^{100}$	$\mathbf{x},\mathbf{s} \in [0,10]^{100}$	$\mathbf{x},\mathbf{s} \in [0,100]^{100}$
$\varphi_1(t)$	51.85	56.1	55.15
$\varphi_2(t)$	51.7	55.7	55.0
$\varphi_3(t)$	52.15	56.2	55.45
$\Psi_{p,\sigma}$	62.2	67.0	65.75

Table 2. Average iteration numbers on problems of size 100x100.

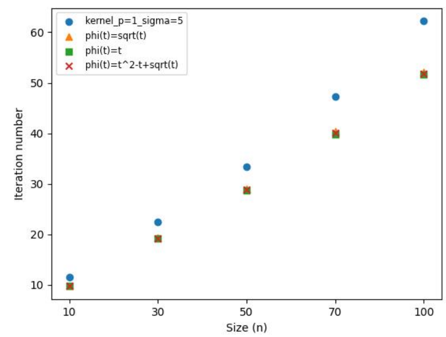

Figure 1. Iteration numbers for all different variants are far from the theoretical ones.

It can be seen from Table 3, Figure 2 and Figure 3 that if the values of $\mathbf{x}$ are significantly smaller than the values of $\mathbf{s}$, then the problem can be solved with almost constant iteration number, even in case of larger-sized problems. This observation can be seen in Table 3 in case of problems with size 100x100 and 200x200.

IPAProblems	$\mathbf{x} \in [0,1]^{100}, \mathbf{s} \in [9,11]^{100}$	$\mathbf{x} \in [0,100]^{100}, \mathbf{s} \in [9900,11000]^{100}$	$\mathbf{x} \in [0,1]^{200}, \mathbf{s} \in [9,11]^{200}$	$\mathbf{x} \in [0,100]^{200}, \mathbf{s} \in [9900,11000]^{200}$
$\varphi_1(t)$	6.0	4.45	7.0	4.0
$\varphi_2(t)$	6.0	5.05	7.0	4.8
$\varphi_3(t)$	6.5	5.55	8.1	6.8
$\Psi_{p,\sigma}$	7.0	5.05	9.7	6.0

Table 3. Average iteration numbers on problems with size 100x100 and 200x200.

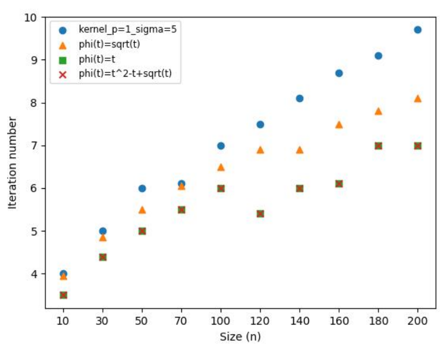

Figure 2. Iteration numbers for all different variants are slightly increasing.

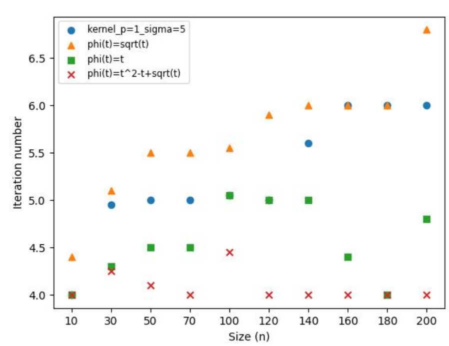

Figure 3. Iteration numbers for all different variants are almost constant.

However, if the values of $\mathbf{s}$ are the smaller ones (see in Table 4 and Figure 4), the solution can be computed with more iteration numbers than in the case when the values of $\mathbf{x}$ are similar or even smaller than the values of $\mathbf{s}$. We showed this result in Table 4 in the cases of problems with size 100x100.

IPAProblems	$\mathbf{x} \in [9,11]^{100}, \mathbf{s} \in [0,1]^{100}$	$\mathbf{x} \in [9900,11000]^{100}, \mathbf{s} \in [0,100]^{100}$
$\varphi_1(t)$	201.1	377.7
$\varphi_2(t)$	200.65	370.3
$\varphi_3(t)$	201.4	379.8
$\Psi_{p,\sigma}$	249.5	490.4

Table 4. Average iteration numbers on problems with size 100x100.

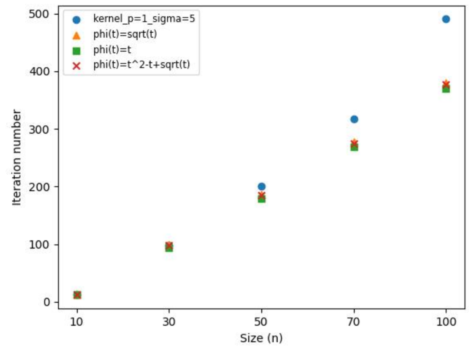

Figure 4. Iteration numbers are larger than in the case shown in Figure 2

Finally, we showed that, the used kernel function has slightly worse iteration numbers than the shown three AET functions in most of the cases. Furthermore, the new AET function $\varphi_1(t)$ gives slightly better results than $\varphi_3(t)$, as well. In Table 3 and Figure 3 it can be seen that in these cases the function $\varphi_1(t)$ works slightly better than the other variants of the implemented algorithm, where the used search directions are different.

7. Conclusion

We proposed a new PC IPA for solving $P_*(\kappa)$-linear complementarity problems. In the AET technique, we used the function $\varphi(t) = t^2 - t + \sqrt{t}$ which has inflection point. It is interesting that the kernel corresponding to this AET function is neither self-regular, nor eligible. We showed that the iteration bound of the PC IPA matches the best known iteration bound for this type of PC IPAs. We also provided a set of parameters for which the PC IPA is well defined. In order to show the efficiency of the algorithm we presented numerical results, as well. From the numerical results, we can conclude that in several cases, the choice of the AET function plays important role in the number of iterations. As a future research we would like to understand which AET functions should be used in the different LCP problems in order to obtain better complexity bounds and numerical results, as well.

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