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EXTENDED EFFICIENT HIGH CONVERGENCE ORDER SCHEMES FOR EQUATIONS

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Dedicated to the memory of Dr. Ram U. Verma

Abstract. Several methods are used to develop iterative schemes for solving equations. Higher-order derivatives, on the other hand, are often considered to be used in the calculation of convergence order. But the derivatives are not on the schemes. More significantly, there are no bounds on the error and uniqueness information for the solution to be generated either. So the advantages of these algorithms are restricted in their use of equations with operators that are at least seven times differentiable. We investigate the ball of convergence analysis using only the first derivative for two sixth-order algorithms that are run under an equal set of circumstances. In addition, we provide a calculable ball comparison between the two schemes under consideration. Our technique is based on the first derivative that only appears on the schemes. This way, we can make these schemes more useful for addressing equations involving Banach space-valued operators. Hence, the applicability is extended for these schemes. The technique can be used on other schemes.

Keywords. Banach spaces; Convergence order; Convergence ball; Fréchet derivative; Local convergence.

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1. Introduction

Consider a Fréchet derivable operator $\Phi : \Omega \subseteq X \to Y$, where X and Y are Banach spaces, and $\Omega(\neq \emptyset)$ is convex and open. In science and other practical fields, the equations of the type

$$\Phi(v) = 0 \tag{1.1}$$

are regularly used to address a wide range of complicated problems. It is necessary to point that it is a tough process to obtain the solutions to these equations. The solutions were only obtained analytically in a small number of cases. As a result, iterative procedures are often

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employed to solve these equations. It is, however, a challenging job to create an effective iterative strategy for addressing (1.1). The traditional Newton iterative approach is the one that was most often considered to solve this problem. In addition, a large number of research on higher-order modifications of conventional processes, such as Newton's, Chebyshev's, Jarratt's, etc. have been utilized.

Various higher-order iterative strategies for calculating a solution of (1.1) have been described in the literature. These are based on Newton-like or Newton's iterative step. A number of authors, for example, Homeier [18], Frontini and Sormani [15], Cordero and Torregrosa [11, 12, 13], Noor and Waseem [20] and Grau et al. [16, 17] have created third convergence order techniques, each of which requires one Φ and two Φ' evaluations. In [13], two cubically convergent iterative procedures were designed by Cordero and Torregrosa. Another third-order convergent scheme based on the evaluations of two Φ, one Φ', and one inversion of a matrix was presented by Darvishi and Barati [14]. In addition, Cordero et al. [11, 12] extended Jarratt's scheme [19] for addressing nonlinear systems. Grau et al. [16, 17] and Darvishi and Barati [14] also suggested schemes with convergence order four. Sharma et al. [21, 22] composed two weighted-Newton steps to generate an efficient fourth-order weighted-Newton scheme for nonlinear systems. Also, fourth and sixth-order convergent iterative algorithms were developed by Sharma and Arora [21] to solve nonlinear systems. Related research results on other iterative processes with their ball of convergence and dynamical behaviors were discussed in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

The main work of this paper is to increase the usefulness of the sixth convergence order schemes that we chosen from [21] and [23], respectively. In addition, we compare their convergence balls.

$$\begin{aligned} y_n &= v_n - \gamma\Phi'(v_n)^{-1}\Phi(v_n) \\ z_n &= v_n - \left(\frac{23}{8}I - 3\Phi'(v_n)^{-1}\Phi'(y_n) + \frac{9}{8}\left(\Phi'(v_n)^{-1}\Phi'(y_n)\right)^2\right)\Phi'(v_n)^{-1}\Phi(v_n) \\ v_{n+1} &= z_n - \left(\frac{5}{2}I - \frac{3}{2}\Phi'(v_n)^{-1}\Phi'(y_n)\right)\Phi'(v_n)^{-1}\Phi(z_n), \end{aligned} \tag{1.2}$$

$$\begin{aligned} y_n &= v_n - \gamma\Phi'(v_n)^{-1}\Phi(v_n) \\ z_n &= v_n - \Big(I + \frac{21}{8}\Phi'(v_n)^{-1}\Phi'(y_n) - \frac{9}{2}(\Phi'(v_n)^{-1}\Phi'(y_n))^2 \\ &\quad + \frac{15}{8}(\Phi'(v_n)^{-1}\Phi'(y_n))^3\Big)\Phi'(v_n)^{-1}\Phi(v_n) \\ v_{n+1} &= z_n - (3I - \frac{5}{2}\Phi'(v_n)^{-1}\Phi'(y_n) + (\frac{1}{2}\Phi'(v_n)^{-1}\Phi'(y_n))^2)\Phi'(v_n)^{-1}\Phi(z_n). \end{aligned} \tag{1.3}$$

If $\gamma = \frac{2}{3}$, then schemes (1.2) and (1.3) are reduced to the schemes designed in [21] and [23], respectively. The convergence of these schemes was shown under the application of the expensive Taylor formula, which reduces their scope of utility. We consider the following function to help us explain our viewpoint.

$$\Phi(v) = \begin{cases} v^3\ln(v^2) + v^5 - v^4, & \text{if } v \neq 0 \\ 0, & \text{if } v = 0 \end{cases},$$

where $X = Y = \mathbb{R}$ and F is defined on $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Then, the unboundedness of Φ''' makes the earlier convergence theorems ineffective for schemes (1.2) and (1.3). Also, current results provide little information regarding the bounds of the error, the domain of convergence, or the location of the solution. It is critical to investigate the ball analysis of an iterative scheme in detail to determine convergence radii, approximate error bounds, and calculate the region where x_* is the only solution. Another benefit of this analysis is that it simplifies the very difficult task of selecting v_0. Consequently, we are inspired to investigate and compare the convergence balls of (1.2) and (1.3) when subjected to an identical set of constraints. In addition to providing an error estimate $||v_n - x_*||$ and the convergence radii, the convergence theorems that we presented also offer a precise location for the solution.

The following is a summary of the contents of this paper. Section 2 contains the ball of convergence of schemes (1.2) and (1.3). Section 3 includes the numerical experiments. The last section, Section 4, of this paper contains the conclusions.

2. Ball of Convergence

Some scalar parameters and functions are developed for the ball convergence analysis first of scheme (1.2). Set $T = [0, \infty)$. Suppose function

(1) $\omega_0(t) - 1$ has a smallest root $R_0 \in T_0 \setminus \{0\}$ for some function $\omega_0 : T \to T$ continuous and non-decreasing. Set $T_0 = [0, R_0)$.

(2) $g_1(t) - 1$ has a smallest root $r_1 \in T_0 \setminus \{0\}$ for some functions $\omega : [0, 2R_0) \to T$, $\omega_1 : T_0 \to T$ continuous and non-decreasing with $g_1 : T_0 \to T$ given by

$$g_1(t) = \frac{\int_0^1 \omega((1-\theta)t)\, d\theta + |1-\gamma| \int_0^1 \omega_1(\theta t)\, d\theta}{1 - \omega_0(t)}.$$

(3) $g_2(t) - 1$ has a smallest root $r_2 \in T_0 \setminus \{0\}$ with

$$\begin{aligned} g_2(t) = {} & \frac{\int_0^1 \omega((1-\theta)t)\, d\theta}{1 - \omega_0(t)} \\ & + \frac{3}{8}\left(3\left(\frac{\omega_0(t) + \omega_0(g_1(t)t)}{1 - \omega_0(t)}\right)^2 \right. \\ & \left. + 2\left(\frac{\omega_0(t) + \omega_0(g_1(t)t)}{1 - \omega_0(t)}\right)\right) \frac{\int_0^1 \omega_1(\theta t)\, d\theta}{1 - \omega_0(t)}. \end{aligned}$$

(4) $\omega_0(g_2(t)t) - 1$ has a smallest root $R_1 \in T_0 \setminus \{0\}$. Set $R = \min\{R_0, R_1\}$ and $T_1 = [0, R)$.

(5) $g_3 : T_1 \to T$ is such that $g_3(t) - 1$ has a smallest root $r_3 \in T_1 \setminus \{0\}$, with

$$\begin{aligned} g_3(t) = \Bigg[& \frac{\int_0^1 \omega((1-\theta)g_2(t)t)\, d\theta}{1 - \omega_0(g_2(t)t)} + \frac{(\omega_0(t) + \omega_0(g_2(t)t)) \int_0^1 \omega_1(\theta g_2(t)t)\, d\theta}{(1-\omega_0(t))(1-\omega_0(g_2(t))} \\ & \frac{3(\omega_0(t) + \omega_0(g_2(t)t)) \int_0^1 \omega_1(\theta g_2(t)t)\, d\theta}{(1-\omega_0(t))^2} \Bigg] g_2(t). \end{aligned}$$

The parameter r defined by

$$r = \min\{r_i\}, i = 1, 2, 3 \tag{2.1}$$

shall be proved to be a convergence radius for scheme (1.2). Set $M = [0, r)$. By the definition of r it follows that, for all $t \in M$,

$$0 \leq \omega_0(t) < 1, \tag{2.2}$$

$$0 \leq \omega_0(g_2(t)t) < 1, \tag{2.3}$$

and

$$0 \leq g_i(t) < 1. \tag{2.4}$$

The notation $\overline{B}(x_*, \lambda)$ is used for the closure of the ball $B(x_*, \lambda)$ with radius $\lambda > 0$ and center $x_* \in \Omega$. We suppose from now on that x_* is a simple root of Φ, functions "ω" are as defined previously, and the following hypotheses (A) hold. Suppose:

(A_1) For all $x \in \Omega$,

$$||\Phi'(x_*)^{-1}(\Phi'(x) - \Phi'(x_*))|| \leq \omega_0(||x - x_*||).$$

Set $\Omega_0 = \Omega \cap B(x_*, R_0)$.

(A_2) For all $x, y \in \Omega_0$

$$||\Phi'(x_*)^{-1}(\Phi'(x) - \Phi'(y))|| \leq \omega(||x - y||)$$

and

$$||\Phi'(x_*)^{-1}\Phi'(x)|| \leq \omega_1(||x - x_*||).$$

(A_3) $\overline{B}(x_*, \tilde{r}) \subset \Omega$ for some $\tilde{r}$ to be defined later.

(A_4) There exists $r_* \geq \tilde{r}$ satisfying

$$\int_0^1 \omega_0(\theta r_*)\, d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \overline{B}(x_*, r_*)$.

Next, we develop the ball convergence result for scheme (1.2) utilizing conditions A.

Theorem 2.1. *Suppose that conditions (A_1)-(A_4) hold for $\tilde{r} = r$. Then, the sequence $\{v_n\}$ given by scheme (1.2) is well defined in $B(x_*, r)$, stays in $B(x_*, r)$ and converges to x_* provided that the initial guess $v_0 \in B(x_*, r) \setminus \{x_*\}$. Moreover, the following assertions hold*

$$||y_n - x_*|| \leq g_1(||v_n - x_*||)||v_n - x_*|| \leq ||v_n - x_*|| < r, \tag{2.5}$$

$$||z_n - x_*|| \leq g_2(||v_n - x_*||)||v_n - x_*|| \leq ||v_n - x_*||, \tag{2.6}$$

and

$$||v_{n+1} - x_*|| \leq g_3(||v_n - x_*||)||v_n - x_*|| \leq ||v_n - x_*||, \tag{2.7}$$

where the functions g_i and radius r are as defined previously. Furthermore, the only root of $\Phi(v) = 0$ in the set Ω_1 defined in (A_4) is x_*.

Proof. Let $z \in B(x_*, r) \setminus \{x_*\}$. Using (A_1), (2.1), and (2.2), we obtain

$$||\Phi'(x_*)^{-1}(\Phi'(z) - \Phi'(x_*))|| \leq \omega_0(||z - x_*||) \leq \omega_0(r) < 1,$$

which together with a lemma due to Banach on invertible operators [8] implies $\Phi'(z)^{-1} \in L(Y, X)$ with

$$||\Phi'(z)^{-1}\Phi'(x_*)|| \leq \frac{1}{1 - \omega_0(||z - x_*||)}. \tag{2.8}$$

Notice that y_0, z_0, v_1 are well defined by scheme (1.2). We can write in turn by the first substep of scheme (1.2), (A_2) and (2.8) (for $z = v_0$) that

$$\begin{aligned}
||y_0 - x_*|| &= ||v_0 - x_* - \Phi'(v_0)^{-1}\Phi(v_0) + (1-\gamma)\Phi'(v_0)^{-1}\Phi(v_0)|| \\
&\leq ||\Phi'(v_0)^{-1}\Phi'(x_*)|| \, || \int_0^1 \Phi'(x_*)^{-1}(\Phi'(x_* + \theta(v_0 - x_*)) - \Phi'(v_0))\, d\theta|| \, ||v_0 - x_*|| \\
&+ |1-\gamma| \, ||\Phi'(v_0)^{-1}\Phi'(x_*)|| \, ||\Phi'(x_*)^{-1}\Phi(v_0)|| \\
&\leq \frac{(\int_0^1 \omega((1-\theta)||v_0 - x_*||)\, d\theta + |1-\gamma| \int_0^1 \omega_1(\theta||v_0 - x_*||)\, d\theta)||v_0 - x_*||)}{1 - \omega_0(||v_0 - x_*||)} \\
&\leq g_1(||v_0 - x_*||)||v_0 - x_*|| \leq ||v_0 - x_*|| < r. \qquad (2.9)
\end{aligned}$$

Similarly, by the second substep of (1.2) and (2.9), we have

$$\begin{aligned}
&||z_0 - x_*|| \\
&= ||v_0 - x_* - \Phi'(v_0)^{-1}\Phi(v_0) \\
&- \frac{3}{8}(5I - 8\Phi'(v_0)^{-1}\Phi'(y_0) + 3(\Phi'(v_0)^{-1}\Phi'(y_0))^2))\Phi'(v_0)^{-1}\Phi(v_0)|| \\
&= ||v_0 - x_* - \Phi'(v_0)^{-1}\Phi(v_0) \\
&- \frac{3}{8}(3(\Phi'(v_0)^{-1}(\Phi'(y_0) - \Phi'(v_0)))^2 - 2(\Phi'(v_0)^{-1}(\Phi'(y_0) - \Phi'(v_0)))\Phi'(v_0)^{-1}\Phi(v_0)|| \\
&\leq \Bigg[\frac{\int_0^1 \omega((1-\theta)||v_0 - x_*||)\, d\theta}{1 - \omega_0(||v_0 - x_*||)} \\
&+ \frac{3}{8}\left(3\left(\frac{\omega_0(||v_0 - x_*||) + \omega_0(||y_0 - x_*||)}{1 - \omega_0(||v_0 - x_*||)}\right)^2 \right. \\
&\left. + 2\left(\frac{\omega_0(||v_0 - x_*||) + \omega_0(||y_0 - x_*||)}{1 - \omega_0(||v_0 - x_*||)}\right)\right) \frac{\int_0^1 \omega_1(\theta||v_0 - x_*||)\, d\theta}{1 - \omega_0(||v_0 - x_*||)}\Bigg] ||v_0 - x_*|| \\
&\leq g_2(||v_0 - x_*||)||v_0 - x_*|| \leq ||v_0 - x_*||. \qquad (2.10)
\end{aligned}$$

Moreover, by the third substep of (1.2), (2.9) and (2.10), we have

$$\begin{aligned}
&||v_1 - x_*|| \\
&= ||z_0 - x_* - \Phi'(z_0)^{-1}\Phi(z_0) \\
&+ (\Phi'(z_0)^{-1} - \Phi'(v_0)^{-1})\Phi(z_0) - \frac{3}{2}\Phi'(v_0)^{-1}(\Phi'(v_0) - \Phi'(y_0))\Phi'(v_0)^{-1}\Phi(z_0)|| \\
&\leq \Bigg[\frac{\int_0^1 \omega((1-\theta)||z_0 - x_*||)\, d\theta}{1 - \omega_0(||z_0 - x_*||)} \\
&+ \frac{(\omega_0(||v_0 - x_*||) + \omega_0(||z_0 - x_*||)) \int_0^1 \omega_1(\theta||z_0 - x_*||)\, d\theta}{(1 - \omega_0(||v_0 - x_*||))(1 - \omega_0(||z_0 - x_*||))} \\
&+ \frac{3}{2}\frac{(\omega_0(||v_0 - x_*||) + \omega_0(||y_0 - x_*||)) \int_0^1 \omega_1(\theta||z_0 - x_*||)\, d\theta}{(1 - \omega_0(||v_0 - x_*||))^2}\Bigg] ||z_0 - x_*|| \\
&\leq g_3(||v_0 - x_*||)||v_0 - x_*|| \leq ||v_0 - x_*||, \qquad (2.11)
\end{aligned}$$

where we also used (2.1), (2.4) (for $i = 1,2,3$), (2.8) (for $z = v_0, z_0$), (A_2), and (2.9)-(2.11). Hence, items (2.5)-(2.7) hold if $n = 0$. Simply replace v_0, y_0, z_0, and v_1 by v_j, y_j, z_j, and v_{j+1} in the previous calculations to complete the mathematical induction for items (2.5)-(2.7). Then, from the estimation

$$||v_{j+1} - x_*|| \le b||v_j - x_*|| < r, \tag{2.12}$$

where $b = g_3(||v_0 - x_*||) \in [0,1)$, we deduce $v_{j+1} \in B(x_*, r)$ and $\lim_{j\to\infty} v_j = x_*$. Next, we show the uniqueness of x_*. Set $Q = \int_0^1 \Phi'(x_* + \theta(q - x_*))\, d\theta$ for some $q \in \Omega_1$ with $\Phi(q) = 0$. Then, using (A_1) and (A_4), we obtain

$$\begin{aligned} ||\Phi'(x_*)^{-1}(Q - \Phi'(x_*))|| &\le \int_0^1 \omega_0(\theta||q - x_*||)\, d\theta \\ &\le \int_0^1 \omega_0(\theta r_*)\, d\theta < 1. \end{aligned} \tag{2.13}$$

So, we conclude $x_* = q$ from $0 = \Phi(q) - \Phi(x_*) = Q(q - x_*)$ and the invertability of Q. □

Next, we develop the ball convergence analysis of scheme (1.3) in an analogous way. Define

$$\begin{aligned} \overline{g_1}(t) &= g_1, \\ \overline{g_2}(t) &= \frac{\int_0^1 \omega((1-\theta)t)\, d\theta}{1 - \omega_0(t)} \\ &\quad + \frac{3}{8}\left(5\left(\frac{\omega_0(t) + \omega_0(\overline{g_1}(t)t)}{1 - \omega_0(t)}\right)^2 \right. \\ &\quad \left. + 2\left(\frac{\omega_0(t) + \omega_0(\overline{g_1}(t)t)}{1 - \omega_0(t)}\right)\right) \frac{\int_0^1 \omega_1(\theta t)\, d\theta}{1 - \omega_0(t)}, \end{aligned}$$

and

$$\begin{aligned} \overline{g_3}(t) = \Bigg[& \frac{\int_0^1 \omega((1-\theta)\overline{g_2}(t)t)\, d\theta}{1 - \omega_0(g_2(t)t)} \\ & + \frac{(\omega_0(t) + \omega_0(\overline{g_2}(t)t)) \int_0^1 \omega_1(\theta \overline{g_2}(t)t)\, d\theta}{(1 - \omega_0(t))(1 - \omega_0(\overline{g_2}(t)t))} \\ & + \frac{1}{2}\left(\left(\frac{\omega_0(t) + \omega_0(\overline{g_2}(t)t)}{1 - \omega_0(t)}\right)^2 \right. \\ & \left. + 3\left(\frac{\omega_0(t) + \omega_0(\overline{g_2}(t)t)}{1 - \omega_0(t)}\right)\right) \frac{\int_0^1 \omega_1(\theta \overline{g_2}(t)t)\, d\theta}{1 - \omega_0(t)} \Bigg] g_2(t). \end{aligned}$$

Suppose that functions $\overline{g_i}(t) - 1$ have a smallest root in $T_0 \setminus \{0\}$ denoted by $\overline{r_i}$. Set $\overline{r} = \min\{\overline{r_i}\}$. Moreover, suppose that hypotheses (A_1)-(A_4) hold with $\tilde{r} = \overline{r}$. Following estimates (2.9)-(2.11), we show that functions $\overline{g_i}$ are motivated by calculations:

$$\begin{aligned} ||y_n - x_*|| &\le g_1(||v_n - x_*||)||v_n - x_*|| = \overline{g_1}(||v_n - x_*||)||v_n - x_*|| \\ &\le ||v_n - x_*|| < \overline{r}. \end{aligned}$$

Moreover, the second substep gives

$$\begin{aligned}
&||z_n - x_*||\\
&= ||v_n - x_* - \Phi'(v_n)^{-1}\Phi(v_n)\\
&- \frac{3}{8}(7\Phi'(v_n)^{-1}\Phi'(y_n) - 12(\Phi'(v_n)^{-1}\Phi'(y_n))^2\\
&+ 5(\Phi'(v_n)^{-1}\Phi'(y_n))^3)\Phi'(v_n)^{-1}\Phi(v_n)||\\
&= ||v_n - x_* - \Phi'(v_n)^{-1}\Phi(v_n)\\
&- \frac{3}{8}(5(\Phi'(v_n)^{-1}(\Phi'(y_n) - \Phi'(v_n)))^2\\
&- 2(\Phi'(v_n)^{-1}(\Phi'(y_n) - \Phi'(v_n))))\Phi'(v_n)^{-1}\Phi'(y_n)\Phi'(v_n)^{-1}\Phi(v_n)||\\
&\leq \left[\frac{\int_0^1 \omega((1-\theta)||v_n - x_*||\, d\theta}{1 - \omega_0(||v_n - x_*||)}\right.\\
&+ \frac{3}{8}\left(5\left(\frac{\omega_0(||v_n - x_*||) + \omega_0(||y_n - x_*||)}{1 - \omega_0(||v_n - x_*||)}\right)^2\right.\\
&\left.\left.+ 2\left(\frac{\omega_0(||v_n - x_*||) + \omega_0(||y_n - x_*||)}{1 - \omega_0(||v_n - x_*||)}\right)\right)\frac{\int_0^1 \omega_1(\theta||v_n - x_*||)\, d\theta}{1 - \omega_0(||v_n - x_*||)}\right]||v_n - x_*||\\
&\leq \overline{g_2}(||v_n - x_*||)||v_n - x_*|| \leq ||v_n - x_*||.
\end{aligned}$$

Furthermore, the third substep leads to

$$\begin{aligned}
&||v_{n+1} - x_*||\\
&= ||z_n - x_* - \Phi'(z_n)^{-1}\Phi(z_n)\\
&+ \Phi'(z_n)^{-1}(\Phi'(v_n) - \Phi'(z_n))\Phi'(v_n)^{-1}\Phi(z_n)\\
&- \frac{1}{2}(4I - 5\Phi'(v_n)^{-1}\Phi'(y_n) + (\Phi'(v_n)^{-1}\Phi'(y_n))^2)\Phi'(v_n)^{-1}\Phi(z_n)||\\
&= ||z_n - x_* - \Phi'(z_n)^{-1}\Phi(z_n)\\
&+ \Phi'(z_n)^{-1}(\Phi'(v_n) - \Phi'(z_n))\Phi'(v_n)^{-1}\Phi(z_n)\\
&- \frac{1}{2}((\Phi'(v_n)^{-1}(\Phi'(y_n) - \Phi'(v_n)))^2 - 3\Phi'(v_n)^{-1}(\Phi'(y_n) - \Phi'(v_n)))\Phi'(v_n)^{-1}\Phi(z_n)||\\
&\leq \left[\frac{\int_0^1 \omega((1-\theta)||z_n - x_*||\, d\theta}{1 - \omega_0(||z_n - x_*||)}\right.\\
&+ \frac{(\omega_0(||v_n - x_*||) + \omega_0(||z_n - x_*||))\int_0^1 \omega_1(\theta||z_n - x_*||)\, d\theta}{(1 - \omega_0(||v_n - x_*||))(1 - \omega_0(||z_n - x_*||))}\\
&+ \frac{1}{2}\left(\left(\frac{\omega_0(||v_n - x_*||) + \omega_0(||y_n - x_*||)}{1 - \omega_0(||v_n - x_*||)}\right)^2\right.\\
&\left.\left.+ 3\left(\frac{\omega_0(||v_n - x_*||) + \omega_0(||y_n - x_*||)}{1 - \omega_0(||v_n - x_*||)}\right)\right)\frac{\int_0^1 \omega_1(\theta||z_n - x_*||)\, d\theta}{1 - \omega_0(||v_n - x_*||)}\right]||z_n - x_*||\\
&\leq \overline{g_3}(||v_n - x_*||)||v_n - x_*|| \leq ||v_n - x_*||.
\end{aligned}$$

Hence, we obtain the ball convergence result for scheme (1.3).

Theorem 2.2. *Suppose that hypotheses (A_1)-(A_4) hold with $\tilde{r} = \bar{r}$. Then, the conclusion of theorem 2.1 hold for scheme (1.3) with r, g_i replaced by $\bar{r}$, $\overline{g_i}$, respectively.*

3. Numerical Examples

We use the suggested approaches to estimate the convergence radii for schemes ((1.2) and (1.3)) provided that $\gamma = \frac{2}{3}$.

Example 3.1. Consider $X = Y = C[0,1]$ and $\Omega = \overline{B}(0,1)$. Define Φ on Ω by

$$\Phi(v)(a) = v(a) - 5\int_0^1 au\, v(u)^3\, du,$$

where $v(a) \in C[0,1]$. We have $x_* = 0$. Conditions (A_1)-(A_4) are satisfied for $\omega_0(t) = 7.5t$, $\omega(t) = 15t$, and $\omega_1(t) = 2$. Then, the values of r and $\bar{r}$ are produced by using formulas (2.1) and (2.4), respectively. These results are summarized in Table 1.

Table 1. Comparison of convergence radii for Example 3.1

Scheme (1.2)	*Scheme (1.3)*
$r_1 = 0.022222$	$\overline{r_1} = 0.022222$
$r_2 = 0.018948$	$\overline{r_2} = 0.017369$
$r_3 = 0.013823$	$\overline{r_3} = 0.014756$
$r = 0.013823$	$\bar{r} = 0.014756$

Example 3.2. Let $X = Y = \mathbb{R}^3$ and $\Omega = \overline{B}(0,1)$. Consider Φ on Ω for $v = (v_1, v_2, v_3)^t$ as

$$\Phi(v) = (e^{v_1} - 1, \frac{e-1}{2}v_2^2 + v_2, v_3)^t$$

Then $x_* = (0,0,0)^t$. Conditions (A_1)-(A_4) are satisfied for $\omega_0(t) = (e-1)t$, $\omega(t) = e^{\frac{1}{e-1}}t$ and $\omega_1(t) = 2$. The results are displayed in Table 2.

Table 2. Comparison of convergence radii for Example 3.2

Scheme (1.2)	*Scheme (1.3)*
$r_1 = 0.127564$	$\overline{r_1} = 0.127564$
$r_2 = 0.088919$	$\overline{r_2} = 0.080947$
$r_3 = 0.064030$	$\overline{r_3} = 0.068190$
$r = 0.064030$	$\bar{r} = 0.068190$

Finally, the motivating issue stated in Section 1 is solved if for $x_* = 1$.

Example 3.3. Choose $\omega_0(t) = \omega(t) = 96.662907t$ and $\omega_1(t) = 2$. The radius can be found in Table 3.

It is found that scheme (1.3) has a larger radius of convergence in all three examples. But we cannot conclude that scheme (1.3) is always better to use than the scheme (1.2).

TABLE 3. Comparison of convergence radii for Example 3.3

Scheme (1.2)	*Scheme* (1.3)
$r_1 = 0.002299$	$\overline{r_1} = 0.002299$
$r_2 = 0.001586$	$\overline{r_2} = 0.001443$
$r_3 = 0.001141$	$\overline{r_3} = 0.001215$
$r = 0.001141$	$\bar{r} = 0.001215$

4. CONCLUSIONS

The usability of schemes (1.2) and (1.3) was extended by investigating their ball convergence properties, which are based only on the first derivative and generalized Lipschitz criteria. In addition, the convergence ball comparison between them was discussed in detail.

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