Commun. Optim. Theory 2023 (2023) 6 https://doi.org/10.23952/cot.2023.6

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

OPTIMALITY CONDITIONS IN DISCRETE-TIME INFINITE-HORIZON OPTIMAL CONTROL PROBLEM WITH DISCOUNTING

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Dedicated to Professor Jack Warga on the occasion of his 100th birthday

Abstract. We consider a discrete-time infinite horizon optimal control problem with time discounting criterion when the discounting factor approaches 1 and derive sufficient optimality conditions.

Keywords. Abel limit; Infinite horizon; Optimal control; Optimality conditions.

2020 Mathematics Subject Classification. 49K15, 49N15.

1. Introduction and Preliminaries

In this paper, we consider discrete time controlled dynamical system

$$\begin{aligned} &y(t+1) = f(y(t), u(t)),\ t = 0, 1, \ldots \\ &y(0) = y_0, \\ &y(t) \in Y, \\ &u(t) \in U(y(t)), \end{aligned} \tag{1.1}$$

where Y is a given subset of $\mathbb{R}^m$, which plays the role of a state constraint, $U(\cdot) : Y \rightsquigarrow U_0$ is a mapping to a given metric space U_0, and $f(\cdot,\cdot) : \mathbb{R}^m \times U_0 \to \mathbb{R}^m$ is a continuous function.

We denote by $\mathscr{U}(y_0)$ the sets of controls such that

$$y(t, y_0, u) \in Y$$

for any t. Everywhere in what follows, it is assumed that the set $\mathscr{U}(y_0)$ is not empty for any $y_0 \in Y$, that is, there exists at least one admissible control for any initial condition. (Systems that satisfy this property are called *viable* on Y.)

In this paper, we establish sufficient optimality conditions for the optimal control problem

$$\inf_{u(\cdot)\in\mathscr{U}(y_0)} \limsup_{\alpha\to 1^-} (1-\alpha) \sum_{t=0}^{\infty} \alpha^t k(y(t, y_0, u(\cdot)), u(t)) =: V(y_0), \tag{1.2}$$

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Received August 16, 2022; Accepted October 23, 2022.

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where $\alpha \in (0,1)$, and $k(y,u) : \mathbb{R}^m \times U_0 \to \mathbb{R}$ is a continuous function.

The limit

$$\lim_{\alpha \to 1^-} \inf_{u(\cdot)\in\mathscr{U}(y_0)} (1-\alpha)\sum_{t=0}^{\infty} \alpha^t k(y(t,y_0,u(\cdot)),u(t)), \tag{1.3}$$

if exists, is called Abel limit. (Note that the limit and the infinum in are interchanged in (1.3) compared to (1.2).) A closely related limit of the long-run averages (the so-called Cesàro limit) is

$$\lim_{T\to\infty} \inf_{u(\cdot)\in\mathscr{U}_T(y_0)} \frac{1}{T}\sum_{t=0}^{T-1} k(y(t,y_0,u(\cdot)),u(t)), \tag{1.4}$$

where $\mathscr{U}_T(y_0)$ is the set of admissible controls on the interval $0 \le t \le T-1$. The limits of Cesàro and Abel types have been studied in various contexts since the work of Hardy and Littlewood [18]. There is an extensive literature devoted to the existence and equality of Cesàro and Abel limits in problems of dynamic programming and optimal control in discrete and continuous time, see, e.g., [1, 6, 8, 11, 16, 17, 20, 22, 23]. It was shown in [10] in continuous time setting that these limits are equal when the limiting functions in (1.3) and (1.4) are continuous functions of y_0 (with convergence not necessarilty being uniform) and some other non-restrictive assumptions hold.

A problem of minimizing the long-run average

$$\inf_{u(\cdot)\in\mathscr{U}(y_0)} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} k(y(t,y_0,u(\cdot)),u(t)),$$

was considered in Chapter 4 of [3] and in [4]. It has been established in [3] that this problem is related to an infinite dimensional linear programming problem and its dual. The linear programming approach to problems of control of nonlinear dynamical systems is based on the fact that the so-called occupational measures generated by state-control trajectories satisfy certain *linear* equations that represent the dynamics of the system in integral form. This makes it possible to reformulate various optimal control problems as infinite-dimensional linear programming (IDLP) problems considered on the spaces of occupational measures. Solutions of the dual problems to these IDLP problems can be used to construct feedback controls that ensure optimality of the corresponding trajectories. This approach has been used in many works in deterministic and stochastic settings in continuous and discrete time, see, e.g., [2, 9, 10, 11, 12, 13, 14, 15, 19, 21, 24, 25, 26] and references therein. Other techniques for dealing with deterministic optimal control problems on infinite time horizon have been studied, e.g., in [1, 5, 7, 27, 28, 29].

In the present paper, we use some of the results of [3] to establish sufficient optimality conditions for problem (1.2) and illustrate them with examples. However, unlike the results in [3], we do not impose the compactness assumption on Y.

2. Sufficient Optimality Conditions

Denote

$$G := \{(y,u) | \, y \in Y,\, u \in U(y),\, f(y,u) \in Y\};$$

any admissible trajectory stays in this set.

Consider the problem

$$\sup_{(\mu,\psi,\eta)} \mu =: d^*(y_0), \tag{2.1}$$

where supremum is taken over $\mu \in I\!R$, real-valued $\psi : Y \to I\!R$ and bounded $\eta : Y \to I\!R$ that for all $(y,u) \in G$ satisfy the inequalities

$$\begin{aligned} &k(y,u) + (\psi(y_0) - \psi(y)) + \eta(f(y,u)) - \eta(y) - \mu \geq 0, \\ &\psi(f(y,u)) - \psi(y) \geq 0. \end{aligned} \tag{2.2}$$

(Below we denote the class of bounded functions on Y by $B(Y)$.) The optimal value of problem (2.1)-(2.2) can be equivalently represented as

$$d^*(y_0) = \sup_{\psi,\eta} \inf_{(y,u)\in G} \{k(y,u) + (\psi(y_0) - \psi(y)) + \eta(f(y,u)) - \eta(y)\}, \tag{2.3}$$

where supremum is taken over ψ satisfying the second inequality in (2.2) and $\eta \in B(Y)$.

Denote

$$h_\alpha(y_0) := (1-\alpha) \inf_{u(\cdot)\in\mathscr{U}(y_0)} \sum_{t=0}^{\infty} \alpha^t k(y(t),u(t)), \tag{2.4}$$

which is a problem with a fixed discounting factor α. (Here and below we denote $y(t) := y(t,y_0,u(\cdot))$.) It is proved in [3], Proposition 5.1 that

$$\liminf_{\alpha\to 1^-} h_\alpha(y_0) \geq d^*(y_0). \tag{2.5}$$

(It is assumed throughout [3] that Y is compact, but compactness is not used in the proof of Propostion 5.1.)

Denote the limit of the value functions in (2.4) as $\alpha \to 1^-$ by $h(y_0)$, that is,

$$h(y_0) := \lim_{\alpha\to 1^-} h_\alpha(y_0), \tag{2.6}$$

if the limit exists.

Proposition 2.1. *Let the pointwise limit (2.6) exist for all $y_0 \in Y$ and $\bar\eta(\cdot) \in B(Y)$ be such that*

$$\inf_{(y,u)\in G} \{k(y,u) - h(y) + \bar\eta(f(y,u)) - \bar\eta(y)\} = 0. \tag{2.7}$$

Then $h(y_0) = d(y_0)$ and supremum in (2.3) is reached at the functions $\psi = h$ and $\eta = \bar\eta$.

Proof. The fact that h satisfies the second inequality in (2.2) is proved in [3]. (See (4.39) in [3]; as menitoned above, it is assumed throughout [3] that Y is compact, but the proof of (4.39) is based solely on the dynamic programming principle and does not require compactness of Y). Furthermore, it follows from (2.5) that

$$h(y_0) \geq d^*(y_0), \tag{2.8}$$

therefore, from (2.3) and (2.8),

$$\sup_{\eta(\cdot)\in B(Y)} \inf_{(y,u)\in G} \{k(y,u) + (h(y_0) - h(y)) + \eta(f(y,u)) - \eta(y)\} \leq d^*(y_0) \leq h(y_0), \tag{2.9}$$

hence,

$$\sup_{\eta(\cdot)\in B(Y)} \inf_{(y,u)\in G} \{k(y,u) - h(y) + \eta(f(y,u)) - \eta(y)\} \leq 0. \tag{2.10}$$

From (2.10) and (2.7) it follows that

$$\sup_{\eta(\cdot)\in B(Y)} \inf_{(y,u)\in G} \{k(y,u) - h(y) + \eta(f(y,u)) - \eta(y)\} = 0,$$

that is, the supremum with respect to η is reached at $\eta = \bar{\eta}$. Furthermore,

$$\sup_{\eta(\cdot)\in B(Y)} \inf_{(y,u)\in G} \{k(y,u) + (h(y_0) - h(y)) + \eta(f(y,u)) - \eta(y)\} = h(y_0),$$

which implies via (2.9) that

$$h(y_0) = d^*(y_0)$$

and that supremum in (2.3) with respect to ψ is reached at $\psi = h$. The proposition is proved. □

The following theorem provides sufficient optimality conditions for problem (1.2).

Theorem 2.2. *Assume that a pair $(\bar{\psi}, \bar{\eta})$ of maximizers in problem (2.3) exists and for some admissible process $(y^*(\cdot), u^*(\cdot))$ and all $t \geq 0$,*

$$(y^*(t), u^*(t)) = \operatorname{argmin}_{(y,u)\in G}\{k(y,u) - \bar{\psi}(y) + \bar{\eta}(f(y,u)) - \bar{\eta}(y)\}. \tag{2.11}$$

Then

(a) *there exists the limit $h(y_0) = \lim_{\alpha\to 1^-} h_\alpha(y_0)$;*

(b) *there is equality*

$$V(y_0) = h(y_0) = d^*(y_0) \tag{2.12}$$

(here V is the value function in (1.2));

(c) *the process $(y^*(\cdot), u^*(\cdot))$ is optimal in (1.2).*

Remark 1. For (2.11) to hold it is necessary that two conditions hold. The first is

$$u^*(t) = \operatorname{argmin}_{u\in A(y)}\{k(y^*(t), u) + \bar{\eta}(f(y^*(t), u))\} \quad \forall t, \tag{2.13}$$

where

$$A(y) := \{u \in U(y) | f(y,u) \in Y\}, \tag{2.14}$$

which implies the optimal feedback control law

$$u^f[y] = \operatorname{argmin}_{u\in A(y)}\{k(y,u) + \bar{\eta}(f(y,u))\}. \tag{2.15}$$

This law can be used to construct optimal control in (1.2) if $\bar{\eta}$ is known. Developing methods of approximating $\bar{\eta}$ can be a subject of further research. A problem of this type is addressed, e.g., in [14].

The second condition that must hold if (2.11) does is that the function

$$t \mapsto k(y^*(t), u^*(t)) - \bar{\psi}(y^*(t)) + \bar{\eta}(f(y^*(t), u^*(t))) - \bar{\eta}(y^*(t))$$

remains constant. (Since the right hand side of (2.11) does not depend on t, neither does the left hand side, which implies the statement above.)

Remark 2. Functions $\bar{\psi}$ and $\bar{\eta}$ may depend on y_0, see Example 2 in Section 3.

Proof of Theorem 2.2. Relation (2.11) means via (2.3) that for all t

$$k(y^*(t), u^*(t)) + (\bar{\psi}(y_0) - \bar{\psi}(y^*(t))) + \bar{\eta}(f(y^*(t), u^*(t))) - \bar{\eta}(y^*(t)) = d^*(y_0). \tag{2.16}$$

Multiplying both sides of (2.16) by α^t, adding up for all t and taking into account that $\bar{\psi}(y_0)-\bar{\psi}(y^*(t))\leq 0$ due to (2.2), we obtain

$$\sum_{t=0}^{\infty}\alpha^t k(y^*(t),u^*(t))+\sum_{t=0}^{\infty}\alpha^t(\bar{\eta}(y^*(t+1)-\bar{\eta}(y^*(t)))\leq \frac{d^*(y_0)}{1-\alpha}. \tag{2.17}$$

Taking into account that

$$\begin{aligned}\sum_{t=0}^{\infty}\alpha^t(\bar{\eta}(y^*(t+1)-\bar{\eta}(y^*(t))) &= -\bar{\eta}(y^*(0))+\sum_{t=1}^{\infty}\alpha^{t-1}\bar{\eta}(y^*(t))-\alpha\sum_{t=1}^{\infty}\alpha^{t-1}\bar{\eta}(y^*(t))\\ &= -\bar{\eta}(y^*(0))+(1-\alpha)\sum_{t=1}^{\infty}\alpha^{t-1}\bar{\eta}(y^*(t)),\end{aligned} \tag{2.18}$$

we obtain from (2.17) after multiplying both sides by $(1-\alpha)$ that

$$(1-\alpha)\sum_{t=0}^{\infty}\alpha^t k(y^*(t),u^*(t))-(1-\alpha)\bar{\eta}(y^*(0))+(1-\alpha)^2\sum_{t=1}^{\infty}\alpha^{t-1}\bar{\eta}(y^*(t))\leq d^*(y_0). \tag{2.19}$$

Since $h_\alpha(y_0)\leq(1-\alpha)\sum_{t=0}^{\infty}\alpha^t k(y^*(t),u^*(t))$ and since the second and the third terms on the left side of the last formula vanish as $\alpha\to 1^-$ due to $\bar{\eta}$ being bounded, the latter implies that

$$\limsup_{\alpha\to 1^-}h_\alpha(y_0)\leq d^*(y_0).$$

Taking into account (2.5), we conclude that the limit $h(y_0)=\lim_{\alpha\to 1^-}h_\alpha(y_0)$ exists and is equal to $d^*(y_0)$, that is, part (a) of the theorem and the second equality in (2.12) are true. We also obtain from (2.19) and the equality $d^*(y_0)=h(y_0)$ that

$$\limsup_{\alpha\to 1^-}(1-\alpha)\sum_{t=0}^{\infty}\alpha^t k(y^*(t),u^*(t))\leq h(y_0). \tag{2.20}$$

Let $u(\cdot)\in\mathscr{U}(y_0)$ and let $y(\cdot)$ be the corresponding trajectory. Then

$$(1-\alpha)\sum_{t=0}^{\infty}\alpha^t k(y(t),u(t))\geq h_\alpha(y_0).$$

Therefore,

$$\limsup_{\alpha\to 1^-}(1-\alpha)\sum_{t=0}^{\infty}\alpha^t k(y(t),u(t))\geq \limsup_{\alpha\to 1^-}h_\alpha(y_0)=h(y_0),$$

hence,

$$V(y_0)\geq h(y_0). \tag{2.21}$$

Let us prove the opposite inequality. Since

$$\limsup_{\alpha\to 1^-}(1-\alpha)\sum_{t=0}^{\infty}\alpha^t k(y^*(t),u^*(t))\geq \inf_{u(\cdot)\in\mathscr{U}(y_0)}\limsup_{\alpha\to 1^-}(1-\alpha)\sum_{t=0}^{\infty}\alpha^t k(y(t),u(t)),$$

we get from (2.20) that

$$h(y_0)\geq \inf_{u(\cdot)\in\mathscr{U}(y_0)}\limsup_{\alpha\to 1^-}(1-\alpha)\sum_{t=0}^{\infty}\alpha^t k(y(t),u(t)),$$

that is,

$$h(y_0) \geq V(y_0).$$

Along with (2.21), this implies $h(y_0) = V(y_0)$, which is the first equality in (2.12). From (2.20) we have

$$\limsup_{\alpha \to 1^-} (1-\alpha) \sum_{t=0}^{\infty} \alpha^t k(y^*(t), u^*(t)) \leq V(y_0). \tag{2.22}$$

Since the opposite inequality follows from the definition of $V(y_0)$, (2.22) holds as equality, that is, the process $(y^*(\cdot), u^*(\cdot))$ is optimal. The theorem is proved. $\square$

3. Examples

In this section, we show applications of Theorem 2.2.

Example 1. Consider the problem

$$\inf_{u(\cdot) \in \mathscr{U}(y_0)} \limsup_{\alpha \to 1^-} (1-\alpha) \sum_{t=0}^{\infty} \alpha^t (1 - y(t))^2 \tag{3.1}$$

over the trajectories of the system

$$\begin{aligned} & y(t+1) = y(t) + (1-y(t))^2 u(t),\ t = 0, 1, \dots \\ & y(0) = y_0, \\ & y(t) \in (0,2), \\ & u \in [-1,1]. \end{aligned} \tag{3.2}$$

In this example, $k(y) = (1-y)^2$ and $Y = (0,2)$.

From the first equation in (3.2) it follows that

$$y(t+1) - 1 = (y(t) - 1)(1 + (y(t) - 1)u).$$

Since the second factor is positive for all $y \in (0,2)$ and $u \in [-1,1]$, this implies that the sign of $y(t) - 1$ remains the same for all t, that is, $0 < y_0 < 1 \implies 0 < y(t) < 1$ and $1 < y_0 < 2 \implies 1 < y(t) < 2$ for all t.

It is clear that the control that makes the system approach $y = 1$ as quicky as possible is optimal, that is,

$$u^*(t) = \begin{cases} 1, & 0 < y_0 < 1, \\ -1, & 1 < y_0 < 2, \\ \text{any}, & y_0 = 1, \end{cases} \tag{3.3}$$

and $y^*(t) \to 1$ as $t \to \infty$ for the corresponding trajectory. Our goal is to show that $(y^*(t), u^*(t))$ satisfies (2.11), consistent with the observation.

First, let us show that $h(y) = 0$ for all $y \in Y$. Consider the case when $y_0 \in (1,2)$. Denote $w = y - 1$, then $y(t+1) = y(t) + (1-y(t))^2 u(t)$ can be written as

$$w(t+1) = w(t) + w^2(t)u.$$

Along the optimal control $u \equiv -1$ we have

$$w(t+1) = w(t) - w^2(t),$$

which can be written as

$$-\frac{w(t+1)-w(t)}{w^2(t)}=1.$$

Taking into account that $w(\cdot)$ is positive and decreasing, we have

$$1=-\frac{w(t+1)-w(t)}{w^2(t)}<-\frac{w(t+1)-w(t)}{w(t+1)w(t)}=\frac{1}{w(t+1)}-\frac{1}{w(t)}.$$

Taking summation with respect to t from 0 to $T-1$ we get

$$T<\frac{1}{w(T)}-\frac{1}{w(0)},$$

hence,

$$w(T)<\frac{1}{T+1/(w(0))},$$

which implies that

$$y(t)-1<\frac{1}{t+1/(y_0-1)} \quad \forall t.$$

Therefore, the sum

$$\sum_{t=0}^{T-1}(y(t)-1)^2$$

is uniformly bounded with respect to T, hence

$$h(y_0)=\lim_{\alpha\to 1^-}(1-\alpha)\sum_{t=0}^{\infty}\alpha^t(1-y(t))^2=0.$$

If $y_0\in(0,1]$, the same equality can be proved similarly. We have shown that $h(y)=0$ for all $y\in Y$.

Let us show that (2.7) holds with $\bar{\eta}(y)=|1-y|$. We have

$$\begin{aligned}
&k(y)-h(y)+\bar{\eta}(f(y,u))-\bar{\eta}(y)=(1-y)^2+|1-(y+(1-y)^2u)|-|1-y|\\
&=(1-y)^2+(1-(1-y)u)|1-y|-|1-y|=(1-y)^2-(1-y)u|1-y|\\
&=\begin{cases}(1-y)^2(1-u), & y\in(0,1],\\ (1-y)^2(1+u), & y\in(1,2).\end{cases}
\end{aligned}\tag{3.4}$$

Therefore,

$$\min_{(y,u)\in G}\{k(y)-h(y)+\bar{\eta}(f(y,u))-\bar{\eta}(y)\}=0,$$

that is, (2.7) holds. Due to Proposition 2.1, maximizing functions in (2.3) are $\bar{\psi}=0$ and $\bar{\eta}=|1-y|$. Furthermore, as seen from (3.3) and (3.4),

$$(y^*(t),u^*(t))=\mathrm{argmin}_{(y,u)\in G}\{k(y)-\bar{\psi}(y)+\bar{\eta}(f(y,u))-\bar{\eta}(y)\}\text{ for all }t,$$

that is (2.11) holds, hence, the process (y^*,u^*) is optimal due to Theorem 2.2, consistent with our earlier observation.

Example 2. Consider the problem

$$\inf_{u(\cdot)\in\mathscr{U}(y_0)}\limsup_{\alpha\to 1^-}(1-\alpha)\sum_{t=0}^{\infty}\alpha^t(-y(t))\,dt \tag{3.5}$$

over the trajectories of the system

$$
\begin{aligned}
&y(t+1) = y(t)u(t),\ t = 0,1,\dots\\
&y(0) = y_0,\\
&y(t) \in [0,1],\\
&u \in [0,2].
\end{aligned}
$$

In this example, $k(y) = -y$ and $Y = [0,1]$.

It is clear that for any $\alpha \in (0,1)$, in the problem of minimizing $\sum_{t=0}^{\infty} \alpha^t(-y(t))\,dt$, the optimal feedback law is

$$
u^f[y] = \begin{cases} \text{any } u, & y = 0,\\ 2, & y \in (0,1/2],\\ 1/y, & y \in (1/2,1] \end{cases} \tag{3.6}
$$

and the optimal control is unique for $y_0 \in (0,1]$. The same feedback control law is optimal in problem (3.5), although it is not unique. (Any control that brings the system to $y = 1$ in finite time is optimal in (3.5).) It is clear that

$$
h(y) = \begin{cases} 0, & y = 0,\\ -1, & y \in (0,1], \end{cases} \tag{3.7}
$$

since, if $y_0 > 0$, the optimal trajectory stays as $y = 1$ starting from some moment of time. For a given $y_0 \in (0,1]$ let $n \geq 1$ be such integer that $y_0 \in ((1/2)^n,(1/2)^{n-1}]$. For $y \in (y_0,1]$ let $l = l(y)$, $1 \leq l \leq n$, be such integer that $y \in ((1/2)^l,(1/2)^{l-1}]$, and define

$$
\bar{\eta}_{y_0}(y) := \begin{cases} n-(2^n-1)y_0, & y \in [0,y_0],\\ l-(2^l-1)y, & y \in (y_0,1]. \end{cases} \tag{3.8}
$$

This function is constant on $[0,y_0]$ and monotonically decreasing on $(y_0,1]$. In fact, it is also continuous since, as shown below, $\lim_{y\to((1/2)^l)^+} \bar{\eta}_{y_0}(y) = \lim_{y\to((1/2)^l)^-} \bar{\eta}_{y_0}(y)$ for all $l = 1,\dots,n$:

$$
\begin{aligned}
&\lim_{y\to((1/2)^l)^+} \bar{\eta}_{y_0}(y) = (l-(2^l-1)(1/2)^l) = l-1+(1/2)^l,\\
&\lim_{y\to((1/2)^l)^-} \bar{\eta}_{y_0}(y) = ((l+1)-(2^{(l+1)}-1)(1/2)^l) = l-1+(1/2)^l.
\end{aligned}
$$

For $y_0 = 0$ define $\bar{\eta}_0(y) \equiv 0$.

Denote

$$
g(y,u) := k(y)-h(y)+\bar{\eta}_{y_0}(f(y,u))-\bar{\eta}_{y_0}(y) = k(y)-h(y)+\bar{\eta}_{y_0}(yu)-\bar{\eta}_{y_0}(y)
$$

and let us show that

$$
\min_{(y,u)\in G} g(y,u) = 0, \tag{3.9}
$$

where

$$
G = \{(y,u) \mid y \in [0,1],\ u \in [0,2] \text{ is s.t. } yu \in [0,1]\}.
$$

If $y_0 = 0$, then $\eta_0 = 0$ and $g(y,u) = k(y)-h(y) = -y-h(y) \geq 0$ for all $y \in [0,1]$ due to (3.7).

Assume now that $y_0 > 0$. Since $\bar{\eta}_{y_0}$ is non-increasing, minimization of g with respect to u occurs when $u = u^f[y]$ is given by (3.6). Let us see that for any y_0 and y such that $0 < y_0 \le y \le 1$ we have

$$g(y, u^f[y]) = 0.$$

Consider the following cases which exhaust all possibilities when $0 < y_0 \le y \le 1$: (a) $0 < y_0 \le y < 1/2$, and (b) $y_0 \in (0,1]$, $y_0 \le y$ and $y \in [1/2, 1]$.

(a) If $y_0 \in (0, 1/2)$ and $y \in [y_0, 1/2)$ there exists $l : 2 \le l \le n$ such that $y \in [(1/2)^l, (1/2)^{l-1})$, consequently, $2y \in [(1/2)^{l-1}, (1/2)^{l-2})$, and we have

$$\begin{aligned} g(y, u^f[y]) &= -y + 1 + \bar{\eta}_{y_0}(2y) - \bar{\eta}_{y_0}(y) \\ &= -y + 1 + ((l-1) - (2^{l-1} - 1)(2y)) - (l - (2^l - 1)y) = 0. \end{aligned} \tag{3.10}$$

(b) If $y_0 \in (0,1]$, $y_0 \le y$ and $y \in [1/2, 1]$, then $l = 1$ in (3.8), hence, $\bar{\eta}_{y_0}(y) = 1 - y$; $\bar{\eta}_{y_0}(yu^f[y]) = \bar{\eta}_{y_0}(1) = 0$ and

$$g(y, u^f[y]) = -y + 1 + \bar{\eta}_{y_0}(1) - \bar{\eta}_{y_0}(y) = -y + 1 + 0 - (1 - y) = 0. \tag{3.11}$$

For y_0 and y such that $0 \le y < y_0 \le 1$ it can be similarly shown that $g(y, u^f[y]) \ge 0$ by considering cases (c) $2y \le y_0$, (d) $y_0 < 2y \le 1$, and (e) $2y > 1$. In case (c) we have $\bar{\eta}_{y_0}(2y) - \bar{\eta}_{y_0}(y) = 0$ and $g(y, u^f[y]) = -y + 1 \ge 0$. Calculations in case (d) are very similar to (3.10) and calculations in case (e) coincide with those in (3.11).

Thus, (3.9) holds and, due to Proposition 2.1, maximizing functions in (2.3) are $\bar{\psi} = h$ and $\eta = \bar{\eta}_{y_0}$. If $y_0 > 0$, then for any admissible trajectory $(y^*(t), u^*(t))$ with $u^*(t) = u^f[y^*(t)]$ we have $y^*(t) \ge y_0$ and, as follows from (3.10)-(3.11), $g(y^*(t), u^*(t)) \equiv 0$. Finally, if the initial condition is $y_0 = 0$, then $y^*(t) = 0$, and, again, $g(y^*(t), u^*(t)) \equiv 0$. Thus, due to (3.9),

$$(y^*(t), u^*(t)) = \operatorname{argmin}_{(y,u) \in G} g(y, u) \text{ for all } t.$$

Therefore, any process $(y^*(t), u^*(t))$ with $u^*(t) = u^f[y^*(t)]$ is optimal due to Theorem 2.2, consistent with our earlier observation.

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