Commun. Optim. Theory 2023 (2023) 9 https://doi.org/10.23952/cot.2023.9

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

OPTIMAL CONTROL STRATEGIES WITH MULTIPLE CLOSING INSTANTS FOR LINEAR SYSTEMS WITH DISTURBANCES

NATALIA M. DMITRUK*, DZMITRY A. KASTSIUKEVICH

Optimal Control Methods Department, Belarusian State University, Minsk, Belarus

Dedicated to the memory of Professor Rafail Gabasov

Abstract. This paper deals with an optimal control problem for a linear discrete-time system subject to input and state constraints and unknown bounded disturbances, where the control goal is to minimize a cost function used in linear explicit model predictive control. We define a solution to the problem under consideration in terms of optimal control strategies under the assumption that the state measurements of the system will become available at several future time instants (closing instants), the control loop at these instants will be closed and a new control input will be calculated. Such control strategies provide a compromise between a conservative optimal open-loop worst-case control and computationally demanding dynamic programming. A method for constructing optimal control strategies with one and multiple closing instants is proposed. The method reduces a multilevel optimization problem that arises from the definition of the control strategy to a number of linear programs resulting in low computational demands for the optimal strategy construction and its suitability for applications such as model predictive control, where the optimal control problem is solved online.

Keywords. Control strategy; Disturbance; Linear systems; Model predictive control; Optimal control.

2020 Mathematics Subject Classification. 93C05, 93B45.

INTRODUCTION

Optimal control has received a significant attention of the research community over the recent years. This is driven by growing complexity of control systems and their applications and substantial advances with respect to the computational power which allows solving challenging problems. One of the widely used applications of the optimal control theory and methods is model predictive control (MPC), see e.g. [29, 39] that by now has thousands of successful industrial applications [43].

The main application area of MPC is stabilization of linear and nonlinear plants. MPC, also referred to as receding horizon control, is a control technique that is based on real time solution of the so called predictive optimal control problem that is formulated on a finite control horizon

*Corresponding author.

E-mail address: dmitrukn@bsu.by (N.M. Dmitruk).

Received December 9, 2021; Accepted October 31, 2022.

for the state space model of the plant with the initial condition equal to the actual plant's state measurement. Input and state constraints are naturally taken into account in such formulation and the performance index of the predictive optimal control problem is chosen in such a way that its optimal value can serve as a Lyapunov function for the closed-loop system. The optimal input of the predictive problem is fed to the plant until the next state or output measurement is obtained and the new optimal control problem is formed and solved. For control problems on a finite horizon the MPC scheme is referred to as shrinking horizon MPC [13, 40].

In Minsk the research group headed by R. Gabasov proposed similar ideas for the classical problem of optimal feedback synthesis and called it *optimal real-time feedback control* [16, 17]. The method was developed for various classes of dynamical systems: linear [1, 16, 20], piecewise linear [2], nonlinear ODE's [3], linear time-delay systems [4], systems with distributed parameters [27]. The cornerstone of the optimal real-time feedback control as well as MPC is the availability of efficient numerical methods that can solve optimal control problems at the same rate as the measurements are obtained. Since predictive optimal control problems solved during each control process are of the same type and only the initial state and the control horizon change in the process, the numerical method should also adequately take into account some characteristics of the solution obtained at the previous time instant to find an initial guess for the current instant and speed up the solution in real time. Despite significant advances made by the group in previous years with respect to developing numerical methods for optimal control [15, 17], in the early 2000's new methods were proposed [1, 21]. They are based on two main ideas, namely, utilizing sampled-data inputs for control of continuous systems and the so called dynamic realization of the dual adaptive method for solving linear programs [14, 17]. The system studied in this paper is discrete-time, therefore the use of the adaptive method proved to be natural and very efficient in our numerical experiments.

Previously mentioned papers were devoted to the classical synthesis problem that is formulated for deterministic systems. The systems under uncertainties were studied in [11, 12, 18, 19, 22, 23, 24, 25, 26]. A set-membership uncertainty, as opposed to the stochastic uncertainty, was chosen. In such uncertainty model all unknown values are elements of a given compact set, then the control objective is to guarantee constraints satisfaction for all realizations of the unknown values and estimate the cost under the worst-case realization, i.e. the problems are formulated as optimal guaranteed (robust) control problems [36, 42]. Methods for optimal robust real-time state feedbacks construction for systems under set-membership disturbances were studied in [12, 18, 19, 22, 24] and optimal measurement feedbacks for systems with incomplete and inexact state measurements were considered in [10, 11, 23, 25, 26].

While for deterministic systems the solution of predictive optimal control problems is constructed in open-loop sense and in a particular control process the control input coincides with the dynamic programming solution, for systems under uncertainties different formulations of the optimal control problem can be proposed depending on a priory information about the system's behavior. When no information about the future states is taken into account the open-loop worst-case solution is obtained. Then the optimal feedback defined on the base of open-loop worst-case controls is usually referred to as *open-loop optimal feedback control*. In robust model predictive control the corresponding approach is called the open-loop MPC. It is well known that open-loop worst-case controls are very conservative, see e.g. [38] and discussion in

Section 2. A possible remedy is a formulation of the optimal control problem in a dynamic programming sense which yields a feedback that is optimal with respect to all possible feedbacks [6, 38, 41]. The corresponding MPC is referred to as the *closed-loop or min-max feedback MPC*. Its shortcomings are also well known and result from the curse of dimensionality.

Taking into account the drawbacks of the open-loop solution and the dynamic programming solution, a reasonable idea is to include only "some feedback elements" to achieve a compromise between the performance of the feedback and the numerical simplicity of the open-loop control. Realization of this idea in practice includes e.g. solutions that are a weighted sum of a fixed structure feedback and an open-loop control with optimization over the latter and a feedback gain matrix [30, 35], formulations of the problem as a single linear program as in explicit MPC [6, 31], or defining optimal control strategies based on the assumption that the loop is only closed at a finite number of time instants, e.g. not at all sampling instants [9, 12, 18, 19, 32, 33, 34]. The latter approach was originally proposed by R.Gabasov's group and called *optimal multiple closable feedback*.

This work presents a combination of approaches reported in papers [9, 12, 18, 19, 32, 33, 34, 37] applied to a predictive optimal control problem formulated in linear explicit model predictive control [5, 6, 7]. The problem in [6] is formulated for a linear discrete-time system subject to disturbances taking values in a polyhedral set, input and state constraints and with the objective to minimize the deviation of the trajectory from the steady-state in mixed $1/\infty$-norms. While [5, 6, 7] formulate the optimal control problem as a single multiparametric linear program (mpLP) depending on the initial state as a parameter and use the corresponding mpLP methods [7] to obtain an optimal feedback solution (mainly for problems on a very short control horizon), we assume only a small number of time instants when the loop is closed (*closing instants*) and also obtain a number of non-parametric linear programs only one of which has to be solved online.

The first work where closing instants were introduced was devoted to linear terminal optimal control problems (problems with linear terminal constraints and linear Mayer performance index) [18, 19]. It presents a general, rather conceptual, formulation of the approach, which practical realization was done only for a well-chosen example. An attempt to improve applicability of the approach for optimal real-time feedback control was made in paper [22], where the so called *closure sets* were introduced and their polyhedral approximations were proposed. The method in [22] is iterative and each iteration imply refining the closure sets and then constructing the open-loop control input that steers the system into the refined set corresponding to the closest future closing instant. This approach numerically is quite consuming since a large number of optimization problems is solved on each iteration and online. Therefore, its application in real time is still questionable. Starting from paper [12] and then [32] we established that the closure sets can be parametrized offline in a special way. The obtained parametrization allows to abandon the refinement of closure sets and the iterations, since the problem to be solved online can be formulated as a single linear program easily solved by existing solvers. This paper presents a development of advances made in [12, 32] for the optimal control problem from [6]. We emphasize, that only a predictive optimal control problem is solved in terms of optimal control strategies with closing instants, but the optimal closable feedback based on the strategies is not discussed in this paper. The latter is easily constructed as in [12, 22, 33] once the method, proposed here, is effectively implemented for real-time computations.

The overall paper is structured as follows. In Section 1 we outline the problem formulation and review some results related to optimal open-loop control in Section 2. Section 3 introduces the optimal control strategy with one closing instant and presents an effective method for its construction. A generalization for the case of multiple closing instants is discussed in Section 4. In Sections 3.4 and 4.2, we illustrate the proposed approach by numerical examples and in Section 3.5, we discuss how to implement the method efficiently in order to use it in optimal real-time feedback control or MPC.

1. Problem formulation

We consider a linear discrete-time time-invariant control system subject to unknown bounded disturbances and finite control horizon

$$\begin{gathered} x(t+1) = Ax(t) + Bu(t) + Mw(t), \;\; x(0) = x_0, \\ t = 0, 1, \dots, T-1, \end{gathered} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^r$ is the control, $w(t) \in \mathbb{R}^p$ is the disturbance at time t, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $M \in \mathbb{R}^{n\times p}$ are given matrices and x_0 is a given initial state.

System (1.1) is subject to input and state constraints

$$\begin{gathered} u(t) \in U = \{u \in \mathbb{R}^r : ||u||_\infty \le u_{\max}\},\; t = 0, 1, \dots, T-1, \\ x(t) \in X(t) = \{x \in \mathbb{R}^n :\; H(t)x \le g(t)\},\; t = 1, 2, \dots, T, \end{gathered} \tag{1.2}$$

where $||z||_\infty = \max_i |z_i|$, $H(t) \in \mathbb{R}^{m\times n}$, $g(t) \in \mathbb{R}^m$ are such that $X(t)$ are bounded, $t = 1, 2, \dots, T$.

The disturbance is unknown, bounded

$$w(t) \in W = \{w \in \mathbb{R}^p : ||w||_\infty \le w_{\max}\}, \;\; t = 0, 1, \dots, T-1,$$

and we are interested in robust constraint satisfaction, i.e. (1.2) should be satisfied for all possible disturbances.

The control objective is to minimize the following cost function

$$\sum_{t=0}^{T-1} \big(||Qx(t)||_\infty + ||Ru(t)||_\infty\big) + ||Px(T)||_\infty, \tag{1.3}$$

where $Q, P \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{r\times r}$ are given nonsingular matrices, see [6].

The following notations are used throughout the paper: $u(\cdot) = (u(t), t = 0, 1, \dots, T-1)$, $w(\cdot) = (w(t), t = 0, 1, \dots, T-1)$, $U^t = \underbrace{U \times \ldots \times U}_{t \text{ times}}$, same for W^t.

A trajectory of system (1.1) corresponding to a control input $u(\cdot) \in U^T$ and a disturbance $w(\cdot) \in W^T$ is denoted by $x(t|x_0, u, w)$, $t = 0, 1, \dots, T$.

For any $\Delta_j = \{T_j, T_j+1, \dots, T_{j+1}-1\}$ with $0 \le T_j < T_{j+1} \le T$, the control input and the disturbance on Δ_j are denoted by $u_j(\cdot) = (u_j(t), t \in \Delta_j)$, $w_j(\cdot) = (w_j(t), t \in \Delta_j)$, and $x(t|x_j, u_j, w_j)$ is the state at time t of system (1.1) with the initial state $x(T_j) = x_j$, input $u_j(\cdot)$ and disturbance $w_j(\cdot)$.

The simplest approach for controlling system (1.1) and achieving the stated objectives is to construct the optimal open-loop (worst-case) control $u^0(\cdot)$. It is an input that depends on time t only and is a minimizer for all possible realizations of the disturbance, i.e. for all $w(\cdot) \in W^T$. The optimal open-loop control is constructed before the control process starts, based only on a priori information about the initial state and disturbances. It doesn't take into account that there may be state measurements available in the future and the possibility to make corrections to the control according to the measurements. As a result it is easy to construct the optimal open-loop control or show that it does not exist, however, it is very conservative, since one open-loop input $u^0(\cdot)$ is supposed to steer all possible trajectories of $x(t|x_0,u^0,w)$, $w(\cdot) \in W^T$, in the state constraints sets. This might even lead to infeasibility, if the set W is large. The details of the open-loop solution of the problem under consideration will be discussed in Section 2.

On the other hand the problem can be formulated as the dynamic programming. Bellman equations for the problem under consideration have the form

$$V(t,x) = \min_{u \in U} \max_{w \in W} \left\{ ||Qx||_\infty + ||Ru||_\infty + V(t+1, Ax+Bu+Mw) \right\},$$
$$t = 0,1,\ldots,T-1,$$
$$V(T,x) = ||Px||_\infty, \;\; x \in \mathbb{R}^n.$$

The resulting solution is a feedback $u^0(t,x)$, $t = 0,1,\ldots,T-1$, $x \in \mathbb{R}^n$, that is optimal with respect to all possible feedbacks. This is the solution one would like to construct, however, the computational demands for solving the dynamic programming problem are too high. The optimal feedback in problem under consideration is constructed numerically in [6] for low dimensional examples with short control horizons.

As emphasized in the introduction, in the frame of this work, we consider an intermediate approach between the dynamic programming solution and the optimal open-loop control that was first proposed in [18, 19]. The idea of [18, 19], later developed in [12, 22, 32, 33, 34], is that in the optimal control problem formulation the loop is only closed at a small number of future *closing instants*, i.e. not at all control instants $t = 0,1,\ldots,T-1$ as in the dynamic programming. When the loop is closed, a new state measurement becomes available, a new optimization problem for the rest of the control interval is formulated and solved to construct a new control input that takes into account the obtained measurement. As we will show in numerical experiments, this leads to a trade-off between the computational demands of the dynamical programming and the conservatism of the open-loop solution. We will also show that the performance of the control process can be tuned by the number and the position of the closing instants. The case of one closing instant will be addressed in Section 3 and than the technique will be developed for multiple closing instants in Section 4.

2. Optimal open-loop control

In this section minimization of cost (1.3) subject to system's dynamics (1.1) and constraints (1.2) is defined in terms of open-loop (worst-case) control.

The input $u(\cdot)$ is called a feasible open-loop control if the corresponding trajectory robustly satisfies the state constraints, i.e.

$$x(t|x_0,u,w) \in X(t), \; t = 1,2,\ldots,T, \; \forall w(\cdot) \in W^T.$$

A feasible open-loop control guarantees the worst-case cost

$$\max_w \left\{ \sum_{t=0}^{T-1} \left(||Qx(t)||_\infty + ||Ru(t)||_\infty \right) + ||Px(T)||_\infty \right\}.$$

Then the optimal open-loop control is the solution to a min-max problem

$$J(x_0) = \min_u \max_w \left\{ \sum_{t=0}^{T-1} \left(||Qx(t)||_\infty + ||Ru(t)||_\infty \right) + ||Px(T)||_\infty \right\} \tag{2.1}$$

subject to

$$\begin{gathered} x(t+1) = Ax(t) + Bu(t) + Mw(t), \;\; x(0) = x_0, \\ ||u(t)||_\infty \le u_{\max}, \;\; t = 0, 1, \ldots, T-1, \\ H(t)x(t) \le g(t), \; t = 1, 2, \ldots, T, \; \forall w(\cdot) \in W^T. \end{gathered}$$

Problem (2.1) can be rewritten as a linear program after a number of equivalent standard reformulations, see [6, 8]. We outline them here because same arguments will be used in presentation of new results for optimal control strategies in Sections 3 and 4.

First we discuss how to guarantee robust satisfaction of the i-th state constraint, $i = 1, 2, \ldots, m$, at time t. Obviously if the worst-case state satisfies the constraint under consideration then it is robustly satisfied for all disturbances:

$$h_i(t)^\top x(t|x_0, u, w) \le \max_w h_i(t)^\top x(t|x_0, u, w) \le g_i(t),$$

where $h_i(t)^\top$ is the i-th row of the matrix $H(t)$. Then due to linearity of the system (1.1) we derive the nominal dynamics

$$\begin{gathered} x_0(t+1) = Ax_0(t) + Bu(t), \; x_0(0) = x_0, \\ t = 0, 1, \ldots, T-1, \end{gathered} \tag{2.2}$$

and represent the i-th constraint in the form

$$h_i(t)^\top x_0(t) + \max_w h_i(t)^\top x(t|0, 0, w) \le g_i(t).$$

Denoting

$$\gamma_i(t) = \max_w h_i(t)^\top x(t|0, 0, w) = w_{\max} \sum_{s=0}^{t-1} ||h_i(t)^\top A^s M||_1,$$

and introducing the vector of estimates of the worst-case disturbances $\gamma(t) = (\gamma_i(t), i = 1, \ldots, m)$, we construct the so called tightened sets

$$\bar{X}(t) = \left\{ x \in \mathbb{R}^n : H(t)x \le g(t) - \gamma(t) \right\}, \;\; t = 1, 2, \ldots, T,$$

for the state constraints.

Now if the nominal system (2.2) satisfies the tightened constraints, i.e.,

$$x_0(t) \in \bar{X}(t), \;\; t = 1, 2, \ldots, T,$$

then system (1.1) satisfies (1.2) under all possible realizations of disturbances.

Secondly, we exploit the fact that maximum in (2.1) is attained at one of the vertices of the hypercube W^T. Therefore, enumerating the vertices in an index set L and denoting the l-th vertex by $w^l(\cdot)$, $l \in L$, we rewrite the cost of problem (2.1) in an equivalent form

$$J(x_0) = \min_u \max_{l \in L} \left\{ \sum_{t=0}^{T-1} \left(||Qx(t|x_0, u, w^l)||_\infty + ||Ru(t)||_\infty \right) + ||Px(T|x_0, u, w^l)||_\infty \right\}.$$

Finally, using equivalent epigraph form [8, p. 134, 150] we obtain the open-loop optimal control problem (2.1) in the deterministic form

$$\min_{u,\alpha} \ \alpha, \tag{2.3}$$

subject to

$$\begin{gathered} x_0(t+1) = Ax_0(t) + Bu(t),\ x_0(0) = x_0, \\ x^l(t+1) = Ax^l(t) + Bu(t) + Mw^l(t),\ x^l(0) = x_0,\ l \in L, \\ ||u(t)||_\infty \le u_{\max},\ \ t = 0,1,\dots,T-1, \\ H(t)x_0(t) \le g(t) - \gamma(t),\ t = 1,2,\dots,T, \\ \sum_{t=0}^{T-1} \big(||Qx^l(t)||_\infty + ||Ru(t)||_\infty\big) + ||Px^l(T)||_\infty \le \alpha,\ l \in L. \end{gathered}$$

Problem (2.3) is a convex problem. It can be further reformulated as a linear program, see [6]. The following slack variables are used

$$||Ru(t)||_\infty \le \varepsilon_u(t),\ \ ||Qx^l(t)||_\infty \le \varepsilon_x^l(t),\ \ t = 0,1,\dots,T-1,\ \ ||Px^l(T)||_\infty \le \varepsilon_x^l(T),\ \ l \in L,$$

and the state variables are replaced according to $x(t) = A^t x_0 + \sum\limits_{s=0}^{t-1} A^{t-s-1}\big(Bu(s) + Mw(s)\big)$. The resulting linear program has the form:

$$\begin{gathered} \min_{u,\alpha,\varepsilon_u,\varepsilon_x} \ \alpha, \\ -\varepsilon_u(t)\mathbb{1}_r \pm Ru(t) \le 0,\ \ \pm u(t) \le u_{\max}\mathbb{1}_r, \\ t = 0,\dots,T-1, \\ \sum_{s=0}^{t-1} H(t)A^{t-s-1}Bu(s) \le g(t) - \gamma(t) - H(t)A^t x_0, \\ t = 1,2,\dots,T, \\ -\varepsilon_x^l(t)\mathbb{1}_n \pm \sum_{s=0}^{t-1} QA^{t-s-1}Bu(s) \le \mp\Big(QA^t x_0 + \sum_{s=0}^{t-1} QA^{t-s-1}Mw^l(s)\Big), \\ t = 0,\dots,T-1, \\ -\varepsilon_x^l(T)\mathbb{1}_n \pm \sum_{s=0}^{T-1} PA^{T-1-s}Bu(s) \le \mp\Big(PA^T x_0 + \sum_{s=0}^{T-1} PA^{T-s-1}Mw^l(s)\Big), \\ \sum_{t=0}^{T} \varepsilon_x^l(t) + \sum_{t=0}^{T-1} \varepsilon_u(t) - \alpha \le 0, \\ l \in L, \end{gathered} \tag{2.4}$$

where $\mathbb{1}_n$, $\mathbb{1}_r$ are the n-vector and the r-vector of ones; $\pm$ and $\mp$ are used to shorten the notations and mean that the corresponding constraint is taken into account twice, with $+$ sign as well as with $-$ sign.

Problem (2.4) is a linear program in an inequality form. It has $(4r+m)T + (2n(T+1)+1)|L|$ constraints and $(r+1)T + (T+1)|L| + 1$ variables. Obviously, the dimensions here substantially depend on the number $|L|$ of vertices of W^T, hence, on the control horizon T. In [6] only short horizons are chosen to guarantee that problem (2.4) is solved online.

Other shortcomings of the open-loop solution are well described in [33, 38, 41] and were discussed in Section 1. Here we mention that infeasibility of the open-loop problem (2.1) results from the possible emptiness of the set $\bar{X}(t)$ for some t.

3. Optimal control strategy with one closing instant

To overcome the conservatism and feasibility problems of open-loop controls we follow [12, 18, 22, 33] and assume that at one future time instant $T_1 \in \{1,2,\dots,T-1\}$ a state measurement is taken into account, i.e. the control loop is closed and a new control is calculated for the rest of the time interval. This assumption leads us to a definition of a *control strategy with one closing instant* T_1.

Suppose that on the interval $\Delta_0 = \{0,1,\dots,T_1-1\}$ a control input $u_0(\cdot) = u_0(\cdot|x_0)$ is chosen and denote by

$$X(T_1|x_0,u_0) = \left\{x = x(T_1|x_0,u_0,w_0), w_0(\cdot) \in W^{T_1}\right\}$$

the set of all possible states, that can be reached by time T_1 if the process starts from the initial state $x(0) = x_0$ under the input $u_0(\cdot)$. In a particular control process, depending on the actual disturbance, some state $x^*(T_1)$ will be measured, $x^*(T_1) \in X(T_1|x_0,u_0)$.

We formulate the main assumption as follows:

Assumption 3.1. *Before the control process starts it is known that at time T_1 we can*

1) measure the actual state $x^*(T_1)$;

2) choose a new control input $u_1(\cdot) = u_1(\cdot|x^*(T_1))$ on the interval $\Delta_1 = \{T_1, T_1+1,\dots,T-1\}$.

According to Assumption 3.1 we choose a control input $u_0(\cdot)$ and use it on the interval Δ_0. At time T_1 we measure $x^*(T_1)$ and calculate a new control $u_1(\cdot|x^*(T_1))$. Since $x^*(T_1)$ is not known in advance (at time $t=0$) we have to define the control on Δ_1 for each $x_1 \in X(T_1|x_0,u_0)$. Therefore, we look for a solution of the optimal control problem in terms of a control strategy (with one closing instant T_1) that has the form

$$\pi_1 = \pi_1(0,x_0) = \{u_0(\cdot|x_0); u_1(\cdot|x_1), x_1 \in X(T_1|x_0,u_0)\}.$$

The input $u_0(\cdot|x_0)$ is further referred to as an *initial control* for π_1.

A trajectory of system (1.1) corresponding to a strategy π_1 and a disturbance $w(\cdot) = (w_0(\cdot), w_1(\cdot))$ is defined as a sequential solution of two systems, see [12, 33]:

$$x(t+1) = Ax(t) + Bu_0(t) + Mw_0(t),\ \ x(0) = x_0,\ t \in \Delta_0,$$
$$x(t+1) = Ax(t) + Bu_1(t|x(T_1)) + Mw_1(t),\ \ x(T_1) = x(T_1|x_0,u_0,w_0),\ t \in \Delta_1.$$

3.1. **Feasible and optimal control strategies.** To determine a feasible control strategy π_1 we use the dynamic programming arguments and consequently consider time intervals Δ_1 (after the closing instant) and Δ_0 (before the closing instant).

On the time interval Δ_1 the control input has to be feasible with respect to constraints (1.2). Therefore, for a fixed x_1 the control $u_1(\cdot|x_1) \in U^{T-T_1}$ is chosen in such a way that

$$x(t|x_1,u_1,w_1) \in X(t),\ t = T_1+1, T_1+2,\dots,T,\ \forall w_1(\cdot) \in W^{T-T_1}. \tag{3.1}$$

Let X_1 denote a set of all states $x_1 \in \mathbb{R}^n$ for which a feasible control $u_1(\cdot|x_1)$, as defined above, exists. This set is referred to as the *closure set* at time T_1.

Assumption 3.2. *The closure set* $X_1 \neq \varnothing$.

On the time interval Δ_0 the control input $u_0(\cdot)$ has to steer the system robustly in the state constraint sets $X(t)$:

$$x(t|x_0,u_0,w_0) \in X(t),\ t = 1,2,\dots,T_1,\ \forall w_0(\cdot) \in W^{T_1}, \tag{3.2}$$

and guarantee that for all $x_1 \in X(T_1|x_0,u_0)$ there exists a feasible control input $u_1(\cdot|x_1)$, so that control process can be continued after T_1. The latter condition is satisfied if

$$X(T_1|x_0,u_0) \subseteq X_1. \tag{3.3}$$

Assumption 3.3. *There exists* $u_0(\cdot)$ *such that (3.2), (3.3) are satisfied.*

Under Assumptions 3.2 and 3.3 there exists a feasible control strategy π_1. A priori verification of Assumptions 3.2 and 3.3 seems to be impossible, therefore we will propose a constructive approach to do it in Sections 3.2, 3.3.

Remark 3.4. If there exists no closing instant T_1 such that both assumptions are satisfied, then one should try to find control strategies with multiple closing instants as in Section 4.

For each $x_1 \in X_1$ we define a cost-to-go function at time T_1:

$$V_1(x_1) = \min_{u_1} \max_{w_1} \left\{ \sum_{t=T_1}^{T-1} \left(||Qx(t|x_1,u_1,w_1)||_\infty + ||Ru_1(t)||_\infty\right) + ||Px(T|x_1,u_1,w_1)||_\infty \right\} \tag{3.4}$$

subject to state constraints (3.1). If $x_1 \notin X_1$ then $V_1(x_1) := +\infty$ by definition.

On the base of the strategy π_1 define another feasible strategy

$$\bar{\pi}_1 = \{u_0(\cdot|x_0); u_1^0(\cdot|x_1), x_1 \in X(T_1|x_0,u_0)\},$$

where $u_0(\cdot|x_0)$ is the initial control of π_1 and $u_1^0(\cdot|x_1)$ is a minimizer of problem (3.4). The cost of $\bar{\pi}_1$ is equal to

$$\max_{w_0} \left\{ \sum_{t=0}^{T_1-1} \left(||Qx(t|x_0,u_0,w_0)||_\infty + ||Ru_0(t)||_\infty\right) + V_1(x(T_1|x_0,u_0,w_0)) \right\}. \tag{3.5}$$

Minimizing (3.5) over $u_0(\cdot)$ subject to (3.2) we obtain the optimal initial control $u_0^0(\cdot|x_0)$ and the optimal control strategy

$$\pi_1^0 = \pi_1^0(0,x_0) = \{u_0^0(\cdot|x_0); u_1^0(\cdot|x_1), x_1 \in X(T_1|x_0,u_0^0)\}.$$

In this formulation we omitted the inclusion (3.3) since it is implicitly satisfied for finite values of $V_1(x_1)$.

From the above discussion we conclude that the optimal control strategy π_1^0 consists of

1) *the optimal initial control* $u_0^0(\cdot|x_0)$ that is a solution to the min-max problem

$$V_0(x_0) = \min_{u_0} \max_{w_0} \left\{ \sum_{t=0}^{T_1-1} \left(||Qx(t)||_\infty + ||Ru_0(t)||_\infty\right) + V_1(x(T_1)) \right\} \tag{3.6}$$

subject to

$$x(t+1) = Ax(t) + Bu_0(t) + Mw_0(t),\ \ x(0) = x_0,$$
$$||u_0(t)||_\infty \le u_{\max},\ t = 0,1,\dots,T_1-1,$$
$$H(t)x(t) \le g(t),\ t = 1,2,\dots,T_1,\ \ \forall w_0(\cdot) \in W^{T_1},$$

2) *the optimal open-loop controls* $u_1^0(\cdot|x_1)$, $x_1 \in X(T_1|x_0,u_0^0)$, that are minimizers of problems (3.4) that have the detailed form:

$$V_1(x_1) = \min_{u_1} \max_{w_1} \left\{ \sum_{t=T_1}^{T-1} \left(||Qx(t)||_\infty + ||Ru_1(t)||_\infty \right) + ||Px(T)||_\infty \right\} \tag{3.7}$$

subject to

$$\begin{gathered} x(t+1) = Ax(t) + Bu_1(t) + Mw_1(t), \;\; x(0) = x_0, \\ ||u_1(t)||_\infty \le u_{\max}, \;\; t = T_1, T_1+1, \dots, T-1, \\ H(t)x(t) \le g(t), \; t = T_1+1, T_1+2, \dots, T, \; \forall w_1(\cdot) \in W^{T-T_1}. \end{gathered}$$

Remark 3.5. Note that problem (3.6) implies that the value of the optimal strategy π_1^0 is equal to

$$V_0(x_0) = \min_{u_0} \max_{w_0} \min_{u_1} \max_{w_1} \left\{ \sum_{\substack{t\in\Delta_k \\ k=0,1}} \left(||Qx(t|x_k,u_k(\cdot|x_k),w_k)||_\infty + ||Ru_k(t|x_k)||_\infty \right) + \right.$$
$$\left. + ||Px(T|x_1,u_1(\cdot|x_1),w_1)||_\infty \right\},$$

while the value of the optimal open-loop control $u^0(\cdot)$ (if it exists) is

$$J(x_0) = \min_{u_0} \min_{u_1} \max_{w_0} \max_{w_1} \left\{ \sum_{\substack{t\in\Delta_k \\ k=0,1}} \left(||Qx(t|x_k,u_k,w_k)||_\infty + ||Ru_k(t)||_\infty \right) + ||Px(T|x_1,u_1,w_1)||_\infty \right\}.$$

Obviously, $V_0(x_0) \le J(x_0)$. In the example section 3.4, we will show that performance improvement when applying π_1^0 depends on the position of the closing instant T_1. On the other hand, there are examples, where $V_0(x_0) = J(x_0)$ for all choices of T_1, especially if the horizon T is short. In Section 4.2, we will provide an example, where the open-loop problem is infeasible, however the optimal strategy π_1^0 exists.

To start the control process, we need to know only the optimal initial control $u_0^0(\cdot|x_0)$. The family of optimal open-loop controls $u_1^0(\cdot|x_1)$, $x_1 \in X(T_1|x_0,u_0)$, is not needed at the beginning of the process. Moreover, at time T_1 only one representative of the family, namely $u_1^0(\cdot|x^*(T_1))$, will be calculated. Since (3.7) is an open-loop optimal control problem its solution is easy to calculate as in Section 2. Therefore, in the next section, we discuss how to solve problem (3.6) efficiently. The solution method uses some approximation of the closure set and as a result yields the suboptimal initial control $\bar{u}_0^0(\cdot|x_0)$.

3.2. **Calculating the suboptimal initial control.** Problem (3.6) depends on the solution of problem (3.7) and thus has form of a bilevel optimization problem. We represent it in an equivalent deterministic and epigraph form, see Section 2 and [8]:

$$\min_{u_0,\alpha_0} \alpha_0 \tag{3.8}$$

subject to

$$
\begin{gathered}
x_0(t+1) = Ax_0(t) + Bu_0(t),\ \ x_0(0) = x_0,\\
x^l(t+1) = Ax^l(t) + Bu_0(t) + Mw_0^l(t),\ x^l(0) = x_0,\ l \in L_0,\\
||u_0(t)||_\infty \le u_{\max},\ t = 0,1,\dots,T_1-1,\\
H(t)x_0(t) \le g(t) - \gamma(t),\ t = 1,2,\dots,T_1,\\
\sum_{t=0}^{T_1-1}\Big(||Qx^l(t)||_\infty + ||Ru_0(t)||_\infty\Big) + V_1(x^l(T_1)) \le \alpha_0,\ \ l \in L_0,
\end{gathered}
$$

where $w_0^l(\cdot)$ is the l-th vertex of the hypercube W^{T_1}, $l \in L_0$.

Problem (3.8) is similar to problem (2.3), however, the presence of $V_1(x^l(T_1))$ in the last group of constraints prevents straigthforward reduction of problem (3.8) to a linear program as was the case for (2.3). More specifically, the last group of constraints will be represented as

$$
\begin{gathered}
||Qx^l(t)||_\infty \le \varepsilon_x^l(t),\ \ ||Ru_0(t)||_\infty \le \varepsilon_u(t),\ \ t = 0,1,\dots,T_1-1,\\
V_1(x^l(T_1)) \le \alpha_1^l,\ \ \sum_{t=0}^{T_1-1}\big(\varepsilon_x^l(t) + \varepsilon_u(t)\big) + \alpha_1^l \le \alpha_0,\ \ l \in L_0,
\end{gathered}
\tag{3.9}
$$

and our focus is on function $V_1(x_1)$, $x_1 \in X_1$, in (3.9). We outline the idea how to tackle these constraints here and prove the main result in Section 3.3.

For a fixed x_1 the value $V_1(x_1)$ is the optimal value of problem (3.7) that similarly to problem (2.1) in Section 2 can be rewritten as a linear program. If the state x_1 is considered as a parameter, then (3.7) is a multiparametric linear program. It is well known [28, p. 180], that the optimal value of a multiparametric linear program is piecewise linear and convex with respect to the parameter, i.e. x_1. Then the α-sublevel set

$$X_1(\alpha) = \{x_1 \in X_1 : V_1(x_1) \le \alpha\} \subseteq X_1$$

is a convex polytope for any fixed α such that $X_1(\alpha) \ne \varnothing$. Since state constraints sets $X(t)$ are bounded, all $X_1(\alpha)$ are also bounded. Obviously, $X_1(\alpha') \subseteq X_1(\alpha) \subseteq X_1$ for $\alpha' < \alpha$.

In the sequel, we replace Assumption 3.2 with a more practical one:

Assumption 3.6. *An interval* $[\alpha_{\min}, \alpha_{\max}]$, $\alpha_{\min} < \alpha_{\max}$, *such that* $X_1(\alpha) \ne \varnothing$, $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, *is known.*

Values $\alpha_{\min}$, $\alpha_{\max}$ should be the lower and the upper bounds for possible values of α_1^l, $l \in L_0$, in (3.9). We will discuss the choice of such bound for $\alpha_{\max}$ in Remark 3.8 and show how to compute $\alpha_{\min}$ in Section 3.3.

In simple examples the polytopes $X_1(\alpha)$, $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, can be found explicitly, but in general some approximation of $X_1(\alpha)$ is inevitable. We follow the approach in [12, 22] and approximate the set $X_1(\alpha)$ by a convex polytope $\bar{X}_1(\alpha)$ of a simpler structure. The main contribution of this paper is a "linear" form of the approximation with respect to the parameter α:

$$\bar{X}_1(\alpha) = \{x_1 \in \mathbb{R}^n :\ P_1 x_1 \le g_1 + \lambda_1 \alpha\}, \tag{3.10}$$

where the rules for $P_1 \in \mathbb{R}^{m_1 \times n}$, $g_1, \lambda_1 \in \mathbb{R}^{m_1}$ will be discussed in Section 3.3.

Representation (3.10) is the key to reducing problem (3.8) to a linear program, thus making it computationally attractive.

In terms of the control problem (3.7) the set $X_1(\alpha)$ bounds the worst-case cost-to-go V_1, and then the constraint $V_1(x^l(T_1)) \le \alpha_1^l$ is rewritten first as $x^l(T_1) \in X_1(\alpha_1^l)$, $l \in L_0$, and then, due to (3.10), as

$$P_1 x^l(T_1) \le g_1 + \lambda_1 \alpha_1^l, \ \ l \in L_0.$$

Obviously, the latter allows us to approximate (3.8) by a linear program

$$\begin{gathered}
\min_{u_0,\varepsilon_x,\varepsilon_u,\alpha_1,\alpha_0} \ \alpha_0, \\
-\varepsilon_u(t)\mathbb{1}_r \pm R u_0(t) \le 0, \ \ \pm u_0(t) \le u_{\max}\mathbb{1}_r, \\
t = 0,1,\dots,T_1-1, \\
\sum_{s=0}^{t-1} H(t)A^{t-s-1}Bu_0(s) \le g(t) - \gamma(t) - H(t)A^t x_0, \\
t = 1,2,\dots,T_1, \\
-\varepsilon_x^l(t)\mathbb{1}_n \pm \sum_{s=0}^{t-1} QA^{t-s-1}Bu_0(s) \le \mp\big(QA^t x_0 + \sum_{s=0}^{t-1} QA^{t-s-1}Mw_0^l(s)\big), \\
t = 0,1,\dots,T_1-1, \\
-\lambda_1\alpha_1^l + \sum_{s=0}^{T_1-1} P_1 A^{T_1-s-1}Bu_0(s) \le g_1 - \big(P_1 A^{T_1} x_0 + \sum_{s=0}^{T_1-1} P_1 A^{T_1-s-1} M w_0^l(s)\big), \\
\sum_{t=0}^{T_1-1}\big(\varepsilon_x^l(t) + \varepsilon_u(t)\big) + \alpha_1^l - \alpha_0 \le 0, \ \ \alpha_{\min} \le \alpha_1^l \le \alpha_{\max}, \\
l \in L_0.
\end{gathered} \tag{3.11}$$

Problem (3.11) has $(4r+m)T_1 + (2nT_1 + 2m_1 + 3)|L_0|$ constraints and $(r+1)T_1 + (T_1 + 1)|L_0| + 1$ variables. It is worth mentioning that, while m_1 (the number of rows in P_1, see (3.10)) depends on approximation and can be quite large, the critical dimension of problem (3.11) still depends on the number of vertices of a hypercube. This time, however, it is the hypercube W^{T_1} and $|L_0| < |L|$, leading to comparable or even smaller dimensions than the ones of the open-loop control problem (2.4).

If problem (3.11) is infeasible, problem (3.8) is infeasible as well, optimal control problem under consideration has no solution in the class of control strategies with closing instant T_1. This happens when Assumption 3.3 is violated.

3.3. **Approximating the sets $X_1(\alpha)$.** In this section we justify representation (3.10) for the approximation of the sets $X_1(\alpha)$.

Choose the system of normal vectors $p_i \in \mathbb{R}^n$, $i = 1, 2, \dots, \bar{m}_1$, $||p_i|| = 1$, independent of α. Here $\bar{m}_1 \le m_1$ in (3.10) and the difference will become clear later.

Let

$$f_i(\alpha) = \max_{x_1 \in X_1(\alpha)} p_i^\top x_1. \tag{3.12}$$

Then the approximation of the set $X_1(\alpha)$ has the form

$$\bar{X}_1(\alpha) = \Big\{x_1 \in \mathbb{R}^n : \ p_i^\top x_1 \le f_i(\alpha), \ i = 1,2,\dots,\bar{m}_1\Big\}. \tag{3.13}$$

Maximization problem (3.12) for a fixed α can be rewritten as

$$f_i(\alpha) = \max_{x_1,u_1} p_i^\top x_1 \tag{3.14}$$

subject to

$$\begin{gathered}
x_0(t+1) = Ax_0(t) + Bu_1(t), \;\; x_0(T_1) = x_1, \\
x^l(t+1) = Ax^l(t) + Bu_1(t) + Mw_1^l(t), \;\; x^l(T_1) = x_1, \;\; l \in L_1, \\
||u_1(t)||_\infty \le u_{\max}, \;\; t = T_1, T_1+1, \dots, T-1, \\
H(t)x_0(t) \le g(t) - \gamma_1(t), \;\; t = T_1+1, T_1+2, \dots, T, \\
\sum_{t=T_1}^{T-1} \left(||Qx^l(t)||_\infty + ||Ru_1(t)||_\infty\right) + ||Px^l(T)||_\infty \le \alpha, \; l \in L_1,
\end{gathered}$$

where $w_1^l(\cdot)$ is the l-th vertex of the hypercube W^{T-T_1}, $l \in L_1$; $\gamma_1(t) = (\gamma_{1i}(t), i = 1,2,\dots,m)$: $\gamma_{1i}(t) = w_{\max} \sum_{s=T_1}^{t-1} ||h_i(t)^\top A^s M||_1$.

Problem (3.14) can be reformulated as a linear program:

$$\begin{gathered}
f_i(\alpha) = \max_{x_1,u_1,\varepsilon_x,\varepsilon_u} p_i^\top x_1, \\
-\varepsilon_u(t)\mathbb{1}_r \pm Ru_1(t) \le 0, \;\; \pm u_1(t) \le u_{\max}\mathbb{1}_r, \\
t = T_1, T_1+1, \dots, T-1, \\
H(t)A^{t-T_1}x_1 + \sum_{s=T_1}^{t-1} H(t)A^{t-s-1}Bu_1(s) \le g(t) - \gamma_1(t), \\
t = T_1+1, T_1+2, \dots, T, \\
-\varepsilon_x^l(t)\mathbb{1}_n \pm QA^{t-T_1}x_1 \pm \sum_{s=T_1}^{t-1} QA^{t-s-1}Bu_1(s) \le \mp \sum_{s=T_1}^{t-1} QA^{t-s-1}Mw_1^l(s), \\
t = T_1, T_1+1, \dots, T-1, \\
-\varepsilon_x^l(T)\mathbb{1}_n \pm PA^{T-T_1}x_1 \pm \sum_{s=T_1}^{T-1} PA^{T-s-1}Bu_1(s) \le \mp \sum_{s=T_1}^{T-1} PA^{T-s-1}Mw_1^l(s), \\
\sum_{t=T_1}^{T} \varepsilon_x^l(t) + \sum_{t=T_1}^{T-1} \varepsilon_u(t) \le \alpha, \\
l \in L_1.
\end{gathered} \tag{3.15}$$

Under Assumption 3.6 problem (3.15) is feasible for any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. Moreover, since $X_1(\alpha)$ are bounded the optimal value $f_i(\alpha)$ is finite for all $i = 1,2,\dots,\bar{m}_1$.

Suppose problem (3.15) is solved for a fixed $\alpha = \bar{\alpha}$. All existing linear programming solvers along with the primal solution find the corresponding dual solution. Let $y_i^l(\bar{\alpha})$, $l \in L_1$, denote the optimal duals corresponding to the last group of constraints, i.e $\sum_{t=T_1}^{T} \varepsilon_x^l(t) + \sum_{t=T_1}^{T-1} \varepsilon_u(t) \le \alpha$, $l \in L_1$. Obviously,

$$\left.\frac{df_i}{d\alpha}\right|_{\alpha=\bar{\alpha}} = \sum_{l \in L_1} y_i^l(\bar{\alpha}).$$

On the base of sensitivity analysis (see Section 3.5) an interval $[a_*, a^*]$ can be found such that the dual solution is constant on this interval: $y_i^l(\alpha) \equiv y_i^l(\bar{\alpha})$, $\alpha \in [a_*, a^*]$. Thus, solving the linear program (3.15) only once for a given $\alpha = \bar{\alpha}$ allows us to characterize its optimal value as a function of the parameter α on the whole interval $[a_*, a^*]$:

$$f_i(\alpha) = f_i(\bar{\alpha}) + (\alpha - \bar{\alpha}) \sum_{l \in L_1} y_i^l(\bar{\alpha}), \quad \alpha \in [a_*, a^*]. \tag{3.16}$$

In general problem (3.15) can be treated as a parametric linear program. From [28] it follows that function $f_i(\alpha)$, $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, is piecewise linear and concave (due to maximization of the cost).

Let $A_i^k \subseteq [\alpha_{\min}, \alpha_{\max}]$, $k = 1, 2, \ldots, K_i$, denote the partition of the interval $[\alpha_{\min}, \alpha_{\max}]$ into subintervals of linearity of the function f_i, $\cup_k A_i^k = [\alpha_{\min}, \alpha_{\max}]$, $int\, A_i^k \cap int\, A_i^{k'} = \varnothing$, $k \neq k'$:

$$f_i(\alpha) = g_i^k + \lambda_i^k \alpha, \quad \alpha \in A_i^k, \quad k = 1, 2, \ldots, K_i. \tag{3.17}$$

Comparing (3.16) and (3.17) we obtain

$$\lambda_i^k = \sum_{l \in L_1} y_i^l(\bar{\alpha}), \quad g_i^k = f_i(\bar{\alpha}) - \bar{\alpha}\lambda_i^k \ \text{ for some } \ \bar{\alpha} \in A_i^k.$$

The following **Algorithm** executed for $i = 1, 2, \ldots, \bar{m}_1$ finds the partition A_i^k, the values λ_i^k, g_i^k, $k = 1, 2, \ldots, K_i$, for (3.17), and $\alpha_{\min}$ such that $X_1(\alpha) = \varnothing$ for $\alpha < \alpha_{\min}$:

1. Set $k := 1$, $a_i^1 := \alpha_{\max}$, choose a small tuning parameter $\varepsilon > 0$.
2. Solve problem (3.15) for $\alpha = a_i^k - \varepsilon$.
3. Analyze the solution:
   3.1. If problem (3.15) is feasible, save the values

$$\lambda_i^k = \sum_{l \in L_1} y_i^l(a_i^k - \varepsilon), \quad g_i^k = f_i(a_i^k - \varepsilon) - (a_i^k - \varepsilon)\lambda_i^k \tag{3.18}$$

and go to step 4.

   3.2. If (3.15) is infeasible, update the lower bound for the parameter: $\alpha_{\min} := a_i^k$, set $K_i := k - 1$ and go to step 5.
4. Find $[a_*, a^*]$ be the rules described in subsection 3.5 below, set $a_i^{k+1} = \max\{a_*, \alpha_{\min}\}$:
   4.1. If $a_i^{k+1} = \alpha_{\min}$, $K_i := k$, go to step 5.
   4.2. Otherwise $k := k + 1$, return to step 2.
5. Stop for a given i.

Remark 3.7. On step 3.2 of the Algorithm a new value of $\alpha_{\min}$ can be found. Note that, since $\alpha_{\min}$ is a global parameter for $X_1(\alpha)$ (independent of i), step 3.2 realizes only once, for $i = 1$. After that the algorithm will no longer implement step 3.2 for the remaining i's.

If after step 3.2 it turns out that $\alpha_{\min} = \alpha_{\max}$, then Assumption 3.6 is not satisfied. Choose new, larger $\alpha_{\max}$ and if for the new value the equality $\alpha_{\min} = \alpha_{\max}$ is satisfied again, the problem has no solution in the class of strategies with the closing instant T_1.

Remark 3.8. The value $\alpha_{\max}$ can also be refined by the Algorithm. It is easy to see, that $\alpha_{\max}$ satisfies the equality $X_1(\alpha_{\max}) = X_1$. This means that $f_i(\alpha) \equiv f_i(\alpha_{\max})$ for $\alpha > \alpha_{\max}$. Therefore, if during the implementation of the Algorithm we find that $\lambda_i^1 = 0$ for all $i = 1, 2, \ldots, \bar{m}_1$, then $a_1^2 = \ldots = a_{\bar{m}_1}^2$ and $\alpha_{\max} = a_1^2$. We propose to choose $\alpha_{\max}$ large enough to start the Algorithm and then find a new value $\alpha_{\max}$.

Remark 3.9. It is easy to show that on step 4 $a^* \le a_i^k$. It is necessary to check that $a^* = a_i^k$, otherwise the algorithm tuning parameter $\varepsilon > 0$ is too large. Choose $\varepsilon := (a_i^k - a^*)/2$ and repeat steps 2–4 of the algorithm.

Remark 3.10. If on step 4 one finds that $a_* = a^* = a_i^k - \varepsilon$, then situation of Remark 3.9 realizes and one should choose a smaller parameter ε.

The Algorithm results in the following data: a number K_i, a sequence of points

$$a_i^1 = \alpha_{\max} > a_i^2 > \ldots > a_i^{K_i} > \alpha_i^{K_i+1} = \alpha_{\min},$$

that form the linearity intervals $A_i^k = [a_i^{k+1}, a_i^k]$, $k = 1,2,\ldots,K_i$, and values (3.18) to construct the function f_i according to (3.17).

A piecewise linear concave function f_i can also be represented in the form

$$f_i(\alpha) = \min_{k=1,2,\ldots,K_i} \left\{ g_i^k + \lambda_i^k \alpha \right\},\ \alpha \in [\alpha_{\min}, \alpha_{\max}],$$

which together with (3.13) yields

$$\bar{X}_1(\alpha) = \left\{ x_1 \in \mathbb{R}^n :\ p_i^\top x_1 \le g_i^k + \lambda_i^k \alpha,\ k = 1,2,\ldots,K_i,\ i = 1,2,\ldots,\bar{m}_1 \right\}.$$

The latter representation is equivalent to (3.10) up to notations

$$P_1 = \begin{pmatrix} P_{1i} \\ i = 1,2,\ldots,\bar{m}_1 \end{pmatrix},\ g_1 = \begin{pmatrix} g_{1i} \\ i = 1,2,\ldots,\bar{m}_1 \end{pmatrix},\ \lambda_1 = \begin{pmatrix} \lambda_{1i} \\ i = 1,2,\ldots,\bar{m}_1 \end{pmatrix}, \tag{3.19}$$

where the blocks are

$$P_{1i} = \begin{pmatrix} p_i^\top \\ k = 1,2,\ldots K_i \end{pmatrix},\ g_{1i} = \begin{pmatrix} g_i^k \\ k = 1,2,\ldots K_i \end{pmatrix},\ \lambda_{1i} = \begin{pmatrix} \lambda_i^k \\ k = 1,2,\ldots K_i \end{pmatrix}. \tag{3.20}$$

Here the matrix P_1 has repetitive rows $p_i^\top$ (the i-th normal is repeated K_i times), $P_1 \in \mathbb{R}^{m_1 \times n}$, $g_1, \lambda_1 \in \mathbb{R}^{m_1}$, where $m_1 = \sum_{i=1}^{\bar{m}_1} K_i$.

Summarizing, the following proposition holds:

Proposition 3.11. *Representation (3.10) holds true for the approximations of the sets $X_1(\alpha)$, $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, where P_1, g_1, λ_1 are calculated according to the formulas (3.19), (3.20).*

Remark 3.12. The accuracy of approximation (3.10) is difficult to estimate theoretically. But it can be evaluated numerically comparing the optimal value of problem (3.11) and the value (3.5) with the control $\bar{u}_0^0(\cdot)$, where $V_1(x(T_1|x_0, \bar{u}_0^0, w_0))$ is found directly from (3.7).

3.4. **Example.** Let us illustrate the results by an example with the following matrices:

$$A = \begin{pmatrix} 0.54030 & 0.84147 \\ -0.84147 & 0.54030 \end{pmatrix},\ B = \begin{pmatrix} 0.45970 \\ 0.84147 \end{pmatrix},\ M = B;$$

$$P = Q = 0.1I,\ \ R = 1.$$

Other parameters are $T = 10$, $u_{\max} = 1$, $w_{\max} = 0.3$. The state constraints are imposed only at the terminal instant and have the form $||x(T)||_\infty \le 2$.

We consider the problem with $x_0 = (5,0)$. For this initial state there exists the optimal open-loop control $u^0(\cdot)$, and the optimal value of problem (2.3) is equal to $J(x_0) = 9.510541$. The optimal open-loop control $u^0(\cdot)$, a possible disturbance and the corresponding trajectory are

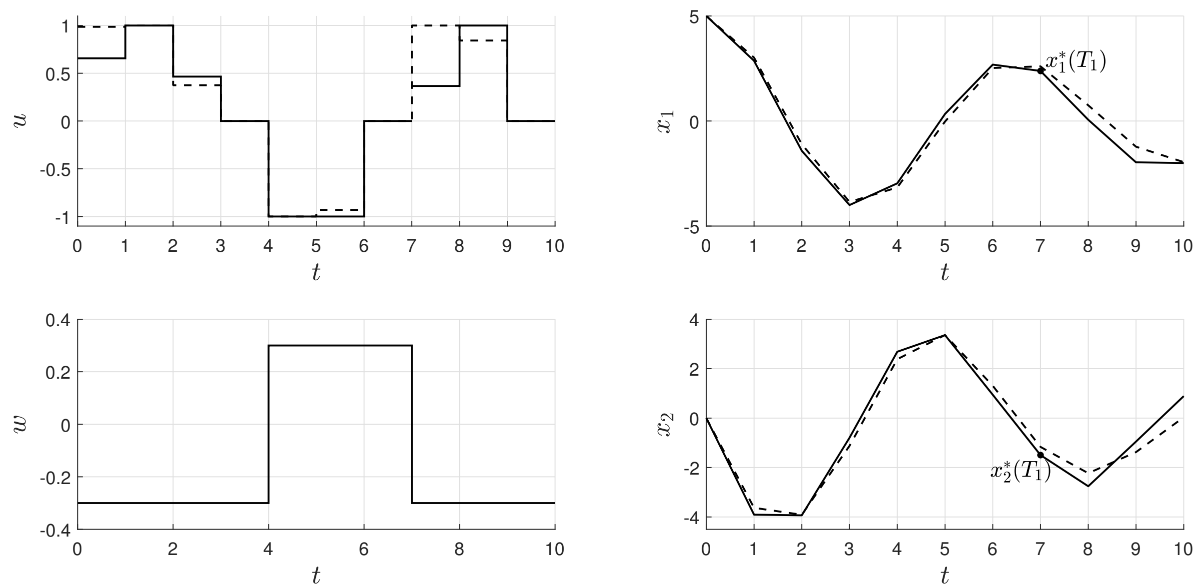

FIGURE 1. Optimal controls, worst-case disturbance and corresponding trajectories in Example 1

shown in Fig.1 (dashed lines). The actual cost is 9.4948 and the chosen disturbance is the worst-case with respect to the optimal strategy π_1^0, see below.

The dimensions of the open-loop problem (2.4) are 46124×11285. Time spend by `linprog` MATLAB procedure to solve the problem was equal to 7.75 sec.

As mentioned in Remark 3.5 the optimal value $V_0(x_0)$ depends on the closing instant T_1. The results are summarized in Table 1. Table 1 also shows the number of vectors to approximate $X_1(\alpha)$ in (3.10) and (3.13) and dimensions of problems (3.11). Since $X_1(\alpha) \subset \mathbb{R}^2$ we were able to find its normals p_i, $i = 1, 2, \ldots, \bar{m}_1$, in (3.13) with the accuracy 10^{-12} for each T_1.

TABLE 1. Dependence of $V_0(x_0)$ and dimensions of optimization problems on the closing instant T_1

T_1	V_0	$\bar{m}_1$ in (3.13)	m_1 in (3.10)	dimensions of (3.11)	$\alpha_{\max}$
2	9.469046	404	4214	16908× 17	11.425706
3	9.359552	210	1818	14676× 39	9.913753
4	9.235468	248	1746	28256× 89	8.386888
5	9.235468	208	1616	52468× 203	7.042253
6	9.175384	136	1044	68568× 461	5.637938
7	8.983898	96	249	67740× 1039	4.215194
8	9.069963	38	498	33568× 2321	2.899477
9	9.303038	14	26	33316× 5139	1.535258

The minimum of V_0 is attained at the closing instant $T_1 = 7$. For this closing instant the optimal initial control $u_0^0(\cdot)$ and the corresponding trajectory are shown in Fig.1 (solid lines until

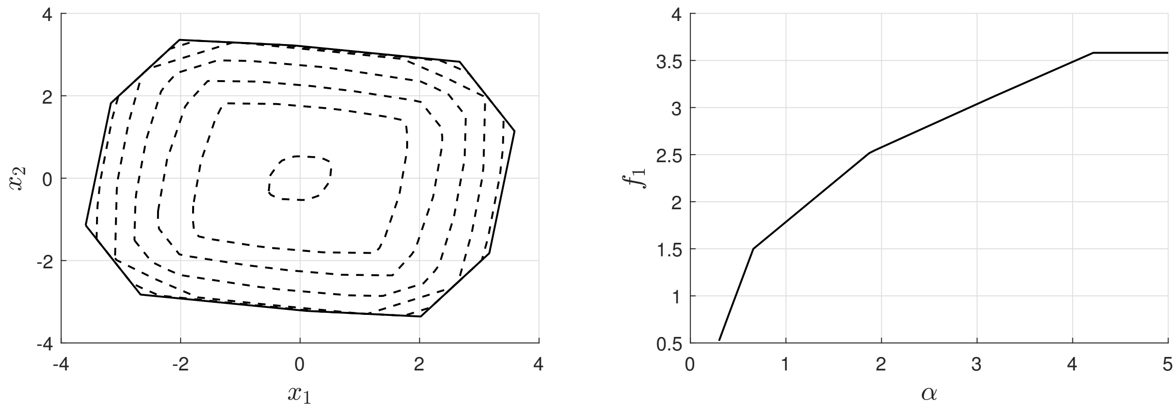

FIGURE 2. Sets $X_1(\alpha)$ and a sample function $f_1(\alpha)$

the closing instant $T_1 = 7$). The solid lines after the closing instant correspond to $u_1^0(\cdot|x^*(7))$, $x^*(7) = (2.382499, -1.496570)$, and its trajectory. The disturbance that is assumed to obtain both trajectories on Fig.1 is the worst with respect to the optimal strategy π_1^0, i.e. (w_0, w_1) consists of w_0 such that maximum in (3.6) is attained at w_0 and maximum in (3.7) is attained at w_1. The actual cost of such control process is, obviously, equal to $V_0(x_0)$.

Fig. 2 (left) shows the closure set X_1 (solid line) and sets $X_1(\alpha)$ for $\alpha \in \{0.3; 1; 1.7; 2.4; 3.1; 3.8; \alpha_{\max}\}$ (dashed lines), as well as a piecewise linear function $f_1(\alpha)$, $\alpha \in [0.3, \alpha_{\max}]$, for a sample vector $p_1 = (-0.999969, 0.007853)$.

It is worth mentioning that in the example under consideration, for the control horizon $T = 8$ there is no performance improvement when the optimal control strategy is used, i.e. $J(x_0) = V_0(x_0)$ for any T_1.

3.5. **On implementation of the algorithm, linear programming solvers for problems (3.11), (3.15) and sensitivity analysis.** In this section we discuss problems that we encountered during the numerical experiments in Section 3.4. We consider three major items in the implementation of the Algorithm proposed in Section 3.3 that have to be addressed to obtain the efficient method for the closure sets approximation construction:

(i) Problem (3.15) is solved many times for different vectors p_i and different values of α. Solving it every time "from scratch", without using any information about the previous solution might lead to very long computation times. Such realization in MATLAB with the standard procedure `linprog` for, e.g. $T_1 = 4$, took 2.5 hours for only 16 first vectors p_i.

(ii) To perform the sensitivity analysis and find the partition A_i^k, $k = 1, 2, \dots, K_i$, one could use the approach based on active constraints of problem (3.15) as proposed in [7, p.16]. However, problem (3.15) is often primal degenerate. In this case the procedure in [7] involves Gauss reduction and is slow for high dimensions of problems (3.15).

(iii) Problem (3.15) is also dual degenerate and standard sensitivity analysis finds only a subinterval $[\bar{a}_*, \bar{a}^*]$ of $[a_*, a^*]$, where not only the optimal dual solution is constant but also the set of active constraints is not changed. In this case the Algorithm is performed as described in Section 3.3 and a post-processing is needed to find a union of subintervals into the maximal interval $[a_*, a^*]$. While the post-processing is easy to organize (the subintervals have equal values of λ_i^k), that procedure involves solving problem (3.15) some extra times resulting in slow performance as mentioned in (i).

Having analyzed (ii) and (iii) we concluded that a simplex-type linear programming solver was needed to obtain the optimal basis along with the primal and dual solutions of (3.15). Then the optimal basis of the problem at the previous iteration (previous α) can be used for hot-start of the solver at the present iteration. Thus, the computation times mentioned in (i) and (ii) will be improved.

In our numerical experiments we used a special method developed by Rafail Gabasov and his co-authors who called it the *adaptive method for solving linear programs* [14, 17]. The method was proposed for linear programs with interval constrains

$$\max\ c^\top z,\ \ b_* \le Gz \le b^*,\ \ d_* \le z \le d^*, \tag{3.21}$$

where $z \in \mathbb{R}^{n_1}$, $G \in \mathbb{R}^{m_1 \times n_1}$, and is also suitable for linear programs in inequality form, which is problem (3.15).

To use the adaptive method and perform the sensitivity analysis afterwards we present problem (3.15) in the general parametric form

$$\max\ c^\top z,\ \ Gz \le b + d\alpha, \tag{3.22}$$

where $z = (x_1, u_1(\cdot), \varepsilon_x^1(\cdot), \dots \varepsilon_x^l(\cdot), \varepsilon_u(\cdot)) \in \mathbb{R}^{n_1}$, $c = (p_1, 0) \in \mathbb{R}^{n_1}$, $G \in \mathbb{R}^{m_1 \times n_1}$, b, $d \in \mathbb{R}^{m_1}$ are the block matrix and vectors with the following structure (in the order of constraints in (3.15)):

$$G = \begin{pmatrix} * & & 0 & & -E_u \\ G_1 & & 0 & & 0 \\ * & -E_x & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ * & 0 & \dots & -E_x & 0 \\ 0 & & E_\alpha & & \mathbb{1} \end{pmatrix},\ b = \begin{pmatrix} 0 \\ b_1 \\ * \\ \vdots \\ * \\ 0 \end{pmatrix},\ d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \mathbb{1}_{|L_1|} \end{pmatrix}, \tag{3.23}$$

$*$ corresponds to nonzero matrices and vectors, $E_x = \operatorname{diag}(\underbrace{\mathbb{1}_{2n}, \dots, \mathbb{1}_{2n}}_{T-T_1+1\ \text{times}})$, $E_u = \operatorname{diag}(\underbrace{\mathbb{1}_{2r}, \dots, \mathbb{1}_{2r}}_{T-T_1\ \text{times}})$, $E_\alpha = \operatorname{diag}(\underbrace{\mathbb{1}^\top_{(T-T_1+1)}, \dots, \mathbb{1}^\top_{(T-T_1+1)}}_{|L_1|\ \text{times}})$ are block diagonal matrices, $\mathbb{1} \in \mathbb{R}^{|L_1| \times (T-T_1)}$ is a matrix of ones, and G_1, b_1 are the constraint matrix and vector of the linear program

$$\begin{gathered} \max_{x_1, u_1} p_i^\top x_1, \\ H(t)A^{t-T_1}x_1 + \sum_{s=T_1}^{t-1} H(t)A^{t-s-1}Bu_1(s) \le g(t) - \gamma_1(t),\ \ t = T_1+1, \dots, T, \\ \pm u_1(t) \le u_{\max}\mathbb{1}_r,\ \ t = T_1, \dots, T-1. \end{gathered} \tag{3.24}$$

The main instrument of the adaptive method is a *support* (corresponds to a basis in the simplex method). For the problem with interval constraints (3.21) the support is a pair $\{I_b, J_b\}$, $I_b \subseteq I = \{1, 2, \dots, m_1\}$, $J_b \subseteq J = \{1, 2, \dots, n_1\}$, $|I_b| = |J_b|$, such that the support matrix $G(I_b, J_b)$ is nonsingular. An empty support can be used for problem (3.21).

For the problem in inequality form (3.22) the support (depending on the parameter α) is $I_b(\alpha) \subset I$ such that $|I_b(\alpha)| = n_1$, $G(I_b(\alpha),J)$ is nonsingular. To every support there corresponds the vector of Lagrange multipliers $y \in \mathbb{R}^{m_1}$: $y_s = 0$, $s \in I \setminus I_b^0(\alpha)$, $y_b = y(I_b^0(\alpha))$: $y_b^\top = c^\top G(I_b(\alpha),J)^{-1}$. The optimality criterion for $z^0(\alpha)$ is as follows: a feasible solution $z^0(\alpha)$ is optimal in problem (3.22) iff there exists a support $I_b^0(\alpha)$ such that the corresponding vector of Lagrange multipliers satisfies

$$y_s \geq 0 \text{ if } g_s^\top z^0(\alpha) = b_s + d_s\alpha, \quad s \in I_b^0(\alpha),$$

where $g_s^\top$ is the s-th row of the matrix G. The support $I_b^0(\alpha)$ is called optimal.

The adaptive method is an iterative method of changing supports until the optimal support is found. For problem (3.22) it needs a valid initial support to start iterations. Taking into account the structure of the matrices (3.23) it is easy to suggest such an initial support, independently of α. It has the form $I_b = I_1^0 \cup I_2$, where I_1^0 is the optimal support in problem (3.24), $|I_1^0| = |J_1|$, $J_1 = \{1,2,\dots,n+r(T-T_1)\}$ are the columns of the matrix G_1. The set I_2 consists of $(T-T_1+1)|L_1| + (T-T_1)$ rows from the blocks 1 and 3 of the matrix G such that $G(I_2,J) = (*|-I)$. Then the initial support matrix in problem (3.22) has the form

$$G(I_b,J) = \left(\begin{array}{c:c} G_1(I_1^0,J_1) & 0 \\ \hdashline * & -I \end{array}\right)$$

and is nonsingular. Moreover, the vector of Lagrange multipliers, corresponding to this initial support is $y = (y_1, 0) \geq 0$, where $y_1 \geq 0$ is the optimal dual solution in problem (3.24).

Problem (3.24) is significantly smaller, than (3.15) since its dimension is independent of $|L_1|$ and α. It can be solved very fast by a version of the adaptive method for linear programs with interval constraints (3.21) with an empty initial support. The adaptive method will find the optimal support $\{I_b^0 = I_1^0, J_b^0 = J_1\}$. Note that the optimal support here depends on the index i, since the cost of problem (3.24) depends on p_i.

When problem (3.15) (same as (3.22) with matrices (3.23)) is solved for a given i for the first time by the Algorithm proposed in Section 3.3, $k = 1$, I_b is taken as the initial support. All other iterations of the Algorithm, $k > 1$, the optimal support of (3.15) from the iteration $(k-1)$ is used as the initial one to hot-start the iterations of the adaptive method. From our experience, in most cases only one iteration is performed to find the optimal solution of problem (3.15) for $k > 1$.

Knowing the optimal support $I_b^0(\alpha)$ of problem (3.22) we can now address items (ii) and (iii) mentioned at the beginning of this section and concerning the sensitivity analysis. From the primal feasibility and the optimality conditions it follows that

$$G_b z = b_b + d_b\alpha, \quad g_s^\top z \leq b_s + d_s\alpha, \quad s \in I \setminus I_b^0(\alpha),$$

where $G_b = G(I_b^0(\alpha),J)$, $b_b = b(I_b^0(\alpha))$, $d_b = d(I_b^0(\alpha))$.

Finding z from the system of linear equations, $z = z(\alpha) = G_b^{-1}(b_b + d_b\alpha)$ and substituting it into inequalities, we obtain

$$(g_s^\top G_b^{-1} d_b - d_s)\alpha \leq b_s - g_s^\top G_b^{-1} b_b, \quad s \in I \setminus I_b^0(\alpha),$$

and conclude that the optimal support $I_b^0(\alpha)$ is unchanged for all $\alpha \in [a_*, a^*]$, where

$$a_* = \max_s \left(b_s - g_s^\top G_b^{-1} b_b\right) / \left(g_s^\top G_b^{-1} d_b - d_s\right), \quad s \in \{s : g_s^\top G_b^{-1} d_b - d_s < 0\},$$
$$a^* = \min_s \left(b_s - g_s^\top G_b^{-1} b_b\right) / \left(g_s^\top G_b^{-1} d_b - d_s\right), \quad s \in \{s : g_s^\top G_b^{-1} d_b - d_s > 0\}.$$

Using the adaptive method as a linear programming solver in the numerical experiments in Section 3.4, we achieved a significant improvement in solution times and in the Algorithm performance. The results are presented in Table 2 (first row).

TABLE 2. Times (in seconds) spent on the construction of the parameters (3.18) of the closure sets $X_1(\alpha)$ by the Algorithm in Section 3.3, on the construction of the suboptimal initial control $u_0^0(\cdot)$ and a new control $u_1(\cdot|x^*(T_1))$ at time instant T_1 depending on the closing instant T_1

T_1	2	3	4	5	6	7	8	9
$X_1(\alpha)$	2345.65	240.72	67.24	17.08	4.58	1.29	0.2	0.07
$u_0^0(\cdot)$	0.01	0.01	0.05	0.16	0.45	1.72	2.98	7.94
$u_1^0(\cdot\|x^*(T_1))$	0.09	0.10	0.02	0.005	0.005	0.003	0.003	0.001

Now consider problem (3.11) for suboptimal initial control construction. This problem is solved before the control process starts, i.e. offline, however, problems of the same type are solved in real time when the optimal control strategy with multiple closing instants is constructed (see Section 4). Moreover, problem (3.7) that has to be solved online is also of the same type as (3.11). Therefore, we need an efficient method to solve all these problems.

Problem (3.11) is a linear program in the inequality form

$$\min\ c^\top z, \;\; Gz \le b,$$

where $z = (u_0(\cdot), \varepsilon_x^1(\cdot), \ldots \varepsilon_x^l(\cdot), \varepsilon_u(\cdot), \alpha_1, \alpha_0) \in \mathbb{R}^{n_0}$, $c = (0, \ldots, 0, 1) \in \mathbb{R}^{n_0}$, $G \in \mathbb{R}^{m_0 \times n_0}$, $b \in \mathbb{R}^{m_0}$ are the block matrix and vector with the following structure

$$G = \left(\begin{array}{c|ccc|c|ccc|c}
* & & 0 & & -E_u & & 0 & & 0 \\ \hdashline
G_0 & & 0 & & 0 & & 0 & & 0 \\ \hdashline
* & -E_x & \ldots & 0 & 0 & & & & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & 0 & & \vdots \\
* & 0 & \ldots & -E_x & 0 & & & & \\ \hdashline
D_0 & 0 & \ldots & 0 & 0 & -\lambda_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
D_0 & 0 & \ldots & 0 & 0 & 0 & \ldots & -\lambda_1 & 0 \\ \hdashline
0 & & E_\alpha & & \mathbb{1} & & I & & -\mathbb{1}_{|L_0|}
\end{array}\right), \quad b = \left(\begin{array}{c} 0 \\ \hdashline b_0 \\ \hdashline * \\ \vdots \\ * \\ \hdashline q_0^1 \\ \vdots \\ q_0^l \\ \hdashline 0 \end{array}\right), \tag{3.25}$$

$E_x = \operatorname{diag}(\underbrace{\mathbb{1}_{2n}, \ldots, \mathbb{1}_{2n}}_{T_1 \text{ times}})$, $E_u = \operatorname{diag}(\underbrace{\mathbb{1}_{2r}, \ldots, \mathbb{1}_{2r}}_{T_1 \text{ times}})$, $E_\alpha = \operatorname{diag}(\underbrace{\mathbb{1}_{T_1}^\top, \ldots, \mathbb{1}_{T_1}^\top}_{|L_0| \text{ times}})$ are block diagonal matrices, $\mathbb{1} \in \mathbb{R}^{|L_0| \times T_1}$ is a matrix of ones, $G_0 \in \mathbb{R}^{(m+2r)T_1 \times r_1^T}$, $b_0 \in \mathbb{R}^{(m+2r)T_1}$ correspond to the constraints

$$\sum_{s=0}^{t-1} H(t)A^{t-s-1}Bu_0(s) \le g(t) - \gamma(t) - H(t)A^t x_0,\ \ t = 1,2,\dots,T_1,$$
$$\pm u_0(t) \le u_{\max}\mathbb{1}_r,\ \ t = 0,1,\dots,T_1-1$$

and $q_0^l = g_1 - P_1A^{T_1}x_0 - \sum_{s=0}^{T_1-1} P_1A^{T_1-s-1}Mw_0^l(s) \in \mathbb{R}^{m_1}$, $l \in L_0$, $D_0 = (P_1A^{T_1-1}B,\dots,P_1AB,P_1B) \in \mathbb{R}^{m_1\times rT_1}$.

Taking into account the block structure of (3.25) it is easy to construct the initial support for problem (3.11). To this end we first solve the following optimal control problem

$$\begin{gathered}\min_{\alpha,u_0} \alpha,\\ x(t+1) = Ax(t) + Bu_0(t) + Mw_0(t),\ \ x_0(0) = x_0,\\ u_0(t) \in U,\ t = 0,1,\dots,T_1-1,\\ x(t) \in X(t),\ t = 1,2,\dots,T_1,\ \ x(T_1) \in X_1(\alpha)\ \forall w_0(\cdot) \in W^{T_1}.\end{gathered} \tag{3.26}$$

Problem (3.26) calculates a feasible control $u_0(\cdot)$ that steers the system robustly in the state constraints sets and, in addition, to a closure set $X_1(\alpha)$ with a minimal α. The corresponding linear program in inequality form (according to the notations introduced above) is as follows

$$\min_{\alpha,u_0} \alpha,\ \ G_0u_0 \le b_0,\ \ -\lambda_1\alpha + D_0u_0 \le q_{\min}. \tag{3.27}$$

where $q_{\min} = (q_s, s = 1,2,\dots,m_1) : q_s = \min_{l\in L_0} q_{0s}^l$.

On the base of the feasible control $u_0(\cdot)$, i.e. the solution of problem (3.27), it is easy to construct a feasible solution of problem (3.11). To this end choose $\varepsilon_x^l(t) = ||Qx(t|x_0,u_0,w_0^l)||_\infty$, $\varepsilon_u(t) = ||Ru_0(t)||_\infty$, $t = 0,1,\dots,T_1-1$, and set all α_1^l equal to the optimal value of the problem (3.27).

As a result of the above construction, optimal support of problem (3.27) provides rT_1+1 rows for the initial support of problem (3.11). We add to them $T_1 + T_1|L_0|$ rows form the first and the third blocks of matrix (3.25) corresponding to active inequalities, and all $|L_0|$ rows in the last block. By construction, the obtained support has exactly $n_0 = (r+1)T_1 + (T_1+1)|L_0| + 1$ elements with a nonsingular support matrix.

Table 2 shows times needed by the adaptive method to calculate the optimal initial control $u_0^0(\cdot|x_0)$ (the second row of the table) and the control $u_1^0(\cdot|x^*(T_1))$ at time T_1, depending on $x^*(T_1)$, in the example of Section 3.4. Obviously, the method is suitable for fast online calculations that is needed at time T_1.

4. Optimal control strategy with multiple closing instants

In this section we develop the ideas of Section 3 for the case of N closing instants T_j, $j = 1,2,\dots,N$: $T_j \in \{1,2,\dots,T-1\}$, $0 = T_0 < T_1 < T_2 < \ldots < T_N < T_{N+1} = T$.

A strategy $\pi_N(0,x_0)$ is defined recursively on the base of strategies $\pi_{N-j}(T_j,x_j)$ with $N-j$ closing instants T_{j+1}, …, T_N; $j = N-1, N-2,\dots,1$:

$$\begin{gathered}\pi_1(T_{N-1},x_{N-1}) = \{u_{N-1}(\cdot|x_{N-1});\ \ u_N(\cdot|x_N),\ x_N \in X(T_N|x_{N-1},u_{N-1})\},\\ \pi_{N-j}(T_j,x_j) = \{u_j(\cdot|x_j);\ \ \pi_{N-j-1}(T_{j+1},x_{j+1}),\ x_{j+1} \in X(T_{j+1}|x_j,u_j)\},\\ \pi_N(0,x_0) = \{u_0(\cdot|x_0);\ \ \pi_{N-1}(T_1,x_1),\ x_1 \in X(T_1|x_0,u_0)\}.\end{gathered} \tag{4.1}$$

Feasibility of (4.1) is also determined recursively by robustly satisfying inclusions

$$x(t|x_j,u_j,w_j) \in X(t),\ t = T_j+1, T_j+2, \dots, T_{j+1},\ \forall w_j(\cdot) \in W^{T_{j+1}-T_j},$$
$$X(T_{j+1}|x_j,u_j) \subseteq X_{j+1},$$

where $X_j,\ j = 1,2,\dots,N$, are the closure sets at times T_j, each of them consists of all states $x_j \in \mathbb{R}^n$ for which a control strategy $\pi_{N-j}(T_j,x_j)$ with $N-j$ closing instants exists, $X_{N+1} = \mathbb{R}^n$.

Optimal control strategy π_N^0 is defined by the *optimal controls* $u_j^0(\cdot|x_j),\ j = 0,1,\dots,N$, that are solutions of Bellman equations

$$V_j(x_j) = \min_{u_j} \max_{w_j} \left\{ \sum_{t=T_j}^{T_{j+1}-1} \left(||Qx(t|x_j,u_j,w_j)||_\infty + ||Ru_j(t)||_\infty \right) + V_{j+1}(x(T_{j+1}|x_j,u_j,w_j)) \right\},$$
$$x_j \in X_j,\quad j = 0,\dots,N,$$
$$V_{N+1}(x) = ||Px||_\infty,\quad x \in \mathbb{R}^n,$$

where minimization is subject to state and input constraints.

4.1. **Calculating the optimal strategy with multiple closing instants.** To calculate the optimal controls $u_j^0(\cdot|x_j),\ j = 0,1,\dots,N$, for optimal strategy π_N^0 we follow the arguments of Sections 3.2 and 3.3. First, we define sets

$$X_j(\alpha) = \left\{ x_j \in \mathbb{R}^n : V_j(x_j) \le \alpha \right\},\quad j = 1,2,\dots,N,\quad \alpha \in [\alpha_{\min}^j, \alpha_{\max}^j].$$

Then each $X_j(\alpha)$ is approximated by an outer polytope

$$\bar{X}_j(\alpha) = \left\{ x_j \in \mathbb{R}^n :\ p_{ji}^\top x_j \le f_{ji}(\alpha),\ \ i = 1,2,\dots,\bar{m}_j \right\},$$

where

$$f_{ji}(\alpha) = \max_{x_j \in X_j(\alpha)} p_{ji}^\top x_j, \tag{4.2}$$

and normal vectors $p_{ji}^\top$, $i = 1,2,\dots,\bar{m}_j$, are independent of α, $j = 1,2,\dots,N$.

Maximization problems (4.2) have the form:

$$f_{ji}(\alpha) = \max_{x_j,u_j} p_{ji}^\top x_j, \tag{4.3}$$

subject to

$$x_0(t+1) = Ax_0(t) + Bu_j(t),\ \ x_0(T_j) = x_j,$$
$$x^l(t+1) = Ax^l(t) + Bu_j(t) + Mw_j^l(t),\ \ x^l(T_j) = x_j,\ l \in L_j,$$
$$||u_j(t)||_\infty \le u_{\max},\ \ t = T_j, T_j+1, \dots, T_{j+1}-1,$$
$$H(t)x_0(t) \le g(t) - \gamma_j(t),\ \ t = T_j+1, T_j+2, \dots, T_{j+1},$$
$$\sum_{t=T_j}^{T_{j+1}-1} \Big(||Qx^l(t)||_\infty + ||Ru_j(t)||_\infty \Big) + V_{j+1}(x^l(T_{j+1})) \le \alpha,\ \ l \in L_j.$$

Here $\gamma_j(t) = (\gamma_{ji}(t), i = 1,2,\dots,m)$: $\gamma_{ji}(t) = w_{\max} \sum_{s=T_j}^{t-1} ||h_i(t)^\top A^s M||_1$, $w_j^l(\cdot)$ is the l-th vertex of the hypercube $W^{T_{j+1}-T_j}$, $l \in L_j$.

Results of Section 3.3 allow us to assume that, as in (3.10),

$$\bar{X}_j(\alpha) = \left\{x_j \in \mathbb{R}^n : P_j x_j \leq g_j + \lambda_j \alpha\right\}, \quad j = 1,2,\ldots,N. \tag{4.4}$$

Representation (4.4) is proved by induction if we rewrite (4.3) as a linear program, approximating each constraint $V_{j+1}(x^l(T_{j+1})) \leq \alpha^l_{j+1}$, $l \in L_j$, by $x^l(T_{j+1}) \in \bar{X}_{j+1}(\alpha^l_{j+1})$, $l \in L_j$, and using (4.4) for $\bar{X}_{j+1}$:

$$\begin{gathered}
f_{ji}(\alpha) = \max_{x_j,u_j,\varepsilon_x,\varepsilon_u} p_{ji}^\top x_j, \\
-\varepsilon_u(t)\mathbb{1}_r \pm Ru_j(t) \leq 0, \;\; \pm u_j(t) \leq u_{\max}\mathbb{1}_r, \\
t = T_j, T_j+1, \ldots, T_{j+1}-1, \\
H(t)A^{t-T_j}x_j + \sum_{s=T_1}^{t-1} H(t)A^{t-s-1}Bu_j(s) \leq g(t) - \gamma_j(t), \\
t = T_j+1, T_j+2, \ldots, T_{j+1}, \\
-\varepsilon^l_x(t)\mathbb{1}_n \pm QA^{t-T_j}x_j \pm \sum_{s=T_j}^{t-1} QA^{t-s-1}Bu_j(s) \leq \mp \sum_{s=T_j}^{t-1} QA^{t-s-1}Mw^l_j(s), \\
t = T_j, T_j+1 \ldots, T_{j+1}-1, \\
-\lambda_{j+1}\alpha^l_{j+1} + P_{j+1}A^{T_{j+1}-T_j}x_j + \sum_{s=T_j}^{T_{j+1}-1} P_{j+1}A^{T_{j+1}-s-1}Bu_j(s) \leq g_{j+1} - \sum_{s=T_j}^{T_{j+1}-1} P_{j+1}A^{T_{j+1}-s-1}Mw^l_j(s), \\
\sum_{t=T_j}^{T_{j+1}-1} \left(\varepsilon^l_x(t) + \varepsilon_u(t)\right) + \alpha^l_{j+1} \leq \alpha, \;\; \alpha^{j+1}_{\min} \leq \alpha^l_{j+1} \leq \alpha^{j+1}_{\max}, \\
l \in L_j.
\end{gathered}$$

Here $j = 1,2,\ldots,N$, and for general representation we set

$$P_{N+1} = \begin{pmatrix} P \\ -P \end{pmatrix}, \;\; g_{N+1} = \begin{pmatrix} 0_n \\ 0_n \end{pmatrix}, \;\; \lambda_{N+1} = \begin{pmatrix} \mathbb{1}_n, \\ \mathbb{1}_n \end{pmatrix}.$$

Applying Algorithm of Section 3.3 to all $j = N, N-1, \ldots, 1$, we obtain partitions $A^k_{ji} = [a^{k+1}_{ji}, a^k_{ji}]$, $k = 1,2,\ldots,K_{ji}$, of the intervals $[\alpha^j_{\min}, \alpha^j_{\max}]$, and the values

$$\lambda^k_{ji} = \sum_{l \in L_j} y^l_{ji}(a^k_{ji} - \varepsilon), \;\; g^k_{ji} = f_{ji}(a^k_{ji} - \varepsilon) - (a^k_{ji} - \varepsilon)\lambda^k_{ji}.$$

Then construct

$$P_j = \begin{pmatrix} P_{ji} \\ i = 1,2,\ldots,\bar{m}_j \end{pmatrix}, \;\; g_j = \begin{pmatrix} g_{ji} \\ i = 1,2,\ldots,\bar{m}_j \end{pmatrix}, \;\; \lambda_j = \begin{pmatrix} \lambda_{ji} \\ i = 1,2,\ldots,\bar{m}_j \end{pmatrix}, \tag{4.5}$$

with blocks

$$P_{ji} = \begin{pmatrix} p^\top_{ji} \\ k = 1,2,\ldots K_{ji} \end{pmatrix}, \;\; g_{ji} = \begin{pmatrix} g^k_{ji} \\ k = 1,2,\ldots K_{ji} \end{pmatrix}, \;\; \lambda_{ji} = \begin{pmatrix} \lambda^k_{ji} \\ k = 1,2,\ldots K_{ji} \end{pmatrix},$$

for approximation (4.4).

Finally, the suboptimal controls $\bar{u}(\cdot|x_j)$, $j = 0,1,\ldots,N$, are found as the solution of the following linear program:

$$\min_{u_j,\varepsilon_x,\varepsilon_u,\alpha_{j+1},\alpha_j} \alpha_j,$$

$$-\varepsilon_u(t)\mathbb{1}_r \pm Ru_j(t) \le 0, \quad \pm u_j(t) \le u_{\max}\mathbb{1}_r,$$

$$t = T_j, T_j+1, \ldots, T_{j+1}-1,$$

$$\sum_{s=T_j}^{t-1} H(t)A^{t-s-1}Bu_j(s) \le g(t) - \gamma_j(t) - H(t)A^t x_j,$$

$$t = T_j+1, T_j+2, \ldots, T_{j+1},$$

$$-\varepsilon_x^l(t)\mathbb{1}_n \pm \sum_{s=T_j}^{t-1} QA^{t-s-1}Bu_j(s) \le \mp\big(QA^t x_j + \sum_{s=T_j}^{t-1} QA^{t-s-1}Mw_j^l(s)\big), \tag{4.6}$$

$$t = T_j, T_j+1, \ldots, T_{j+1}-1,$$

$$-\lambda_{j+1}\alpha_{j+1}^l + \sum_{s=T_j}^{T_{j+1}-1} P_{j+1}A^{T_{j+1}-s-1}Bu_j(s) \le g_{j+1} - \big(P_{j+1}A^{T_{j+1}}x_j + \sum_{s=T_j}^{T_{j+1}-1} P_{j+1}A^{T_{j+1}-s-1}Mw_j^l(s)\big),$$

$$\sum_{t=T_j}^{T_{j+1}-1} \big(\varepsilon_x^l(t) + \varepsilon_u(t)\big) + \alpha_{j+1}^l - \alpha_j \le 0, \quad \alpha_{\min}^{j+1} \le \alpha_{j+1}^l \le \alpha_{\max}^{j+1},$$

$$l \in L_j.$$

Note that problems (4.6) for $j = 1,2,\ldots,N$, are solved at times T_j for the current measurement $x_j = x^*(T_j)$. All matrices (4.5) are constructed beforehand and are not updated during the control process.

4.2. **Example.** We consider an example from section 3.4, however with $T = 12$, since for $T = 10$ adding a second closing instant does not yield improvement in strategy performance.

For $T = 12$ there is no feasible open-loop control, since $||\gamma(T)||_\infty \ge 2$ and the terminal constraint cannot be satisfied robustly. The optimal strategy with one closing instant, however, exists and $V_0(x_0) = 9.997150$ for $T_1 = 7$ and $V_0(x_0) = 10.114121$ for $T_1 = 10$.

Let $T_1 = 7$, $T_2 = 10$. The cost of the optimal control strategy π_2^0 with two closing instants is equal to $V_0(x_0) = 9.788062$.

The closure sets $X_1(\alpha)$ and $X_2(\alpha)$ were approximated with the accuracy 10^{-12} and their description required 324 vectors p_{1i} and 40 vectors p_{2i}. After constructing (4.5) we obtained $m_2 = 130$, $m_1 = 3186$.

Problem (4.6) for $j = 0$ had 411804 constraints and 1039 variables. Its solution took 3.7 sec by the adaptive method.

The optimal initial control $u_0^0(\cdot)$ is presented in Fig. 3. We applied it to the system under the worst-case disturbance and then calculated $u_1^0(\cdot|x^*(7))$ and $u_2^0(\cdot|x^*(10))$. Time spent on online calculations of these optimal controls was equal to 0.0014 and 0.0011 sec, correspondingly. The optimal trajectory is also shown in Fig. 3.

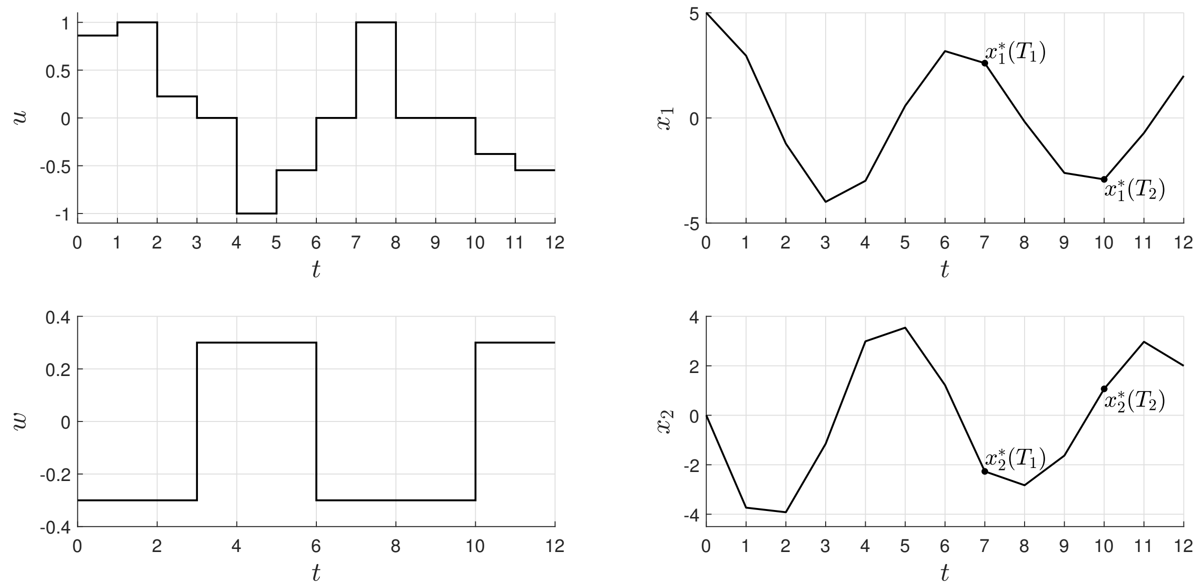

FIGURE 3. Optimal controls, worst-case disturbance and corresponding trajectories in Example 2

5. CONCLUSION

This paper presents an efficient method for constructing optimal control strategies with multiple closing instants in a predictive optimal control problem originating from linear explicit model predictive control. Formulating the problem via dynamic programming arguments however not for all time instants but only for the closing instants T_j we obtain a multilevel optimization problem that was reduced to a number of linear programs that are solved offline to construct the approximations of the closure sets, and a single linear program that is solved online, at the closing instants T_j after the state measurement $x^*(T_j)$ becomes available. The method can be utilized within model predictive control schemes or optimal real-time feedback control since online computations are very fast.

Acknowledgements
The first author is deeply indebted to Professor Rafail Gabasov and Professor Faina Kirillova for their invaluable guidance and support throughout her career.

REFERENCES

[1] N.V. Balashevich, R. Gabasov, F.M. Kirillova, Numerical methods for open-loop and closed-loop optimization of linear control systems, Comput. Math. Math. Phys. 40 (2000) 799-819.
[2] N.V. Balashevich, R. Gabasov, F.M. Kirillova, Numerical methods for open loop and closed loop optimization of piecewise linear systems, Comput. Math. Math. Phys. 41 (2001) 1578-1593.
[3] N.V. Balashevich, R. Gabasov, A.I. Kalinin, F.M. Kirillova, Optimal control of nonlinear systems, Comput. Math. Math. Phys. 42 (2002) 931-956.
[4] N.V. Balashevich, R. Gabasov, F.M. Kirillova, Optimal control of linear systems with delay, Dokl. Math. 74 (2006) 780-785.
[5] A. Bemporad, F. Borrelli, M. Morari, Model predictive control based on linear programming – the explicit solution, IEEE Trans. Auto. Control 47 (2002) 1974-1985.

[6] A. Bemporad, F. Borrelli, M. Morari, Min-max control of constrained uncertain discrete-time linear systems, IEEE Trans. Autom. Control 48 (2003) 1600-1606.
[7] F. Borrelli, Constrained Optimal Control for Hybrid Systems, Springer, Berlin Heidelberg, 2003.
[8] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[9] K.T. Chong, O. Kostyukova, M. Kurdina, Guaranteed control policy with arbitrary set of correction points for linear-quadratic system with delay, Control and Cybernetics 39 (2010) 1723-1733.
[10] N.M. Dmitruk, R. Findeisen, F. Allgöwer, Optimal measurement feedback control of finite-time continuous linear systems, IFAC Proceedings Volumes 41 (2008) 15339-15344.
[11] N.M. Dmitruk, Optimal robust control of linear systems by inaccurate measurements of output signals, J. Automation Info. Sci. 43 (2011) 11.
[12] N.M. Dmitruk, Optimal strategy with one closing instant for a linear optimal guaranteed control problem, Comp. Math. Math. Phys. 58 (2018) 642-658.
[13] A. Dontchev, I. Kolmanovsky, M. Krastanov, V. Veliov, P. Vuong, Approximating optimal finite horizon feedback by model predictive control. Sys.& Control Lett. 139 (2020) 104666.
[14] R. Gabasov, F.M. Kirillova, A.I. Tyatyushkin, Constructive Methods of Optimization. Part 1. Linear Problems, University Press, Minsk, 1984.
[15] R. Gabasov, F.M. Kirillova, Constructive Methods of Optimization. Part 2. Control Problems, University Press, Minsk, 1984.
[16] R. Gabasov, F.M. Kirillova, O.I. Kostyukova, Construction of optimal controls of feedback type in a linear problem, Dokl. Math. 44 (1992) 608-613
[17] R. Gabasov, F.M. Kirillova, S.V. Prischepova, Optimal Feedback Control, Springer, 1995.
[18] R. Gabasov, F.M. Kirillova, E.A. Kostina, Subtended Feedback with Respect to State for Optimization of Uncertain Control Systems. I. Single Loop, Autom. Remote Control 57 (1996) 1008-1015
[19] R. Gabasov, F.M. Kirillova, E.A. Kostina, Closed-Loop State Feedback for Optimization of Uncertain Control Systems. II. Multiply Closed Feedback, Autom. Remote Control 57 (1996) 1137-1145
[20] R. Gabasov, F.M. Kirillova, N.V. Balashevich Open-loop and closed-loop optimization of linear control systems, Asian J. Control 2 (2000) 155-168.
[21] R. Gabasov, N.M. Dmitruk, F.M. Kirillova, Optimization of the Multidimensional Control Systems with Parallelepiped Constraints, Auto. Remote Control 63 (2002) 345-366
[22] R. Gabasov, F. Kirillova, N. Balashevich, Guaranteed on-line control for linear systems under disturbances, Funct. Diff. Equ. 11 (2004) 341-361.
[23] R. Gabasov, N.M. Dmitruk, F.M. Kirillova, Optimal control of multidimensional systems by inaccurate measurements of their output signals, Proc. Steklov Inst. Math. (Suppl.) suppl.2 (2004) S52-S75.
[24] R. Gabasov, N.M. Dmitruk, F.M. Kirillova, Optimal guaranteed control of delay systems, Proc. Steklov Inst. Math. (Suppl.) 255: suppl.2 (2006) S26-S46.
[25] R. Gabasov, F.M. Kirillova, E.I. Poyasok, Optimal control based on a preposteriori estimates of set-membership uncertainty, Auto. Remote Control 72 (2011) 74-87.
[26] R. Gabasov, F.M. Kirillova, E.I. Poyasok, Optimal control of a dynamic system with multiple uncertainty in the initial state as based on imperfect measurements of input and output signals, Comput. Math. Math. Phys. 52 (2012) 992-1006.
[27] R. Gabasov, F.M. Kirillova, D.S. Kuzmenkov, Real-time optimal control of a special distributed parameter system, Comput. Math. Math. Phys. 54 (2014) 1765-1775.
[28] T. Gal, Postoptimal Analyses, Parametric Programming and Related Topics: Degeneracy, Multicriteria Decision Making Redundancy, De Gruyter, Berlin, 1995.
[29] L. Grüne, J. Pannek, Nonlinear Model Predictive Control, Springer, Cham., 2017.
[30] P.J. Goulart, E.C. Kerrigan, J.M. Maciejowski, Optimization over state feedback policies for robust control with constraints, Automatica 42 (2006) 523-533.
[31] E.C. Kerrigan, J.M. Maciejowski, Feedback min-max model predictive control using a single linear program: robust stability and the explicit solution, Int. J. Robust Nonlinear Control 14 (2004) 395-413.
[32] D.A. Kastsiukevich, N.M. Dmitruk, A method for constructing an optimal control strategy in a linear terminal problem. J. Belarus. State Uni. Math. Inf. 2 (2021) 38-50.

[33] O. Kostyukova, E. Kostina, Robust optimal feedback for terminal linear-quadratic control problems under disturbances, Math. Program. B. 107 (2006) 131-153.
[34] O. Kostyukova, E. Kostina, Worst-case control policies for (terminal) linear-quadratic control problems under disturbances, Int. J. Robust Nonlinear Control 19 (2009) 1940-1958.
[35] M. V. Kothare, V. Balakrishnan, M. Morari, Robust constrained model predictive control using linear matrix inequalities, Automatica 32 (1996) 1361-1379.
[36] A.B. Kurzhanski, Control and Observation under Conditions of Uncertainty, Nauka, Moscow, 1977.
[37] A.B. Kurzhanski, P. Varaiya, On reachability under uncertainty, SIAM J. Control Optim. 41 (2002) 181-216.
[38] J. Lee, Z. Yu, Worst-case formulations of model predictive control for systems with bounded parameters, Automatica 33 (1997) 763-781.
[39] J.B. Rawlings, D.Q. Mayne, Model Predictive Control: Theory and Design, Nob Hill Publishing, Madison, 2009.
[40] J. Skaf, S.P. Boyd, A.J. Zeevi, Shrinking-horizon dynamic programming, Int. J. Robust Nonlinear Control 20 (2010) 1993-2002.
[41] P.O.M. Scokaert, D.Q. Mayne, Min-max feedback model predictive control for constrained linear systems, Trans. Autom. Control 43 (1998) 1136-1142.
[42] H. Witsenhausen, A minimax control problem for sampled linear systems, IEEE Trans. Auto. Control 13 (1968) 5-21.
[43] S.J. Qin, T.A. Badgwell, A survey of industrial model predictive control technology, Control Eng. Pract. 11 (2003) 733-764.