Commun. Optim. Theory 2024 (2024) 10 https://doi.org/10.23952/cot.2024.10

MATHRES

Communications in Optimization Theory

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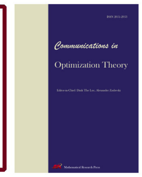

A NOTE ON THE CONTINUITY OF INVARIANT OPERATORS

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Dedicated to the memory of Professor Petr Zabrejko

Abstract. In this paper, we discuss conditions sufficient for the continuity of invariant operators, i.e., generally nonlinear operators commuting with the operator of multiplication by characteristic functions, acting in a large class of spaces of measurable functions over a finite, atomless measure space. Our setting includes, among many others, modular function spaces, Lebesgue spaces, Orlicz and Musielak-Orlicz spaces, variable Lebesgue spaces, and hence extends the scope beyond the normed spaces. Typical examples of invariant operators include superposition operators (also known as Nemytskii operators), multiplication operators and operators constructed from them.

Keywords. Invariant operator; Modular function space; Nonlinear operator; Superposition operator.

2020 Mathematics Subject Classification. 47H30, 45P05, 46A80, 46E30.

1. Preface

After receiving an invitation to contribute to the volume dedicated to the memory of Petr Zabrejko, I researched my mathematical library and found a late article on the superposition operator, published jointly by Zabrejko and Jürgen Appell [3]. In the introduction to this paper, the Authors mentioned how after many years they returned to that topic on the occasion of Gérard Bourdaud's birthday, and in this connection they quoted a famous French saying ”On revient toujours à son premier amour”. Intrigued by this declaration, I dived into their earlier work. In their very interesting 1989 paper [2], I found, rather surprisingly, a reference to my own, very early work [13], in which, among other things, I discussed continuity of invariant operators acting in Banach function spaces. Immediately then I decided, in accordance with the French proverb, to return to the ”first love” and to continue this topic in the context of modular function spaces. And this is how this contribution came to being.

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Received: May 27, 2023; Accepted: August 10, 2023.

2. Introduction

Let (X, Σ, μ) be a finite, atomless, complete measure space. Let $\mathscr{M}$ denote the space of measurable real-valued functions on X.

Definition 2.1. [13] Let E be any linear subspace of $\mathscr{M}$. An operator $T : E \to \mathscr{M}$ is called an invariant operator if

(a) $T(0) = 0$;
(b) $T(1_A f) = 1_A T(f)$, for any $f \in E$ and any $A \in \Sigma$, where 1_A denotes a characteristic function of the set A.

Remark 2.2. It is obvious that the composition, the sum and the product of any number of invariant operators is still an invariant operator.

Remark 2.3. Let E be any linear subspace of $\mathscr{M}$ and let $T : E \to \mathscr{M}$ be any, in general, nonlinear operator. For a $w \in E$ we define the translated operator $T_w : E \to \mathscr{M}$ by

$$T_w(u)(x) = T(w)(x) - T(w+u)(x).$$

Note that $T_w(0) = 0$ and that, in any linear topology (or more generally, with respect to any convergence preserving translation, e.g., convergence a.e., more on such convergence in [12, 17, 18, 24]) T is continuous at w is equivalent to T_w being continuous at zero.

Observe that in view of Remark 2.3, to prove a continuity at w of an operator T satisfying

$$T(w) - T(w + 1_A u) = 1_A (T(w) - T(u+w)) \tag{2.1}$$

for any $u, w \in E$ any $A \in \Sigma$, it suffices to prove a continuity of an associated invariant operator T_w at zero. Hence, our results below can be easily extended to the more general class of operators satisfying (2.1). See [13] for a more formal approach.

Example 2.4. Let F be a real-valued function of $X \times \mathbb{R}$ into $\mathbb{R}$, where $X \subset \mathbb{R}^n$ is a Lebesgue measurable set of finite measure. An operator defined by $S_F(u)(x) = F(x, u(x))$, where $u \in \mathscr{M}, x \in X$ is called a superposition operator (or Nemytskii operator). It is easy to check that S_F is invariant in the sense of Definition (2.1) provided $F(x, 0) = 0$ for all $x \in X$ and F satisfies the following Carathéodory conditions

(1) $F(x, \cdot)$ is continuous for almost every $x \in X$;
(2) $F(\cdot, u)$ is measurable for all $u \in \mathbb{R}$.

A classical result [4] states that S_F satisfying the above conditions is continuous at zero with respect to the convergence in measure. For more discussion on this operator, the reader is referred to the papers by Appell and Zabrejko [2, 3], and the literature referenced there. It is worthwhile noticing that the linear multiplication operator $T(u)(x) = f(x)u(x)$ is an important example of the superposition operator, see e.g. [1], and hence of the invariant operator. By Remark 2.2 many operators constructed from superposition operators remain invariant.

Example 2.5. Let us consider a Hammerstein-type nonlinear integral operator

$$H(u)(x) = \int_{[0,1]} k(x,s) F(s, u(s)) dm(s),$$

and observe that $H = K \circ S_F$, where K is a linear integral operator. Suppose that we want to prove continuity of H at zero. We will show in Theorem 4.4 that under suitable assumptions

S_F is continuous. Hence, we will be able to reduce our problem to the investigation of the continuity of a linear operator K. Typically, it is much simpler to prove continuity of linear than non-linear operators.

3. Preliminaries

Throughout this paper, we always assume that (X,Σ,μ) is a finite, atomless, complete measure space. By $\mathscr{S}\subset\mathscr{M}$ we will denote the space of all Σ-measurable real-valued simple functions (or shortly, simple functions) on X, that is functions of the form $\sum_{i=1}^{n}\alpha_i 1_{A_i}$, where $A_i\in\Sigma$, $\alpha_i\in\mathbb{R}$. As usual, we will identify functions equal μ almost everywhere and sets with a symmetric difference being a μ-null set. In the sequel, equality and other operations on functions and sets will be always understood that way.

Definition 3.1. We say that $\rho:\mathscr{M}\to[0,+\infty]$ is a function modular on (X,Σ,μ) if the following conditions are satisfied:

(i) $\rho(f)=0$ if and only if $f=0$ almost everywhere;
(ii) ρ is an even function, that is, $\rho(-f)=\rho(f)$ for every $f\in\mathscr{M}$;
(iii) ρ is monotone, that is, $|f(x)|\le|g(x)|$ for almost everywhere implies $\rho(f)\le\rho(g)$, where $f,g\in\mathscr{M}$;
(iv) ρ is orthogonally subadditive, that is, $\rho(1_{A\cup B}f)\le\rho(1_A f)+\rho(1_B f)$ for any $A,B\in\Sigma$ such that $A\cap B=\emptyset$, where $f\in\mathscr{M}$;
(v) ρ is order continuous in $\mathscr{S}$, that is, $g_n\in\mathscr{S}$ and $|g_n(x)|\downarrow 0$ almost everywhere implies that $\rho(g_n)\downarrow 0$;
(vi) $\rho(f)=\sup\{\rho(g):g\in\mathscr{S},\ |g(x)|\le|f(x)|,\ x\in X\}$;

The class of all function modulars on (X,Σ,μ) will be denoted by $\mathscr{R}(X,\Sigma,\mu)$. Remark 3.2 below will provide further justification for this terminology.

Remark 3.2. For $f\in\mathscr{M}$ and $A\in\Sigma$, let us denote $\rho(f,A)=\rho(1_A f)$. Also, by convention, we will sometimes write $\rho(\alpha,A)$ instead of $\rho(\alpha 1_A)$. We will use these notations when convenient. Noting also that $\rho(\alpha,A)=0$ for any $\alpha>0$ if and only if A is a μ-null set, it is straightforward to prove that $\rho:\mathscr{M}\times\Sigma\to[0,+\infty]$ is a function modular in the sense of Definition 2.1.1 in the book [16], where ρ-null sets are actually equal to μ-null sets. Therefore, we can use results of the standard theory of modular function spaces as per the framework defined by Kozlowski in [14, 15, 16]. Note that, as proved there, a function modular ρ, as defined in Definition 3.1, is a modular in the sense of a standard definition, see, e.g., [22, 23], that is,

(1) $\rho(f)=0$ if and only if $f=0$;
(2) $\rho(-f)=\rho(f)$;
(3) $\rho(\alpha f+\beta g)\le\rho(f)+\rho(g)$, for $\alpha,\beta\ge 0$, $\alpha+\beta=1$.

Observe also that the approach taken in Definition 3.1 goes along the lines of Definition 3.1 in the book [10]. However, the latter definition assumes convexity and the Fatou property of ρ, while we do not assume either of these properties. As it is demostrated in examples below, from the point of view of applications of the results of this paper, it is important that ρ does not have to be convex.

Example 3.3. Typical examples of modular function spaces as defined in Definition 3.1 are:

(1) Lebesgue spaces: L^p-spaces for $0<p<+\infty$;

(2) variable Lebesgue spaces: $L^{p(t)}$-spaces, where $0 < p(t) < +\infty$;
(3) Orlicz and Musielak-Orlicz spaces;

and many spaces built upon them, see e.g., [10, 16, 19, 21, 22]. Note that these spaces do not have to be normed spaces. They are however F-spaces with the F-norm generated by the modular (see Definition 3.9). Variable Lebesgue spaces, typical examples of Musielak-Orlicz spaces, have found recently a lot of attention because of their wide range of applications, see e.g. [5, 6, 7, 8, 11, 25]. Modular function spaces are closely related to Banach function spaces (studied by Luxemburg and Zaanen, see e.g. [20]), generalised Orlicz spaces (studied by Orlicz and Musielak, see e.g. [22]), and ideal spaces (studied by Zabrejko, see e.g. [26, 27] and by Väth, see [26]).

We need to note an important for this paper property of function modulars, [16, Proposition 2.1.2], which can be also easily proved directly from the above properties of ρ.

Proposition 3.4. *$\rho(f,\cdot) : \Sigma \to [0,+\infty]$ is a sigma-subadditive measure, that is*

$$\rho\Big(f, \bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \rho(f, A_i), \tag{3.1}$$

for any sequence $\{A_i\}$ of measurable sets and any $f \in \mathscr{M}$.

The next result follows easily from Definition 3.1, the proof is standard.

Proposition 3.5. *Under the assumption of this paper, the following two conditions are equivalent:*

(1) $\rho(\alpha, A_n) \to 0$ for every $\alpha > 0$;
(2) $\mu(A_n) \to 0$,

where $A_n \in \Sigma$ for every $n \in \mathbb{N}$.

Using Proposition 3.5 we immediately obtain the next result.

Proposition 3.6. *Under the assumption of this paper, the convergence in measure μ is equivalent to the convergence in submeasure (ρ). Recall that $f_n \to 0(\rho)$ if for every $\alpha, \varepsilon > 0$ $\rho(\alpha, A_n(\varepsilon)) \to 0$, where $A_n(\varepsilon) = \{x \in X : |f_n(x)| \geq \varepsilon\}$; see [10, 16].*

Definition 3.7. Since ρ is a function modular, we can define the modular function space L_ρ in a standard way, that is, as the vector space consisting of all functions $f \in \mathscr{M}$ such that $\rho(\lambda f) \to 0$ if $\lambda \to 0$.

Remark 3.8. Observe that in the context of this paper, $X \in \Sigma$ and hence $\mathscr{S} \subset L_\rho$ because of part (v) of Definition 3.1. Therefore, $1_X \in L_\rho$, and hence every bounded measurable function is a member of L_ρ.

As usual, L_ρ will be equipped with an F-norm $\|\cdot\|_\rho$ defined by

$$\|f\|_\rho = \inf\Big\{\alpha > 0 : \rho\Big(\frac{f}{\alpha}\Big) \leq \alpha\Big\}. \tag{3.2}$$

Let us recall the definition of F-norms; see, e.g., [9, 22]

Definition 3.9. Let $\mathscr{X}$ be a real vector space. A function $\|\cdot\| : \mathscr{X} \to [0,+\infty)$ is called an F-norm if

(i) $\|u\| = 0$ if and only if $u = 0$;
(ii) $\|-u\| = \|u\|$;
(iii) $\|u+w\| \leq \|u\| + \|w\|$;
(iv) $\|\alpha_k u_k - \alpha u\| \to 0$ if $\alpha_k \to \alpha$ and $\|u_k - u\| \to 0$. It is easy to see that $\mathscr{X}$ becomes in this case a linear metric space with a metric $d(u,w) = \|u-w\|$. If this metric space is complete we call $(\mathscr{X}, \|\cdot\|)$ an F-space.

It is well known that every modular function space $(L_\rho, \|\cdot\|_\rho)$ is an F-space (see e.g. [16, Theorem 2.3.7]). We also know that the F-norm $\|\cdot\|_\rho$ is monotone.

Using Proposition 3.6 and the results from the general theory of modular function spaces we can easily prove the following result.

Proposition 3.10. *In the setting of this paper, described in this Section, the following implications are true for any sequence of functions $f_n \in L_\rho$: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$, where*

(a) $\|f_n\|_\rho \to 0$;
(b) $\rho(f_n) \to 0$;
(c) $f_n \overset{\mu}{\to} 0$;
(d) there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to 0$ almost everywhere.

In general, none of these implications can be reversed.

The following property of function modulars will play an important role in this paper.

Definition 3.11. Let $\rho \in \mathscr{R}(X, \Sigma, \mu)$. We say that ρ has the Δ_2 property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$.

The following characteristics of function modulars with the Δ_2-property are well known, see [10, 14, 15, 16] for more discussions on this topic.

Proposition 3.12. *Let $\rho \in \mathscr{R}(X, \Sigma, \mu)$. Then the following statements are equivalent:*

(1) ρ has the Δ_2-property;
(2) $\rho(f_n) \to 0$ if and only if $\|f_n\|_\rho \to 0$;
(3) $\|\cdot\|_\rho$ is order continuous in the whole of L_ρ, that is, $\|f_n\|_\rho \downarrow 0$ if $|f_n| \downarrow 0$ for any sequence $\{f_n\}$ of functions from L_ρ.

The next remark follows immediately from Proposition 3.12 part (3).

Remark 3.13. Let $\rho \in \mathscr{R}(X, \Sigma, \mu)$ be a function modular with the property Δ_2. Then $\|1_{(\cdot)} f\|_\rho$ is order continuous for every $f \in L_\rho$, that is, $\|1_{A_n} f\|_\rho \to 0$, whenever $A_n \downarrow \emptyset$.

4. Main results

Let us start with the following technical result being a function modular version of Lemma 5.1 in [13] proven there in the Banach function space setting (see [20] for foundations of the general theory of Banach function spaces).

Lemma 4.1. *Let $\rho \in \mathscr{R}(X, \Sigma, \mu)$ be a function modular space with the Δ_2-property. Then to every $\varepsilon > 0$ and every $f \in L_\rho$ there corresponds a $\delta > 0$ such that $\|1_D f\|_\rho < \varepsilon$ whenever $\mu(D) < \delta$.*

Proof. Assume to the contrary that there exists an $\varepsilon > 0$ and $f \in L_\rho$ such that for every $\delta_n = \frac{\mu(X)}{2^n}$ (recall that, as always in this paper, $\mu(X) < +\infty$) there exists a set $D_n \in \Sigma$ such that

(i) $\mu(D_n) < \delta_n$;
(ii) $\|1_{D_n} f\|_\rho > \varepsilon$.

Let $E_n = \bigcup_{k=n}^{\infty} D_k$. Hence, $\{E_n\}$ is a nonincreasing sequence of measurable sets and

$$\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(D_k) \le \mu(X) \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = 0. \tag{4.1}$$

Using Remark 3.13, via the Δ_2-property, from (4.1) we conclude that

$$0 < \varepsilon < \|1_{D_n} f\|_\rho \le \|1_{E_n} f\|_\rho \to 0,$$

as $n \to \infty$. The contradiction completes the proof. □

Lemma 4.2. *Let $\rho \in \mathscr{R}(X,\Sigma,\mu)$ be a function modular space with the Δ_2-property. Let $\alpha > 0$ and the sequences of sets $Y_n \in \Sigma$, and of functions $v_n \in L_\rho$ be such that*

(a) $\mu(Y_n) \to 0$;
(b) $\|1_{Y_n} v_n\|_\rho > \frac{2}{3}\alpha$, *for every* $n \in \mathbb{N}$;
(c) $\|1_{Z_n} v_n\|_\rho < \frac{1}{6}\alpha$, *for every* $n \in \mathbb{N}$, *where* $Z_n = \bigcup_{i=n+1}^{\infty} Y_i$.

Let $v = \sum_{k=1}^{\infty} 1_{W_k} v_k$, *where* $W_k = Y_k \setminus Z_k$. *Then* $v \notin L_\rho$.

Proof. Observe that $W_n \cap W_m = \emptyset$ for $n \neq m$. Hence, $1_{W_n} v = 1_{W_n} v_n$ and consequently

$$1_{W_n} v = 1_{Y_n} v_n - 1_{Z_n} v_n + 1_{Z_n \setminus Y_n} v_n. \tag{4.2}$$

Using (4.2) together with subadditivity and monotonicity of the F-norm we get

$$\begin{aligned} \|1_{W_n} v\|_\rho &\ge \|1_{Y_n} v_n\|_\rho - \|1_{Z_n} v_n\|_\rho - \|1_{Z_n \setminus Y_n} v_n\|_\rho \\ &\ge \|1_{Y_n} v_n\|_\rho - \|1_{Z_n} v_n\|_\rho - \|1_{Z_n} v_n\|_\rho \ge \frac{1}{3}\alpha > 0. \end{aligned} \tag{4.3}$$

Assume to the contrary that $v \in L_\rho$. Noting that $\mu(W_n) \le \mu(Y_n) \to 0$ and using Lemma 4.1, we conclude that $\|1_{W_n} v\|_\rho$ must tend to zero, which contradicts (4.3). The proof of the Lemma is complete. □

Lemma 4.3. *Let L_ρ be a modular function space. Let a sequence of functions $\{w_n\}$ be such that $w_n \in L_\rho$ for every $n \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \rho(w_i) < r$ for an $r > 0$. Let $\{W_n\}$ be a sequence of mutually disjoint sets. Let us define $w = \sum_{i=1}^{\infty} 1_{w_i} w_i$. Then $w \in L_\rho$.*

Proof. Let $\varepsilon > 0$ be fixed arbitrarily. There exists a $K \in \mathbb{N}$ such that

$$\sum_{i=K+1}^{\infty} \rho(w_i) < \frac{\varepsilon}{2}. \tag{4.4}$$

Since all functions $w_1, ..., w_K$ belong to L_ρ, there exists $0 < \lambda_0 < 1$ such that

$$\sum_{i=1}^{K} \rho(\lambda w_i) < \frac{\varepsilon}{2K}, \tag{4.5}$$

for every $0 < \lambda < \lambda_0$. For any such a λ, using (4.5) and (4.4), respectively, we have

$$\rho(\lambda w) = \rho\Big(\sum_{i=1}^{\infty} \lambda 1_{w_i} w_i\Big) \leq \sum_{i=1}^{\infty} \rho(\lambda 1_{w_i} w_i) \leq \sum_{i=1}^{K} \rho(\lambda w_i) + \sum_{i=K+1}^{\infty} \rho(w_i) < \varepsilon,$$

which means that $w \in L_\rho$, as claimed. □

We are now ready to prove the main result of this paper, the Continuity Theorem.

Theorem 4.4. *[Continuity Theorem] Let $\rho_E, \rho_H \in \mathscr{R}(X, \Sigma, \mu)$. Assume that ρ_H has Δ_2-property. Let $B_r = \{u \in L_{\rho_E} : \rho_E(u) < r\}$ and let $T : B_r \to L_{\rho_H}$ be an invariant operator in the sense of Definition 2.1. Assume that T is continuous at 0 with respect to the convergence in measure μ. Then T is (ρ_E, ρ_H)-continuous at 0, that is, $\rho_E(u_n) \to 0$ implies that $\rho_H(T(u_n)) \to 0$, where $u_n \in B_r$. Moreover, T is $(\|\cdot\|_{\rho_E}, \|\cdot\|_{\rho_H})$-continuous at 0.*

Proof. In view of Proposition 3.10, it is enough to prove the $(\rho_E, \|\cdot\|_{\rho_H})$-continuity of T at zero. Assume to the contrary that there exists a sequence of functions $z_n \in B_r$ such that $\lim_{n\to\infty} \rho_{\rho_E}(z_n) = 0$ while $\|T(z_n)\|_{\rho_H}$ does not tend to zero. There exists then an $\alpha > 0$ and a subsequence $\{w_n\}$ of $\{z_n\}$ such that

$$\|T(w_n)\|_{\rho_H} > \alpha \tag{4.6}$$

for every $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} \rho_E(w_n) < r.$$

By Definition 3.1 part (v) and the Δ_2-property of ρ_H, it follows that there exists a $\delta > 0$ such that $\|\delta 1_X\|_{\rho_H} \leq \frac{\alpha}{6}$. We will inductively construct a sequence of sets $Y_n \in \Sigma$ and a sequence of functions $u_n \in B_r$ such that, denoting $v_n = T(u_n)$, the sequences $\{Y_n\}$ and $\{v_n\}$ satisfy conditions (a), (b) and (c) from Lemma 4.2. Set $u_1 = w_1$, $Y_1 = X$ and $b_1 = \mu(X) < +\infty$, and assume that u_k, Y_k and b_k have been chosen. Since ρ_H satisfies Δ_2, by Lemma 4.1, choose b_{k+1} such that

$$0 < 2b_{k+1} < b_k,$$

and

$$\|1_D T(u_k)\|_{\rho_H} < \frac{\alpha}{6} \tag{4.7}$$

for any $D \in \Sigma$ such that $\mu(D) < 2b_{k+1}$. Since $\rho_E(w_n) \to 0$, it follows from Proposition 3.10 that $w_n \xrightarrow{\mu} 0$ and, by assumed μ-continuity of T at zero, that $T(w_n) \xrightarrow{\mu} 0$. Hence, there exists a natural number $k_0 > k$ such that $\mu(X \setminus G'_{k_0}) < b_{k+1}$, where $G'_{k_0} = \{x \in X : |T(w_{k_0})(x)| < \delta\}$. Define $G_{k+1} = G'_{k_0}$, $Y_{k+1} = X \setminus G_{k+1}$ and $u_{k+1} = w_{k_0}$. Observe that

$$\|1_{G_{k+1}} T(u_{k+1})\|_{\rho_H} \leq \|\delta 1_{G'_{k_0}}\|_{\rho_H} \leq \|\delta 1_X\|_{\rho_H} \leq \frac{\alpha}{6}. \tag{4.8}$$

By combining (4.8) with (4.6) we get

$$\|1_{Y_{k+1}} T(u_{k+1})\|_{\rho_H} \geq \|T(u_{k+1})\|_{\rho_H} - \|1_{G_{k+1}} T(u_{k+1})\|_{\rho_H} > \frac{2}{3}\alpha,$$

proving condition (b) of Lemma 4.2. To prove (a) let us note that

$$\mu(Y_{k+1}) = \mu(X \setminus G_{k+1}) < b_{k+1} < \frac{b_k}{2} \leq \frac{\mu(X)}{2^k} \to 0,$$

as $k \to \infty$. It remains to prove (c) of Lemma 4.2. To this end let us denote $Z_k = \bigcup_{i=k+1}^{\infty} Y_i$ and calculate

$$\mu(Z_k) \leq \sum_{i=k+1}^{\infty} \mu(Y_i) \leq \sum_{i=k+1}^{\infty} b_i \leq b_{k+1} \sum_{i=0}^{\infty} \frac{1}{2^i} = 2b_{k+1}.$$

Thus, by (4.7), $\|1_{Z_k} T(u_k)\|_{\rho_H} < \frac{\alpha}{6}$, proving (c) and finishing the inductive construction.

Let us define $W_k = Y_k \setminus Z_k$ and $u = \sum_{k=1}^{\infty} 1_{W_k} u_k$. By the σ-additivity of ρ we have

$$\rho_E(u) = \rho_E\Big(\sum_{k=1}^{\infty} 1_{W_k} u_k\Big) \leq \sum_{k=1}^{\infty} \rho_E(1_{W_k} u_k) \leq \sum_{k=1}^{\infty} \rho_E(u_k) \leq \sum_{n=1}^{\infty} \rho_E(w_n) < r,$$

which by Lemma 4.3 means that $u \in B_r \subset L_{\rho_E}$. Hence, $T(u) \in L_{\rho_H}$. On the other hand, since T is invariant and W_k are mutually disjoint, we get

$$\Big|\sum_{k=1}^{\infty} 1_{W_k} v_k\Big| = \Big|\sum_{k=1}^{\infty} 1_{W_k} T(u_k)\Big| = \Big|\sum_{k=1}^{\infty} T(1_{W_k} u_k)\Big|$$
$$= \Big|\sum_{k=1}^{\infty} T(1_{W_k} u)\Big| = \Big|\sum_{k=1}^{\infty} 1_{W_k} T(u)\Big| \leq \Big|T(u)\Big|,$$

almost everywhere, which implies that $\sum_{k=1}^{\infty} 1_{W_k} v_k \in L_{\rho_H}$ because $T(u) \in L_{\rho_H}$. On the other hand, based on our inductive construction, we can infer from Lemma 4.2 that $\sum_{k=1}^{\infty} 1_{W_k} v_k \notin L_{\rho_H}$. The contradiction completes the proof of Theorem 4.4.

□

The following counterexample shows that we cannot dispense in Theorem 4.4 with the assumption of the Δ_2-property of the target space.

Example 4.5. Let $X = [0,1]$ and μ be the Lebesgue measure on X. Let L_{ρ_E} and L_{ρ_H} be two Orlicz spaces over X generated by two φ-functions φ_E and φ_H, respectively. Assume that ρ_E satisfies Δ_2 but ρ_H does not. Since ρ_H does not satisfy Δ_2, it follows from Propositions 3.10 and 3.12 that there exists a sequence $\{v_n\}$ of elements from L_{ρ_H} such that $\rho_H(v_n) \to 0$ but $\|v_n\|_\rho > \alpha > 0$ for all $n \in \mathbb{N}$. Define $u_n(x) = (\varphi_E^{-1} \circ \varphi_H)(v_n(x))$ and observe that

$$\rho_E(u_n) = \int_X \varphi_E(u_n(x))d\mu(x) = \int_X \varphi_H(v_n(x))d\mu(x) = \rho_H(v_n) \to 0.$$

Let us now define a Carathéodory function $f(y) = (\varphi_H^{-1} \circ \varphi_E)(y)$ for $y \in X$ and the corresponding superposition operator S_f. Direct calculation shows that $S_f(u_n)(x) = v_n(x)$ and hence that

$$\|S_f(u_n)\|_{\rho_H} = \|v_n\|_{\rho_H} > \alpha > 0.$$

Therefore S_f cannot be continuous at zero. As an example of a φ-function φ_H such that ρ_{φ_H} does not have the Δ_2-property one can take $\varphi_H(v) = e^{|v|} - 1$; see, e.g., [19, 22].

Assuming the Δ_2-property this time for the domain side of T we obtain the following result.

Theorem 4.6. *Let $\rho_E, \rho_H \in \mathscr{R}(X,\Sigma,\mu)$. Assume that ρ_E has Δ_2-property. Let $T : L_{\rho_E} \to \mathscr{M}$ be an invariant operator. If $T(B_r) \subset L_{\rho_H}$, where $B_r = \{u \in L_{\rho_E} : \rho_E(u) < r\}$, then $T(L_{\rho_E}) \subset L_{\rho_H}$.*

Proof. Let us take an arbitrary $f \in L_{\rho_E}$. By Lemma 4.1 there exists $\delta > 0$ such that $\|1_D f\|_{\rho_E} < r$ provided $\mu(D) < \delta$. Since μ is finite and atomless there exists a sequence $\{X_i\}_{i=1}^p$ of measurable, mutually disjoint sets with $\mu(X_i) < \delta$ for $i = 1,...,p$, and such that $X = \bigcup_{i=1}^p X_i$. Thus $\|1_{X_i} f\|_{\rho_E} < r$, which implies that $T(1_{X_i} f) \in L_{\rho_H}$ for every $i = 1,...,p$. Since T is invariant it follows that

$$T(f) = \sum_{i=1}^p 1_{X_i} T(f) = \sum_{i=1}^p T(1_{X_i} f) \in L_{\rho_H}.$$

□

Combining Theorems 4.4 and 4.6 we easily get our next, very useful result.

Theorem 4.7. *Let $\rho_E, \rho_H \in \mathscr{R}(X,\Sigma,\mu)$. Assume that both ρ_E and ρ_H have the Δ_2-property. Let $T : L_{\rho_E} \to \mathscr{M}$ be an invariant operator which is continuous at 0 with respect to convergence in measure μ. Let $B \subset L_{\rho_E}$ be an open neighbourhood of zero in L_{ρ_E} . If $T(B) \subset L_{\rho_H}$ then $T : L_{\rho_E} \to L_{\rho_H}$ and it is $(\|\cdot\|_{\rho_E}, \|\cdot\|_{\rho_H})$-continuous at 0.*

Proof. Observe that there exists $0 < r < 1$ such that $\{u \in L_{\rho_E} : \|u\|_{\rho_E} < r\} \subset B$. It follows from the general modular space theory that $\rho_E(u) \le \|u\|_{\rho_E} < 1$, see e.g., [22, Theorem 1.5], and hence that $B_r = \{u \in L_{\rho_E} : \rho_E(u) < r\} \subset B$. The rest of the proof follows from Theorem 4.6. □

Remark 4.8. In view of comments in Example 2.4 all results of our paper are valid for an important case of the superposition operator. Recall that the superposition operator S_F is always an invariant operator which is continuous with respect to the convergence in measure provided F satisfies the Carathéodory conditions.

Remark 4.9. [Historical notes] Theorem 4.4 is an extension to modular function spaces of the continuity theorem for invariant operators acting in Banach function spaces, proved by Kozlowski in his 1980 paper [13, Theorem 5.2]. This continuity result was later referenced by Appell and Zabrejko in their 1989 paper [2], where they discussed various properties of the superposition operator.

Acknowledgements

The author would like to thank the anonymous referee for the valuable suggestions to improve the presentation of the paper.

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