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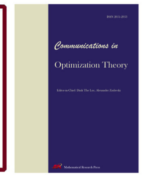

SMALL PARAMETER METHOD FOR OPTIMIZATION OF SINGULARLY PERTURBED DYNAMICAL SYSTEMS

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Dedicated to the memory of Professor Rafail Gabasov

Abstract. An overview of the results obtained for optimization problems of singularly perturbed systems in Minsk by the optimal control research group is given. All the methods reviewed in the paper share the same research methodology that is based on the idea of a finite dimensional parametrization of the solution of the optimal control problem. The range of problems considered include problems with scalar and multidimensional control inputs, time-optimal and terminal control problems, problems with linear terminal state constraints, and quadratic performance indices.

Keywords. Asymptotic approximation; Optimal control; Dynamical systems; Small parameter; Singular perturbations.

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1. INTRODUCTION

The systems of differential equations with small parameters at some derivatives are usually referred to as singularly perturbed systems. In the mathematical theory of optimal processes, considerable attention is given to optimization problems for such systems, which is driven by the effectiveness of asymptotic methods for solving them. It is well known that the numerical solution of optimal control problems involves repetitive integration of direct and conjugate systems. If the problem contains singular perturbations, these dynamical systems are rigid, resulting in serious difficulties during calculations such as unacceptably long computing time and inevitable computation errors accumulation. On the contrary, asymptotic methods not only avoid integration of singularly perturbed systems, but also reduce the original optimal control problem to problems of smaller dimension that are easier to solve.

The most common approach used to investigate singularly perturbed optimal control problems is based on asymptotic decomposition of solutions of singularly perturbed differential

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equations that is applied to the boundary value problem of the maximum principle (see, e.g. [1] – [6]). Such approach can be applied to build the solution asymptotics for problems with an open control domain and smooth control inputs, i.e. variational type problems. In case of geometric input constraints in the inequality form this approach encounters serious difficulties, since the differential equations of the boundary value problem of the maximum principle do not satisfy the smoothness assumption necessary for asymptotic methods application. Probably for that reason the research in this case was mainly focused on the issue of a limiting problem, i.e. the problem which solution is a limit in a certain topology of the solution of a singularly perturbed problem when a small parameter tends to zero [7] – [11].

This paper presents an overview of the results obtained in Minsk by the Optimal Control research group headed by R. Gabasov and F.M. Kirillova on optimization problems for singularly perturbed systems. We concentrate on problems with terminal state constraints, i.e. with the constraints on the right endpoint of the trajectory.

2. RESEARCH METHODOLOGY

We start with a definition of asymptotic approximations to solutions of optimization problems for singularly perturbed systems with a small parameter $\mu > 0$ at some derivatives.

Definition 2.1. A feasible control input satisfying the geometric input constraints is called an asymptotically suboptimal control of order N $(N = 0, 1, ...)$ if it deviates from the optimal control in terms of the performance criterion by a value of order $O(\mu^{N+1})$ and the corresponding trajectory of the dynamical system satisfies the terminal constraints with the same infinitesimal order.

Our research methodology is based on the idea of a *finite dimensional parametrization* of the optimal solution. In many optimal control problems it is possible to specify finite dimensional elements that allow to reconstruct the solution of the problem in a simple way. Importantly, in perturbed problems these elements, as a rule, can be chosen in such a way that they smoothly depend on a small parameter. We refer to these finite dimensional elements as the *defining elements* of the solution.

Depending on a particular optimal control problem the defining elements can be the switching time instants of bang-bang control inputs, the initial and final time instants of singular and quasi-sigular arcs, the Lagrange multipliers, the duration of the control process (in the case when it is not fixed) etc. If we denote by $a_1, a_2, ..., a_k$ the collection of all defining elements of the problem under consideration and apply the maximum principle together with the feasibility conditions to the reconstructed solution we obtain a system of algebraic equations of the form

$$F_i(a_1, a_2, \ldots, a_k, \mu) = 0, \quad i = \overline{1,k}, \tag{2.1}$$

where μ is a small parameter. Similarly to the defining elements, equations (2.1) are called the *defining equations*. They result from the integration of direct and conjugate dynamical systems that are also the perturbed systems. Using the method of boundary functions (see [12]), we expand the functions $F_i(a_1, a_2, ..., a_k, \mu)$, $i = \overline{1,k}$, in terms of the powers of the small parameter μ

$$F_i(a_1, a_2, \ldots, a_k, \mu) \sim F_{i0}(a_1, a_2, \ldots, a_k) + \mu F_{i1}(a_1, a_2, \ldots, a_k) + \ldots, \quad i = \overline{1,k}.$$

Then under the conditions of the implicit function theorem and applying the method of indefinite coefficients we can find the asymptotics of the solution of system (2.1). To construct asymptotically suboptimal controls of a given order, it is sufficient to replace the unknown defining elements $a_i(\mu)$, $i=\overline{1,k}$, by their asymptotic approximations of the corresponding order.

The main difficulty in implementing the described scheme is to find the higher coefficients of the defining elements expansion, i.e. the solutions of the zero-order approximation system

$$F_{i0}(a_1,a_2,\ldots,a_k)=0,\ \ i=\overline{1,k}. \tag{2.2}$$

It turns out that if the original optimal control problem is singularly perturbed, then the roots of system (2.2), as a rule, will be the defining elements of two optimal control problems of smaller dimensions. One of them is degenerate and the other is chosen while analyzing system (2.2), which constitutes an informal stage of the study.

The described approach is convenient for numerical implementation since calculations are reduced to expansions of the finite-dimensional elements.

The asymptotic approximations of the defining elements can by further used to find their exact values by the refining procedure proposed in [13].

3. Problems with scalar control inputs

The methodology described in the previous section was first applied to the time-optimal control problem for a linear time-invariant system with a scalar control input

$$\begin{gathered}\dot y=A_1y+A_2z+b_1u,\ \ y(0)=y_*,\ \ y(T)=0,\\ \mu\dot z=A_3y+A_4z+b_2u,\ \ z(0)=z_*,\ \ z(T)=0,\\ |u(t)|\le 1, t\in[0,T],\ \ J(u)=T\to\min,\end{gathered} \tag{3.1}$$

where $0<\mu\ll 1$, $y\in\mathbb{R}^n$, $z\in\mathbb{R}^m$. It is assumed that A_4 is a stable matrix, i.e. all its eigenvalues have negative real parts.

In [14], an algorithm for constructing asymptotically suboptimal controls of any order in problem (3.1) was proposed. The defining elements in this case are the switching time instants of the optimal control and the optimal final time T. The switching time instants are divided into two groups. The first group contains instants that are close to the corresponding switching time instants in the *first basic problem*

$$\begin{gathered}\dot y=A_0y+b_0u,\ \ y(0)=y_*,\ \ y(T)=0,\\ |u(t)|\le 1,\ \ t\in[0,T],\ \ J_0(u)=T\to\min,\\ A_0=A_1-A_2A_4^{-1}A_3,\ \ b_0=b_1-A_2A_4^{-1}b_2.\end{gathered} \tag{3.2}$$

The second group consists of the time instants that differ from the optimal final time by a margin of order of μ. The number of these elements is equal to the number of switching time instants in the solution of the *second basic problem*

$$\begin{gathered}\frac{dz}{ds}=A_4z+b_2u,\ \ z(-\infty)=A_4^{-1}b_2,\ \ |u(s)|\le 1,\ \ s\le 0,\\ z(0)=0,\ \ J_1(u)=\int_{-\infty}^{0}(u(s)+1)ds\to\min.\end{gathered}$$

The optimal final time in the original singularly perturbed problem is close to the optimal final time in problem (3.2).

The computational procedure of the algorithm includes solving basic problems, integration of systems of linear differential equations and finding the roots of non-degenerate linear algebraic systems. Note that asymptotically suboptimal zero-order control is found immediately after solving these problems.

In [15], an algorithm for constructing asymptotically suboptimal controls in a terminal control problem for a linear time-invariant singularly perturbed system with a fixed final time and a variable right endpoint of the trajectory was developed. The problem studied in [15] has the form

$$\begin{gathered}\dot{y} = A_1 y + A_2 z + b_1 u, \ \ y(0) = y_*, \\ \mu\dot{z} = A_3 y + A_4 z + b_2 u, \ \ z(0) = z_*, \\ |u(t)| \le 1, \ \ t \in T = [0, t^*], \ \ H_1 y(t^*) = g_1, \ \ H_2 z(t^*) = g_2, \\ c_1^T y(t^*) + \mu c_2^T z(t^*) \to \max,\end{gathered} \tag{3.3}$$

where $0 < \mu \ll 1$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $g_1 \in \mathbb{R}^{n_1}$, $g_2 \in \mathbb{R}^{m_1}$. As in the previous problem, the matrix A_4 is stable.

The defining elements in problem (3.3) are the switching time instants of optimal control. As in problem (3.1), they are divided into two groups. The original problem in this case in split into two basic problems, the first of them has the form

$$\begin{gathered}\dot{y} = A_0 y + b_0 u, \ \ y(0) = y_*, \ \ |u(t)| \le 1, \ \ t \in T, \\ H_1 y(t^*) = g_1, \ \ c_1^T y(t^*) \to \max.\end{gathered}$$

The second basic problem is

$$\begin{gathered}\frac{dz}{ds} = A_4 z - u^0(t^*) b_2 u, \ \ z(-\infty) = -u^0(t^*) A_4^{-1} b_2, \\ |u(s)| \le 1, \ \ s \le 0, \ \ H_2 z(0) = H_2 A_4^{-1} A_3 y^0(t^*) + g_2, \\ c^T z(0) - |b_0^T \psi^0(t^*)| \int_{-\infty}^{0} (u(s) + 1) ds \to \max,\end{gathered}$$

where $c = c_2 + (A_2 A_4^{-1})^T \psi^0(t^*)$, $u^0(t)$ is the optimal control in the first basic problem, and $y^0(t)$, $\psi^0(t)$ are the corresponding solutions of the direct and conjugate systems. As in the previous problem, asymptotically suboptimal zero-order control can be constructed immediately after solving the basic problems.

The results obtained in [14, 15] were generalized in [16, 17] to the case of nonlinear singularly perturbed systems of the form

$$\begin{gathered}\dot{y} = a_1(y,t) + A_1(y,t) z + b_1(y,t) u, \\ \mu\dot{z} = a_2(y,t) + A_2(y,t) z + b_2(y,t) u.\end{gathered}$$

The papers [18, 19] were devoted to asymptotic optimization of linear singularly perturbed systems containing at derivatives the parameters of various orders of smallness. The algorithms proposed in these papers developed the results obtained in [14, 15].

In [20], an algorithm was developed that is aimed at construction the of asymptotically suboptimal control of a given order in a minimum force problem for a linear singularly perturbed system

$$\begin{gathered}\dot{y} = A_1 y + A_2 z + b_1 u, \ \ \mu\dot{z} = A_3 y + A_4 z + b_2 u, \\ y(0) = y_* \ne 0, \ \ z(0) = z_*, \ \ y(t^*) = 0, z(t^*) = 0,\end{gathered} \tag{3.4}$$

$$J(u) = \sup_{t\in[0,t^*]} |u(t)| \to \min,$$

with a scalar control and a stable matrix A_4.

The defining elements in problem (3.4) are the switching time instants of the optimal control and the optimal input intensity (the optimal value of the problem). The basic problems in this case have the form

$$\dot{y} = A_0 y + b_0 u,\ \ y(0) = y^0,\ \ y(t^*) = 0,$$

$$J_1(u) = \sup_{t\in[0,t^*]} |u(t)| \to \min;$$

$$\frac{dz}{ds} = A_4 z + b_2 u,\ \ s \le 0,\ \ z(-\infty) = A_4^{-1} b_2,\ \ z(0) = 0, |u(s)| \le 1,$$

$$|u(s)| \le 1,\ J_2(u) = \int_{-\infty}^{0} (u(s)+1)ds \to \min.$$

Paper [21] was devoted to the construction of asymptotically suboptimal controls in a linear-quadratic problem

$$\dot{y} = A_1 y + A_2 z + b_1 u,\ \ y(0) = y_* \ne 0,\ \ y(t^*) = 0,$$

$$\mu \dot{z} = A_3 y + A_4 z + b_2 u, z(0) = z_*, z(t^*) = 0, \tag{3.5}$$

$$|u(t)| \le 1,\ \ t \in [0,t^*],\ \ J(u) = \frac{1}{2}\int_0^{t^*} u^2(t)dt \to \min,$$

where $0 < \mu \ll 1$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, u is a scalar control and matrix A_4 is stable.

The defining elements in this problem are the saturation time instants of optimal control and the initial values (at the time instant t^*) of the conjugate variables. The saturation time instants here are the endpoints of the intervals where the absolute value of the optimal control equals to 1.

Problem (3.5) decomposes into two basic problems of the form

$$\dot{y} = A_0 y + b_0 u,\ \ y(0) = y_* \ne 0,\ \ y(t^*) = 0,$$

$$|u(t)| \le 1,\ \ t \in [0,t^*],\ \ J_1(u) = \frac{1}{2}\int_0^{t^*} u^2(t)dt \to \min;$$

$$\frac{dz}{ds} = A_4 z + b_2 u,\ \ z(0) = 0,\ \ z(-\infty) = -u^0(t^*) A_4^{-1} b_2,$$

$$|u(s)| \le 1,\ \ s \le 0,\ \ J_2(u) = \int_{-\infty}^{0} ((u(s) - b_0^T \psi^0(t^*))^2 - c^2)ds \to \min.$$

Here $c = u^0(t_*) - b_0^T \psi^0(t_*)$, $u^0(t)$ is the optimal control of the first basic problem, $\psi^0(t)$ is the corresponding solution of the conjugate system.

4. Problems with multidimensional control inputs

In the problems discussed in the previous section, the control inputs were scalar, however, the obtained results can be easily generalized to systems with multidimensional controls $u(t) = (u_1(t), \ldots, u_r(t))$ if their values are subject to geometric constraints

$$a_i \leq u_i(t) \leq b_i, i = 1, 2, \ldots, r.$$

At the same time, in many applied problems with multidimensional controls constraints on their values have the form

$$||u(t)|| \leq a,$$

where $||u|| = \sqrt{u_1^2 + \ldots + u_r^2}$ is the Euclidean norm of the vector u. First of all, this applies to control problems for mechanical systems in which the control inputs, as a rule, are bounded in magnitude forces.

In [22], an algorithm was proposed for constructing asymptotically suboptimal controls of a given order in the time-optimal control problem

$$\begin{gathered} \dot{y} = A_1 y + A_2 z + B_1 u, \;\; y(0) = y_*, \;\; y(T) = 0, \\ \mu \dot{z} = A_3 y + A_4 z + B_2 u, \;\; z(0) = z_*, \;\; z(T) = 0, \\ ||u(t)|| \leq 1, \;\; t \in [0, T], \;\; J(u) = T \to \min, \end{gathered} \tag{4.1}$$

where $0 < \mu \ll 1$, $u \in \mathbb{R}^r$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$. It is assumed that the matrix A_4 is stable. The defining elements in problem (4.1) are the optimal final time and the vector of conjugate variables at this time instant. The initial problem in this case splits into two basic problems. The first of them is the degenerate problem

$$\begin{gathered} \dot{y} = A_0 y + B_0 u, \;\; y(0) = y_*, \;\; y(T) = 0, \\ ||u(t)|| \leq 1, \;\; t \in [0, T], \;\; J_0(u) = T \to \min, \\ A_0 = A_1 - A_2 A_4^{-1} A_3, \;\; B_0 = B_1 - A_2 A_4^{-1} B_2, \end{gathered}$$

and the second basic problem has the form

$$\frac{dz}{ds} = A_4 z + B_2 u, \;\; z(-\infty) = -A_4^{-1} B_2 \Delta^0(T_0)/||\Delta^0(T_0)||, \;\; ||u(s)|| \leq 1, \;\; s \leq 0,$$

$$z(0) = 0, \;\; J_1(u) = \int_{-\infty}^{0} (u^T(s)\Delta^0(T_0) - ||\Delta^0(T_0)||) ds \to \max,$$

where $\Delta^0(t) = B_0^T \psi^0(t)$, and $\psi^0(t)$, T_0 are the vector of conjugate variables corresponding to the optimal control and the final time in the first basic problem. The computational procedure of the algorithm, in addition to solving the basic problems, includes integrating the systems of linear differential equations and finding the roots of non-degenerate linear algebraic systems. However, asymptotically suboptimal zero-order control can be formed immediately after solving these basic problems.

In [23], an algorithm was developed for constructing asymptotically suboptimal controls in the problem of terminal control of a linear time-invariant singularly perturbed system on a fixed control interval and subject to a variable right endpoint of the trajectory

$$\begin{gathered} \dot{y} = A_1 y + A_2 z + B_1 u, \;\; y(0) = y_*, \\ \mu \dot{z} = A_3 y + A_4 z + B_2 u, \;\; z(0) = z_*, \end{gathered} \tag{4.2}$$

$$||u(t)|| \le 1,\ \ t \in [0,t^*],\ \ H_1 y(t^*) = g_1,\ \ H_2 z(t^*) = g_2,$$
$$c_1^T y(t^*) + \mu c_2^T z(t^*) \to \max,$$

where $0 < \mu \ll 1$, $u \in \mathbb{R}^r$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $g_1 \in \mathbb{R}^{n_1}$, $g_2 \in \mathbb{R}^{m_1}$. As in the previous problems, the matrix A_4 is stable. The defining elements in problem (4.2) are the Lagrange multipliers. The basic problems in this case have the form

$$\dot{y} = A_0 y + B_0 u,\ \ y(0) = y_*,\ \ ||u(t)|| \le 1,\ \ t \in [0,t^*],$$
$$H_1 y(t^*) = g_1,\ \ J_0(u) = c_1^T y(t^*) \to \max;$$
$$\frac{dz}{ds} = A_4 z + B_2 u,\ \ z(-\infty) = -A_4^{-1} B_2 u^0(t^*),\ \ ||u(s)|| \le 1,\ \ s \le 0,$$
$$H_2 z(0) = H_2 A_4^{-1} A_3 y^0(t^*) + g_2,\ \ J_1(u) = c_0^T z(0) + \int_{-\infty}^{0} (u^T(s)\Delta^0(t^*) - ||\Delta^0(t^*)||)ds \to \max,$$

where $c_0 = c_2 + (A_2 A_4^{-1})^T \psi^0(t^*)$, $u^0(t)$ is the optimal control in the first basic problem, $y^0(t)$, $\psi^0(t)$ are the corresponding solutions of direct and conjugate systems.

The results of papers [22, 23] are generalized in [24, 25] to the control problems for nonlinear singularly perturbed systems

$$\dot{y} = a_1(y,t) + A_1(y,t)z + B_1(y,t)u,$$
$$\mu \dot{z} = a_2(y,t) + A_2(y,t)z + B_2(y,t)u.$$

In paper [26], the results obtained in [20] were generalized to systems with multidimensional controls.

In [27], an optimization problem for a linear singularly perturbed system with an integral quadratic performance index is investigated

$$\begin{gathered} \dot{y} = A_1(t)y + A_2(t)z + B_1(t)u,\ \ y(t_*) = y_*, \\ \mu \dot{z} = A_3(t)y + A_4(t)z + B_2(t)u,\ \ z(t_*) = z_*, \\ y(t^*) = 0,\ \ z(t^*) = 0,\ \ J(u) = \frac{1}{2}\int_{t_*}^{t^*} (y^T M(t) y + \mu z^T L(t) z + u^T P(t) u)dt \to \min, \end{gathered} \tag{4.3}$$

where $0 < \mu \ll 1$, $u \in \mathbb{R}^r$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $M(t)$, $L(t)$ are positive semi-definite matrices, and $P(t)$ is a positive definite matrix for all $t \in [t_*, t^*]$. It is assumed that the eigenvalues of the matrix $A_4(t)$ are negative at every time instant $t \in [t_*, t^*]$.

The defining elements in problem (4.3) are the initial values (at time instant t^*) of the conjugate variables. The basic problems in this case are formulated as

$$\dot{y} = A_0(t)y + B_0(t)u,\ \ y(t_*) = y_*,\ \ y(t^*) = 0,$$
$$J_1(u) = \frac{1}{2}\int_{t_*}^{t^*} (y^T M(t) y + u^T P(t) u)dt \to \min,$$
$$A_0(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t),\ \ B_0(t) = B_1(t) - A_2(t)A_4^{-1}(t)B_2(t);$$
$$\frac{dz}{ds} = A_4(t^*)z + B_2(t^*)u,\ \ z(0) = A_4^{-1}(t^*)B_2(t^*)u^0(t^*),$$
$$z(-\infty) = 0,\ \ J_2(u) = \frac{1}{2}\int_{-\infty}^{0} (u^T(s)P(t^*)u(s))ds \to \min,$$

where $u^0(t)$, $t \in [t_*, t^*]$, is the optimal control in the first basic problem.

In addition to asymptotic approximations to the optimal open-loop control, asymptotic approximations of zero and first order for the optimal feedback are constructed in [27].

Definition 4.1. A vector function $u^{(N)}(y,z,t,\mu)$ is called an asymptotically suboptimal feedback of order N if for any initial state $(t_*, y_*, z_*))$, $t_* < t^*$, the equality holds

$$u^{(N)}(y_*, z_*, t_*, \mu) = u^{(N)}(t_*, \mu),$$

where $u^{(N)}(t,\mu)$, $t \in [t_*, t^*]$, is the asymptotically suboptimal open-loop control of order N.

In [28], the results obtained in [27] were generalized to problems with a variable endpoint of the trajectory

$$\dot{y} = A_1(t)y + A_2(t)z + B_1(t)u, \;\; y(t_*) = y_*,$$
$$\mu\dot{z} = A_3(t)y + A_4(t)z + B_2(t)u, \;\; z(t_*) = z_*, \tag{4.4}$$
$$H_1 y(t^*) = g_1, \;\; H_2 z(t^*) = g_2, \;\; J(u) = \frac{1}{2}\int_{t_*}^{t^*} (y^T M(t)y + \mu z^T L(t)z + u^T P(t)u)dt \to \min,$$

where $0 < \mu \ll 1$, $u \in \mathbb{R}^r$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $g_1 \in \mathbb{R}^{n_1}$, $g_2 \in \mathbb{R}^{m_1}$, $M(t)$, $L(t)$ are positive semi-definite and $P(t)$ is a positive definite symmetric matrices for all $t \in [t_*, t^*]$. As in the previous problem, it is assumed that all eigenvalues of the matrix $A_4(t)$ are negative at every time instant $t \in [t_*, t^*]$. The defining elements in problem (4.4) are the Lagrange multipliers. The basic problems in this case have the form

$$\dot{y} = A_0(t)y + B_0(t)u, \;\; y(t_*) = y_*, \;\; H_1 y(t^*) = g_1,$$

$$J_1(u) = \frac{1}{2}\int_{t_*}^{t^*} (y^T M(t)y + u^T P(t)u)dt \to \min,$$

$$A_0(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t), \;\; B_0(t) = B_1(t) - A_2(t)A_4^{-1}(t)B_2(t);$$

$$\frac{dz}{ds} = A_4(t^*)z + B_2(t^*)u, \;\; z(0) = H_2 A_4^{-1}(t^*)(A_3(t^*)y^0(t^*) + B_2(t^*)u^0(t^*)) + g_2,$$

$$z(-\infty) = 0, \;\; J_2(u) = \frac{1}{2}\int_{-\infty}^{0} (u^T(s)P(t^*)u(s))ds \to \min,$$

where $u^0(t)$, $y^0(t)$, $t \in [t_*, t^*]$, are the optimal control and trajectory in the first basic problem. As in [27], asymptotically suboptimal feedbacks of zero and first order were also constructed for this problem.

5. Conclusion

The paper presents an overview of asymptotic methods for solving a wide range of optimization problems for singularly perturbed systems that were developed in Minsk by the Optimal Control research group. These methods utilize a unified methodology based on the idea of the finite-dimensional parametrization of the optimal control. The principal advantage of the proposed algorithms is a decomposition of the original singularly perturbed problems into two optimal control problems of smaller dimensions. This decomposition allows to solve problems with a large number of state variables efficiently.

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