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BROWNIAN MOTION APPROXIMATION BY NEURAL NETWORKS

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Dedicated to the memory of Professor Ram U. Verma

Abstract. The first author recently derived several approximation results by neural network operators. There, the activation functions are induced by the arctangent, algebraic, Gudermannian and generalized symmetrical sigmoid functions. The results we apply here are univariate on a compact interval, regular and fractional. The outcome is the quantitative approximation of Brownian motion over the two dimensional sphere. We derive several Jackson type inequalities estimating the degree of convergence of our neural network operators to a general expectation function of Brownian motion. We give a detailed list of approximation applications regarding the expectation of well known functions of Brownian motion. Smoothness of our functions is taken into account producing higher speeds of approximation.

Keywords. Brownian motion, Expectation; Quantitative approximation; Neural network operators.

2020 Mathematics Subject Classification. 26A33, 41A17, 41A25.

1. Introduction

The first author of this paper in [1, 2] first established neural network approximation to continuous functions with rated by very specifically defined neural network operators of Cardaliagnet-Euvrard and 'Squashing' types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators 'bell-shaped' and 'squashing' functions are assumed to be compact support. Also the first author inspired by [17], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [4, 5, 6, 7, 9], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [8, 10, 12].

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In [16], the first author continued similar studies for Banach space valued functions for activation functions deriving from the arctangent, algebraic, Gudermanian and generalized symmetrical sigmoid functions. The authors based and inspired by [22] perform here neural network quantitative approximations to Brownian motion over the two dimensional sphere.

They present a series of Jackson type inequalities estimating the error of approximation to a general expectation function of the Brownian motion and its derivative. They produce regular and fractional calculus results. They finish with a lot of important applications.

2. About Neural Network Operators

2.1. **About the arctangent activation function neural networks.** We consider the function $arctan(x) = \int_0^x \frac{dz}{1+z^2}$ for all $x \in \mathbb{R}$ and use the function

$$h(x) := \frac{2}{\pi} arctan\left(\frac{\pi}{2}x\right) = \frac{2}{\pi}\int_0^{\frac{\pi x}{2}} \frac{dz}{1+z^2}, x \in \mathbb{R}.$$

which is a sigmoid type function and it is strictly increasing. We have that $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$, and $h'(x) = \frac{4}{4+\pi^2x^2}$ for all $x \in \mathbb{R}$. We consider the activation function $\psi_1(x) = \frac{1}{4}(h(x+1) - h(x-1))$, $x \in \mathbb{R}$, and we notice that $\psi_1(x) = \psi_1(-x)$, i.e., it is an even function. Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi_1(x) > 0$ for all $x \in \mathbb{R}$. We see that $\psi_1(0) = \frac{1}{\pi} arctan\left(\frac{\pi}{2}\right) \cong 0.319$. Letting $x > 0$, we have that

$$\psi_1'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) = \frac{-4\pi^2 x}{(4+\pi^2(x+1)^2))(4+\pi^2(x-1)^2))} < 0.$$

Hence, $\psi_1'(x) < 0$ for $x > 0$, That is ψ_1 is strictly decreasing on $[0,+\infty)$, strictly increasing on $(-\infty,0]$, and $\psi_1'(0) = 0$. Observe that $\lim_{x\to+\infty} \psi_1(x) = \frac{1}{4}(h(+\infty) - h(+\infty)) = 0$ and $\lim_{x\to-\infty} \psi_1(x) = \frac{1}{4}(h(-\infty) - h(-\infty)) = 0$. That is the x-axis is the horizontal asymptote on ψ_1. Thus ψ_1 is a bell symmetric function with maximum $\psi_1(0) \cong 0.319$.

Theorem 2.1. ([11, p. 286]) *We have that* $\sum_{i=-\infty}^{+\infty} \psi_1(x-i) = 1$ *for every* $x \in \mathbb{R}$.

Theorem 2.2. ([11, p. 287]) *It holds* $\int_{-\infty}^{+\infty} \psi_1(x)dx = 1$.

Hence, $\psi_1(x)$ is a density function on $\mathbb{R}$.

Definition 2.3. Letting $f \in C([a,b])$, we call the first modulus of continuity of f at $\delta > 0$ the following $\omega_1(f,\delta) = \sup_{x,y\in[a,b]:|x-y|\le\delta} |f(x) - f(y)|$. We have that $\omega_1(f,\delta) \longrightarrow 0$ if and only if $\delta \longrightarrow 0$.

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

Definition 2.4. ([11]) Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na\rceil \le \lfloor nb\rfloor$. Define the real positive linear network operator

$${}_1A_n(f,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f(\frac{k}{n})\psi_1(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi_1(nx-k)}, x \in [a,b].$$

Clearly here ${}_1A_n(f,x) \in C([a,b])$. In [11], we studied the pointwise and uniform convergence of ${}_1A_n(f,x)$ to $f(x)$ with rates

Theorem 2.5. ([11]) *Let* $f \in C([a,b]), 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, x \in [a,b]$. *Then*

(1)

$$\left|{}_1A_n(f,x)-f(x)\right| \leq 4.9737\left[\omega_1\left(f,\frac{1}{n^\alpha}\right)+\frac{4\|f\|_\infty}{\pi^2\left(n^{1-\alpha}-2\right)}\right]=:\rho_1(f),$$

and

(2) $\|{}_1A_n(f)-f\|_\infty \leq \rho_1(f)$. *We notice that* $\lim_{n\to\infty} {}_1A_n(f)=f$, *pointwise and uniformly.*

Definition 2.6. Let $\nu \geq 0, n=\lceil \nu \rceil, f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo functional derivative (see [18, 49-52], [21, 23]) the function $D^\nu_{*\alpha}f(x)=\frac{1}{\Gamma(n-\nu)}\int_\alpha^x(x-t)^{n-\nu-1}f^{(n)}(t)dt$ for every $x\in[a,b]$, where Γ is the gamma function $\Gamma(\nu)=\int_0^\infty e^{-t}t^{\nu-1}dt, \nu>0$. Notice $D^\nu_{*\alpha}f \in L_1([a,b])$ and $D^\nu_{*\alpha}f(x)$ exists a.e. on $[a,b]$. We set $D^0_{*\alpha}f(x)=f(x)$ for every $x\in[a,b]$.

Definition 2.7. (see also [3, 20, 21]) Let $f\in AC^m([a,b]), m=\lceil\beta\rceil, \beta>0$. The right Caputo functional derivative of order $\beta>0$ is given by $D^\beta_{b-}f(x)=\frac{(-1)^m}{\Gamma(m-\beta)}\int_x^b(\zeta-x)^{m-\beta-1}f^{(m)}(\zeta)d\zeta$, for every $x\in[a,b]$. We set $D^0_{b-}f(x)=f(x)$. Notice that $D^\beta_{b-}f\in L_1([a,b])$ and D^β_{b-} exists a.e. on $[a,b]$.

We also mention the real valued fractional approximation result by neural networks.

Theorem 2.8. *([11]) Let* $0<\alpha<1, f\in C^1([a,b]), 0<\beta<1, x\in[a,b], n\in\mathbb{N}: n^{1-\beta}>2$. *Then*

$$\left|{}_1A_n(f,x)-f(x)\right| \leq \frac{4.9737}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D^\alpha_{x-}f,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D^\alpha_{*x}f,\frac{1}{n^\beta}\right)_{[x,b]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\frac{2}{\pi^2\left(n^{1-\beta}-2\right)}\left(\left\|D^\alpha_{x-}f\right\|_{\infty,[a,x]}(x-a)^\alpha+\left\|D^\alpha_{*x}f\right\|_{\infty,[x,b]}(b-x)^\alpha\right)\right\}.$$

As we see here that we obtain the real valued fractionally type pointwise convergence with rates of ${}_1A_n \longrightarrow I$ the unit opertor as $n\longrightarrow\infty$.

2.2. **About the algebraic activation function neural networks.** Here see also [13].

We consider the generator algebraic function $\varphi(x)=\frac{x}{\sqrt[2m]{1+x^{2m}}}$, $m\in\mathbb{N}$, $x\in\mathbb{R}$, which is a sigmoidal type of function and is a strictly increasing function. We see that $\varphi(-x)=-\varphi(x)$ with $\varphi(0)=0$. We obtain that

$$\varphi'(x)=\frac{1}{\left(1+x^{2m}\right)^{\frac{2m+1}{2m}}}>0, \text{ for every } x\in\mathbb{R},$$

proving φ as strictly increasing over $\mathbb{R}$, $\varphi'(-x)=\varphi'(x)$. We easily find that

$$\lim_{x\to+\infty}\varphi(x)=1, \varphi(+\infty)=1, \text{ and } \lim_{x\to-\infty}\varphi(x)=-1, \varphi(-\infty)=-1.$$

We consider the activation function $\psi_2(x)=\frac{1}{4}[\varphi(x+1)-\varphi(x-1)]$. Clearly, it is $\psi_2(x)=\psi_2(-x)$ for every $x\in\mathbb{R}$, so ψ_2 is an even function and symmetric with respect to the y-axis. Clearly $\psi_2(x)>0$, for every $x\in\mathbb{R}$. Also it is $\psi_2(0)=\frac{1}{2\sqrt[2m]{2}}$, $m\in\mathbb{N}$. By [13], we have that

$\psi_2'(x) < 0$, for $x > 0$. That is ψ_2 is strictly decreasing over $[0, +\infty)$. Clearly ψ_2 is strictly increasing over $(-\infty, 0]$ and $\psi_2'(0) = 0$. Furthermore, we obtain that

$$\lim_{x\to+\infty} \psi_1(x) = \frac{1}{4}(\varphi(+\infty) - \varphi(+\infty)) = 0,$$

and

$$\lim_{x\to-\infty} \psi_2(x) = \frac{1}{4}(\varphi(-\infty) - \varphi(-\infty)) = 0,$$

which is the x-axis is the horizontal asymptote on ψ_2 . Conclusion, ψ_2 is a bell shape symmetric function with maximum $\psi_2(0) = \frac{1}{2\sqrt[2m]{2}}, m \in \mathbb{N}$.

Theorem 2.9. *([13]) It holds $\sum_{i=-\infty}^{+\infty} \psi_2(x-i) = 1$ for every $x \in \mathbb{R}$.*

Theorem 2.10. *([13, p. 287]) It holds $\int_{-\infty}^{+\infty} \psi_2(x)dx = 1$.*

So ψ_2 is a density function.

Definition 2.11. ([13]) Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the real positive valued linear network operator

$$_2A_n(f,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f(\frac{k}{n})\psi_2(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi_2(nx-k)}, x \in [a,b].$$

Clearly here $_2A_n(f,x) \in C([a,b])$. We mention here about the pointwise and uniform convergence of $_2A_n(f,x)$ to $f(x)$ with rates.

Theorem 2.12. *([13]) Let $f \in C([a,b]), 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, x \in [a,b], m \in \mathbb{N}$. Then*

(1)

$$|_2A_n(f,x) - f(x)| \leq 2\left(\sqrt[2m]{1+4^m}\right)\left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{2m(n^{1-\alpha}-2)^{2m}}\right] =: \rho_2(f),$$

and

(2) $\|_2A_n(f) - f\|_\infty \leq \rho_2(f)$.

Hence, $\lim_{n\to+\infty} {}_2A_n(f) = f$, pointwise, and uniformly.

Theorem 2.13. *([13]) Let $0 < \alpha < 1, f \in C^1([a,b]), 0 < \beta < 1, m \in \mathbb{N}, x \in [a,b], n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

$$|_2A_n(f,x) - f(x)| \leq \frac{2\sqrt[2m]{1+4m}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\omega_1\left(D_{x^-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]}}{n^{\alpha\beta}} + \right.$$

$$\left. + \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\left\|D_{x^-}^\alpha f\right\|_{\infty,[a,x]}(x-a)^\alpha + \left\|D_{*x}^\alpha f\right\|_{\infty,[x,b]}(b-x)^\alpha\right)\right\}.$$

As we see here that we obtain the real valued functionally type pointwise convergence with rates of $_2A_n \to I$ the unit operator as $n \to \infty$.

2.3. **About the Gudermannian activation function neural networks.** We consider $gd(x)$ the Guardemannian function [14], which is a sigmoid function, as a generator function:

$$\sigma(x) = 2arctan\left(tanh\left(\frac{x}{2}\right)\right) = \int_0^x \frac{dt}{cosht} =: gd(x), x \in \mathbb{R}.$$

Let the normalized generator sigmoid function

$$f(x) := \frac{4}{\pi}\sigma(x) = \frac{4}{\pi}\int_0^x \frac{dt}{cosht} = \frac{8}{\pi}\int_0^x \frac{1}{e^t + e^{-t}}dt,\ x \in \mathbb{R}.$$

Here $f'(x) = \frac{4}{\pi cosh(x)} > 0$ for every $x \in \mathbb{R}$. Hence f is strictly increasing on $\mathbb{R}$. Notice that $tanh(-x) = -tanh(x)$ and $arctan(-x) = -arctan(x), x \in \mathbb{R}$. So, here the neural network activation function is $\psi_3(x) = \frac{1}{4}[f(x+1) - f(x-1)]$ for all $x \in \mathbb{R}$. By [14], we see that $\psi_3(-x) = \psi_3(x)$ for every $x \in \mathbb{R}$, i.e., it is even and symmetric with respect to the $y-$axis. Here, we have $f(+\infty) = 1, f(-\infty) = 1$ and $f(0) = 0$. Clearly, it is $f(-x) = -f(x)$ for every $x \in \mathbb{R}$, an odd function, symmetric with respect to the origin. Since $x+1 > x-1$ and $f(x+1) > f(x-1)$, we obtain $\psi_3(x) > 0$ for all $x \in \mathbb{R}$. By [14], we have that $\psi_1(0) = \frac{2}{\pi}gd(1) \cong 0.551$. By [14], ψ_3 is strictly decreasing on $[0, +\infty)$, strictly increasing on $(-\infty, 0]$, and $\psi_3'(0) = 0$. Also we have that $\lim\limits_{x\to+\infty}\psi_3(x) = \lim\limits_{x\to-\infty}\psi_3(x) = 0$, which is the $x-$axis is the horizontal asymptote on ψ_3. Conclusion, ψ_3 is a bell symmetric function with maximum $\psi_3(0) \cong 0.551$.

Theorem 2.14. *([14]) It holds that* $\sum_{i=-\infty}^{+\infty}\psi_3(x-i) = 1$ *for every* $x \in \mathbb{R}$

Theorem 2.15. *([14]) It holds that* $\int_{-\infty}^{+\infty}\psi_3(x)dx = 1$.

So $\psi_3(x)$ is a density function.

Definition 2.16. ([14]) Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \le \lfloor nb \rfloor$. Define the real positive valued linear network operator

$${}_3A_n(f,x) := \frac{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\psi_3(nx-k)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\psi_3(nx-k)}, x \in [a,b].$$

Clearly, here ${}_3A_n(f,x) \in C([a,b])$. We mention here about the pointwise and uniform convergence of ${}_3A_n(f,x)$ to $f(x)$ with rates.

Theorem 2.17. *([14]) Let* $f \in C([a,b]), 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, x \in [a,b]$. *Then*

(1)

$$|{}_3A_n(f,x) - f(x)| \le 2.412\left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{8\|f\|_\infty}{\pi e^{\left(n^{1-\alpha}-2\right)}}\right] =: \rho_3(f),$$

and

(2) $\|{}_3A_n(f) - f\|_\infty \le \rho_3(f)$.

We obtain that $\lim\limits_{n\to+\infty}{}_3A_n(f) = f$, *pointwise and uniformly.*

We also mention the following real valued functional approximation result by neural networks.

Theorem 2.18. *([14]) Let* $0<\alpha<1, f\in C^{1}([a,b]), 0<\beta<1, x\in[a,b], n\in\mathbb{N}: n^{1-\beta}>2$. *Then*

$$|{}_{3}A_{n}(f,x)-f(x)|\leq\frac{2.412}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\frac{1}{\pi^{2}\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}.$$

As we see here that we obtain the real valued functionally type pointwise convergence with rates of ${}_{3}A_{n}\to I$ the unit operator as $n\to\infty$.

2.4. **About the generalized symmetrical activation function neural networks.** Here we consider the generalized symmetrical sigmoid function ([15, 19]) $f_{1}(x)=\frac{x}{\left(1+|x|^{\mu}\right)^{\frac{1}{\mu}}}$, $\mu>0, x\in\mathbb{R}$. This has applications in immunology and protection from disease together with probability theory. It is also called a symmetrical protection curve. The parameter μ is a shape parameter controling how fast the curve approaches the asymptotes for a given slope at the inflection point. When $\mu=1$ f_{1} is the absolute sigmoid function, and when $\mu=2$ f_{1} is the square root sigmoid function. When $\mu=1.5$ the function approximates the arctangent function, when $\mu=2.9$ it approximates the logistic function, and when $\mu=3.4$ it approximates the error function. Parameter μ is estimated in the likelihood maximization ([19]) For more details, see [19]. Next, we study the particular generator sigmoid function

$$f_{2}(x)=\frac{x}{\left(1+|x|^{\lambda}\right)^{\frac{1}{\lambda}}}, \lambda \text{ is an odd number, } x\in\mathbb{R}$$

We have that $f_{2}(0)=0$, and $f_{2}(-x)=-f_{2}(x)$. So f_{2} is symmetric with respect to zero. When $x\geq 0$, we obtain that ([15])

$$f_{2}'(x)=\frac{1}{\left(1+x^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}>0,$$

that is f_{2} is strictly increasing on $[0,+\infty)$, strictly increasing on $(-\infty,0]$. Hence f_{2} is strictly increasing on $\mathbb{R}$. We have that $f_{2}(+\infty)=f_{2}(-\infty)=1$. Let us consider the activation function ([15]):

$$\psi_{4}(x)=\frac{1}{4}\left[f_{2}(x+1)-f_{2}(x-1)\right]=$$

$$\psi_{4}(x)=\frac{1}{4}\left[\frac{x+1}{\left(1+|x+1|^{\lambda}\right)^{\frac{1}{\lambda}}}-\frac{x-1}{\left(1+|x-1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right].$$

Clearly, it holds ([15]) $\psi_{4}(-x)=\psi_{4}(x)$ for every $x\in\mathbb{R}$, $\psi_{4}(0)=\frac{1}{2\sqrt[\lambda]{2}}$, and $\psi_{4}(x)>0$, for every $x\in\mathbb{R}$. Following [15], we have that ψ_{4} is strictly decreasing on $[0,+\infty)$, strictly increasing on $(-\infty,0]$, by $\psi_{4}-$symmetry with respect to $y-$axis, and $\psi_{4}'(0)=0$. Clearly, it is $\lim\limits_{x\to+\infty}\psi_{4}(x)=$

$\lim_{x\to-\infty}\psi_4(x)=0$. Therefore the $x-$axis is the horizontal asymptote on $\psi_4(x)$. The value $\psi_4(0)=\frac{1}{2\sqrt[\lambda]{2}}$, where λ is an odd number, is the maximum of ψ_4, which is a bell shaped function.

Theorem 2.19. *([15]) It holds that $\sum_{i=-\infty}^{+\infty}\psi_4(x-i)=1$ for every $x\in\mathbb{R}$.*

Theorem 2.20. *([15]) We have that $\int_{-\infty}^{+\infty}\psi_4(x)dx=1$.*

So $\psi_4(x)$ is a density function on $\mathbb{R}$.

Definition 2.21. ([15]) Let $f\in C([a,b])$ and $n\in\mathbb{N}:\lceil na\rceil\le\lfloor nb\rfloor$. Define the real positive valued linear network operator

$$ {}_4A_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\psi_4(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_4(nx-k)},x\in[a,b]. $$

Clearly here ${}_4A_n(f,x)\in C([a,b])$. We mention here about the pointwise and uniform convergence of ${}_4A_n(f,x)$ to $f(x)$ with rates.

Theorem 2.22. *([15]) Let $f\in C([a,b]),0<\alpha<1,n\in\mathbb{N}:n^{1-\alpha}>2,x\in[a,b],\lambda\in\mathbb{N}$ is odd. Then*

(1)

$$ |{}_4A_n(f,x)-f(x)|\le 2\sqrt[\lambda]{1+2^\lambda}\left[\omega_1\left(f,\frac{1}{n^\alpha}\right)+\frac{\|f\|_\infty}{\lambda\left(n^{1-\alpha}-2\right)^\lambda}\right]=:\rho_4(f), $$

and

(2) $\|{}_4A_n(f)-f\|_\infty\le\rho_4(f)$. Hence, $\lim_{n\to\infty}{}_4A_n(f)=f$, pointwise and uniformly.

Next, we mention the corresponding real valued fractional approximation result by neural networks.

Theorem 2.23. *([15]) Let $0<\alpha<1,f\in C^1([a,b]),0<\beta<1,\lambda$ is odd $x\in[a,b],n\in\mathbb{N}:n^{1-\beta}>2$. Then*

$$ |{}_4A_n(f,x)-f(x)|\le\frac{2\sqrt[\lambda]{1+2^\lambda}}{\Gamma(\alpha+1)} $$

$$ \left\{\frac{\omega_1\left(D_{x-}^\alpha f,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D_{*x}^\alpha f,\frac{1}{n^\beta}\right)_{[x,b]}}{n^{\alpha\beta}}+\right. $$

$$ \left.+\frac{1}{2\lambda\left(n^{1-\beta}-2\right)^\lambda}\left(\left\|D_{x-}^\alpha f\right\|_{\infty,[a,x]}(x-a)^\alpha+\|D_{*x}^\alpha f\|_{\infty,[x,b]}(b-x)^\alpha\right)\right\}. $$

As we see here that we obtain real valued fractionally type pointwise convergence with rates of ${}_4A_n\to I$ the unit operator, as $n\to\infty$. We give the following unified definition.

Definition 2.24. Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. Define the following positive linear network operators ($j = 1,2,3,4$)

$$ {}_jA_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi_j(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_j(nx-k)}, x \in [a,b]. $$

Set

$$ c_j := \begin{cases} 4.9737, & j = 1 \\ 2\sqrt[2m]{1+4^m}, & j = 2, m \in \mathbb{N} \\ 2.412, & j = 3 \\ 2\sqrt[\lambda]{1+2^\lambda}, & j = 4, \lambda \in \mathbb{N} \text{ is odd} \end{cases}. $$

Furthermore, let $n, m \in \mathbb{N}, 0 < \alpha < 1$ and $\lambda \in \mathbb{N}$ is odd. Define,

$$ \gamma_j := \begin{cases} \frac{4}{\pi^2 \left(n^{1-\alpha}-2\right)}, & j = 1 \\ \frac{1}{2m\left(n^{1-\alpha}-2\right)^{2m}}, & j = 2 \\ \frac{8}{\pi e^{\left(n^{1-\alpha}-2\right)}}, & j = 3 \\ \frac{1}{\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}, & j = 4 \end{cases}. $$

We present the unified basic result.

Theorem 2.25. *Let $f \in C([a,b]), 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, x \in [a,b], j = 1,2,3,4$. Then*

(1)

$$ \left| {}_jA_n(f,x) - f(x) \right| \leq c_j \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \gamma_j(n) \|f\|_\infty \right] =: \rho_j(f), $$

and

(2)

$$ \left\| {}_jA_n(f) - f \right\|_\infty \leq \rho_j(f). $$

Hence $\lim_{n\to\infty} {}_jA_n(f) = f$, pointwise and uniformly.

Proof. From Theorems 2.5, 2.12, 2.17, and 2.22, we have the desired conclusion immediately. □

Remark 2.26. Let $m, n \in \mathbb{N}, 0 < \beta < 1$ and $\lambda \in \mathbb{N}$ is odd. Set

$$ \varepsilon_j(n) := \begin{cases} \frac{2}{\pi^2\left(n^{1-\beta}-2\right)}, & j = 1 \\ \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}, & j = 2 \\ \frac{4}{\pi^2\left(n^{1-\beta}-2\right)}, & j = 3 \\ \frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}, & j = 4. \end{cases}. $$

Next, we present the following unified result.

Theorem 2.27. *Let* $0<\alpha<1, f\in C^1([a,b]), 0<\beta<1, x\in[a,b], n\in\mathbb{N}: n^{1-\beta}>2,\ j=1,2,3,4$. *Then*

$$\left|{}_jA_n(f,x)-f(x)\right|\leq\frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\varepsilon_j(n)\left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\|D^{\alpha}_{*x}f\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}.$$

where c_j *as in (2.4). We have* ${}_jA_n(f,x)\to f(x)$, *as* $n\to\infty$.

Proof. From Theorems 2.8, 2.13, 2.18, and 2.23, we obtain the desired conclusion immediately. ∎

3. About Brownian Motion on 2−Dimensional Sphere

3.1. **Describing the Brownian motion on** S^2**.** ([22]) The Brownian motion on S^n is a diffusion (Markov) process $W_t, t\geq 0$, on S^n whose transition density is a function $P(t,x,y)$ on $(0,\infty)\times S^n\times S^n$ satisfying $\frac{\partial P}{\partial t}=\frac{1}{2}\Delta_n P$, and $P(t,x,y)\to\delta_x(y)$ as $t\to 0^+$, where Δ_n is the Laplace-Beltrami operator of S^n acting on the x-variables and $\delta_x(y)$ is the delta mass at x, i.e. $P(t,x,y)$ is the **heat kernel** of S^n. The heat kernel exists, it is unique, positive, and smooth in (t,x,y).

Remark 3.1. The heat kernel $P(t,x,y)$ satisfies the following properties

1. Symmetry: $P(t,x,y)=P(t,y,x)$.
2. The semigroup identity, for any $s\in(0,t)$, $P(t,x,y)=\int_{S^n}P(s,x,z)P(t-s,z,y)d\mu(z)$, where $d\mu$ is the area measure element of S^n.
3. For all $t>0$ and $x\in S^n$ $\int_{S^n}P(t,x,y)d\mu(y)=1$.
4. As $t\to\infty$, $P(t,x,y)$ approaches the uniform density on S^n, i.e. $\lim_{t\to\infty}P(t,x,y)=\frac{1}{A_n}$, where A_n is the area of the S^n with radius a. It is also well known that

$$A_n=\frac{2\pi^{\frac{n+1}{2}}a^n}{(\frac{n-1}{2})!},\text{ for } n \text{ odd}$$

$$A_n=\frac{2^n(\frac{n}{2}-1)!\pi^{\frac{n}{2}}a^n}{(n-1)!},\text{ for } n \text{ even.}$$

Finally, the symmetry of S^n implies that $P(t,x,y)$ depends only on t and $d(x,y)$, the distance between x and y. Thus in spherical coordinates it depends on t and the angle φ between x and y. Hence, $P(t,x,y)=p(t,\varphi)$, where $p(t,\varphi)$ satisfies

$$\frac{\partial p}{\partial t}=\frac{1}{2}\Delta_n p=\frac{1}{2a^2}\left[(n-1)\cot\varphi\cdot\frac{\partial p}{\partial\varphi}+\frac{\partial^2 p}{\partial\varphi^2}\right]$$

and $\lim_{t\to 0^+}aA_{n-1}p(t,\varphi)\cdot\sin^{n-1}\varphi=\delta(\varphi)$, where $\delta(\cdot)$ is the standard Dirac delta function on $\mathbb{R}$.

3.2. **Explicit form of the heat kernel of S^2.** Let $W_t,\ t \geq 0$ be the Brownian motion on a 2-dimensional sphere S^2 of radius a. The transition density function $p(t,\varphi)$ of X_t is the unique solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2a^2 \sin\varphi}\left(\frac{\partial^2 p(t,\varphi)}{\partial \varphi^2}\sin\varphi + \frac{\partial p}{\partial \varphi}\cos\varphi\right)$$

and $\lim_{t\to 0^+} 2\pi a^2 \sin\varphi \cdot p(t,\varphi) = \delta(\varphi)$. The solution to the diffusion equation

$$\frac{\partial K(t,\varphi)}{\partial t} = \frac{1}{\sin\varphi}\left(\cos\varphi \frac{\partial K(t,\varphi)}{\partial \varphi} + \sin\varphi \frac{\partial^2 K(t,\varphi)}{\partial \varphi^2}\right)$$

with initial condition $\lim_{t\to 0^+} 2\pi \sin(\varphi) K(t,\varphi) = \delta(\varphi)$ is given by the function

$$K(t,\varphi) = \frac{1}{4\pi}\sum_{n\in\mathbb{N}} (2n+1)\exp\left(-n(n+1)\sqrt{2t}\right) P_n^0(\cos\varphi).$$

Here $P_n^0,\ \ n = 0,1,2,\ldots$ is the associated Legendre polynomials of order zero, i.e.,

$$P_n^0(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

This fact implies the following result.

Proposition 3.2. *([22]) The transition density function of the Brownian motion W_t, $t \geq 0$ on S^2 with radius a it is given by the function*

$$p(t,\varphi) = \frac{1}{4\pi a^2}\sum_{n\in\mathbb{N}} (2n+1)\exp\left(-\frac{n(n+1)\sqrt{t}}{a}\right) P_n^0(\cos\varphi).$$

Theorem 3.3. *Consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $[0,\pi]$, i.e. there exists $M > 0$ such that $|g(\phi)| \leq M$, for every $\phi \in [0,\pi]$, and Lebesgue measurable on $\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E(|g(W)|)(t) = \int_0^\pi |g(\phi)|\, p(t,\phi)d\phi$ is continuous in t, and $E(|g(W)|)(t) \leq \pi M p(t_o,\phi_0)$, where $p(t_0,\phi_0) = \max_{(t,\phi)\in[t_1,t_2]\times[0,\pi]} p(t,\phi)$ with $0 < t_1 < t_2$, where $p(t,\phi)$ is the transition density function of the Brownian motion W_t, $t \geq 0$ on S^2 given by (3.2).*

Proof. It is known that the transition density function of the Brownian motion $W_t, t \geq 0$ on $S^2, p(t,\phi)$ is continuous in $(t,\phi) \in [t_1,t_2]\times[0,\pi], t_1 > 0$. By the extreme value theorem, there exists $(t_0,\phi_0) \in [t_1,t_2]\times[0,\pi]$ such that $p(t_0,\phi_0) = \max_{(t,\phi)\in[t_1,t_2]\times[0,\pi]} p(t,\phi)$. So we have $0 \leq p(t,\phi) \leq p(t_0,\phi_0)$ for every $(t,\phi) \in [t_1,t_2]\times[0,\pi]$. Let $N \in \mathbb{N}, t_N, t \in [t_1,t_2] : t_N \to t$, as $N \to \infty$. Then, $p(t_N,\phi) \to p(t,\phi)$ for every $\phi \in [0,\pi]$. The function $g : \mathbb{R} \to \mathbb{R}$, is bounded on $[0,\pi]$, i.e. there is a $M > 0$ such that $|g(\phi)| \leq M$, for every $\phi \in [0,\pi]$, and Lebesgue measurable on $\mathbb{R}$. Furthermore, we have that $|g(\phi)|\, p(t_N,\phi) \to |g(\phi)|\, p(t,\phi)$ as $N \to \infty$ and $|g(\phi)|\, p(t_N,\phi) \leq |g(\phi)|\, p(t_o,\phi_0)$ for all $\phi \in [0,\pi]$ and $N \in \mathbb{N}$. So, by dominated convergence theorem, we obtain that $E(|g(W)|)(t_N) \to E(|g(W)|)(t)$ as $N \to \infty$. Thus $E(|g(W)|)(t)$ is proved to be continuous in t. Moreover, $|g(\phi)|\, p(t,\phi) \leq M p(t_o,\phi_0)$ for all $t \in [t_1,t_2]$ and $\phi \in [0,\pi]$. Thus, $E(|g(W)|)(t) = \int_0^\pi |g(\phi)|\, p(t,\phi)d\phi \leq \pi M p(t_o,\phi_0)$. ∎

Proposition 3.4. *Consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $[0,\pi]$ and Lebesgue measurable on $\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E(|g(W)|)(t) = \int_0^\pi |g(\phi)|\, p(t,\phi)d\phi$ is differentiable in t, and $\frac{\partial E(|g(W)|)}{\partial t} = \int_0^\pi |g(\phi)| \frac{\partial(p(t,\phi))}{\partial t} d\phi$, which is continuous in t.*

Proof. As it is mentioned above, the transition density function of the Brownian motion $W_t, t \geq 0$ on S^2, $p(t,\phi)$ is continuous in $(t,\phi) \in [t_1,t_2] \times [0,\pi], t_1 > 0$. We have

$$E(|g(W)|) = \int_0^\pi |g(\phi)| \, p(t,\phi) d\phi, \text{ for every } t \in [t_1,t_2].$$

We apply differentiation under the integral sign. We notice

$$|g(\phi)| \frac{\partial p(t,\phi)}{\partial t} \leq M \left\| \frac{\partial p(t,\phi)}{\partial t} \right\|_{\infty,[t_1,t_2]\times[0,\pi]}.$$

Therefore, there exists

$$\frac{\partial E(|g(W)|)}{\partial t} = \int_0^\pi |g(\phi)| \frac{\partial (p(t,\phi))}{\partial t} d\phi,$$

which is continuous in t (same proof as in Theorem 3.3). $\square$

4. Main Results

We present the following general approximation results of Brownian Motion by neural network operators.

Theorem 4.1. *Let* $0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, t \in [t_1,t_2]$, *where* $t_1 > 0, j = 1,2,3,4$. *Then,*

(1)

$$\begin{aligned} &\left| {}_jA_n(E(|g(W)|))(t) - (E(|g(W)|))(t) \right| \\ &\leq c_j \left[\omega_1 \left(E(|g(W)|), \frac{1}{n^\alpha} \right) + \gamma_j(n) \|E(|g(W)|)\|_{\infty,[t_1,t_2]} \right] \\ &=: \rho_j(E(|g(W)|)), \end{aligned}$$

and

(2) $\left\| {}_jA_n(E(|g(W)|)) - E(|g(W)|) \right\|_{\infty,[t_1,t_2]} \leq \rho_j(E(|g(W)|))$. *Then* $\lim_{n\to\infty} {}_jA_n(E(|g(W)|)) = E(|g(W)|)$, *pointwise and uniformly.*

Proof. By Theorem 2.25, we have the desired conclusion immediately. $\square$

Theorem 4.2. *Let* $0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, t \in [t_1,t_2]$, *where* $t_1 > 0, j = 1,2,3,4$. *Then,*

(1)

$$\left| {}_jA_n\left(\frac{\partial E(|g(W)|)}{\partial t} \right)(t) - \left(\frac{\partial E(|g(W)|)}{\partial t} \right)(t) \right| \leq$$

$$c_j \left[\omega_1 \left(\left(\frac{\partial E(|g(W)|)}{\partial t} \right), \frac{1}{n^\alpha} \right) + \gamma_j(n) \left\| \frac{\partial E(|g(W)|)}{\partial t} \right\|_{\infty,[t_1,t_2]} \right] =: \rho_j \left(\frac{\partial E(|g(W)|)}{\partial t} \right),$$

and

(2)

$$\left\| {}_jA_n\left(\frac{\partial E(|g(W)|)}{\partial t}\right) - \frac{\partial E(|g(W)|)}{\partial t} \right\|_{\infty,[t_1,t_2]} \leq \rho_j\left(\frac{\partial E(|g(W)|)}{\partial t}\right).$$

Then $\lim_{n\to\infty} {}_jA_n\left(\frac{\partial E(|g(W)|)}{\partial t}\right) = \frac{\partial E(|g(W)|)}{\partial t}$, *pointwise and uniformly.*

Proof. By Theorem 2.25, we have the desired conclusion immediately. $\square$

Next, we give the following fractional calculus related result.

Theorem 4.3. *Let* $0<\alpha,\beta<1, t\in[t_1,t_2], t_1>0, n\in\mathbb{N}: n^{1-\beta}>2, j=1,2,3,4$. *Then*

$$\left|{}_jA_n\left(E\left(|g(W)|\right)(t)\right)-E\left(|g(W)|\right)(t)\right|\leq\frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D_{t-}^{\alpha}E\left(|g(W)|\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E\left(|g(W)|\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\varepsilon_j(n)\left(\left\|D_{t-}^{\alpha}E\left(|g(W)|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E\left(|g(W)|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

Then ${}_jA_n(E\left(|g(W)|\right))(t)\to E\left(|g(W)|\right)(t)$, *as* $n\to\infty$.

Proof. By Theorem 2.27, we have the desired conclusion immediately. □

5. Applications

For a function $g:\mathbb{R}\to\mathbb{R}$, which is bounded on $[0,\pi]$ and Lebesgue measurable on $\mathbb{R}$ and $W(t,\phi)$ the Brownian motion on S^2, we use the following notations $E\left(|g(W)|\right):=E\left(|g(W)|\right)^{(0)}$ and $\frac{\partial E(|g(W)|)}{\partial t}:=E\left(|g(W)|\right)^{(1)}$. We can apply our main results to function $g(W)=W$. Consider the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=x$ for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(|W|\right)(t)=\int_0^{\pi}\phi p(t,\phi)d\phi$ is continuous in t. Moreover,

Corollary 5.1. *Let* $0<\alpha<1, n\in\mathbb{N}: n^{1-\alpha}>2, t\in[t_1,t_2]$, *where* $t_1>0, i=0,1$ *and* $j=1,2,3,4$. *Then*

(1)

$$\left|{}_jA_n\left(E\left(|W|\right)^{(i)}\right)(t)-\left(E\left(|W|\right)^{(i)}\right)(t)\right|\leq$$

$$c_j\left[\omega_1\left(E\left(|W|\right)^{(i)},\frac{1}{n^{\alpha}}\right)+\gamma_j(n)\left\|E\left(|W|\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_j(E\left(|W|\right)^{(i)}),$$

and

(2)

$$\left\|{}_jA_n\left(E\left(|W|\right)^{(i)}\right)-E\left(|W|\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\leq\rho_j(E\left(|W|\right)^{(i)}).$$

Then $\lim_{n\to\infty}{}_jA_n(E\left(|W|\right)^{(i)})=E\left(|W|\right)^{(i)}$, *pointwise and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.2. *Let* $0<\alpha,\beta<1, t\in[t_1,t_2], t_1>0, n\in\mathbb{N}: n^{1-\beta}>2, j=1,2,3,4$. *Then*

$$\left|{}_jA_n\left(E\left(|W|\right)(t)\right)-E\left(|W|\right)(t)\right|\leq\frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D_{t-}^{\alpha}E\left(|W|\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E\left(|W|\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\varepsilon_j(n)\left(\left\|D_{t-}^{\alpha}E\left(|W|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E\left(|W|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

Hence, ${}_jA_n(E\left(|W|\right))(t)\to E\left(|W|\right)(t)$, *as* $n\to\infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. □

For the next corollaries, we consider the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=cosx$ for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(|cosW|\right)(t)=\int_0^{\pi}|\cos\phi|\,p(t,\phi)d\phi$ is continuous in t.

Corollary 5.3. *Let $0<\alpha<1, n\in\mathbb{N}: n^{1-\alpha}>2, t\in[t_1,t_2]$, where $t_1>0, i=0,1$ and $j=1,2,3,4$. Then*

(1)

$$\left|{}_jA_n\left(E\left(|\cos W|\right)^{(i)}\right)(t)-\left(E\left(|\cos W|\right)^{(i)}\right)(t)\right|\leq$$
$$c_j\left[\omega_1\left(E\left(|\cos W|\right)^{(i)},\frac{1}{n^{\alpha}}\right)+\gamma_j(n)\left\|E\left(|cosW|\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_j(E\left(|\cos W|\right)^{(i)}),$$

and

(2)

$$\left\|{}_jA_n\left(E\left(|\cos W|\right)^{(i)}\right)-E\left(|\cos W|\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\leq\rho_j(E\left(|\cos W|\right)^{(i)}).$$

Then $\lim\limits_{n\to\infty}{}_jA_n(E\left(|\cos W|\right)^{(i)})=E\left(|\cos W|\right)^{(i)}$*, pointwise and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.4. *Let $0<\alpha,\beta<1, t\in[t_1,t_2], t_1>0, n\in\mathbb{N}: n^{1-\beta}>2, j=1,2,3,4$. Then*

$$\left|{}_jA_n\left(E\left(|\cos W|\right)(t)\right)-E\left(|\cos W|\right)(t)\right|\leq\frac{c_j}{\Gamma(\alpha+1)}$$
$$\left\{\frac{\omega_1\left(D^{\alpha}_{t-}E\left(|\cos W|\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D^{\alpha}_{*t}E\left(|\cos W|\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}}+\right.$$
$$\left.+\varepsilon_j(n)\left(\left\|D^{\alpha}_{t-}E\left(|\cos W|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D^{\alpha}_{*t}E\left(|\cos W|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

Then ${}_jA_n(E\left(|\cos W|\right))(t)\to E\left(|\cos W|\right)(t)$*, as* $n\to\infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. □

Let the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=\sin x$ for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(|\sin W|\right)(t)=\int_0^{\pi}\sin(\phi)p(t,\phi)d\phi$ is continuous in t.

Corollary 5.5. *Let $0<\alpha<1, n\in\mathbb{N}: n^{1-\alpha}>2, t\in[t_1,t_2]$, where $t_1>0, i=0,1$ and $j=1,2,3,4$. Then*

(1)

$$\left|{}_jA_n\left(E\left(|\sin W|\right)^{(i)}\right)(t)-\left(E\left(|\sin W|\right)^{(i)}\right)(t)\right|\leq$$
$$c_j\left[\omega_1\left(E\left(|\sin W|\right)^{(i)},\frac{1}{n^{\alpha}}\right)+\gamma_j(n)\left\|E\left(|\sin W|\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_j(E\left(|\sin W|\right)^{(i)}),$$

and

(2)

$$\left\| {}_jA_n\left(E\left(|\sin W|\right)^{(i)}\right) - E\left(|\sin W|\right)^{(i)}\right\|_{\infty,[t_1,t_2]} \leq \rho_j(E\left(|\sin W|\right)^{(i)}).$$

Then $\lim_{n\to\infty} {}_jA_n(E\left(|\sin W|\right)^{(i)}) = E\left(|\sin W|\right)^{(i)}$, *pointwise and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.6. *Let* $0 < \alpha, \beta < 1, t \in [t_1,t_2], t_1 > 0, n \in \mathbb{N} : n^{1-\beta} > 2, j = 1,2,3,4$. *Then*

$$\left| {}_jA_n\left(E\left(|\sin W|\right)(t)\right) - E\left(|\sin W|\right)(t)\right| \leq \frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\omega_1\left(D^{\alpha}_{t-}E\left(|\sin W|\right), \frac{1}{n^{\beta}}\right)_{[t_1,t]} + \omega_1\left(D^{\alpha}_{*t}E\left(|\sin W|\right), \frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}} + \right.$$

$$\left. +\varepsilon_j(n)\left(\left\|D^{\alpha}_{t-}E\left(|\sin W|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha} + \left\|D^{\alpha}_{*t}E\left(|\sin W|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

Then ${}_jA_n(E\left(|\sin W|\right))(t) \to E\left(|\sin W|\right)(t)$, *as* $n \to \infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. □

Let the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = \tanh x$ for every $x \in \mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(|\tanh W|\right)(t) = \int_0^{\pi} |\tanh(\phi)|\, p(t,\phi)d\phi$ is continuous in t.

Corollary 5.7. *Let* $0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, t \in [t_1,t_2]$, *where* $t_1 > 0, i = 0,1$ *and* $j = 1,2,3,4$. *Then*

(1)

$$\left| {}_jA_n\left(E\left(|\tanh W|\right)^{(i)}\right)(t) - \left(E\left(|\tanh W|\right)^{(i)}\right)(t)\right| \leq$$

$$c_j\left[\omega_1\left(E\left(|\tanh W|\right)^{(i)}, \frac{1}{n^{\alpha}}\right) + \gamma_j(n)\left\|E\left(|\tanh W|\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\right] =: \rho_j(E\left(|\tanh W|\right)^{(i)}),$$

and

(2)

$$\left\| {}_jA_n\left(E\left(|\tanh W|\right)^{(i)}\right) - E\left(|\tanh W|\right)^{(i)}\right\|_{\infty,[t_1,t_2]} \leq \rho_j(E\left(|\tanh W|\right)^{(i)}).$$

Then $\lim_{n\to\infty} {}_jA_n(E\left(|\tanh W|\right)^{(i)}) = E\left(|\tanh W|\right)^{(i)}$, *pointwise and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.8. *Let* $0 < \alpha, \beta < 1, t \in [t_1,t_2], t_1 > 0, n \in \mathbb{N} : n^{1-\beta} > 2, j = 1,2,3,4$. *Then*

$$\left| {}_jA_n\left(E\left(|\tanh W|\right)(t)\right) - E\left(|\tanh W|\right)(t)\right| \leq \frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\omega_1\left(D^{\alpha}_{t-}E\left(|\tanh W|\right), \frac{1}{n^{\beta}}\right)_{[t_1,t]} + \omega_1\left(D^{\alpha}_{*t}E\left(|\tanh W|\right), \frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}} + \right.$$

$$+\varepsilon_j(n)\left(\left\|D_{t-}^{\alpha}E\left(|\tanh W|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E\left(|\tanh W|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\Bigg\}.$$

Then ${}_jA_n(E(|\tanh W|))(t)\to E(|\tanh W|)(t)$, *as* $n\to\infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. □

In the following let as consider the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=e^{-\ell x},\ell>0$ for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(e^{-\ell x}\right)(t)=\int_0^{\pi}e^{-\ell\phi}p(t,\phi)d\phi$ is continuous in t.

Corollary 5.9. *Let* $0<\alpha<1,n\in\mathbb{N}:n^{1-\alpha}>2,\ell>0$ *and* $t\in[t_1,t_2]$, *where* $t_1>0,i=0,1$ *and* $j=1,2,3,4$. *Then*

(1)

$$\left|{}_jA_n\left(E\left(e^{-\ell w}\right)^{(i)}\right)(t)-\left(E\left(e^{-\ell w}\right)^{(i)}\right)(t)\right|\le$$

$$c_j\left[\omega_1\left(E\left(e^{-\ell w}\right)^{(i)},\frac{1}{n^{\alpha}}\right)+\gamma_j(n)\left\|E\left(e^{-\ell w}\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_j(E\left(e^{-\ell w}\right)^{(i)}),$$

and

(2)

$$\left\|{}_jA_n\left(E\left(e^{-\ell w}\right)^{(i)}\right)-E\left(e^{-\ell w}\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\le\rho_j(E\left(e^{-\ell w}\right)^{(i)}).$$

We have that $\lim_{n\to\infty}{}_jA_n(E\left(e^{-\ell w}\right)^{(i)})=E\left(e^{-\ell w}\right)^{(i)}$, *pointwise and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.10. *Let* $0<\alpha,\beta<1,\ell>0,\ t\in[t_1,t_2],t_1>0,n\in\mathbb{N}:n^{1-\beta}>2,j=1,2,3,4$. *Then*

$$\left|{}_jA_n\left(E\left(e^{-\ell w}\right)(t)\right)-E\left(e^{-\ell w}\right)(t)\right|\le\frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D_{t-}^{\alpha}E\left(e^{-\ell w}\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E\left(e^{-\ell w}\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\varepsilon_j(n)\left(\left\|D_{t-}^{\alpha}E\left(e^{-\ell w}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E\left(e^{-\ell w}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

We have ${}_jA_n(E\left(e^{-\ell w}\right))(t)\to E\left(e^{-\ell w}\right)(t)$, *as* $n\to\infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. □

In the following we consider the logistic sigmoid function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=\frac{1}{1+e^{-x}}$ for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(\frac{1}{1+e^{-W}}\right)(t)=\int_0^{\pi}\frac{1}{1+e^{-\phi}}p(t,\phi)d\phi$ is continuous in t.

Corollary 5.11. *Let* $0<\alpha<1,n\in\mathbb{N}:n^{1-\alpha}>2$ *and* $t\in[t_1,t_2]$, *where* $t_1>0,i=0,1$ *and* $j=1,2,3,4$. *Then*

(1)

$$\left| {}_jA_n\left(E\left(\frac{1}{1+e^{-W}}\right)^{(i)}\right)(t)-\left(E\left(\frac{1}{1+e^{-W}}\right)^{(i)}\right)(t)\right|\leq$$

$$c_j\left[\omega_1\left(E\left(\frac{1}{1+e^{-W}}\right)^{(i)},\frac{1}{n^{\alpha}}\right)+\gamma_j(n)\left\|E\left(\frac{1}{1+e^{-W}}\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_j(E\left(\frac{1}{1+e^{-W}}\right)^{(i)}),$$

and

(2)

$$\left\| {}_jA_n\left(E\left(\frac{1}{1+e^{-W}}\right)^{(i)}\right)-E\left(\frac{1}{1+e^{-W}}\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\leq\rho_j(E\left(\frac{1}{1+e^{-W}}\right)^{(i)}).$$

We have that $\lim_{n\to\infty}\ {}_jA_n(E\left(\frac{1}{1+e^{-W}}\right)^{(i)})=E\left(\frac{1}{1+e^{-W}}\right)^{(i)}$*, pointwise and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.12. *Let* $0<\alpha,\beta<1,,\ t\in[t_1,t_2],t_1>0,n\in\mathbb{N}:n^{1-\beta}>2,j=1,2,3,4.$ *Then*

$$\left| {}_jA_n\left(E\left(\frac{1}{1+e^{-W}}\right)(t)\right)-E\left(\frac{1}{1+e^{-W}}\right)(t)\right|\leq\frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D_{t-}^{\alpha}E\left(\frac{1}{1+e^{-W}}\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E\left(\frac{1}{1+e^{-W}}\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\varepsilon_j(n)\left(\left\|D_{t-}^{\alpha}E\left(\frac{1}{1+e^{-W}}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E\left(\frac{1}{1+e^{-W}}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

We have ${}_jA_n\left(E\left(\frac{1}{1+e^{-W}}\right)\right)(t)\to E\left(\frac{1}{1+e^{-W}}\right)(t)$*, as* $n\to\infty$*.*

Proof. By Theorem 4.3, we have the desired conclusion immediately. □

Let now as consider the generalised logistic sigmoid function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=(1+e^{-x})^{-\delta}$, where $\delta>0$, for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(\left(1+e^{-W}\right)^{-\delta}\right)(t)=\int_0^{\pi}\left(1+e^{-\phi}\right)^{-\delta}p(t,\phi)d\phi$ is continuous in t.

Corollary 5.13. *Let* $0<\alpha<1,\delta>0,n\in\mathbb{N}:n^{1-\alpha}>2$ *and* $t\in[t_1,t_2]$*, where* $t_1>0,i=0,1$ *and* $j=1,2,3,4$*. Then*

(1)

$$\left| {}_jA_n\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}\right)(t)-\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}\right)(t)\right|\leq$$

$$c_j\left[\omega_1\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)},\frac{1}{n^{\alpha}}\right)+\gamma_j(n)\left\|E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\right]$$

$$=:\rho_j\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}\right),$$

and

(2)

$$\left\| {}_jA_n\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}\right) - E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}\right\|_{\infty,[t_1,t_2]} \leq \rho_j(E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}).$$

We have that $\lim_{n\to\infty} {}_jA_n(E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}) = E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(i)}$*, pointwise and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.14. *Let* $0<\alpha,\beta<1, \delta>0$, $t\in[t_1,t_2], t_1>0, n\in\mathbb{N}: n^{1-\beta}>2, j=1,2,3,4$. *Then*

$$\left| {}_jA_n\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)(t)\right) - E\left(\left(1+e^{-W}\right)^{-\delta}\right)(t)\right| \leq \frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\omega_1\left(D^{\alpha}_{t-}E\left(\left(1+e^{-W}\right)^{-\delta}\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]} + \omega_1\left(D^{\alpha}_{*t}E\left(\left(1+e^{-W}\right)^{-\delta}\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}} + \right.$$

$$\left. +\varepsilon_j(n)\left(\left\|D^{\alpha}_{t-}E\left(\left(1+e^{-W}\right)^{-\delta}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha} + \left\|D^{\alpha}_{*t}E\left(\left(1+e^{-W}\right)^{-\delta}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

We have ${}_jA_n\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)\right)(t) \to E\left(\left(1+e^{-W}\right)^{-\delta}\right)(t)$*, as* $n\to\infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. □

The Gompertz function $g:\mathbb{R}\to\mathbb{R}$, with $g(x)=e^{\mu e^{-x}}, \mu<0$ is a sigmoid function which describes growth as being slowest at the start and end of a given time period. Let $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(e^{\mu e^{-W}}\right)(t)=\int_0^{\pi}e^{\mu e^{-\phi}}p(t,\phi)d\phi$ is continuous in t.

Corollary 5.15. *Let* $0<\alpha<1, \mu<0, n\in\mathbb{N}: n^{1-\alpha}>2$ *and* $t\in[t_1,t_2]$*, where* $t_1>0, i=0,1$ *and* $j=1,2,3,4$. *Then*

(1)

$$\left| {}_jA_n\left(E\left(e^{\mu e^{-W}}\right)^{(i)}\right)(t) - \left(E\left(e^{\mu e^{-W}}\right)^{(i)}\right)(t)\right| \leq$$

$$c_j\left[\omega_1\left(E\left(\left(e^{\mu e^{-W}}\right)^{(i)},\right)\frac{1}{n^{\alpha}}\right) + \gamma_j(n)\left\|E\left(\left(e^{\mu e^{-W}}\right)^{(i)}\right)\right\|_{\infty,[t_1,t_2]}\right] =: \rho_j\left(E\left(e^{\mu e^{-W}}\right)^{(i)}\right),$$

and

(2)

$$\left\| {}_jA_n\left(E\left((e^{\mu e^{-W}}\right)^{(i)}\right) - E\left(\left(e^{\mu e^{-W}}\right)^{(i)}\right)\right\|_{\infty,[t_1,t_2]} \leq \rho_j(E\left(e^{\mu e^{-W}}\right)^{(i)}).$$

We have that $\lim_{n\to\infty} {}_jA_n(E\left(e^{\mu e^{-W}}\right)^{(i)}) = E\left(e^{\mu e^{-W}}\right)^{(i)}$*, pointwise, and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.16. *Let* $0<\alpha,\beta<1, \mu<0,\ t\in[t_1,t_2], t_1>0, n\in\mathbb{N}: n^{1-\beta}>2, j=1,2,3,4$. *Then*

$$\left|{}_jA_n\left(E\left(e^{\mu e^{-W}}\right)(t)\right)-E\left(e^{\mu e^{-W}}\right)(t)\right|\leq\frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D_{t-}^{\alpha}E\left(e^{\mu e^{-W}}\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E\left(e^{\mu e^{-W}}\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\varepsilon_j(n)\left(\left\|D_{t-}^{\alpha}E\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

We have ${}_jA_n\left(E\left(e^{\mu e^{-W}}\right)\right)(t)\to E\left(e^{\mu e^{-W}}\right)(t)$, *as* $n\to\infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. □

Another special case of The Gompertz functions is $g:\mathbb{R}\to\mathbb{R}$, with $g(x)=e^{-e^{\kappa x}}, \kappa<0$. Let $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(e^{-e^{\kappa W}}\right)(t)=\int_0^{\pi}e^{-e^{-\kappa\phi}}p(t,\phi)d\phi$ is continuous in t.

Corollary 5.17. *Let* $0<\alpha<1, \kappa<0, n\in\mathbb{N}: n^{1-\alpha}>2$ *and* $t\in[t_1,t_2]$, *where* $t_1>0, i=0,1$ *and* $j=1,2,3,4$. *Then*

(1)

$$\left|{}_jA_n\left(E\left(e^{-e^{\kappa W}}\right)^{(i)}\right)(t)-\left(E\left(e^{-e^{\kappa W}}\right)^{(i)}\right)(t)\right|\leq$$

$$c_j\left[\omega_1\left(E\left(\left(e^{-e^{\kappa W}}\right)^{(i)},\right)\frac{1}{n^{\alpha}}\right)+\gamma_j(n)\left\|E\left(\left(e^{-e^{\kappa W}}\right)^{(i)}\right)\right\|_{\infty,[t_1,t_2]}\right]=:\rho_j\left(E\left(e^{-e^{\kappa W}}\right)^{(i)}\right)$$

and

(2)

$$\left\|{}_jA_n\left(E\left(e^{-e^{\kappa W}}\right)^{(i)}\right)-E\left(e^{-e^{\kappa W}}\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\leq\rho_j\left(E\left(e^{-e^{\kappa W}}\right)^{(i)}\right).$$

We have that $\lim_{n\to\infty}{}_jA_n(E\left(e^{-e^{\kappa W}}\right)^{(i)})=E\left(e^{-e^{\kappa W}}\right)^{(i)}$, *pointwise, and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. □

Next, we give the following fractional calculus related result.

Corollary 5.18. *Let* $0<\alpha,\beta<1, \kappa<0,\ t\in[t_1,t_2], t_1>0, n\in\mathbb{N}: n^{1-\beta}>2, j=1,2,3,4$. *Then*

$$\left|{}_jA_n\left(E\left(e^{-e^{\kappa W}}\right)(t)\right)-E\left(e^{-e^{\kappa W}}\right)(t)\right|\leq\frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D_{t-}^{\alpha}E\left(e^{-e^{\kappa W}}\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E\left(e^{-e^{\kappa W}}\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\varepsilon_j(n)\left(\left\|D_{t-}^{\alpha}E\left(e^{-e^{\kappa W}}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E\left(e^{-e^{\kappa W}}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}.$$

We have ${}_jA_n\left(E\left(e^{-e^{\kappa W}}\right)\right)(t)\to E\left(e^{-e^{\kappa W}}\right)(t)$, *as* $n\to\infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. ☐

Finally, we consider the function $g:\mathbb{R}\to\mathbb{R}$, with $g(x)=P_m^0(\cos x)$, where $P_m^0(x)$ is the Legendre Polynomial of degree m. Let $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation $E\left(\left|P_m^0(\cos W)\right|\right)(t)=\int_0^\pi\left|P_m^0(\cos\phi)\right|p(t,\phi)d\phi$ is continuous in t.

Corollary 5.19. *Let* $0<\alpha<1, n\in\mathbb{N}:n^{1-\alpha}>2$ *and* $t\in[t_1,t_2]$, *where* $t_1>0, i=0,1$ *and* $j=1,2,3,4$. *Then*

(1)

$$\left|{}_jA_n\left(E\left(\left|P_m^0(\cos W)\right|\right)^{(i)}\right)(t)-\left(E\left(\left|P_m^0(\cos W)\right|\right)^{(i)}\right)(t)\right|\le$$

$$c_j\left[\omega_1\left(E\left(\left(\left|P_m^0(\cos W)\right|\right)^{(i)},\right)\frac{1}{n^\alpha}\right)+\gamma_j(n)\left\|E\left(\left(\left|P_m^0(\cos W)\right|\right)^{(i)}\right)\right\|_{\infty,[t_1,t_2]}\right]$$

$$=:\rho_j\left(E\left(\left|P_m^0(\cos W)\right|\right)^{(i)}\right),$$

and

(2)

$$\left\|{}_jA_n\left(E\left(\left|P_m^0(\cos W)\right|\right)^{(i)}\right)-E\left(\left|P_m^0(\cos W)\right|\right)^{(i)}\right\|_{\infty,[t_1,t_2]}\le\rho_j\left(E\left(\left|P_m^0(\cos W)\right|\right)^{(i)}\right).$$

We have that $\lim_{n\to\infty}{}_jA_n(E\left(\left|P_n^0(\cos W)\right|\right)^{(i)})=E\left(\left|P_n^0(\cos W)\right|\right)^{(i)}$, *pointwise, and uniformly.*

Proof. By Theorems 4.1 and 4.2, we have the desired conclusion immediately. ☐

Next, we give the following fractional calculus related result.

Corollary 5.20. *Let* $0<\alpha,\beta<1$, $t\in[t_1,t_2], t_1>0, n\in\mathbb{N}:n^{1-\beta}>2, j=1,2,3,4$. *Then*

$$\left|{}_jA_n\left(E\left(\left|P_m^0(\cos W)\right|\right)(t)\right)-E\left(\left|P_m^0(\cos W)\right|\right)(t)\right|\le\frac{c_j}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\omega_1\left(D_{t-}^\alpha E\left(\left|P_m^0(\cos W)\right|\right),\frac{1}{n^\beta}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^\alpha E\left(\left|P_m^0(\cos W)\right|\right),\frac{1}{n^\beta}\right)_{[t,t_2]}}{n^{\alpha\beta}}+\right.$$

$$\left.+\varepsilon_j(n)\left(\left\|D_{t-}^\alpha E\left(\left|P_m^0(\cos W)\right|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^\alpha+\left\|D_{*t}^\alpha E\left(\left|P_m^0(\cos W)\right|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^\alpha\right)\right\}.$$

We have ${}_jA_n\left(E\left(\left|P_m^0(\cos W)\right|\right)\right)(t)\to E\left(\left|P_m^0(\cos W)\right|\right)(t)$, *as* $n\to\infty$.

Proof. By Theorem 4.3, we have the desired conclusion immediately. ☐

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